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"O'RAIFEARTAIGH SYMPOSIUM

ON

NON-PERTURBATIVE AND SYMMETRY METHODS IN FIELD THEORY"

Lattice Field Theory with the Sign Problem

and

the Maximum Entropy Method

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in collaboration with

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based on

M. Imachi , Y. Shinno and H.Y., Prog.Theor.Phys.,111, 387-411,2004

M. Imachi , Y. Shinno and H.Y., hep-lat/0506032

M. Imachi , Y. Shinno and H.Y., Prog.Theor.Phys.,115, 931-949,2006

1 Introduction

- Lattice Field Theory (LFT)
  - as a powerful tool to study non-perturbative dynamics of Quantum Field Theory
  - Monte Carlo (MC) study combined with Renormalization Group idea to reach the continuum limit of LFT
- Notorious sign problem is a big obstacle to study
  - finite density QCD
  - LFT with the θ term
- Study of LFT with a θ term:
  - strong CP problem
    Schierholz's work (1994) :
    flattening of $f(\theta)$ at $\theta \geq \theta_c (< \pi)$ for finite volume $V \Rightarrow$ a signal of a first order phase transition, and θ_c moves to 0 as V increases (#).
  - Possible rich phase structure
    Cardy and Rabinovici's work in Z(N) theory: oblique confinement phase
- However, (#) turns out to be spurious, coming form the sign problem
  - A standard way to circumvent it is to use the Fourier transform method (FTM):

$$\mathcal{Z}(\theta) = \sum_Q e^{i\theta Q} P(Q), \tag{1}$$

$\mathcal{Z}(\theta)$: the partition function

$P(Q)$: topological charge distribution calculated with the real Boltzmann weight

- It turns out in the FTM that flattening is due to the error in $P(Q)$ and then exhibits spurious phase transition.

- Study of the sign problem
  - Factorization method (Anagnostopulos and Nishimura)
  - Imaginary θ (Azcoiti et.al.)
  - Our approach: using the maximum entropy method (MEM)

    $\Rightarrow$ main subject of this talk

2 Sign Problem and flattening of the free energy

- Partition function is given by the complex action

$$\mathcal{Z}(\theta) = \frac{\int [d\bar{z}dz] e^{-S+i\theta\hat{Q}(\bar{z},z)}}{\int [d\bar{z}dz] e^{-S}}.$$

impossible to generate configurations according to the complex Boltzmann weight
(complex action problem or sign problem)

- $\mathcal{Z}(\theta)$ is obtained by Fourier-transforming $P(Q)$.

$$\mathcal{Z}(\theta) = \frac{\int [d\bar{z}dz] e^{-S+i\theta\hat{Q}(\bar{z},z)}}{\int [d\bar{z}dz] e^{-S}} \equiv \sum_Q e^{i\theta Q} P(Q), \qquad (2)$$

where

$$P(Q) \equiv \frac{\int [d\bar{z}dz]_Q e^{-S}}{\int d\bar{z}dz e^{-S}}. \qquad (3)$$

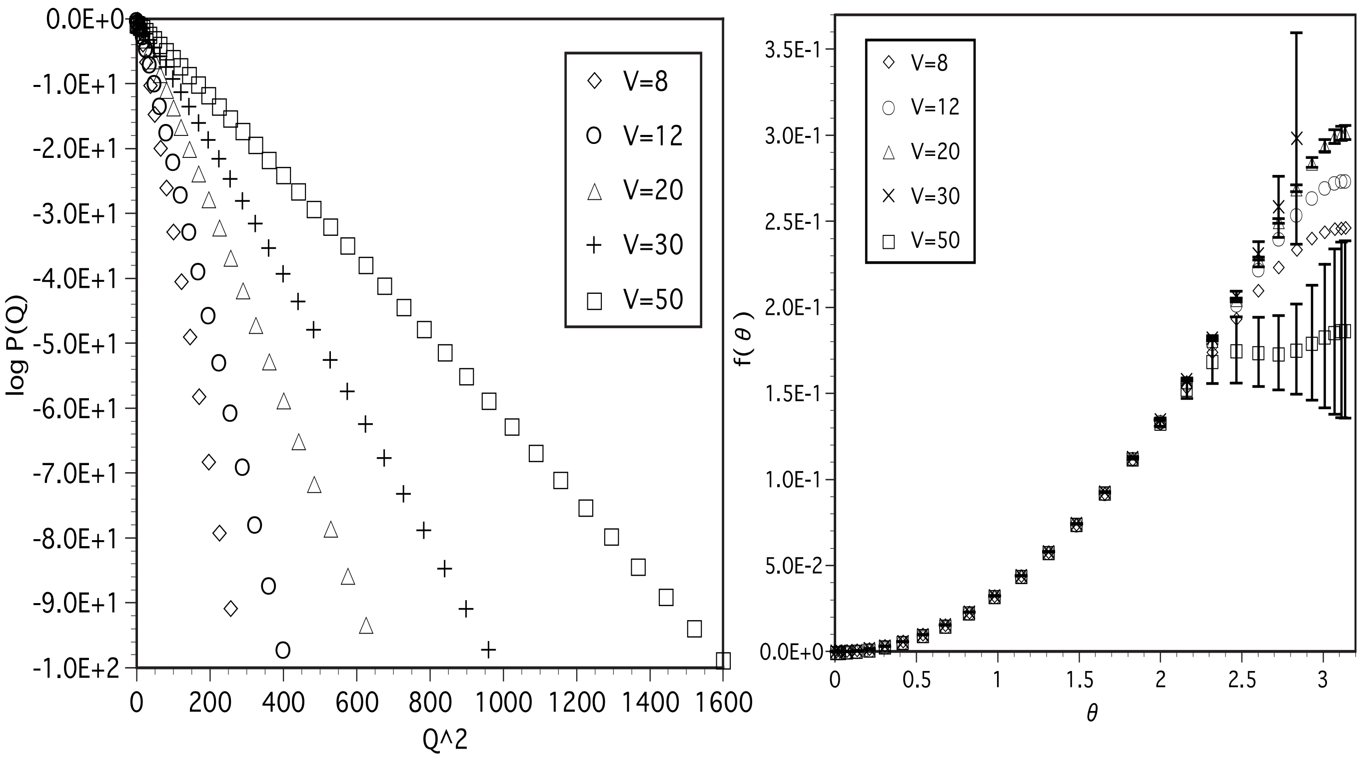


Figure 1: **Gaussian topological charge distribution and corresponding $f(\theta)$ for various lattice volumes. The parameter δ is chosen to be 1/400.**

– $P(Q)$ is calculated by counting the number of the configurations and constructing a histogram.

– Numerical Fourier transform of $P(Q) \Rightarrow$ flattening of $f(\theta)$ at large θ in $V = 50$ case in Fig.1

- Reason for flattening

  – obtained $P(Q)$ consists of two parts, a true value $\tilde{P}(Q)$ and an error of $P(Q)$.

$$P(Q) = \tilde{P}(Q) + \delta P(Q), \tag{4}$$

  – Suppose that the error at $Q = 0$ dominates. Then the free energy density looks like

$$\begin{aligned} f(\theta) &= -\frac{1}{V}\log \mathcal{Z}(\theta) \approx -\frac{1}{V}\log\{\sum_Q \tilde{P}(Q)e^{i\theta Q} + \delta P(0)\} \\ &= -\frac{1}{V}\log\{e^{-V\tilde{f}(\theta)} + \delta P(0)\}, \end{aligned} \tag{5}$$

  where $\tilde{f}(\theta)$ is the true free energy density and $\delta P(0)$ is error at $Q = 0$. If

$$e^{-V\tilde{f}(\theta)} \approx \delta P(0) \qquad \text{at} \quad \theta = \theta_f \quad (< \pi), \tag{6}$$

  then

$$f(\theta) = \begin{cases} \tilde{f}(\theta) & \theta \leq \theta_f \\ -\frac{1}{V}\log\{\delta P(0)\} = \text{const.} & \theta_f < \theta \quad (< \pi) \end{cases} \tag{7}$$

- comments

  – $\delta P(Q) < 0$ could leads to negative values of $\mathcal{Z}(\theta)$ $\Rightarrow$ also referred to as flattening (e.g., $V = 30$ case in Fig.1)

  – To overcome such flattening, need statistics proportional to e^V.

Summary of this part:

- **sign problem appears as flattening of $f(\theta)$**
- **could lead to a spurious first order phase transition at $\theta \approx \theta_{\rm f}$**

3 Maximum Entropy Method (MEM)

- **What is the MEM?**
  - **based on the Bayesian statistics**
  - **a method for the parameter inference in data analysis**
  - **suited to ill-posed problems (# of data ≪ # of parameters)**
  - **a wide variety of applicability**

Motivation for using the MEM

- **Can a true signal be extracted from data, contaminated by errors?**
- **Have a look at the sign problem from a different point of view**

- **Bayes theorem**

$$\mathrm{prob}(A|B) = \mathrm{prob}(A)\frac{\mathrm{prob}(B|A)}{\mathrm{prob}(B)}, \qquad (8)$$

  - $\mathrm{prob}(A)$: **the probability that an event A occurs**
  - $\mathrm{prob}(A|B)$: **the conditional probability that A occurs under the condition that B occurs.**

- **Bayesian approach for parameter (ξ) inference for data x**

$$\text{prob}(\xi|x) \propto \text{prob}(\xi)\text{prob}(x|\xi)$$

  - **Posterior probability** $\text{prob}(\xi|x)$ **that ξ realizes for a given data x**
  - **Prior probability** $\text{prob}(\xi)$ **that reflects the knowledge of ξ before x is given**
  - **likelihood** $\text{prob}(x|\xi)$

- **What we do is to apply the above idea to determine the behavior of $\mathcal{Z}(\theta)$ for given data $P(Q)$ in**

$$P(Q) = \int_{-\pi}^{\pi} d\theta \frac{e^{-i\theta Q}}{2\pi} \mathcal{Z}(\theta). \tag{9}$$

$$\text{prob}(\mathcal{Z}(\theta)|P(Q), I) = \text{prob}(\mathcal{Z}(\theta)|I) \frac{\text{prob}(P(Q)|\mathcal{Z}(\theta), I)}{\text{prob}(P(Q)|I)}, \tag{10}$$

  - **Likelihood function**

$$\text{prob}(P(Q)|\mathcal{Z}(\theta), I) = \frac{e^{-\frac{1}{2}\chi^2}}{X_L}, \tag{11}$$

$$\chi^2 \equiv \sum_{Q,Q'} (P^{(\mathcal{Z})}(Q) - \bar{P}(Q)) C^{-1}_{Q,Q'} (P^{(\mathcal{Z})}(Q') - \bar{P}(Q')) \tag{12}$$

$P^{(\mathcal{Z})}(Q)$: **constructed from $\mathcal{Z}(\theta)$ through**

$$P^{(\mathcal{Z})}(Q) = \int_{-\pi}^{\pi} d\theta \frac{e^{-i\theta Q}}{2\pi} \mathcal{Z}(\theta). \tag{13}$$

$$\bar{P}(Q) = \frac{1}{N_d} \sum_{l=1}^{N_d} P^{(l)}(Q), \tag{14}$$

N_d: **the number of sets of data.**

C^{-1} **: inverse covariance matrix obtained by the data set** $\{P(Q)\}$**.**

– **Prior probability**

$$\mathrm{prob}(\mathcal{Z}(\theta)|I,\alpha) = \frac{e^{\alpha S}}{X_S(\alpha)}, \qquad (15)$$

S: **Shannon-Jaynes entropy**

$$S = \int_{-\pi}^{\pi} d\theta \left[\mathcal{Z}(\theta) - m(\theta) - \mathcal{Z}(\theta)\log\frac{\mathcal{Z}(\theta)}{m(\theta)} \right], \qquad (16)$$

$m(\theta)$: **"default model"**
reflecting the prior knowledge about $\mathcal{Z}(\theta)$
α: **a real positive parameter**

Posterior probability

$$\mathrm{prob}(\mathcal{Z}(\theta)|P(Q),I,\alpha,m) = \frac{\exp\left(-\frac{1}{2}\chi^2 + \alpha S\right)}{X_L X_S(\alpha)}. \qquad (17)$$

two limitting cases:

$\alpha \to 0$: **usual maximum likelihood method (MLM), or** $\min\{\chi^2\}$ **gives** $\mathcal{Z}(\theta)$
$\alpha \to \infty$: $\mathcal{Z}(\theta) = m(\theta)$

– **ill posed problem** $(\alpha = 0)$

# of data (discrete variable Q**)** $\ll$ **# of parameters (continuous varable** θ**)**
MLM yields degenerate solutions

$\Rightarrow$ **MEM** $(\alpha \neq 0)$ **gives an unique solution**

– **For the information** I**, we impose the criterion**

$$\mathcal{Z}(\theta) > 0 \qquad (18)$$

so that $\mathrm{prob}(\mathcal{Z}(\theta) \leq 0|I,m) = 0$**.**

MEM

maximizes the probability $\mathrm{prob}(\mathcal{Z}(\theta)|P(Q),I)$ **for obtaining the best image of** $\mathcal{Z}(\theta) \equiv \hat{\mathcal{Z}}(\theta)$

Procedure for obtaining the best image $\hat{\mathcal{Z}}(\theta)$

(1) Maximizing W **for a fixed** α **to obtain** $\mathcal{Z}_n^{(\alpha)}$

(2) Averaging $\mathcal{Z}_n^{(\alpha)}$

(3) Error estimation

(1) Maximizing W **for a fixed** α **to obtain** $\mathcal{Z}_n^{(\alpha)}$**:**

Find the image that maximizes W

$$W \equiv -\frac{1}{2}\chi^2 + \alpha S \tag{19}$$

in functional space of $\mathcal{Z}_n$ **for a given** α**:**

$$\frac{\delta}{\delta \mathcal{Z}(\theta)}(-\frac{1}{2}\chi^2 + \alpha S)\Big|_{\mathcal{Z}=\mathcal{Z}^{(\alpha)}} = 0. \tag{20}$$

(2) Averaging $\mathcal{Z}_n^{(\alpha)}$**:**

The α**-independent final image** $\hat{\mathcal{Z}}_n$ **can be calculated according to the probability.**

$$\hat{\mathcal{Z}}_n = \int d\alpha \ \mathrm{prob}(\alpha|P(Q),I,m)\mathcal{Z}_n^{(\alpha)}, \tag{21}$$

$$\mathrm{prob}(\alpha|P(Q),I,m) = \exp\left\{\Lambda(\alpha) + W(\mathcal{Z}^{(\alpha)})\right\}, \tag{22}$$

$$\Lambda(\alpha) \equiv \frac{1}{2}\sum_k \log\frac{\alpha}{\alpha+\lambda_k},$$

where λ_k are eigenvalues of the matrix in θ space;

$$\frac{1}{2}\sqrt{\mathcal{Z}_m}\frac{\partial^2\chi^2}{\partial\mathcal{Z}_m\partial\mathcal{Z}_n}\sqrt{\mathcal{Z}_n}\,\Big|_{\mathcal{Z}=\mathcal{Z}^{(\alpha)}}. \tag{23}$$

(3) Error estimation:

Error of the final output image $\hat{\mathcal{Z}}_n$ is calculated based on the uncertainty of the image:

$$\langle(\delta\hat{\mathcal{Z}}_n)^2\rangle \equiv \int d\alpha \langle(\delta\mathcal{Z}_n^{(\alpha)})^2\rangle \mathrm{prob}(\alpha|P(Q), I, m), \tag{24}$$

where

$$\begin{aligned}\langle(\delta\mathcal{Z}_m^{(\alpha)})^2\rangle &\equiv \frac{\int[d\mathcal{Z}]\int_\Theta d\theta_n d\theta_{n'}\delta\mathcal{Z}_n\delta\mathcal{Z}_{n'}\mathrm{prob}(\mathcal{Z}_m|P(Q), I, m, \alpha)}{\int[d\mathcal{Z}]\int_\Theta d\theta_n d\theta_{n'}\mathrm{prob}(\mathcal{Z}_m|P(Q), I, m, \alpha)} \\ &\simeq -\frac{1}{\int_\Theta d\theta_n d\theta_{n'}}\int_\Theta d\theta_n d\theta_{n'}\left(\frac{\partial^2 W}{\partial\mathcal{Z}_n\partial\mathcal{Z}_{n'}}\Big|_{\mathcal{Z}=\mathcal{Z}^{(\alpha)}}\right)^{-1}\end{aligned} \tag{25}$$

4 Results

MEM analysis

1. MEM analysis using mock data
   (a) Gaussian $P(Q)$
   (b) 'True' flattening
2. MEM analysis using MC data (CP^3 model)

4.1 preparation

- Mock data: Gaussian case
  - $P(Q) = A\exp\left(-c\, Q^2/V\right)$

- Add to $P(Q)$ Gaussian noise generated with the variance $\delta \times P(Q)$ for each value of Q.
- δ is varied (from 1/10 to 1/600), and the results are stable for $\delta > 1/300$ for $V = 50$.

- Covariance matrix:

$$C_{Q,Q'} = \frac{1}{N_d(N_d-1)} \sum_{l=1}^{N_d} (P^{(l)}(Q) - \bar{P}(Q))(P^{(l)}(Q') - \bar{P}(Q')), \tag{26}$$

  - Each set of data consists of $P(Q)$ from $Q = 0$ to $Q = N_q - 1$.
  - N_d: the number of sets of the data
  - $P^{(l)}(Q)$: the l-th set of data for $P(Q)$
  - $\bar{P}(Q)$: the average
  - N_d is varied, and the outcome is stable for $30 \lesssim N_d$.

- Techinical details
  - Use of Singular Value Decomposition (SVD) and the Newton method
  - Inverting $C_{Q,Q'}$ requires the quadruple precision.

Various Default models $m(\theta)$

(i) constant default model:
$m(\theta) = 0.1, 0.3, 1.0$.

(ii) strong coupling limit of the CP^{N-1} model:
$m(\theta) = (\sin(\theta/2)/(\theta/2))^V$

(iii) Gaussian default :
$m(\theta) = \exp(-\frac{\ln 10}{\pi^2}\gamma\theta^2)$
(The parameter γ is varied in the analysis.)

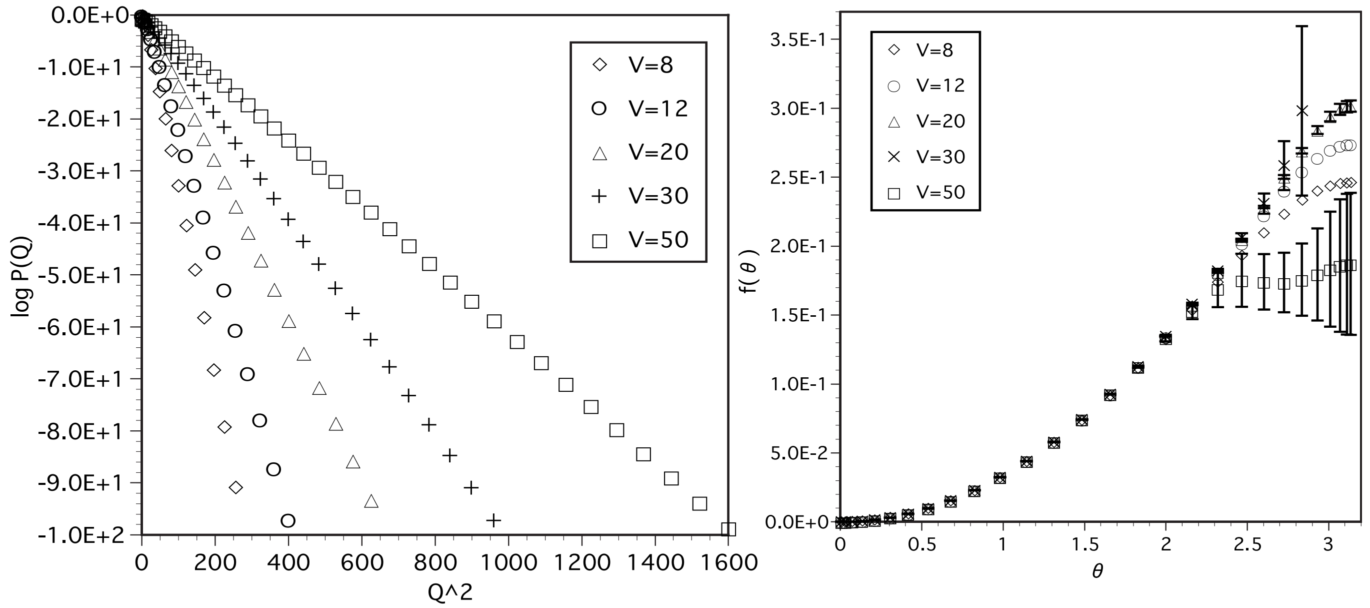


Figure 2: **Gaussian topological charge distribution and corresponding $f(\theta)$ obtained by using the Fourier method for various lattice volumes. The parameter δ is chosen to be $1/400$.**

4.2 Analysis of the models

4.2.1 Gaussian $P(Q)$ as a mock data

In order to check the MEM results, we choose

$$P(Q) = \exp[-\frac{c}{V}Q^2], \tag{27}$$

because $\mathcal{Z}(\theta)$ is analytically known

$$\mathcal{Z}_{\rm pois}(\theta) = \sqrt{\frac{\pi V}{c}} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{V}{4c}(\theta - 2\pi n)^2\right]. \tag{28}$$

- **Two types of data (c is fixed to 7.42):**
  1. **no flattening case ($V = 8, \mathbf{12}, 20$)**
  2. **flattening case ($V = 30, \mathbf{50}$)**

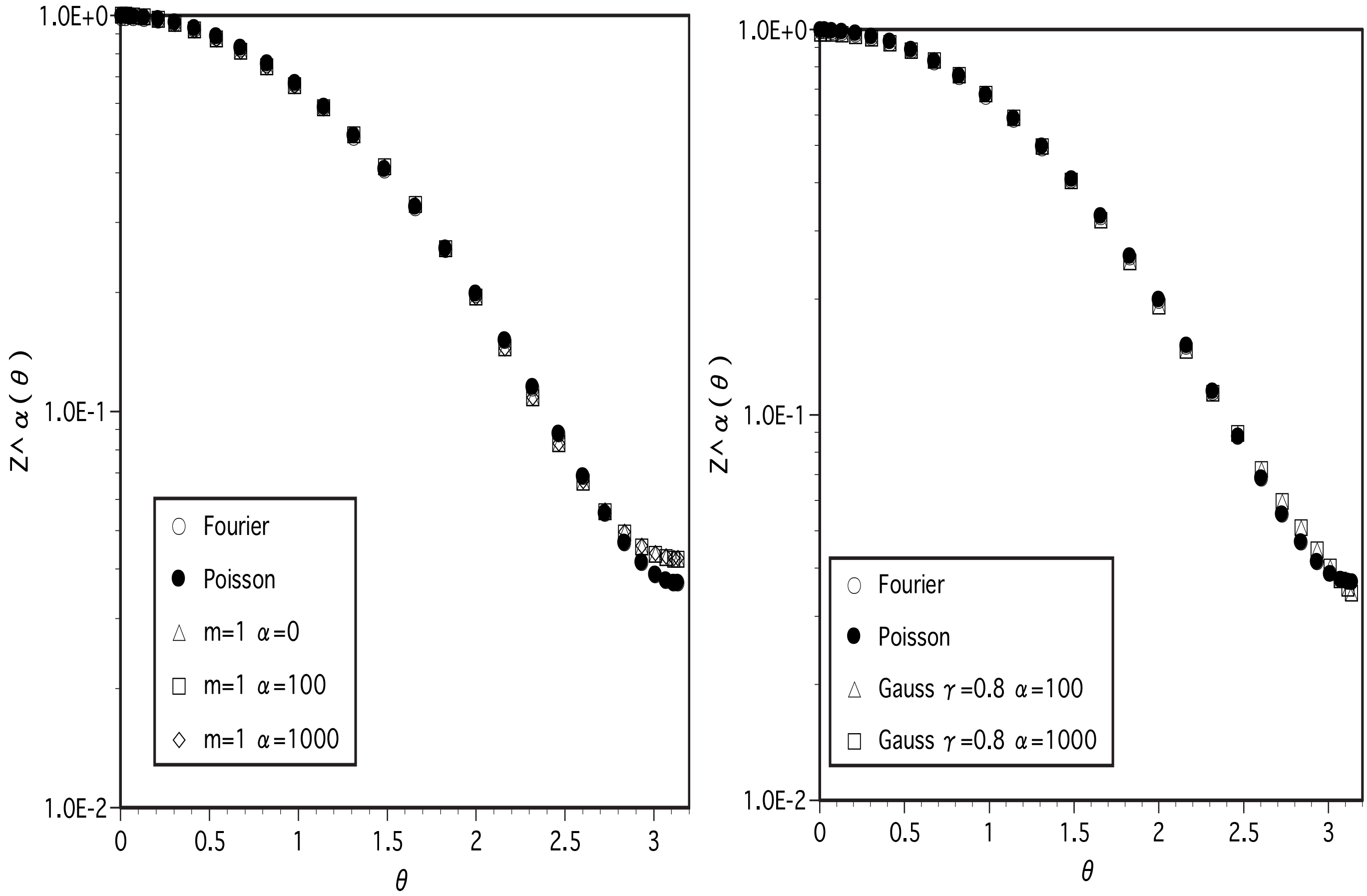


Figure 3: $\mathcal{Z}^{(\alpha)}(\theta)$ **for the data without flattening. Here,** $V = 12$**. The default model** $m(\theta)$ **is chosen to be the constant 1.0 and the Gaussian function with** $\gamma = 0.8$**.**

Remember the three step-procedure

(i) Calculate $\mathcal{Z}_n^{(\alpha)}$ **by maximizing** W **for a fixed** α**.**

(ii) Calculate $\hat{\mathcal{Z}}_n$ **by averaging** $\mathcal{Z}_n^{(\alpha)}$ **with the weight** $\mathrm{prob}(\alpha|P(Q), I, m) \equiv \mathrm{prob}(\alpha)$ **according to**

$$\hat{\mathcal{Z}}_n = \int d\alpha \ \mathrm{prob}(\alpha|P(Q), I, m)\mathcal{Z}_n^{(\alpha)}. \tag{29}$$

(iii) Error estimation

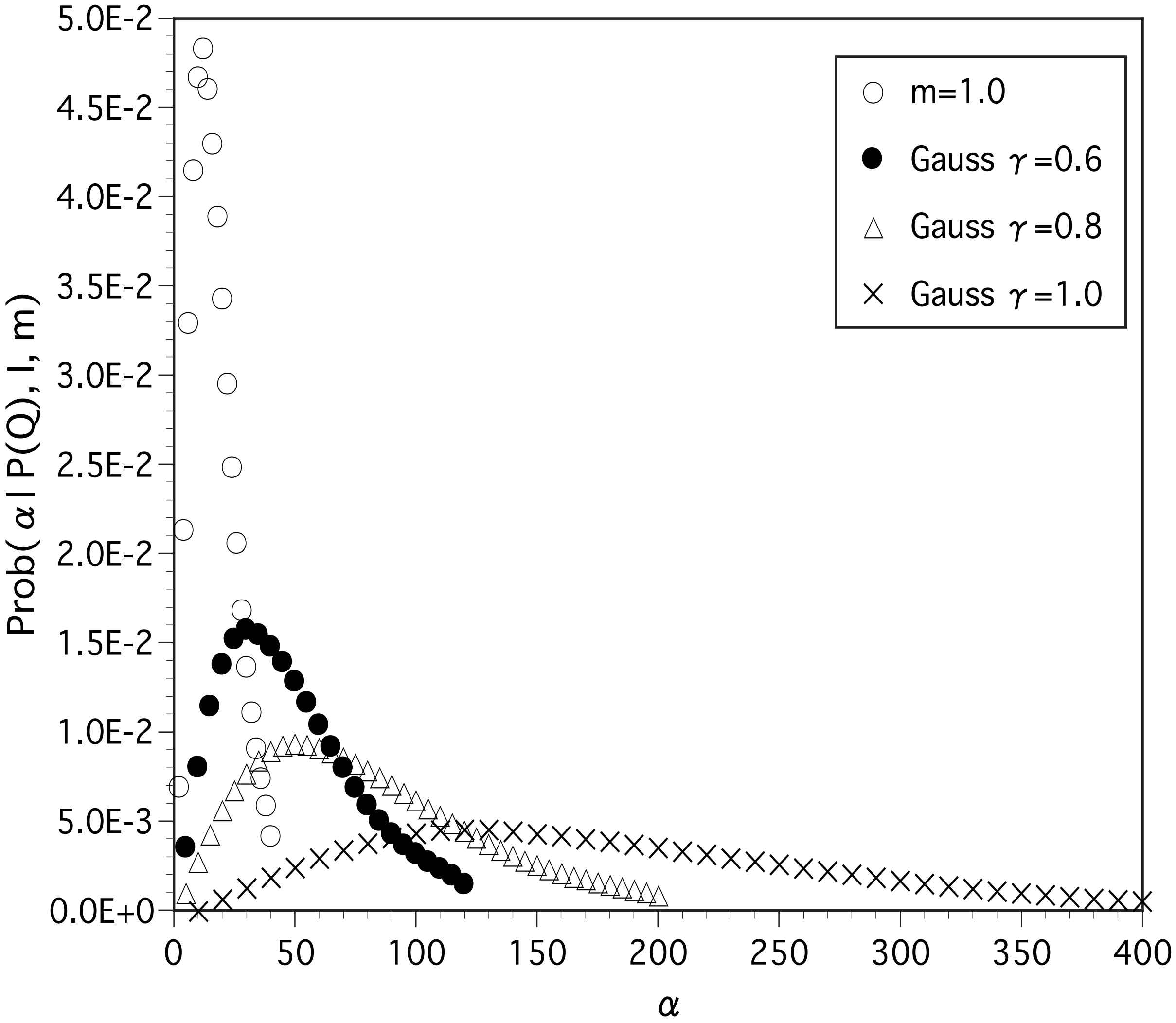


Figure 4: $\text{prob}(\alpha|P(Q), I, m)$ **for the data without flattening. Here,** $V = 12$**. The default model is chosen to be the constant** $m(\theta) = 1.0$ **and the Gaussian function with** $\gamma = 0.6$**, 0.8 and 1.0.**

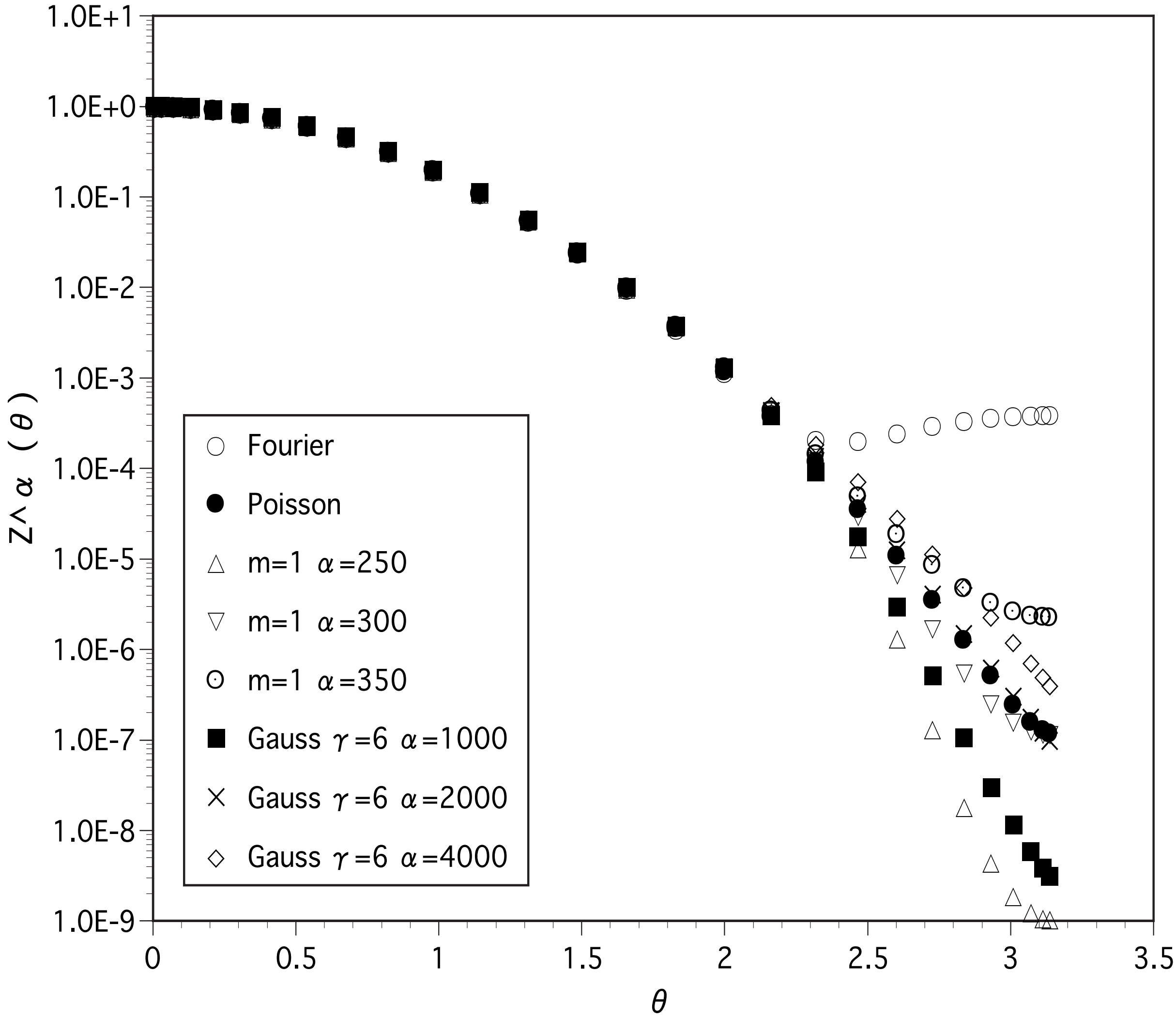


Figure 5: $\mathcal{Z}^{(\alpha)}(\theta)$ **for the data with flattening. Here,** $V = 50$**. The default model** $m(\theta)$ **is chosen to be the constant 1.0 and the Gaussian function with** $\gamma = 6$**. Comparing with the result of the Fourier transform (circles), the result of the MEM for certain values of** α **(**$\alpha \approx 300$ **for** $m(\theta) = 1.0$ **and** $\alpha \approx 2000$ **for the Gaussian) approximately reproduces that of the exact partition function (filled circles).**

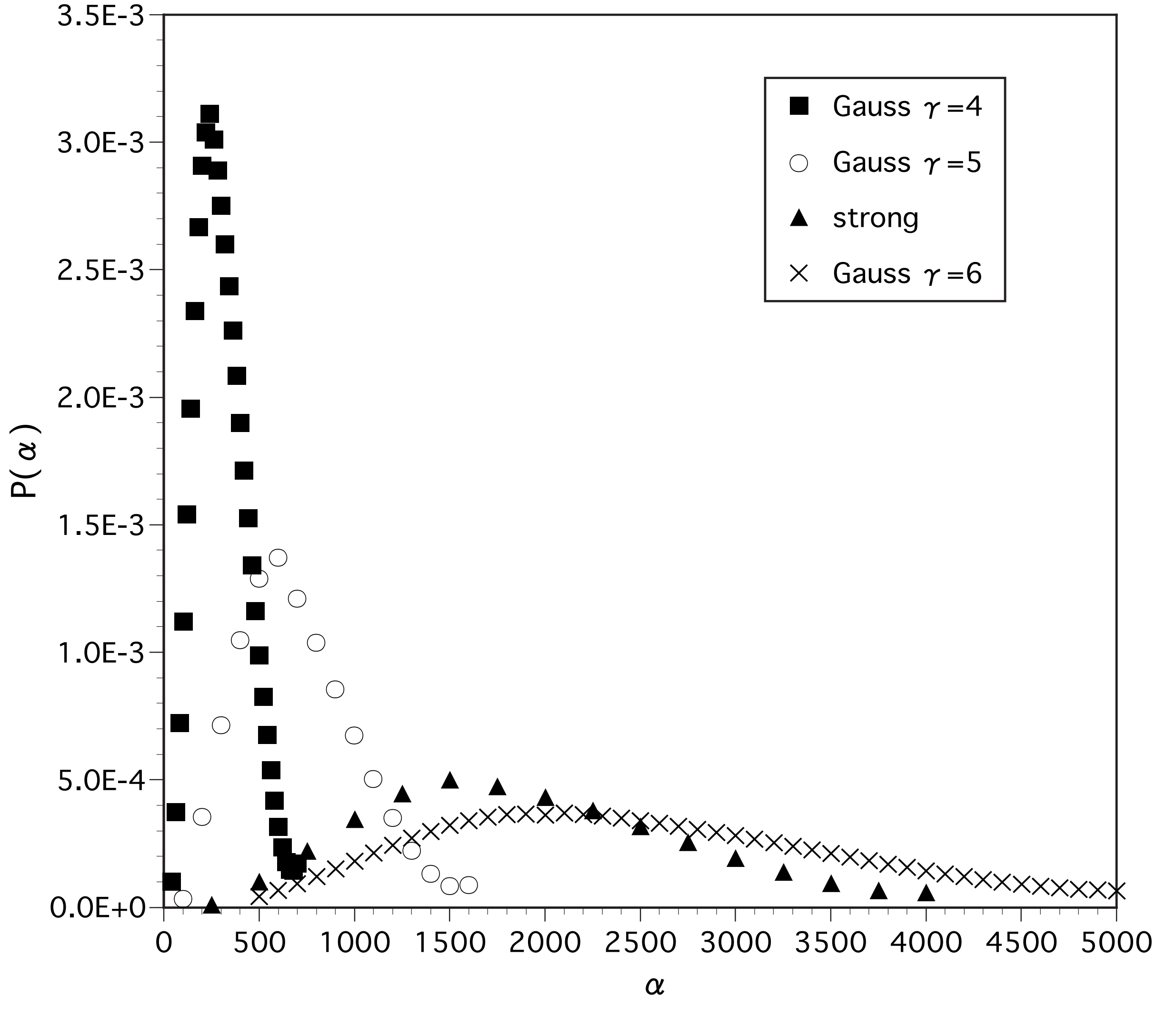


Figure 6: prob$(\alpha|P(Q), I, m)$ **for various** $m(\theta)$**.** V **is fixed to 50.**

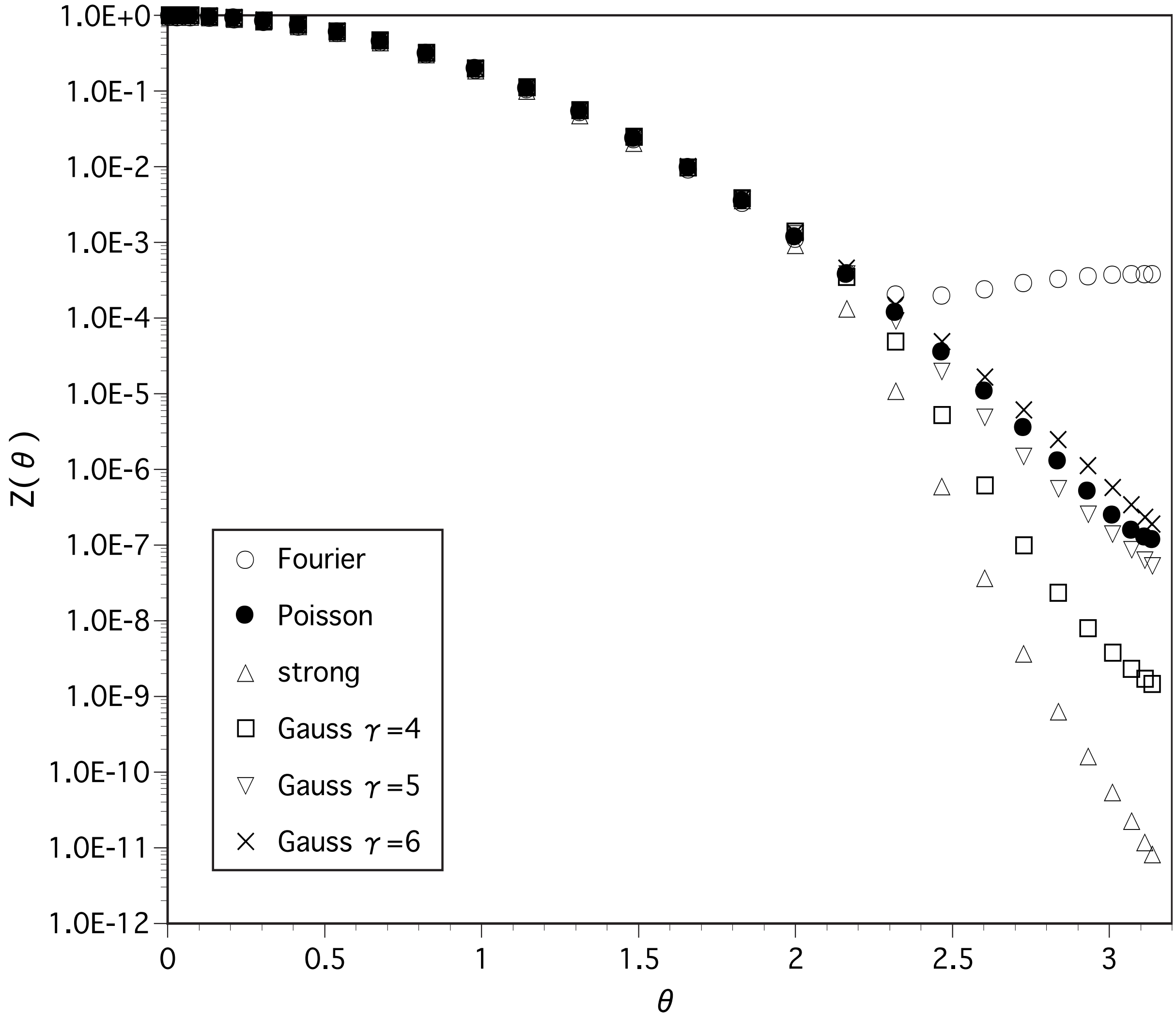


Figure 7: **Averaged partition function $\hat{\mathcal{Z}}(\theta)$ for various $m(\theta)$. $\hat{\mathcal{Z}}_n = \int d\alpha\ \mathrm{prob}(\alpha|P(Q), I, m)\mathcal{Z}_n^{(\alpha)}$. V=50. The result of the Fourier transform is also included. Those for the Gaussian default models with $\gamma = 5$ and 6 agree reasonably with the exact result, $\mathcal{Z}_{\mathrm{pois}}(\theta)$. The result for $m_{\mathrm{strg}}(\theta)$ shows a large deviation from $\mathcal{Z}_{\mathrm{pois}}(\theta)$.**

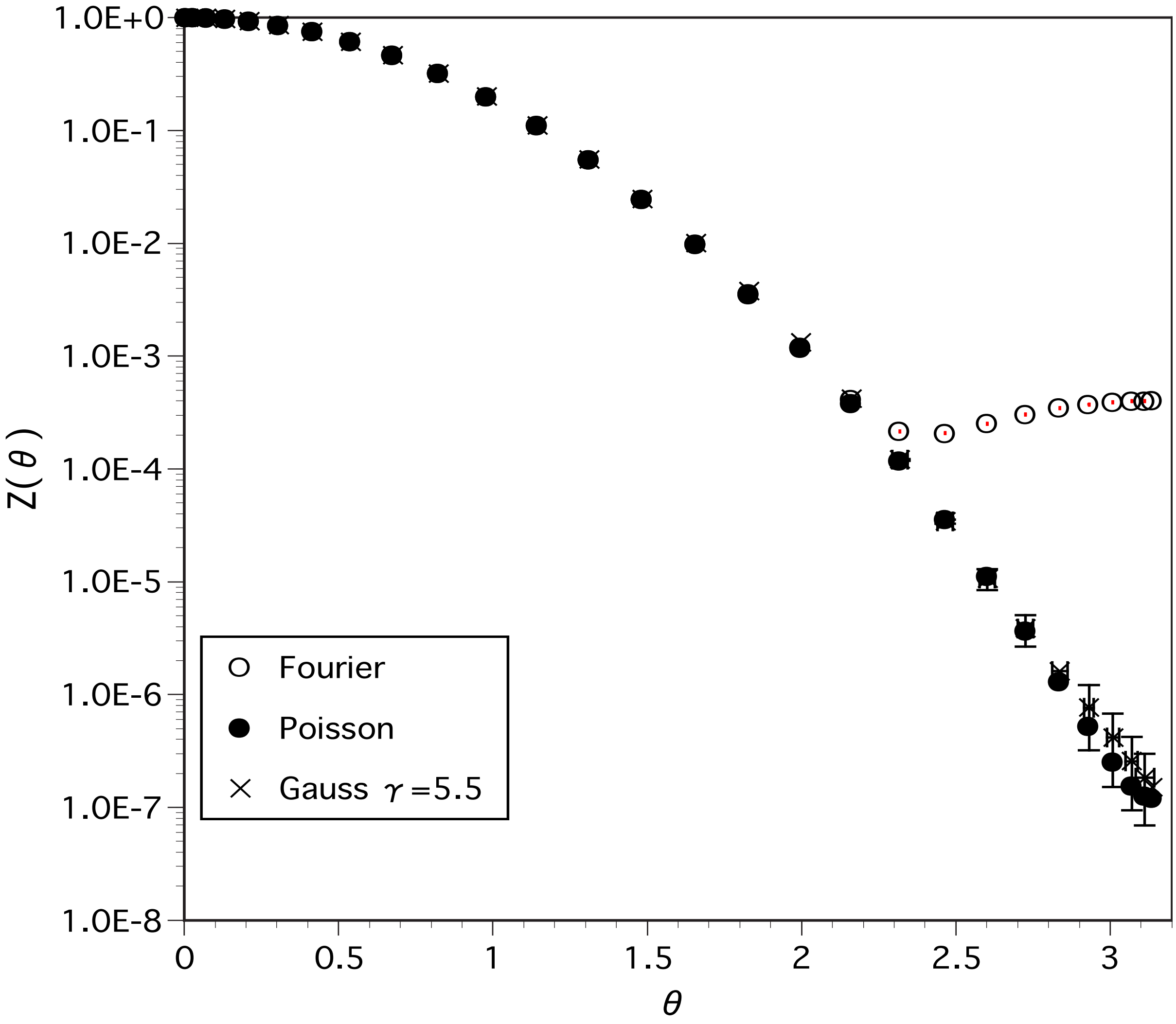


Figure 8: $\hat{\mathcal{Z}}(\theta)$ **(crosses) with the error bars for the Gaussian default model with** $\gamma = 5.5$**. Here,** $V = 50$**. Compared to the result of the Fourier transform (circles), a remarkable improvement is clearly seen.**

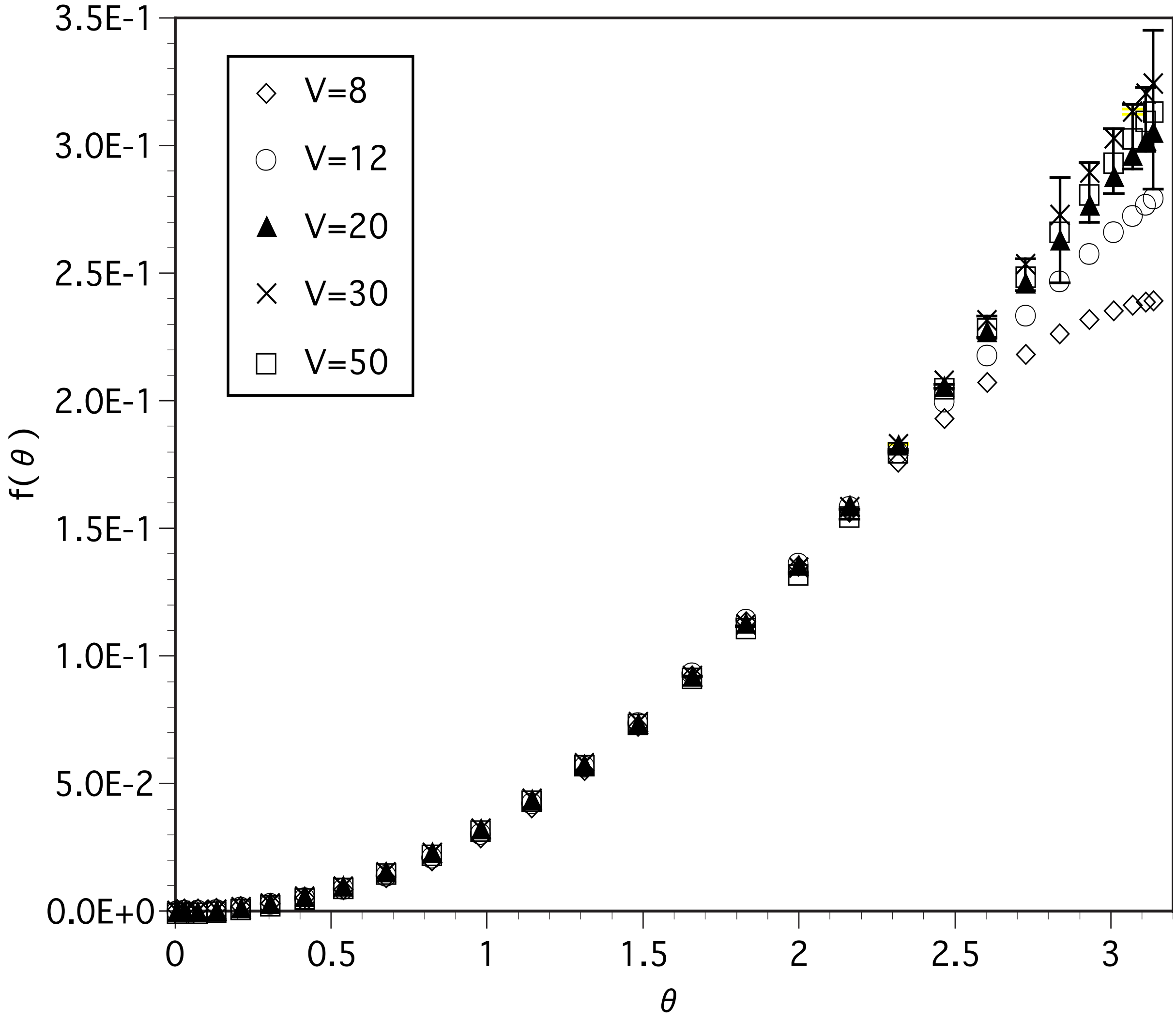


Figure 9: **Free energy density $f(\theta)$ calculated from $\hat{\mathcal{Z}}(\theta)$ for various volumes.**

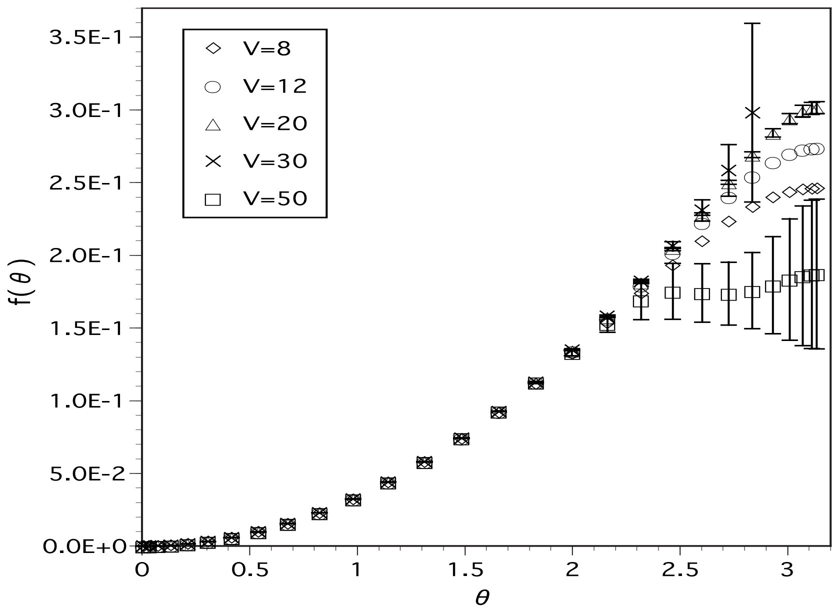


Figure 10: $f(\theta)$ **calculated by using the Fourier method. (The same as Fig.1.)**

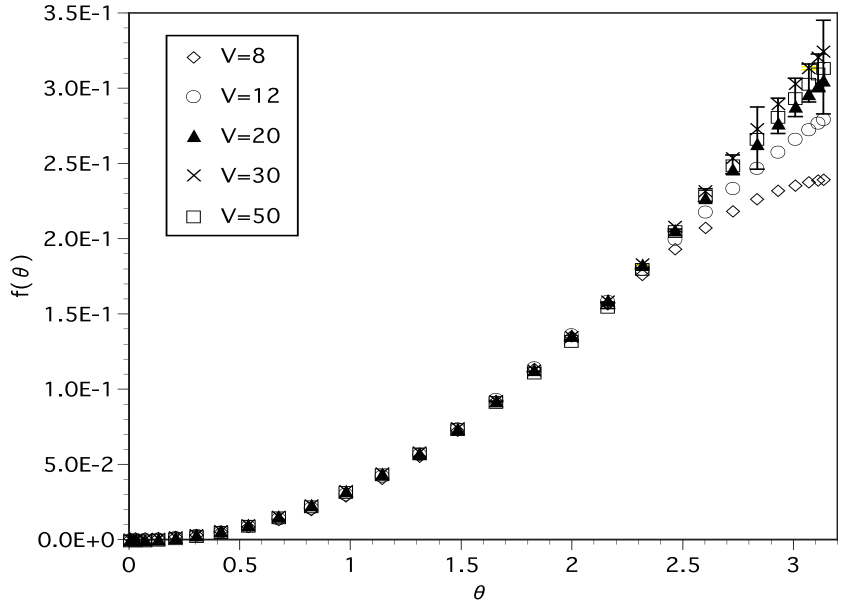


Figure 11: $f(\theta)$ **calculated from** $\hat{\mathcal{Z}}(\theta)$**:MEM.**

Summary of this section

We applied the MEM to two types of data for $P(Q)$:

(1) the data without flattening:
The MEM reproduces $f(\theta)$, which agree with those obtained by using the FTM.

(2) the data with flattening:
The best image with the least errors could reproduce smooth behavior of $f(\theta)$ in contrary to the FTM.

4.2.2 MEM analysis using MC data (CP^3 model)

- **CP^{N-1} model exhibits common dynamical properties with QCD**
- **Application of the MEM to MC data**
  - **Simulation of the 2-d CP^3 model with the fixed point action**
  - **$f(\theta)$ for various values of volume $L \times L$**

Table 1: Parameter values used in the MC simulations of the CP^3 model with the FP action. For the MEM analysis, new MC simulations were performed for $L = 38$ and 50.

β	L :	$Q_{\min}$–$Q_{\max}$:	total number of measurements (M/set)
3.0	12 :	0–30 :	10.0
	24 :	0–18 :	10.0
	32 :	0–24 :	3.0
	38 :	0–27 :	5.0
	46 :	0–33 :	1.0
	50 :	0–15 :	30.0
	56 :	0–18 :	5.0

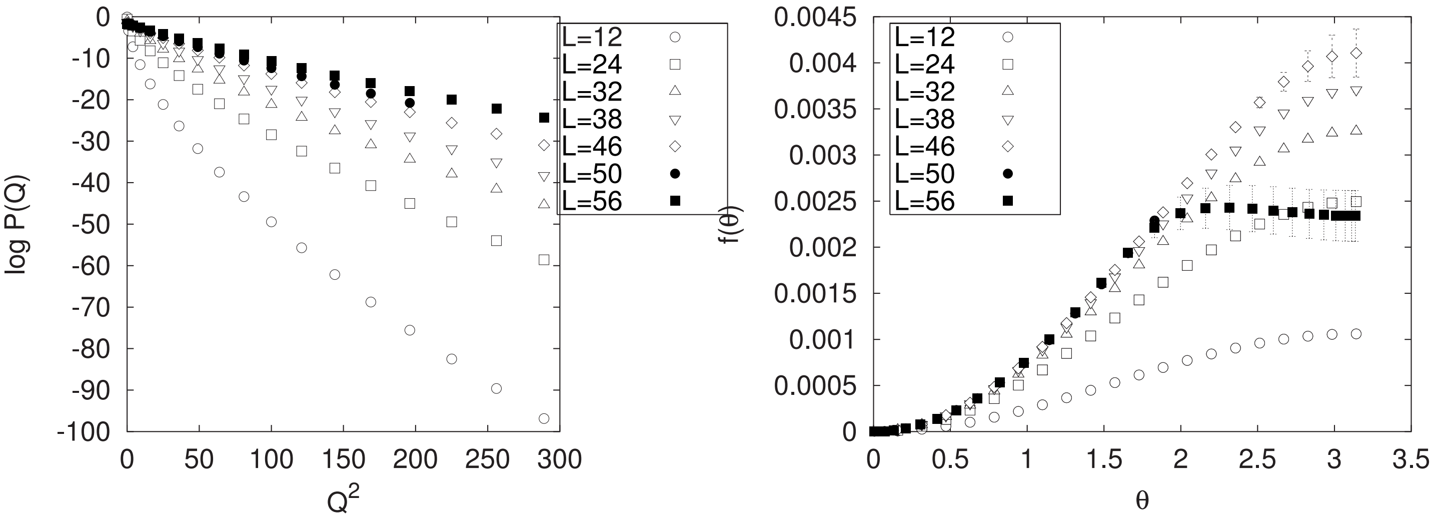


Figure 12: **Topological charge distributions $P(Q)$ (left panel) and free energy densities $f(\theta)$ (right panel) of the CP^3 model for $\beta = 3.0$ and various lattice sizes L.**

– $\beta = 3.0$ ($\xi \approx 7$)

– **Concentrate on two values of L**

1. $L = 38$ **in the non flattening case**
2. $L = 50$ **in the flattening case**

– **default models**

(i) **Gaussian with various values of γ**
$m_{\mathrm{G}}(\theta) = \exp\left[-\gamma \frac{\ln 10}{\pi^2}\theta^2\right]$

(ii) $m(\theta) = \hat{\mathcal{Z}}(\theta)$ **for smaller volumes**
For the analysis of $L_0 = 50$, $\hat{\mathcal{Z}}(\theta)$ for $L = 24,\ 32$ and 38 are employed as the default models. $\equiv$ $m_{L/L_0}(\theta) = m_{L/50}(\theta)$.

Results
1. non-flattening (L=38)
the similar behavior to that in the Gaussian mock data
2. flattening (L=50)

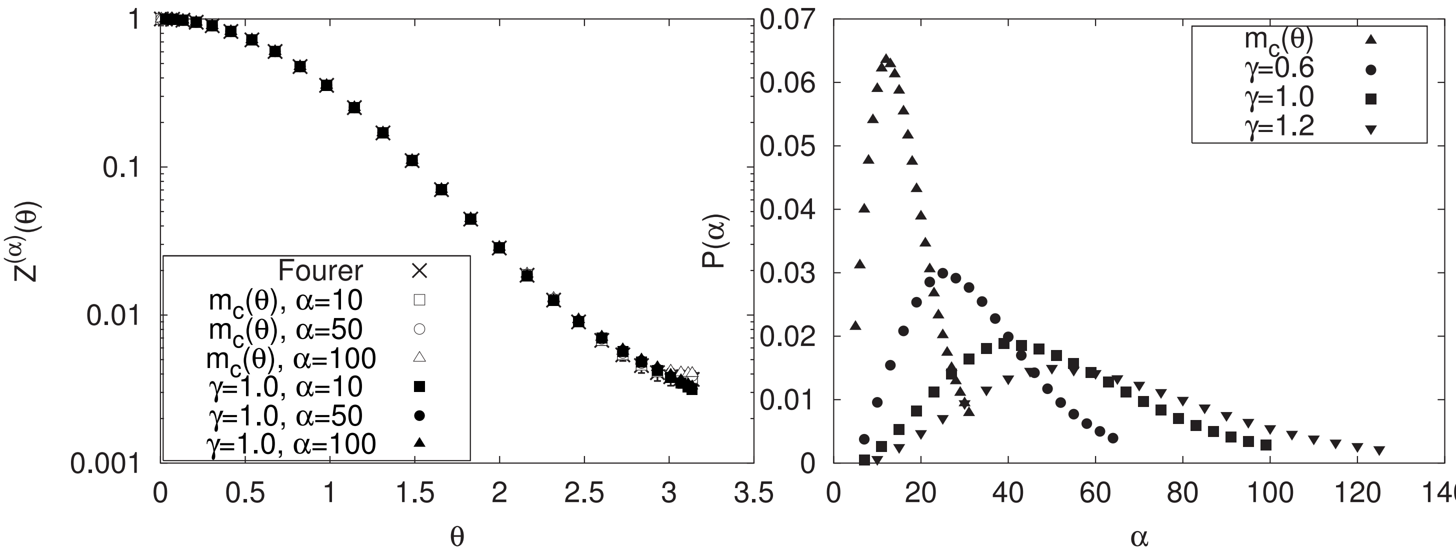


Figure 13: **Images $\mathcal{Z}^{(\alpha)}(\theta)$ for a given α (left panel) and probabilities $P(\alpha)$ in the non-flattening case ($L = 38$). In the left panel, $m_c(\theta)$ and $m_G(\theta)$ with $\gamma = 1.0$ are used. In addition, $m_G(\theta)$ with $\gamma = 0.6$ and 1.2 are used in the right panel.**

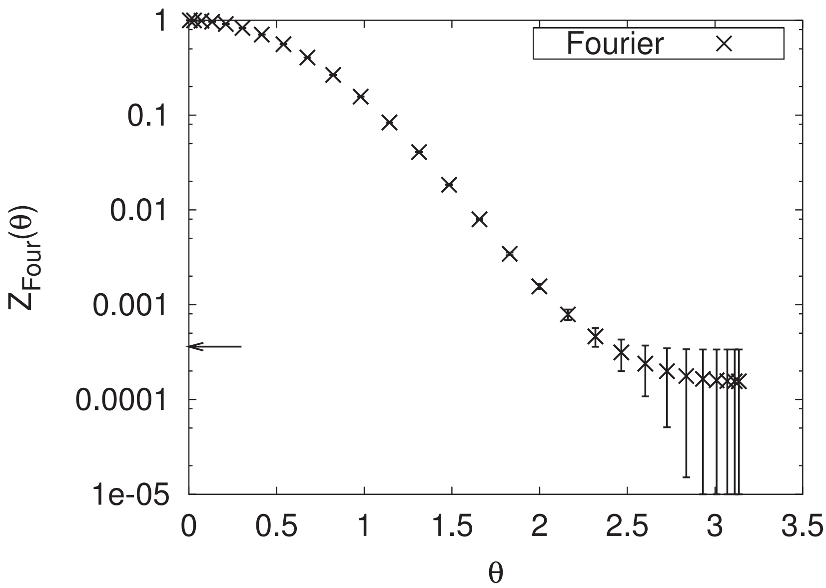


Figure 14: **Partition function $\mathcal{Z}_{\rm Four}(\theta)$ obtained using the FTM in the case with flattening ($L = 50$). The number of measurements is 30.0M/set. The arrow indicates the value of ϵ ($= 3.610 \times 10^{-4}$).**

- **statistics dependence**

  varied the # of statistics $2.0 - 30.0\times\ 10^6$/set.

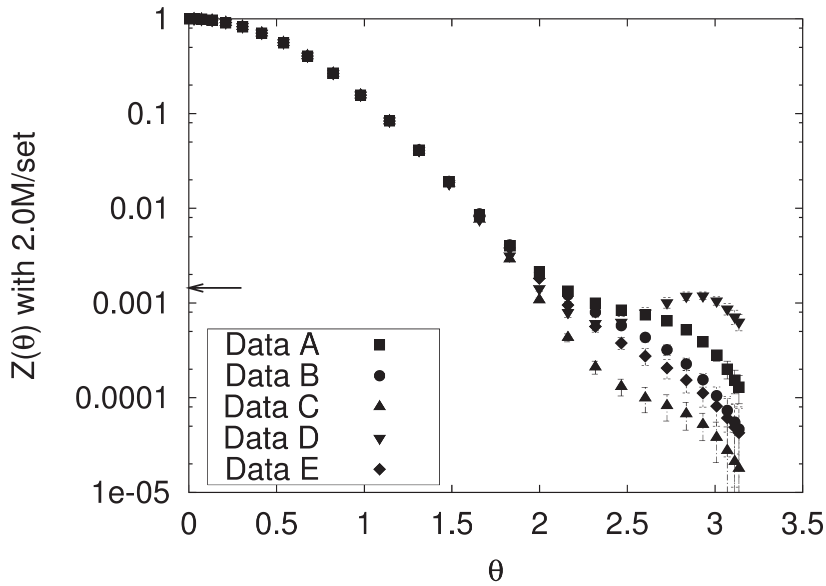
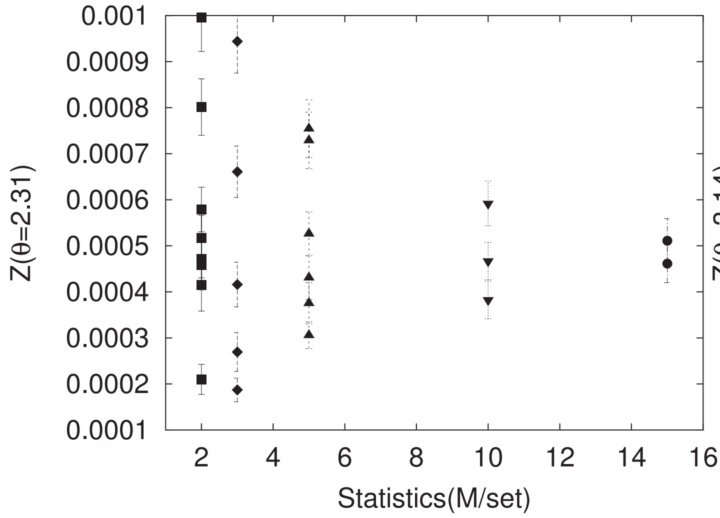


Figure 15: **The five most probable images $\hat{\mathcal{Z}}(\theta)$ for $L = 50$ with 2.0M/set. Here, the Gaussian default model with $\gamma = 5.0$ was used. The arrow indicates the value of $\epsilon\,(= 1.441 \times 10^{-3})$ in Data A.**



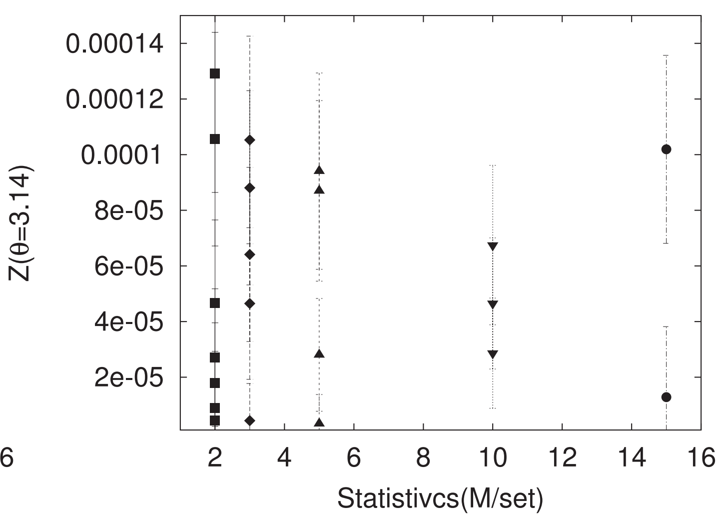


Figure 16: **Values of $\hat{\mathcal{Z}}(\theta)$ for θ=2.31 (left panel) and 3.14 (right panel). $L = 50$. The horizontal axis represents the number of measurements. Here, the Gaussian default model with $\gamma = 5.0$ was used.**

a qualitative difference between $\theta \lesssim$ and $\gtrsim 2.5$.

- **prior dependence**
  - $g(\alpha)$**:prior probability of** α **in**

$$\mathrm{prob}(\alpha|P(Q), I, m) \equiv P(\alpha) \propto g(\alpha)e^{W(\alpha)+\Lambda(\alpha)}, \quad (30)$$

  - $g(\alpha)$ **is chosen according to prior information.**
  - **two types of** $g(\alpha)$**, in general,**

    (i) **Laplace's rule**, $g_{\mathrm{Lap}}(\alpha) = \mathrm{const}$:
    corresponding to no knowledge about the prior information of α

    (ii) **Jeffrey's rule**, $g_{\mathrm{Jef}}(\alpha) = 1/\alpha$:
    $P(\alpha)$ **be invariant with respect to a change in scale**

  - **investigate the sensitivity of** $\hat{\mathcal{Z}}(\theta)$ **to the choice of** $g(\alpha)$ **by studying**

$$\Delta(\theta) \equiv \frac{|\hat{\mathcal{Z}}_{\mathrm{Lap}}(\theta) - \hat{\mathcal{Z}}_{\mathrm{Jef}}(\theta)|}{(\hat{\mathcal{Z}}_{\mathrm{Lap}}(\theta) + \hat{\mathcal{Z}}_{\mathrm{Jef}}(\theta))/2}, \quad (31)$$

$\hat{\mathcal{Z}}_{\mathrm{Lap}}(\theta)$: **the most probable images according to Laplace's rule**
$\hat{\mathcal{Z}}_{\mathrm{Jef}}(\theta)$: **the most probable images according to Jeffrey's rule**

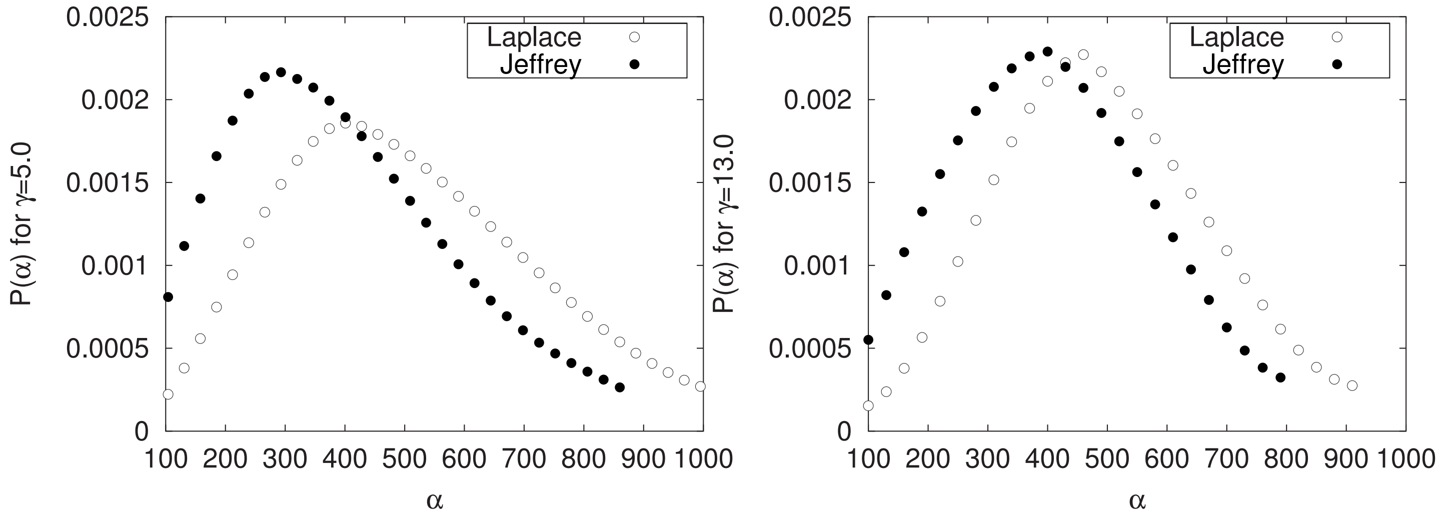


Figure 17: $P_{\mathrm{Lap}}(\alpha)$ and $P_{\mathrm{Jef}}(\alpha)$. **As the default models, Gaussian functions with** $\gamma = 5.0$ **(left panel) and** 13.0 **(right panel) were used.**

- **default model dependence**
  $\Delta(\theta)$ depends on the default models.
  The Gaussian default with $\gamma = 5.0$ is the minimum case. (See Table 2 and Figure 18.)

Table 2: **Values of $\Delta(\theta)$ at $\theta = 2.31, 2.60, 2.83$ and 3.14 for various $m(\theta)$.**

default model	$\Delta(2.31)$	$\Delta(2.60)$	$\Delta(2.83)$	$\Delta(3.14)$
$m_G(\theta)$ with $\gamma = 3.0$	6.30×10^{-3}	1.30×10^{-2}	1.87×10^{-2}	2.24×10^{-2}
$m_G(\theta)$ with $\gamma = 5.0$	5.35×10^{-3}	1.41×10^{-3}	1.12×10^{-2}	1.84×10^{-2}
$m_G(\theta)$ with $\gamma = 8.0$	1.03×10^{-1}	1.50×10^{-1}	1.70×10^{-1}	1.75×10^{-1}
$m_G(\theta)$ with $\gamma = 10.0$	1.01×10^{-1}	4.67×10^{-1}	8.21×10^{-1}	9.60×10^{-1}
$m_G(\theta)$ with $\gamma = 13.0$	1.05×10^{-1}	4.96×10^{-1}	7.12×10^{-1}	7.95×10^{-1}
$m_{24/50}(\theta)$	3.44×10^{-3}	7.67×10^{-3}	1.13×10^{-2}	1.36×10^{-2}
$m_{32/50}(\theta)$	7.30×10^{-3}	1.57×10^{-2}	2.29×10^{-2}	2.72×10^{-2}
$m_{38/50}(\theta)$	1.30×10^{-2}	2.60×10^{-2}	3.65×10^{-2}	4.26×10^{-2}

Summary of this section

We applied the MEM to two types of real data for $P(Q)$ (CP^3 model):

(1) the data without flattening ($L = 32$):
The MEM reproduces $f(\theta)$, which agree with those obtained by using the FTM.

(2) the data with flattening ($L = 50$):

- **studied the prior dependence**
- **studied the default model dependence**

- **The image with the least prior dependence exhibits a smooth behavior of $f(\theta)$ with small errors ($\gamma = 5.0$).**
- **At $\theta \gtrsim 2.5$, the default model dependence is large.**

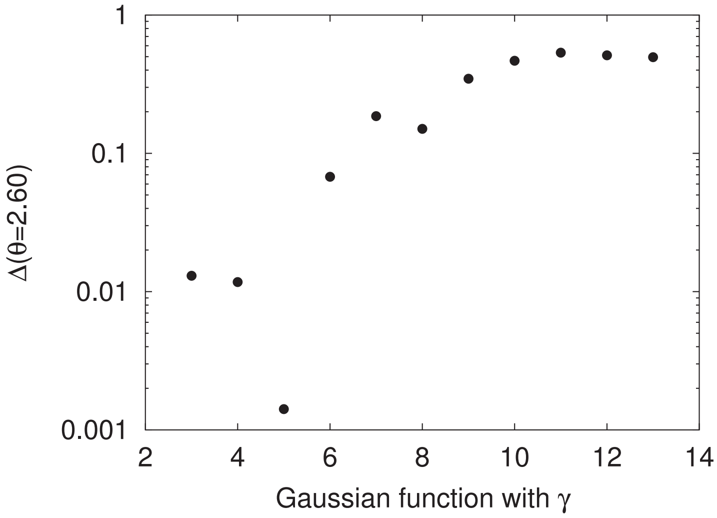


Figure 18: **Values of $\Delta(\theta)$ for $\theta = 2.60$. The Gaussian functions are used as the default models. The horizontal axis represents the value of γ in $m_{\rm G}(\theta)$.**

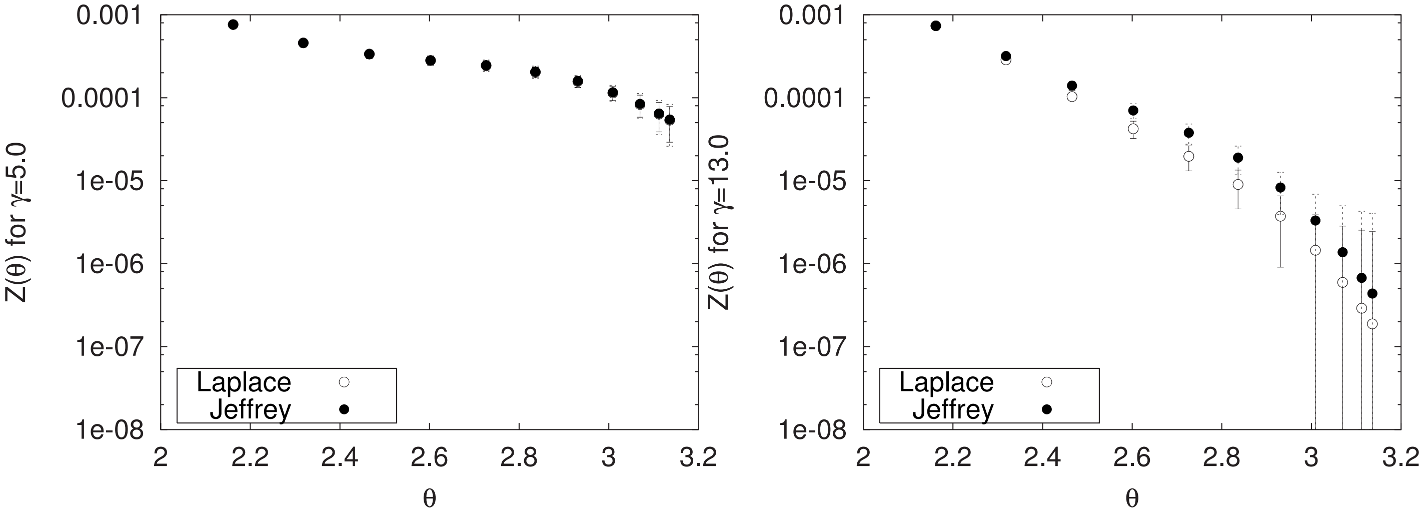


Figure 19: $\hat{\mathcal{Z}}_{\rm Lap}(\theta)$ **and** $\hat{\mathcal{Z}}_{\rm Jef}(\theta)$ **for** $\theta \in [2.0, \pi]$**.**

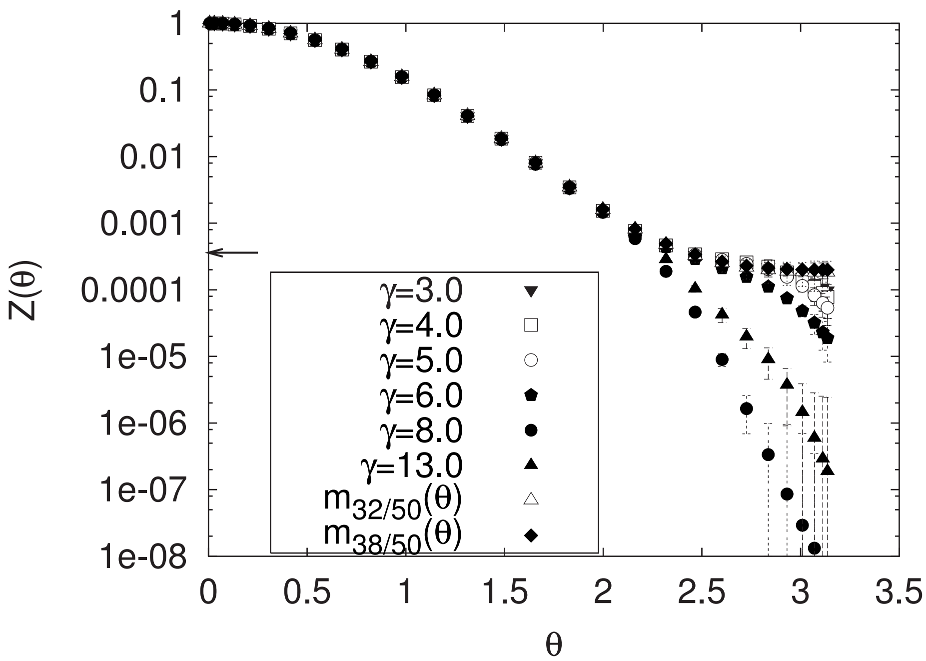


Figure 20: **The most probable images $\hat{\mathcal{Z}}(\theta)$ for various $m(\theta)$. The arrow indicates the value of ϵ $(= 3.610 \times 10^{-4})$.**

5.Conclusions

We studied the sign problem in terms of the MEM by using the LFT with the θ term.
We analyzed the following models:

- **using mock data**
  - **Gaussian $P(Q)$**
  - **'true' flattening**
- **using MC data (CP(3) model)**
  - **default model dependence**
  - **prior dependence**

Further study

- **favorable prior ?**
- **application of the MEM to finite density QCD**