Phases of generalized Potts-Models and their Relevance for Gauge Theories

A. Wipf

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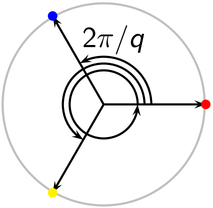
Potts-Models



generalized Ising models:

$$\theta_x \in \{2\pi k/q\}\,,\quad 1 \le k \le q$$

$$H = -J\sum_{\langle xy\rangle} \cos\left(\theta_x - \theta_y\right)$$

$$\mathbb{Z}_q : \theta_x \to \theta_x + 2\pi n/q$$

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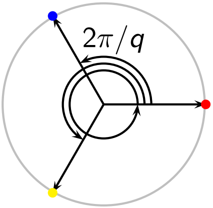
- 

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  phase transition symmetric $\leftrightarrow$ ferromagnetic
  $d = 2$: second order $q \leq 4$, first order $q > 4$
  $d = 3$: second order $q \leq 2$, first order $q > 2$

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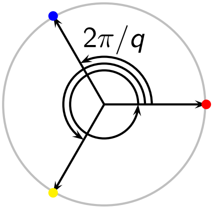

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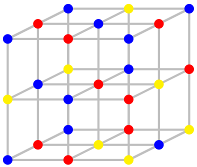

- anti-ferromagnetic phase:
  rich vacuum structures
  symmetric $\leftrightarrow$ antiferrom:
  $d = 3, q = 3$: second order
  entropy of ground states?

- entropy $S_B(p) = -\sum p(w) \log p(w) \Rightarrow$ free energy

$$\beta F = \inf_p \left(\beta \langle H \rangle_\rho - S_B\right) \Rightarrow p_{\mathrm{Gibbs}} \sim e^{-\beta H}$$

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- variational characterization of (convex) effective action:

$$\Gamma[m] = \inf_p \Big(\beta \langle H \rangle_p - S(p) \,\Big|\, \langle e^{i\theta(x)} \rangle_p = m(x)\Big)$$

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- mean field approximation:

$$\rho(w) = \prod_x \rho_x(\theta_x) \Rightarrow \Gamma_{\text{MF}}[m]$$

translational invariance: $p_x = p \Rightarrow m(x) = m$
effective potential: $\Gamma_{\text{MF}}[m] = V\, u_{\text{MF}}(m)$

$$u_{\text{MF}}(m) = \inf_p \left(-Kmm^* + \sum_\theta p(\theta) \log p(\theta)\right)$$

$$m = \sum_\theta p(\theta)\, e^{i\theta}, \quad K = dJ.$$

- antiferromagnetic phase:
  translational invariance on sublattices $\Lambda = \Lambda_1 \cup \Lambda_2$
  two neighbours in different sublattices
  $p(x) = p_i \Rightarrow m(x) = m_i$ for $x \in \Lambda_i$

$$u_{\mathrm{MF}}\,(m_1, m_2) = \frac{1}{2}\left(K|m_1 - m_2|^2 + \sum_i u_{\mathrm{MF}}(m_i)\right),$$

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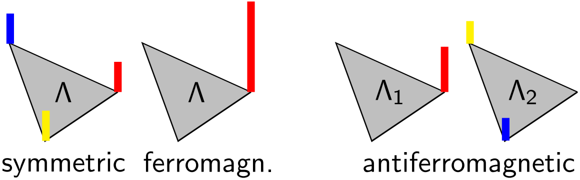
$$u_{\rm MF}(m_1, m_2) = \frac{1}{2}\left(K|m_1 - m_2|^2 + \sum_i u_{\rm MF}(m_i)\right),$$

- $K > K_{f,c} > 0 \Rightarrow m_1 = m_2 \neq 0$, $\mathbb{Z}_q$-broken
  $K < K_{a,c} < 0 \Rightarrow m_1 \neq m_2 \neq 0$, $\mathbb{Z}_{2q}$-broken



Polyakov-Loop Dynamics

- finite temperature gluodynamics
  order parameter for confinement: Polyakov loop
  effective action:

$$e^{-S_{\rm eff}[\mathcal{P}]} = \int \mathcal{D}U\delta\left(\mathcal{P}_{\boldsymbol{x}}, \prod_{t=0}^{N_t} U_{t,\boldsymbol{x};0}\right) e^{-S_{\rm w}[U]}$$

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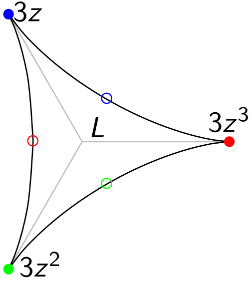

gauge invariance:

$$S_{\rm eff} = S_{\rm eff}[L], \quad L_{\boldsymbol{x}} = \mathrm{Tr}\,\mathcal{P}_{\boldsymbol{x}}$$

global Z_3 center symmetry:

$$S_{\rm eff}[L] = S_{\rm eff}[z \cdot L]$$

- good ansatz for $S_{\rm eff}$?

- strong coupling expansion for $S_{\text{eff}}[\mathcal{P}]$
  $\Rightarrow \mathbb{Z}_3$-invariant character expansion
  nearest neighbour interaction

$$S_{\text{eff}} = \lambda_{10}S_{10} + \lambda_{21}S_{21} + \lambda_{20}S_{20} + \lambda_{11}S_{11} + \dots$$
$$S_{10} = \sum \left(\chi_{10}(\mathcal{P}_{\boldsymbol{x}})\chi_{01}(\mathcal{P}_{\boldsymbol{y}}) + h.c\right), \; S_{21} = \dots$$

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- center-transformation:

$$\chi_{pq}(z\mathcal{P}) = z^{p-q}\chi_{pq}(\mathcal{P}), \quad z^3 = z^*z = 1$$

With $L = \mathrm{Tr}\,\mathcal{P}$: leading terms

$$\begin{aligned} S_{\text{eff}} &= (\lambda_{10} - \lambda_{21})\sum \left(L_{\boldsymbol{x}}L^*_{\boldsymbol{y}} + \text{h.c.}\right) \\ &+ \lambda_{21}\sum \left(L^2_{\boldsymbol{x}}L_{\boldsymbol{y}} + L^2_{\boldsymbol{y}}L_{\boldsymbol{x}} + \text{h.c.}\right) \end{aligned}$$

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- complex field with compact target space, $\prod$(reduced Haar measures), close relation to 3-state Potts model

Gluodynamics and Pott-Models

- naive reduction to Potts: $\mathcal{P}_{\boldsymbol{x}} \longrightarrow e^{i\theta_{\boldsymbol{x}}}\mathbb{1} \in$ centre

$$S_{\text{eff}} \longrightarrow H \quad \text{with} \quad J = 18(\lambda_{01} + 4\lambda_{21})$$

true for all $S_{\text{eff}} \Rightarrow S_{\text{eff}}$ is extension of $\mathbb{Z}_3$ model.

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  effective finite-temperature $SU(N)$-gluodynamics in d dimensions $\cong \mathbb{Z}_N$ spin model in $d-1$ dimensions.
- same critical exponents $SU(2)$ and Ising (Engels et.al) same universality class (symmetric $\leftrightarrow$ ferrom.)

	β/ν	γ/ν	ν
4d SU(2)	0.545	1.93	0.65
3d Ising	0.516	1.965	0.63

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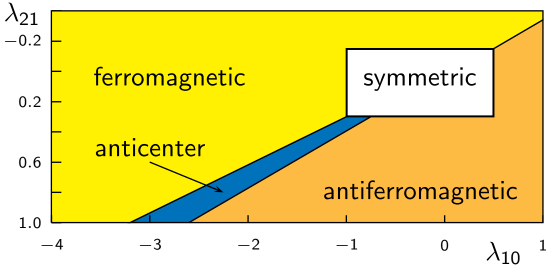

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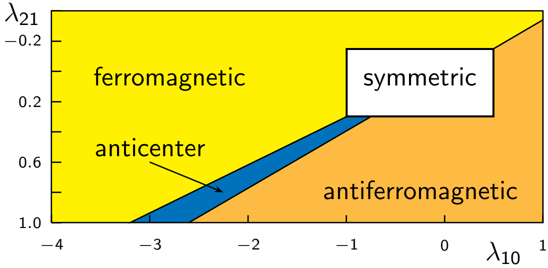

- quantum fluctuations $\Rightarrow$ include symmetric phase new ferromagnetic anti-center phase qualitatively correct phase diagram

Modified mean field approximation

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  ⇒ nontrivial variational problem on two-sites
- most simple effective model (Polonyi)

$$S_{\rm eff} = \lambda S_{10} = \lambda \sum \left(L_{\boldsymbol{x}} L_{\boldsymbol{y}}^{*} + \text{h.c} \right)$$

Lagrangean multiplier for $\bar{L}_i$ on Λ_i

- mean field effective potential for minimal model

$$2u_{\mathrm{MF}}(L_1, L_1^*, L_2, L_2^*) = -d\lambda|L_1 - L_2|^2 + \sum v_{\mathrm{MF}}(L_i, L_i^*)$$

$$v_{\mathrm{MF}}(L, L^*) = d\lambda|L|^2 + \gamma_0(L, L^*)$$

γ_0 Legendre-transform of

$$w_0(j, j^*) = \log \int d\mu_{\mathrm{red}} \exp\left(jL + j^* L^*\right)$$

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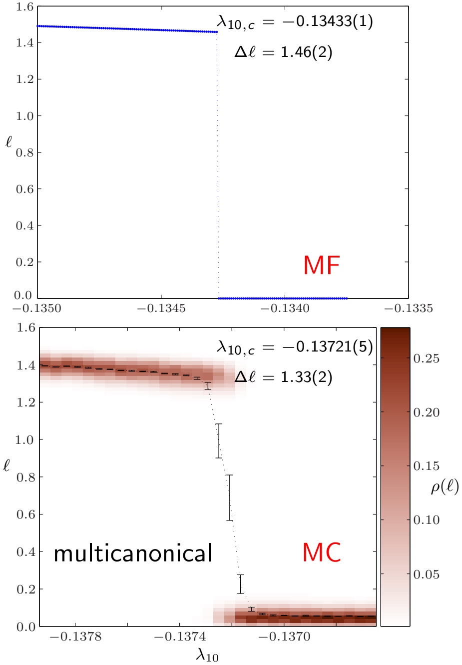
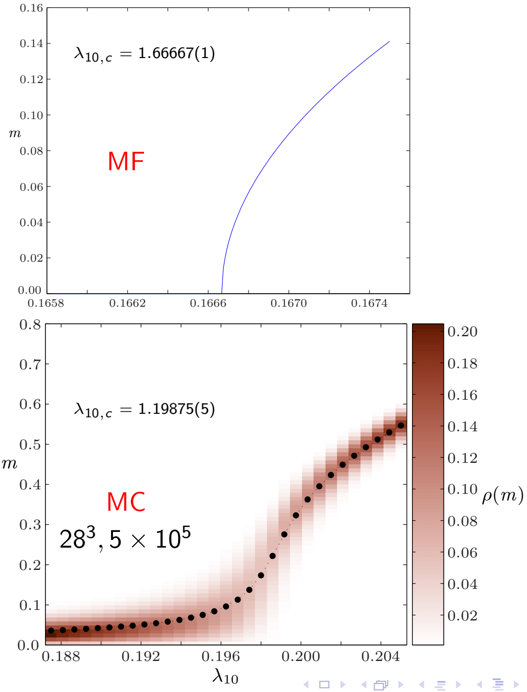
- group integral in closed form not known for $SU(3)$!!
  $\int \exp\left(j \,\mathrm{Tr}\,(U)\right) =$hypergeometric function


$\lambda_{10,c} = -0.13433(1)$
$\Delta\ell = 1.46(2)$
MF
ℓ
$\lambda_{10,c} = -0.13721(5)$
$\Delta\ell = 1.33(2)$
multicanonical
MC
$\rho(\ell)$
λ_{10}



$\lambda_{10,c} = 1.66667(1)$
MF
m
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
0.1658
0.1662
0.1666
0.1670
0.1674
$\lambda_{10,c} = 1.19875(5)$
MC
$28^3, 5 \times 10^5$
m
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.188
0.192
0.196
0.200
0.204
λ_{10}
$\rho(m)$
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
0.18
0.20

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  critical exponents in mean field?
- finite temperature gluodynamics
  $\longrightarrow$ effective $\mathbb{Z}_3$ models with compact target spaces
  $\longrightarrow$ 3-state Potts-model

universality test in 'unphysical region'
(for gluodynamics)

Results of MC-Simulations

- phase diagram and transitions → histograms
  large statistics, expensive → fast algorithms!
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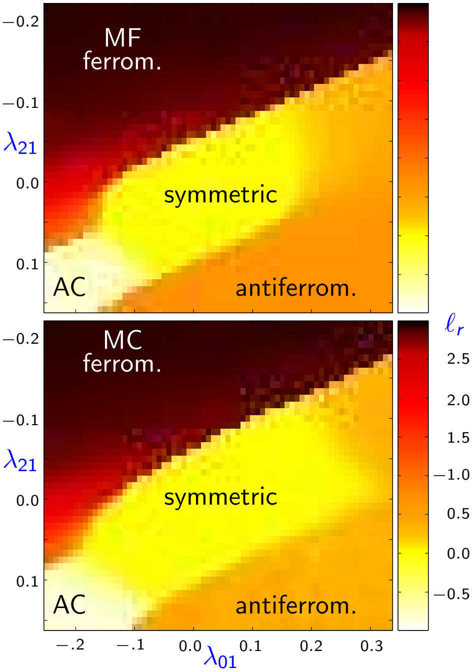
- rich phase structure: 4 different phases, second und first order transitions, tricritical points(?), mean field very good.


MF
ferrom.
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λ_{21}
λ_{21}
λ_{01}
ℓ_r
−0.2
−0.1
0.0
0.1
−.2
−.1
0.0
0.1
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0.3
2.5
2.0
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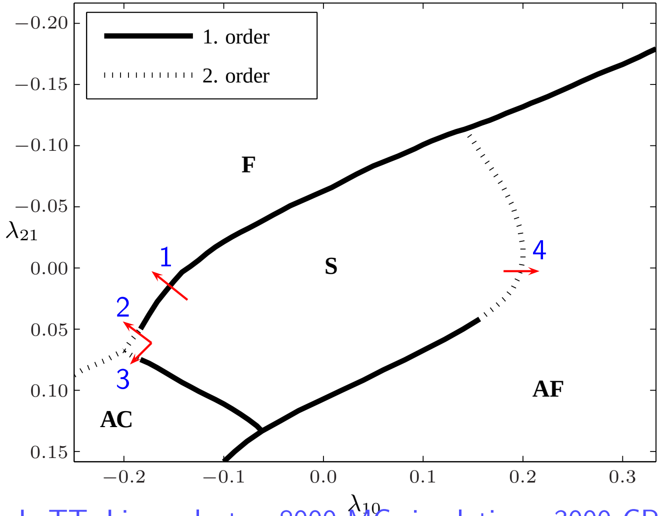
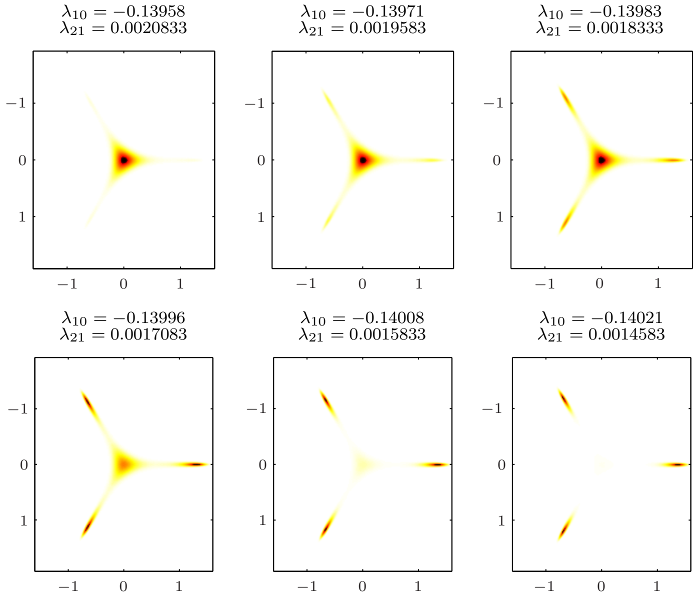
0.5
0.0
−0.5



jenLaTT, Linux cluster, 8000 MC simulations, 3000 CPUh



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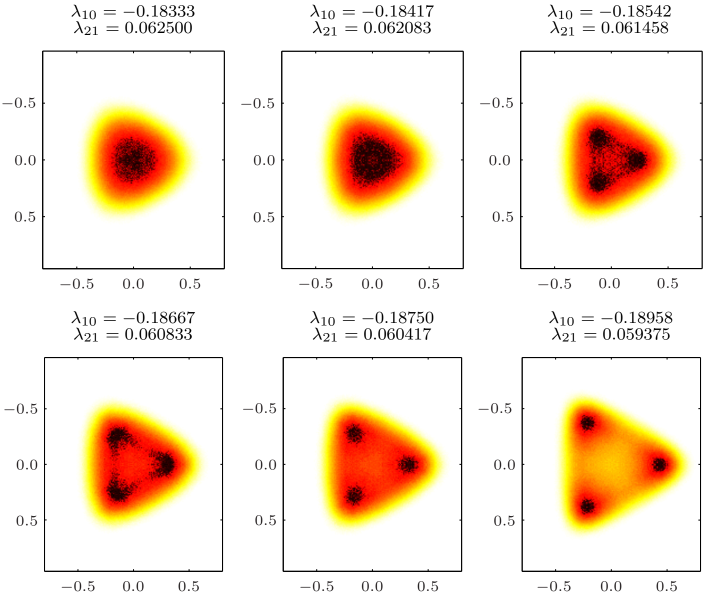
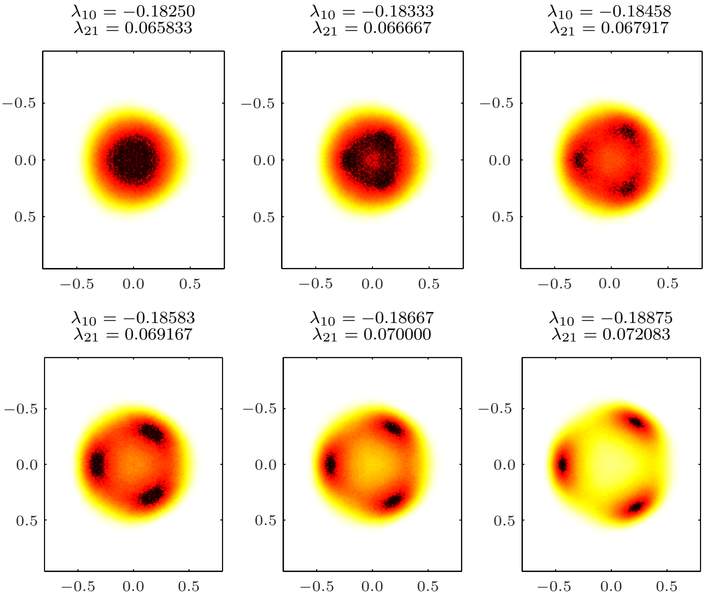
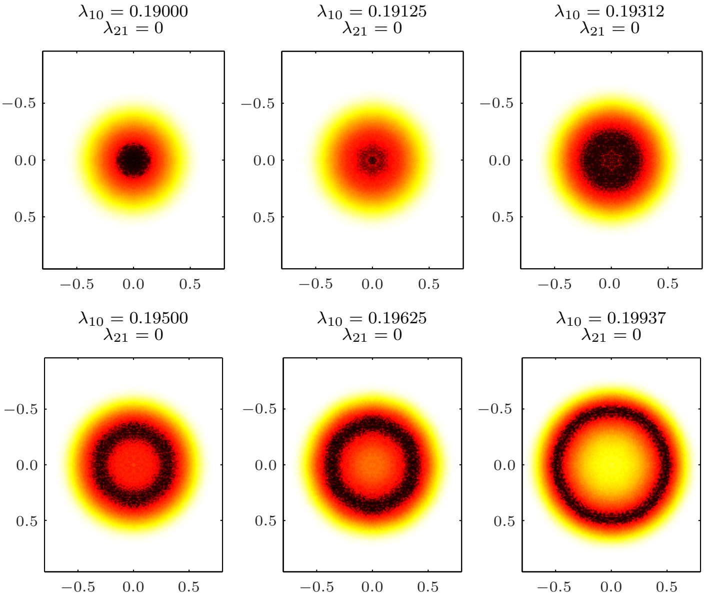


- 2: Histogramm of L, S $\leftrightarrow$ FM, 2nd



- 3: Histogramm of L, S ↔ AC, 2nd



- 4: Histogramm of M, S $\leftrightarrow$ AF, 2nd

Conclusions

- strong coupling for Polykaov-loops effective action

JHEP 06 (2004) 005, PRD 72 (2005) 065005, hep-lat/0605012

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- vacuum-sector of AF phase? $SU(N)$ group integrals?
- is AC-phase relevant for gluodynamics?

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- calculate $\lambda_j(\beta)$ via IMC (cp. $SU(2)$)
- efficient 'group-theoretic' Schwinger-Dyson equations!
- vacuum-sector of AF phase? $SU(N)$ group integrals?
- is AC-phase relevant for gluodynamics?
- include fermions in effective Polyakov-loop dynamics.

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