LOR2006
Budapest, June 22-24, 2006

Fermions on curved spaces, symmetries, and quantum anomalies

Mihai Visinescu

Department of Theoretical Physics

National Institute of Physics and Nuclear Engineering

Bucharest, Romania

Fermions on curved spaces

1. Pseudo-classical approach
2. Dirac equation on curved spaces
3. Gravitational anomalies
4. Axial anomalies

Geometrical objects

1. Symmetric Stäckel-Killing (S-K) tensors

$$K_{(\mu\cdots\nu;\lambda)} = 0.$$

2. Antisymmetric Killing-Yano (K-Y) tensors

$$f_{\mu_1\ldots\mu_{r-1}(\mu_r;\lambda)} = 0.$$

Pseudoclassical approach

Action:

$$S = \int_a^b d\tau \left(\frac{1}{2}\, g_{\mu\nu}(x)\, \dot{x}^\mu\, \dot{x}^\nu + \frac{i}{2}\, g_{\mu\nu}(x)\, \psi^\mu\, \frac{D\psi^\nu}{D\tau} \right).$$

Covariant derivative of ψ^μ

$$\frac{D\psi^\mu}{D\tau} = \dot{\psi}^\mu + \dot{x}^\lambda \Gamma^\mu_{\lambda\nu} \psi^\nu \,.$$

World-line Hamiltonian

$$H = \frac{1}{2} g^{\mu\nu} \Pi_\mu \Pi_\nu$$

covariant momentum

$$\Pi_\mu = g_{\mu\nu} \dot{x}^\nu$$

Constant of motion $\mathcal{J}(x, \Pi, \psi)$, the bracket with H vanishes

$$\{H, \mathcal{J}\} = 0 \,.$$

Expand $\mathcal{J}(x, \Pi, \psi)$ in a power series in the covariant momentum

$$\mathcal{J} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{J}^{(n)\mu_1 \ldots \mu_n}(x, \psi) \Pi_{\mu_1} \ldots \Pi_{\mu_n}$$

Generalized Killing equations:

$$\mathcal{J}^{(n)}_{(\mu_1 \ldots \mu_n;\mu_{n+1})} + \frac{\partial \mathcal{J}^{(n)}_{(\mu_1 \ldots \mu_n}}{\partial \psi^\sigma} \Gamma^\sigma_{\mu_{n+1})\lambda} \psi^\lambda = \frac{i}{2} \psi^\rho \psi^\sigma R_{\rho\sigma\nu(\mu_{n+1}} \mathcal{J}^{(n+1)\nu}{}_{\mu_1 \ldots \mu_n)}$$

For a Killing vector R_μ ($R_{(\mu;\nu)} = 0$) there is a conserved quantity in the spinning case:

$$\mathcal{J} = \frac{i}{2} R_{[\mu;\nu]} \psi^\mu \psi^\nu + R_\mu \dot{x}^\mu$$

Assume that a S-K tensor can be written as a symmetrized product of two K-Y tensors

$$K^{\mu\nu}_{ij} = \frac{1}{2}(f^{\mu}_{i\ \lambda} f^{\nu\lambda}_{j} + f^{\nu}_{i\ \lambda} f^{\mu\lambda}_{j}).$$

The conserved quantity for the spinning space is

$$\mathcal{J}_{ij} = \frac{1}{2!} K_{ij}^{\mu\nu} \dot{x}_\mu \dot{x}_\nu + \mathcal{J}_{ij}^{(1)\mu} \dot{x}_\mu + \mathcal{J}_{ij}^{(0)}$$

where

$$\mathcal{J}_{ij}^{(0)} = -\frac{1}{4} \psi^\lambda \psi^\sigma \psi^\rho \psi^\tau (R_{\mu\nu\lambda\sigma} f_i{}^\mu{}_\rho f_j{}^\nu{}_\tau + \frac{1}{2} c_{i\lambda\sigma}{}^\pi c_{j\rho\tau\pi}),$$

$$\mathcal{J}_{ij}^{(1)\mu} = \frac{i}{2} \psi^\lambda \psi^\sigma (f_i{}^\nu{}_\sigma D_\nu f_j{}^\mu{}_\lambda + f_j{}^\nu{}_\sigma D_\nu f_i{}^\mu{}_\lambda + \frac{1}{2} f_i^{\mu\rho} c_{j\lambda\sigma\rho} + \frac{1}{2} f_j^{\mu\rho} c_{i\lambda\sigma\rho})$$

with

$$c_{i\mu\nu\lambda} = -2 f_{i[\nu\lambda;\mu]}$$

Conserved supercharge

$$\begin{aligned} Q_f &= f_{\mu_1 \ldots \mu_r} \Pi^{\mu_1} \psi^{\mu_2} \ldots \psi^{\mu_r} \\ &\quad + \frac{i}{r+1} (-1)^{r+1} f_{[\mu_1 \ldots \mu_r ; \mu_{r+1}]} \cdot \psi^{\mu_1} \ldots \psi^{\mu_{r+1}}. \end{aligned}$$

This quantity is a superinvariant (supercharge $Q_0 = \Pi_\mu \psi^\mu$)

$$\{Q_f, Q_0\} = 0$$

Dirac equation on a curved background

Dirac operator on a curved background ($\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I$)

$$D_s = \gamma^\mu \hat{\nabla}_\mu .$$

Canonical covariant derivative for spinors

$$\hat{\nabla}_\mu \gamma^\mu = 0,$$

$$\hat{\nabla}_{[\rho} \hat{\nabla}_{\mu]} = \frac{1}{4} R_{\alpha\beta\rho\mu} \gamma^\alpha \gamma^\beta$$

For any isometry with Killing vector R_μ there is an operator

$$X_k = -i (R^\mu \hat{\nabla}_\mu - \frac{1}{4} \gamma^\mu \gamma^\nu R_{\mu;\nu})$$

which commutes with the *standard* Dirac operator.
A K-Y tensor produces a *non-standard* Dirac operator

$$D_f = -i \gamma^\mu (f_\mu{}^\nu \hat{\nabla}_\nu - \frac{1}{6} \gamma^\nu \gamma^\rho f_{\mu\nu;\rho})$$

which anticommutes with the standard Dirac operator D_s.

Euclidean Taub-NUT space

Metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = f(r)(d\vec{x})^2 + \frac{g(r)}{16m^2}(dx^4 + A_i dx^i)^2$$

$\vec{A}$ is the gauge field of a monopole

$$\mathrm{div}\vec{A} = 0\,, \quad \vec{B} = \mathrm{rot}\,\vec{A} = 4m\frac{\vec{x}}{r^3}.$$

$$f(r) = g^{-1}(r) = V^{-1}(r) = \frac{4m + r}{r}$$

Four Killing vectors

$$D_A = R^\mu_A\,\partial_\mu, \qquad A = 1, 2, 3, 4\,.$$

Conservation of angular momentum and "relative electric charge" ($\vec{p} = V^{-1}\dot{\vec{r}}$ is the mechanical momentum):

$$\vec{j} = \vec{r} \times \vec{p} + q\,\frac{\vec{r}}{r} \;\;, \quad q = g(r)(\dot{\theta} + \cos\theta\dot{\varphi})$$

Four K-Y tensors of valence 2.

- Three are covariantly constant

$$
\begin{aligned}
f_i &= 8m(d\chi + \cos\theta d\varphi) \wedge dx_i - \epsilon_{ijk}(1 + \frac{4m}{r})dx_j \wedge dx_k, \\
D_\mu f^{\nu}_{i\lambda} &= 0 \ , \qquad i, j, k = 1, 2, 3.
\end{aligned}
$$

- The fourth K-Y tensor is

$$
f_Y = 8m(d\chi + \cos\theta d\varphi) \wedge dr + 4r(r + 2m)(1 + \frac{r}{4m}) \sin\theta d\theta \wedge d\varphi
$$

having a non-vanishing covariant derivative

Runge-Lenz vector

$$\vec{K} = \frac{1}{2}\,\vec{K}_{\mu\nu}\,\dot{x}^{\mu}\dot{x}^{\nu} = \vec{p} \times \vec{j} + \left(\frac{q^2}{4m} - 4mE\right)\frac{\vec{r}}{r}$$

where

$$E = \frac{1}{2} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$$

is the energy.
The components $K_{i\mu\nu}$ are **Stäckel-Killing tensors**

$$K_{i\mu\nu} - \frac{1}{8m}(R_{4\mu}R_{i\nu} + R_{4\nu}R_{i\mu}) = m\left(f_{Y\,\mu\lambda}f_i{}^{\lambda}{}_{\nu} + f_{Y\,\nu\lambda}f_i{}^{\lambda}{}_{\mu}\right).$$

Spinning Taub-NUT space

Angular momentum, "relative electric charge"

$$\vec{J} = \vec{B} + \vec{j}, \quad J_4 = B_4 + q$$

where $\vec{J} = (J_1, J_2, J_3), \vec{B} = (B_1, B_2, B_3)$ and the spin corrections are

$$B_A = \frac{i}{2} R_{A[\mu;\nu]} \psi^\mu \psi^\nu$$

Supercharges from covariantly constant K-Y and Q_0 realize the $N = 4$ supersymmetry algebra:

$$\{Q_A, Q_B\} = -2i\delta_{AB} H \quad , \quad A, B = 0, \dots, 3$$

Hyper-Kähler geometry of the Taub-NUT manifold !
Runge-Lenz vector in the spinning case

$$\mathcal{K}_i = 2m \left(-i\{Q_Y, Q_i\} + \frac{1}{8m^2} J_i J_4 \right) .$$

Dirac equation in the Taub-NUT space

Dirac matrices $\{\hat{\gamma}^{\hat{\alpha}}, \hat{\gamma}^{\hat{\beta}}\} = 2\delta^{\hat{\alpha}\hat{\beta}}$
Standard Dirac operator

$$D_s = \hat{\gamma}^{\hat{\alpha}}\hat{\nabla}_{\hat{\alpha}} = i\sqrt{V}\vec{\hat{\gamma}} \cdot \vec{P} + \frac{i}{\sqrt{V}}\hat{\gamma}^4 P_4 + \frac{i}{2}V\sqrt{V}\hat{\gamma}^4 \vec{\Sigma}^* \cdot \vec{B}$$

where

$$\hat{\nabla}_i = i\sqrt{V}P_i + \frac{i}{2}V\sqrt{V}\varepsilon_{ijk}\Sigma_j^* B_k\,, \quad \hat{\nabla}_4 = \frac{i}{\sqrt{V}}P_4 - \frac{i}{2}V\sqrt{V}\vec{\Sigma}^* \cdot \vec{B}\,.$$

momentum operators

$$P_i = -i(\partial_i - A_i\partial_4) \ \ , \ \ P_4 = -i\partial_4$$

spin connection ($S^{\hat{\alpha}\hat{\beta}} = -i[\hat{\gamma}^{\hat{\alpha}}, \hat{\gamma}^{\hat{\beta}}]/4.$)

$$\Sigma_i^* = S_i + \frac{i}{2}\hat{\gamma}^4\hat{\gamma}^i\,, \quad S_i = \frac{1}{2}\varepsilon_{ijk}S^{jk}$$

Hamiltonian operator of the massless Dirac field

$$H = \hat{\gamma}^5 D_s = \left(\begin{array}{cc} 0 & V\pi^* \dfrac{1}{\sqrt{V}} \\ \sqrt{V}\pi & 0 \end{array} \right) .$$

where

$$\pi = \sigma_P - \frac{iP_4}{V} , \quad \pi^* = \sigma_P + \frac{iP_4}{V} , \quad \sigma_P = \vec{\sigma} \cdot \vec{P}$$

Klein-Gordon operator

$$\Delta = -\nabla_\mu g^{\mu\nu} \nabla_\nu = V\,\pi^*\pi = V\vec{P}^{\,2} + \frac{1}{V} {P_4}^2 .$$

Total angular momentum $\vec{J} = \vec{L} + \vec{S}$ where the orbital angular momentum is

$$\vec{L} = \vec{x} \times \vec{P} - 4m\frac{\vec{x}}{r} P_4 .$$

Dirac-type operators are constructed from K-Y tensors $f_i (i = 1, 2, 3)$ and f_Y.

- Q_i from covariantly constant K-Y f_i

$$Q_i = -i f_{i\,\hat{\alpha}\hat{\beta}} \hat{\gamma}^{\hat{\alpha}} \hat{\nabla}^{\hat{\beta}}$$

  $N = 4$ superalgebra, including $Q_0 = i D_s = i \hat{\gamma}^5 H$:

$$\{Q_A, Q_B\} = 2 \delta_{AB} H^2 \,, \quad A, B, ... = 0, 1, 2, 3$$

  linked to the hyper-Kähler geometry of the Taub-NUT space.

- Q_Y constructed from f_Y

Runge-Lenz operator

$$N_i = m\,\{Q_Y\,,\,Q_i\} - J_i P_4.$$

Commutation relations

$$\begin{aligned}
[N_i,\,P_4] &= 0\,, \quad [N_i,\,J_j] = i\varepsilon_{ijk}N_k\,, \\
[N_i,\,Q_0] &= 0\,, \quad [N_i,\,Q_j] = i\varepsilon_{ijk}Q_k P_4\,, \\
[N_i,\,N_j] &= i\varepsilon_{ijk}J_k F^2 + \frac{i}{2}\varepsilon_{ijk}Q_i H
\end{aligned}$$

where $F^2 = {P_4}^2 - H^2$.
Redefine the components of the Runge-Lenz operator

$$\mathcal{K}_i = N_i + \frac{1}{2}H^{-1}(F - P_4)Q_i$$

having the desired commutation relation

$$[\mathcal{K}_i,\,\mathcal{K}_j] = i\varepsilon_{ijk}J_k F^2\,.$$

Gravitational anomalies

Classical motions a S-K tensor $K_{\mu\nu}$ generate a quadratic constant of motion

$$K = K_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu\,.$$

Quantum operator

$$\mathcal{K} = D_\mu K^{\mu\nu} D_\nu$$

Scalar Laplacian

$$\mathcal{H} = D_\mu D^\mu$$

Evaluate the commutator

$$\begin{aligned}[D_\mu D^\mu, \mathcal{K}] = \qquad & 2K^{\mu\nu;\lambda} D_{(\mu} D_\nu D_{\lambda)} + 3{K^{(\mu\nu;\lambda)}}_{;\lambda} D_{(\mu} D_{\nu)} \\ & + \{\frac{1}{2} g_{\lambda\sigma}(K_{(\lambda\sigma;\mu);\nu} - K_{(\lambda\sigma;\nu);\mu}) - \frac{4}{3} {K_\lambda}^{[\mu} R^{\nu]\lambda}\}_{;\nu} D_\mu\end{aligned}$$

Hidden symmetry of the quantized system

$$[\mathcal{H}, \mathcal{K}] = -\frac{4}{3}\{{K_\lambda}^{[\mu} R^{\nu]\lambda}\}_{;\nu} D_\mu$$

On a generic curved spacetime there appears a *gravitational quantum anomaly* proportional to a contraction of the S-K tensor $K_{\mu\nu}$ with the Ricci tensor $R_{\mu\nu}$.

Integrability condition for K-Y tensors of valence $r = 2$

$$R_{\mu\nu[\sigma}{}^{\tau} f_{\rho]\tau} + R_{\sigma\rho[\mu}{}^{\tau} f_{\nu]\tau} = 0\,.$$

Contracting this integrability condition on the Riemann tensor

$$f^{\rho}{}_{(\mu} R_{\nu)\rho} = 0.$$

Suppose

$$K_{\mu\nu} = f_{\mu\rho} f_{\nu}{}^{\rho}.$$

Integrability condition becomes

$$K^{\rho}{}_{[\mu} R_{\nu]\rho} = 0.$$

The operators constructed from symmetric S-K tensors are in general a source of gravitational anomalies for scalar fields. However, when the S-K tensor admits a decomposition in terms of K-Y tensors the anomaly disappears. owing to the existence of the K-Y tensors.

Extended Taub-NUT spaces

Extended Taub-NUT metric defined on $\mathbb{R}^4 - \{0\}$

$$ds_K^2 = f(r)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2) + g(r)(d\chi + \cos\theta\, d\varphi)^2$$

$f(r)$ and $g(r)$ are functions given, with constants a, b, c, d, by

$$f(r) = \frac{a + br}{r} \quad , \quad g(r) = \frac{ar + br^2}{1 + cr + dr^2}\,.$$

If one takes the constants

$$c = \frac{2b}{a}, d = \frac{b^2}{a^2}$$

the extended Taub-NUT metric becomes the original Euclidean Taub-NUT metric up to a constant factor.

Extended Taub-NUT space still admits a Runge-Lenz vector

$$\vec{K} = \vec{p} \times \vec{j} + \kappa \frac{\vec{r}}{r}\,.$$

with

$$\kappa = -a\,E + \frac{1}{2} c\,q^2$$

where the conserved energy E is

$$E = \frac{\vec{p}^{\,2}}{2f(r)} + \frac{q^2}{2g(r)}\,.$$

A direct evaluation shows that the commutator $[\mathcal{H}, \mathcal{K}]$ does not vanish implying the presence of the gravitational anomaly.

To illustrate for the third S-K $K_3^{\mu\nu}$ tensor in spherical coordinates

$$
\begin{aligned}
K_3^{rr} &= -\frac{ar\cos\theta}{2(a+br)} \\
K_3^{r\theta} = K_3^{\theta r} &= \frac{\sin\theta}{2} \\
K_3^{\theta\theta} &= \frac{(a+2br)\cos\theta}{2r(a+br)} \\
K_3^{\varphi\varphi} &= \frac{(a+2br)\cot\theta\csc\theta}{2r(a+br)} \\
K_3^{\varphi\chi} = K_3^{\chi\varphi} &= -\frac{(2a+3br+br\cos(2\theta)\csc^2\theta}{4r(a+br)} \\
K_3^{\chi\chi} &= \frac{(a-adr^2+br(2+cr)+(a+2br))\cot^2\theta)\cos\theta}{2r(a+br)}\,.
\end{aligned}
$$

Just to exemplify, we write down from the commutator $[\mathcal{H}, \mathcal{K}]$ the function which multiplies the covariant derivative D_r

$$
\begin{aligned}
&\frac{3r\cos\theta}{4(a+br)^3(1+cr+dr^2)^2}\,\cdot \\
&\{-2bd(2ad-bc)r^3 + \\
&[3bd(2b-ac)-(ad+bc)(2ad-bc)]r^2 + \\
&2(ad+bc)(2b-ac)r + a(2ad-bc) + (b+ac)(2b-ac)\}
\end{aligned}
$$

As it is expected there is no gravitational anomaly for the standard Euclidean Taub-NUT metric $(c = \frac{2b}{a}, d = \frac{b^2}{a^2})$.

Index formulas and axial anomalies

Let (M, g) be a closed Riemannian spin manifold of odd dimension, Σ the spinor bundle and D the (self-adjoint) Dirac operator on M. Let

$$\Pi^{\pm} : \mathcal{C}^{\infty}(M, \Sigma) \to \mathcal{C}^{\infty}(M, \Sigma)$$

be the spectral projections associated to D and the intervals $[0, \infty)$, respectively $(-\infty, 0]$. If ϕ_T is an eigenspinor of D of eigenvalue T, then

$$\Pi^{+}(\phi_T) = \begin{cases} \phi_T & \text{if } T \geq 0; \\ 0 & \text{otherwise;} \end{cases} \qquad \Pi^{-}(\phi_T) = \begin{cases} \phi_T & \text{if } T \leq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Let now g^X be a Riemannian metric on the cylinder $X := [l_1, l_2] \times M$. Endow X with the product orientation, so that $\{l_1\} \times M$ is negatively oriented and $\{l_2\} \times M$ is positively oriented inside X. Let D^+ be the chiral Dirac operator on X. For each $t \in [l_1, l_2]$ let g_t be the metric on M obtained by restricting g^X to $\{t\} \times M$. We denote by Σ_t the spinor bundle over (M, g_t) and by D_t, $\Pi_t^{\pm}$ the Dirac operator and the spectral projections with respect to the metric g_t.

There exist canonical identifications of the spinor bundle Σ_t with $\Sigma^{\pm}(X)_{|\{t\}\times M}$. Denote by ϕ_t the restriction of a positive spinor from X to $\{t\} \times M$.

Theorem 1 *Let $X = [l_1, l_2] \times M$ be a product spin manifold with a smooth metric g^X as above. Set*

$$\mathcal{C}^\infty(X, \Sigma^+, \Pi^+) := \{\phi \in \mathcal{C}^\infty(X, \Sigma^+); \Pi^-_{l_1}\phi_{l_1} = 0, \Pi^+_{l_2}\phi_{l_2} = 0\}.$$

Then the operator

$$D^+ : \mathcal{C}^\infty(X, \Sigma^+, \Pi^+) \to \mathcal{C}^\infty(X, \Sigma^-)$$

is Fredholm, of index equal to the spectral flow of the pair (D_{l_1}, D_{l_2}).

Berger introduced a family of Riemannian metrics on the 3-sphere as follows: The Hopf fibration $h : S^3 \to S^2$ defines a vertical subbundle V in TS^3. Let $H \subset TS^3$ be the orthogonal complement with respect to the standard metric g_{S^3}. Then h becomes a Riemannian submersion when we endow S^3 with its standard metric, and S^2 with 4 times its standard metric. Let g_H, g_V denote the restriction of g_{S^3} to the horizontal, respectively the vertical bundle.
For each constant $\lambda > 0$ the Berger metric g_λ on S^3 is defined by the formula

$$g_\lambda := g_H + \lambda^2 g_V.$$

Lemma 2 *For $\lambda < 2$, D_λ has no harmonic spinors.*

Proof: It is easy to compute the scalar curvature of g_λ. Namely, $\kappa(g_\lambda)$ is constant on S^3, $\kappa(g_\lambda) = (4 - \lambda^2)/12$. In particular $\kappa(g_\lambda)$ is positive for $\lambda < 2$. Lichnerowicz's formula proves then that $\ker D_\lambda = 0$. ■

Theorem 3 *Let*

$$\Lambda(\lambda) := \{(p,q) \in {\mathbb{N}^*}^2; \lambda^2 = 2\sqrt{(p-q)^2 + 4\lambda^2 pq}\}.$$

Then

$$\dim\ker(D_\lambda) = N(\lambda) := \sum_{(p,q)\in\Lambda(\lambda)} p+q.$$

If $N(\lambda) > 0$ there exists $\epsilon > 0$ such that for $|t-\lambda| < \epsilon$, the "small" eigenvalues of D_t are given by families

$$T(t,p,q) := \frac{t}{2} - \sqrt{\frac{(p-q)^2}{t^2} + 4pq}, \quad (p,q) \in \Lambda(\lambda)$$

with multiplicity $p+q$.

In particular, the first harmonic spinors appear for $\lambda = 4$ where the kernel of D_4 is two-dimensional. Moreover, the set of those $\lambda \in (0,\infty)$ for which $N(\lambda) \neq 0$ is discrete. For $l > 0$ set

$$S(l) := \sum_{\lambda \leq l} N(\lambda).$$

Corollary 4 *The spectral flow of the family $\{D_t\}_{t\in[l_1,l_2]}$ of Berger Dirac operators equals $S(l_2) - S(l_1)$.*

Proof: By differentiating $T(t,p,q)$ we see that the function $t \to T(t,p,q)$ is strictly increasing, so the spectral flow of the family $\{D_t\}$ across λ is precisely $N(\lambda)$. ■

For the extended Taub-NUT metric ds_K^2 on $\mathbb{R}^4 \setminus \{0\} \simeq (0,\infty) \times S^3$ in terms of the Berger metrics

$$ds_K^2 = (ar + br^2)\left(\frac{dr^2}{r^2} + 4g_{\lambda(r)}\right),$$

where

$$\lambda(r) := \frac{1}{\sqrt{1 + cr + dr^2}}$$

Axial anomalies translate to Dirac operators with non-vanishing index. We are interested in the chiral Dirac operator on a annular piece of $\mathbb{R}^4 \setminus \{0\}$. First set $X_{l_1,l_2} := [l_1, l_2] \times S^3 \subset \mathbb{R}^4 \setminus \{0\}$ with the induced extended Taub-NUT metric.

Theorem 5 *The index of* D^+ *over* (X_{l_1,l_2}, ds^2_K) *with the APS boundary condition is*

$$\mathrm{index}(D^+) = S(\lambda(l_2)) - S(\lambda(l_1))$$

Proof: By Theorem 1 the index is equal to the spectral flow of the pair of boundary Dirac operators. Now the metrics on the boundary spheres are constant multiples of the Berger metrics $g_{\lambda(l_1)}$, respectively $g_{\lambda(l_2)}$. The spectral flow of a path of conformal metrics (even with non-constant conformal factor) vanishes by the conformal invariance of the space of harmonic spinors . Thus the spectral flow can be computed using the pair of metrics $g_{\lambda(l_1)}$ and $g_{\lambda(l_2)}$. The conclusion follows from Corollary 4. ■

Corollary 6 *If* $c > -\frac{\sqrt{15d}}{2}$ *then the extended Taub-NUT metric does not contribute to the axial anomaly on any annular domain (i.e., the index of the Dirac operator with APS boundary condition vanishes).*

Proof: The hypothesis implies that $\lambda(r) < 4$ for all $r > 0$. From the remark following Theorem 3 we see that $S(\lambda(l_1)) = S(\lambda(l_2)) = 0$. ■

References

Pseudo-classical models

- F. A. Berezin and M. S. Marinov, *Ann. Phys.* **104**, 336 (1977).
- G.W.Gibbons, R.H.Rietdijk and J.W.van Holten, *Nucl. Phys.* **B404**, 42 (1993).
- D. Vaman and M. Visinescu, *Phys. Rev.* **D54**, 1398 (1996).
- D. Vaman and M. Visinescu, *Fortschr. Phys.* **47**, 493 (1999).
- M. Visinescu, *Phys. Lett.* **B339**, 28 (1994).
- M. Visinescu, *Fortschr. Phys.* **B48**, 229 (2000).

Euclidean Taub-NUT space

- G. W. Gibbons and P. J. Ruback, *Commun. Math. Phys.* **115**, 267 (1988).
- G. W. Gibbons and N. S. Manton, *Nucl. Phys.* **B274**, 183 (1986).

Extended Euclidean Taub-NUT metrics

- T. Iwai and N. Katayama, *J. Geom. Phys.* **12**, 55 (1993).
- T. Iwai and N. Katayama, *J. Math. Phys.* **35**, 2914 (1994).
- T. Iwai and N. Katayama, *J. Phys. A: Math. Gen.* **27**, 3179 (1994).
- Y. Miyake, *Osaka J. Math.* **32**, 659 (1995).

Dirac-type operators

- B. Carter and R. G. McLenaghan, *Phys. Rev.* **D19**, 1093 (1979).
- B. Carter, *Phys. Rev.* **D16**, 3395 (1977).
- R. G. McLenaghan and Ph. Spindel, *Phys. Rev.* **D20**, 409 (1979).
- M. Cariglia, *Class. Quantum Grav.* **21**, 1051 (2004).
- I. I. Cotăescu and M. Visinescu, *Mod. Phys. Lett. A* **15**, 145 (2000).
- I. I. Cotăescu and M. Visinescu, *Int. J. Mod. Phys. A* **16**, 1743 (2001).
- I. I. Cotăescu and M. Visinescu, *Phys. Lett.* **B502**, 229 (2001).
- I. I. Cotăescu and M. Visinescu, *Class. Quantum Grav.* **18**, 3383 (2001).
- I. I. Cotăescu and M. Visinescu, *Class. Quantum Grav.* **21**, 11 (2004).

APS - axial anomalies

- M. F. Atiyah, V. K. Patodi and I. M. Singer, *Math. Proc. Cambridge Philos. Soc.* **77**, 43 (1975).
- M. F. Atiyah, V. K. Patodi and I. M. Singer, *Math. Proc. Cambridge Philos. Soc.* **79**, 71 (1976). H. Römer B. Schroer, *Phys. Lett.* **71B**, 182 (1977).
- C. N. Pope, *Nucl. Phys.* **B141**, 432 (1978).
- T. Eguchi, P. B. Gilkey and A. J. Hanson, *Phys. Rev.* **D17**, 423 (1978).
- K. Peeters and A. Waldron, *J. H. E. P.* **992**, 024 (1999).
- I. I. Cotăescu, S. Moroianu and M. Visinescu, *J. Phys. A: Math. Gen.* **38**, 7005 (2005) ; *arXiv: math-ph/0411047.*
- S. Moroianu and M. Visinescu, *J. Phys. A: Math. Gen.* **39**, 6575 (2006) ; *arXiv: math-ph/0511025.*