The Virasoro Algebra and Some Exceptional Lie and Finite Groups

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Overview

Aim of Talk: To explain an interesting connection between properties of the Virasoro algebra and a number of exceptional Lie and finite groups.

- The Virasoro algebra and the vacuum Verma module.
- The Kac determinant and its relationship to certain exceptional Lie and finite groups.
- Vertex Operator Algebras (VOAs) and the Li-Zamalodchikov metric
- VOA automorphism group invariant quadratic Casimirs.
- Expansions of rational matrix elements.

The Virasoro Algebra and Verma Modules

Virasoro Algebra *Vir* **of Central Charge** C

$$[L_m, L_n] = (m-n)L_{m+n} + (m^3 - m)\frac{C}{12}\delta_{m,-n}, \quad [L_m, C] = 0.$$

The Vacuum Verma Module $V(C,0)$**.** Let $\mathbf{1}$ denote the vacuum vector where

$$L_0\mathbf{1} = 0, \quad L_{-1}\mathbf{1} = 0, \quad L_1\mathbf{1} = 0$$

Consider the Virasoro descendents of the vacuum

$$V(C,0) = \{L_{-n_1}L_{-n_2}\ldots L_{-n_k}\mathbf{1} | n_1 \geq n_2 \geq \ldots \geq n_k \geq 2\}$$

$V(C,0)$ is a module for *Vir* graded by L_0 where

$$L_0 L_{-n_1}\ldots L_{-n_k}\mathbf{1} = (n_1 + \ldots + n_k)L_{-n_1}\ldots L_{-n_k}\mathbf{1}$$

$n = n_1 + \ldots + n_k \geq 0$ is the Virasoro level.

Then

$$V(C,0) = \bigoplus_{n\geq 0} V^{(n)}(C,0)$$

where $V^{(n)}(C,0)$ denotes the vectors of level n.

General Verma Module $V(C,h)$. Let v denote a vector such that

$$L_n v = h\delta_{n,0} v \quad \text{for all } n \geq 0$$

v is called a Primary Vector of level h. Then for each primary vector we obtain a module $V(C,h)$ for Vir generated by the Virasoro descendents of v

$$\{L_{-n_1}L_{-n_2}\ldots L_{-n_k}v | n_1 \geq n_2 \geq \ldots \geq n_k \geq 1\}$$

The Kac Determinant

We consider $V = V(C,0)$ only here. V is irreducible provided no descendent vector is itself a primary vector.

Define a symmetric bilinear form $\langle , \rangle$ on V with $\langle \mathbf{1}, \mathbf{1} \rangle = 1$ where

$$\langle L_{-n}u, v \rangle = \langle u, L_n v \rangle.$$

for arbitrary vectors u, v. Note $\langle u, v \rangle = 0$ for u, v of different Virasoro level.

Consider the Gram matrix $(\langle u, v \rangle)$ for all vacuum descendents u, v. Then V is irreducible iff the Gram matrix is invertible i.e. The level n Kac determinant

$$\det_{V^{(n)}}(\langle u, v \rangle)$$

is non-vanishing (Kac, Feigen and Fuchs).

Level 2: $V^{(2)} = \{\omega = L_{-2}\mathbf{1}\}$. ω is called the conformal vector. The Gram matrix is

$$\langle\omega,\omega\rangle = \langle\mathbf{1}, L_2 L_{-2}\mathbf{1}\rangle = \langle\mathbf{1}, (4L_0 + \frac{C}{12}(8-2))\mathbf{1}\rangle = \frac{C}{2}$$

Level 4: $V^{(4)} = \{L_{-2}L_{-2}\mathbf{1}, L_{-4}\mathbf{1}\}$ with Gram matrix

$$\begin{bmatrix} C(4+\frac{1}{2}C) & 3C \\ 3C & 5C \end{bmatrix}$$

and Kac determinant $C^2(5C+22)$.

Level 6: $\dim V^{(6)} = 4$ with Kac determinant $\frac{3}{4}C^4(5C+22)^2(2C-1)(7C+68)$.

Level 8: $\dim V^{(8)} = 7$ with Kac determinant

$$3C^7(5C+22)^4(2C-1)^2(7C+68)^2(3C+46)(5C+3)$$

Level 10: $\dim V^{(10)} = 12$ with Kac determinant

$$\frac{225}{2}C^{12}(5C+22)^8(2C-1)^5(7C+68)^4(3C+46)^2(5C+3)^2(11C+232)$$

Some Exceptional Group Numerology

Consider the prime factors of the Kac determinant for level ≤ 10 for particular values of C. We observe some coincidences with properties of a number of exceptional Lie and finite groups.

Deligne's Exceptional Lie groups: $A_1, A_2, G_2, D_4, F_4, E_6, E_7, E_8$. The dimension of the adjoint representation of each of these groups for dual Coxeter no $h^\vee$ is (Vogel)

$$d = \frac{2(5h^\vee - 6)(h^\vee + 1)}{h^\vee + 6}$$

Compare d to the level 4 Kac det factors C and $5C + 22$ for certain values of C:

	A_1	A_2	G_2	D_4	F_4	E_6	E_7	E_8
$h^\vee$	2	3	4	6	9	12	18	30
d	3	2^3	2.7	$2^2.7$	$2^2.13$	$2.3.13$	7.19	$2^3.31$
C	1	2	$\frac{2.7}{5}$	2^2	$\frac{2.13}{5}$	2.3	7	2^3
$5C+22$	3^3	2^5	$2^2.3^2$	$2.3.7$	$2^4.3$	$2^2.13$	3.19	2.31

Every prime divisor of d is a prime divisor of the numerator of the Kac det.

Some Exceptional Finite Groups. The prime divisors of the order of a number of exceptional finite groups are also related to the Kac determinant factors. We highlight three examples.

The Monster Simple Group ***M*.** The classification theorem of finite simple groups states that a finite simple group is either one of several infinite families of simple groups (e.g. the alternating groups A_n for $n \geq 5$) or else is one of 26 sporadic finite simple groups. The largest sporadic group is the Monster group M of order

$$|\mathrm{M}| = 2^{46}.3^{20}.5^{9}.7^{6}.11^{2}.13^{3}.17.19.23.29.31.41.47.59.71 \simeq 8 \times 10^{53}$$

The two lowest dimensional irreducible representations are of dimension

$$d_1 = 196883 = 47.59.71$$

$$d_2 = 21296876 = 2^2.31.41.59.71$$

Consider the level 10 Kac determinant factors for $C = 24$

C	$5C+22$	$2C-1$	$7C+68$	$3C+46$	$5C+3$	$11C+232$
$2^3.3$	2.71	47	$2^2.59$	2.59	3.41	$2^4.31$

All of the prime divisors $2, 31, 41, 47, 59, 71$ of d_1 and d_2 are divisors of the Kac det!

The Baby Monster Simple Group B. The second largest sporadic group is the Baby Monster group B of order

$$|\mathrm{B}| = 2^{41}.\,3^{13}.\,5^{6}.\,7^{2}.\,11.\,13.\,17.\,19.\,23.\,31.\,47$$

Consider the level 6 Kac determinant factors for $C = 23\frac{1}{2}$

C	$5C+22$	$2C-1$	$7C+68$
$\frac{47}{2}$	$\frac{3^2.31}{2}$	$2.\,23$	$\frac{3.5.31}{2}$

The prime divisors $2, 3, 5, 23, 31, 47$ are divisors of the numerator of the Kac det.

The Simple Group $O^+_{10}(2)$. This group has order

$$|O^+_{10}(2)| = 2^{20}.\,3^{5}.\,5^{2}.\,7.\,17.\,31$$

Consider the level 6 Kac determinant factors for $C = 8$

C	$5C+22$	$2C-1$	$7C+68$
2^3	$2.\,31$	$3.\,5$	$2^2.\,31$

The prime divisors $2, 3, 5, 31$ are divisors of the Kac det.

What is going on?

Vertex Operator Algebras

These observations can be understood in the context of Vertex Operator Algebras (Borcherds, Frenkel, Lepowsky, Meurmann, Goddard,...). The basic idea is that the groups appearing above arise as symmetry groups of particular VOAs. The relationship with the Kac determinant (and many other properties) follows from the existence of particular group invariant vectors which are Virasoro descendents of the vacuum.

A Vertex Operator Algebra (VOA) consists of a Z-graded vector space $V = \bigoplus_{k\geq 0} V^{(k)}$ with $\dim V^{(k)} < \infty$ and with the following properties:

Vacuum. $V^{(0)} = \{\mathbf{1}\}$ for vacuum vector $\mathbf{1}$.

Vertex Operators (State-Field Correspondence). For each $a \in V^{(k)}$ we have a vertex operator

$$Y(a,z) = \sum_{n\in Z} a_n z^{-n-k},$$

with component operators (modes) $a_n \in \mathrm{End}V$ such that

$$Y(a,z).\mathbf{1}|_{z=0} = a_{-k}.\mathbf{1} = a$$

Here z is a formal variable (taken as a complex number in physics).

Virasoro Structure. For the conformal vector $\omega \in V^{(2)}$ we have

$$Y(\omega,z) = \sum_{n\in Z} L_n z^{-n-2}$$

where L_n forms a Virasoro algebra of central charge C.

The Z –grading is determined by L_0 i.e. $V^{(k)} = \{a \in V | L_0 a = ka\}$.

L_{-1} acts as translation operator with

$$Y(L_{-1}a,z) = \partial_z Y(a,z) \quad \text{i.e. } (L_{-1}a)_n = -(n+k)a_n \quad \text{for } a \in V^{(k)}$$

Locality. For any pair of vertex operators we have for integer $N \gg 0$.

$$(x-y)^N [Y(a,x), Y(b,y)] = 0$$

These axioms easily lead to the following basic VOA properties:

Translation. For any $a \in V$ then for $|y| < |x|$ (formally expanding in y/x)

$$e^{yL_{-1}}Y(a,x)e^{-yL_{-1}} = Y(a,x+y)$$

Skew-symmetry. For $a,b \in V$ then

$$Y(a,z)b = e^{zL_{-1}}Y(b,-z)a.$$

Associativity. For $a,b \in V$ then for $|x-y| < |y| < |x|$.

$$Y(a,x)Y(b,y) = Y(Y(a,x-y)b,y)$$

Borcherd's Commutator Formula. For $a \in V^{(k)}$ and $b \in V$ then

$$[a_m,b_n] = \sum_{j\geq 0} \binom{m+k-1}{j} (a_{j-k+1}b)_{m+n}.$$

Example. For $a = \omega \in V^{(2)}$ and $m = 0$ and any b

$$[L_0,b_n] = (L_{-1}b)_n + (L_0 b)_n = -nb_n$$

i.e. $b_n : V^{(m)} \to V^{(m-n)}$. In particular, the zero mode b_0 is a linear operator on $V^{(m)}$.

Similarly for all m and any primary vector $b \in V^{(h)}$

$$[L_m,b_n] = ((h-1)m-n)b_{m+n}.$$

Invariant Bilinear Form - Li-Zamalodchikov metric. Assume that $V^{(0)} = \{\mathbf{1}\}$ and $L_1 v = 0$ for all $v \in V^{(1)}$. Then there exists a unique invariant bilinear form $\langle , \rangle$, which we call the Li-Z metric, with $\langle \mathbf{1}, \mathbf{1} \rangle = 1$ where (Li)

$$\langle L_n a, b \rangle = \langle a, L_{-n} b \rangle \quad \text{for all } a, b \in V$$

$$\langle c_n a, b \rangle = (-1)^k \langle a, c_{-n} b \rangle \quad \text{for all } a, b \in V, \quad \text{ and primary } c \in V^{(k)}$$

$\langle , \rangle$ symmetric (Frenkel Huang Lepowsky). $\langle , \rangle$ non-degenerate iff V is semisimple (Li).

Lie and Kac-Moody Algebras. Consider $a, b \in V^{(1)}$. Define $[a, b] = ad(a)b = a_0 b$ $(= -b_0 a$ by skew-symmetry) and which satisfies the Jacobi identity. Then $V^{(1)}$ is a Lie algebra. Furthermore $\langle a, b \rangle$ is an invariant invertible symmetric bilinear form

$$\langle [a, b], c \rangle = \langle -b_0 a, c \rangle = \langle a, b_0 c \rangle = \langle a, [b, c] \rangle$$

The full commutator formula gives a Kac-Moody algebra.

$$[a_m, b_n] = (a_0 b)_{m+n} + (a_1 b)_{m+n} = [a, b]_{m+n} - m \langle a, b \rangle \delta_{m+n,0}$$

Griess Algebras. Suppose $\dim V^{(1)} = 0$. Consider $a, b \in V^{(2)}$. Then $a_2 b = \langle a, b \rangle \mathbf{1}$. Skew-symmetry implies $a_0 b = b_0 a$. Thus we define $a \bullet b = a_0 b$ to form a commutative non-associative Griess algebra on $V^{(2)}$ with invariant bilinear form

$$\langle a \bullet b, c \rangle = \langle b, a \bullet c \rangle \qquad \text{for all } a, b, c \in V^{(2)}$$

The Automorphism Group of a VOA

$g \in GL(V)$ is an element of the VOA automorphism group $Aut(V)$ iff

$$gY(a,z)g^{-1} = Y(ga,z) \quad \text{for all } a \in V$$

with $g\omega = \omega$ the conformal vector. Thus the grading is preserved by $Aut(V)$. Furthermore, every Virasoro descendent of the vacuum is invariant under $Aut(V)$.

The Li-Z metric is automorphism group invariant

$$\langle ga, gb\rangle = \langle a, b\rangle \quad \text{for all } a, b \in V$$

For VOAs with $\dim V^{(1)} > 0$ then $Aut(V)$ contains continuous symmetries generated by the Lie algebra $V^{(1)}$.

VOAs for which $\dim V^{(1)} = 0$ are of particular interest. Examples include the Moonshine Module $V^{\natural}$ of central charge $C = 24$ where $Aut(V) = M$, the Monster group. In this case, $V^{(2)}$ is the original Griess algebra of dimension $196884 = 1 + 196883$ where ω is M invariant. (Frenkel, Lepowsky and Meurman)

Other examples include VOAs with $C = 23\frac{1}{2}$ with $Aut(V) = B$, the Baby Monster where $\dim V^{(2)} = 1 + 96255$ (Hoen) and $C = 8$ with $Aut(V) = O^{+}_{10}(2)$ and $\dim V^{(2)} = 1 + 155$ (Griess).

Quadratic Casimirs

Consider a VOA with an invertible Li-Z metric and with $d = \dim V^{(1)} > 0$. Let $V^{(1)}$ have a basis $\{a_\alpha | \alpha = 1..d\}$ and dual basis $\{a^\beta | \beta = 1..d\}$ i.e. $\langle a^\alpha, a_\beta \rangle = \delta^\alpha_\beta$. We define the $Aut(V)$ invariant quadratic Casimir vectors (α summed)

$$\lambda^{(n)} = (a^\alpha)_{1-n} a_\alpha \in V^{(n)}$$

In general $\lambda^{(0)} = -d\mathbf{1}$ and $\lambda^{(1)} = 0$. Furthermore, using $[L_m, a^\alpha_n] = -na^\alpha_{m+n}$ it follows that

$$L_m \lambda^{(n)} = (n-1)\lambda^{(n-m)} \quad \text{for all } m > 0$$

Suppose $\lambda^{(n)}$ is a Virasoro descendent of the vacuum (Matsuo). Then $\lambda^{(n)}$ can be determined exactly via the invertible Li-Z metric.

Example. Suppose $\lambda^{(2)}$ is a Virasoro descendent i.e. $\lambda^{(2)} = \alpha\omega$ for some α. Hence

$$\langle \omega, \lambda^{(2)} \rangle = \alpha \langle \omega, \omega \rangle \text{ i.e. } \langle \mathbf{1}, L_2 \lambda^{(2)} \rangle = \alpha \frac{C}{2}$$

But $L_2 \lambda^{(2)} = \lambda^{(0)} = -d\mathbf{1}$ implies $\lambda^{(2)} = -\frac{2d}{C}\omega$ i.e.

$$\omega = -\frac{C}{2d} a^\alpha_{-1} a_\alpha \quad \text{"Sugawara Construction"}$$

Note that the zero mode is then $\lambda^{(2)}_0 = -\frac{2d}{C} L_0$.

Rational Matrix Elements and the $V^{(1)}$ Killing Form

Consider the following matrix element for $a^{\alpha}, a_{\beta}, b, c \in V^{(1)}$

$$F(x,y) = \langle b, Y(a^{\alpha},x)Y(a_{\alpha},y)c\rangle$$

Locality implies $F(x,y)$ must be a rational function of x,y of the form

$$F(x,y) = \frac{g(x,y)}{x^2y^2(x-y)^2}, \qquad g = A(x^4+y^4) + B(x^3y+xy^3) + Cx^2y^2$$

g is a homogeneous, symmetric polynomial of degree 4. Associativity implies

$$\begin{aligned} F(x,y) &= \langle b, Y(Y(a^{\alpha},x-y)a_{\alpha},y)c\rangle \\ &= \sum_{n\geq 0}\langle b, Y(\lambda^{(n)},y)c\rangle(x-y)^{n-2} \\ &= (x-y)^{-2}\sum_{n\geq 0}\langle b, \lambda_0^{(n)}c\rangle\left(\frac{x-y}{y}\right)^n \end{aligned}$$

Assuming $\lambda^{(2)}$ is a Virasoro descendent then $\lambda_0^{(2)} = -\frac{2d}{C}L_0$. Thus expanding in $|x-y| < |y|$ the leading terms are

$$F(x,y) = -d(x-y)^{-2}[1 + 0 + \frac{2}{C}\left(\frac{x-y}{y}\right)^2 + \ldots]\langle b,c\rangle$$

Comparing to g leads to two conditions on A,B,C.

We may alternatively expand $F(x,y)$ as follows

$$\begin{aligned} F(x,y) &= \langle b, Y(a^\alpha, x) e^{yL_{-1}} Y(c,-y) a_\alpha \rangle \quad \text{Skew-symmetry} \\ &= \langle b, e^{yL_{-1}} Y(a^\alpha, x-y) Y(c,-y) a_\alpha \rangle \quad \text{Translation} \\ &= \langle e^{yL_1} b, Y(a^\alpha, x-y) Y(c,-y) a_\alpha \rangle \quad \text{Invariant LiZ metric} \\ &= \langle b, Y(a^\alpha, x-y) Y(c,-y) a_\alpha \rangle \quad \text{Primary } b \end{aligned}$$

Expanding in $|y| < |x-y|$ the leading terms are

$$\begin{aligned} &\langle b, a^\alpha_{-1} c_1 a_\alpha \rangle (-y)^{-2} + \langle b, a^\alpha_0 c_0 a_\alpha \rangle (-y)^1 (x-y)^1 + \ldots \\ &= -\langle b, c \rangle y^{-2} - \langle a^\alpha, b_0 c_0 a_\alpha \rangle y^{-1} (x-y)^{-1} + \ldots \\ &= -\langle b, c \rangle y^{-2} - Tr_{V^{(1)}}(b_0 c_0) y^{-1} (x-y)^{-1} + \ldots \end{aligned}$$

using $c_1 a_\alpha = -\langle c, a_\alpha \rangle \mathbf{1}$ and $a^\alpha_0 b = -b_0 a^\alpha$ etc.

The leading term determines g completely. The subleading term is the Killing form of the Lie algebra $V^{(1)}$

$$K(b,c) = Tr_{V^{(1)}}(ad(b)ad(c)) = -2\,\frac{(d-C)}{C}\langle b, c \rangle$$

For $d \neq C$, K is invertible and $V^{(1)}$ is semi-simple. (Schellekens, Dong and Mason, T)

Deligne's Exceptional Lie Groups

Suppose furthermore that $\lambda^{(4)}$ is a vacuum Virasoro descendent. Then

$$\lambda^{(4)} = \frac{3d}{C(22+5C)}[4L_{-2}L_{-2}\mathbf{1} + (2+C)L_{-4}\mathbf{1}]$$

Expanding $F(x,y)$ in $|x-y| < |y|$ to the next leading terms we obtain

$$d(C) = \frac{C(22+5C)}{10-C}$$

$$K(a,b) = 12\frac{2+\mathrm{C}}{\mathrm{C}-10}\langle a,b\rangle = -2h^{\vee}\langle a,b\rangle$$

Note the necessary appearance of the Kac factors $C(22+5C)$. This is precisely the original Vogel formula for Deligne's exceptional Lie groups for dual Coxeter number

$$h^{\vee} = 6\frac{2+\mathrm{C}}{10-\mathrm{C}}$$

The **only** semi-simple Lie algebras solutions are the Deligne series. (Maruoka, Matsuo and Shimakura - with many more assumptions, T)

If $\lambda^{(6)}$ is a vacuum descendent then only $C = 1$ or $C = 8$ possible i.e. A_1 and E_8. (T)

If $\lambda^{(n)}$ a vacuum descendent for $n > 8$ then A_1 lattice VOA. (T)

Griess Algebras

Consider a VOA with an invertible Li-Z metric with $\dim V^{(1)} = 0$. Let $\hat{V}^{(2)} = V^{(2)} - \{\omega\}$ be the level 2 primary states with basis $\{a_\alpha\}$ and dual basis $\{a^\alpha\}$ with $d = \dim \hat{V}^{(2)} > 0$. We again define $Aut(V)$ invariant quadratic Casimir vectors

$$\lambda^{(n)} = a^{\alpha}_{2-n} a_\alpha \in V^{(n)}$$

with

$$\lambda^{(0)} = d\mathbf{1}, \quad \lambda^{(1)} = 0$$

$$L_m \lambda^{(n)} = (m+n-2)\lambda^{(n-m)} \ \text{ for } \ m > 0$$

Consider the matrix element for $a^\alpha, a_\beta, b, c \ \in V^{(2)}$

$$F(x,y) = \langle b, Y(a^\alpha, x) Y(a_\alpha, y) c \rangle$$

In this case $F(x,y)$ is a rational function

$$F(x,y) = \frac{g(x,y)}{x^4 y^4 (x-y)^4}$$

where $g(x,y)$ is a homogeneous, symmetric polynomial of degree 8 determined by 5 independent parameters.

Associativity implies expanding in $|x-y| < |y|$ that

$$F(x,y) = (x-y)^{-4} \sum_{n\geq 0} \langle b, \lambda_0^{(n)} c\rangle (\frac{x-y}{y})^n$$

Similarly, we may expand in $|x-y| > |y|$ to obtain

$$F(x,y) = \langle b,c\rangle y^{-4} + 0 + Tr_{\hat{V}^{(2)}}(b_0 c_0) y^{-2}(x-y)^{-2} + \ldots$$

In this case it is necessary to assume that $\lambda^{(2)} \ldots \lambda^{(4)}$ are vacuum descendents in order to determine $g(x,y)$.

Find $V^{(2)}$ is a **simple** Griess algebra via the invertible trace form on $V^{(2)}$ (T)

$$Tr_{V^{(2)}}((b \bullet c)_0) = \frac{8(d+1)}{C} \langle b,c\rangle$$

If furthermore, $\lambda^{(6)}$ is a vacuum descendent then d **is determined** (Matsuo,T)

$$d(C) = \frac{1}{2} \frac{(68+7C)(2C-1)(22+5C)}{748-55C+C^2}$$

and $\hat{V}^{(2)}$ is an **irreducible representation** of $Aut(V)$ (if finite) (T)

Examples. Reproduce dimensions of irred reps of M with $d(24) = 196883$, for B with $d(23\frac{1}{2}) = 96255$ and $O_{10}^{+}(2)$ with $d(8) = 155$.

Other results and goals

- If furthermore $\lambda^{(8)}$ (or $\lambda^{(8)}$ and $\lambda^{(10)}$) are vacuum descendents then $C = 24$. (Matsuo, T).
- $\lambda^{(12)}$ cannot be a vacuum descendent. There must exist a primary $Aut(V)$ invariant vector of level 12. This is related to existence of an $SL(2,Z)$ modular cusp form of weight 12. (T)
- Can also consider the Casimirs $\lambda^{(n)}$ for the primary vectors of level 3 $\hat{V}^{(3)} = V^{(3)} - L_{-1}V^{(2)}$ with $d = \dim \hat{V}^{(3)} > 0$. Then if $\lambda^{(2)} \ldots \lambda^{(10)}$ are vacuum descendents then $\hat{V}^{(3)}$ is an irreducible representation of $Aut(V)$ (if finite) and $d = p(C)/q(C)$ with (T)

$$p(C) = 5C(5C+22)(3C+46)(2C-1)(5C+3)(11C+232)(7C+68)$$

$$q(C) = 75\,\mathrm{C}^6 - 9945\,\mathrm{C}^5 + 472\,404\,\mathrm{C}^4 - 9055\,068\,\mathrm{C}^3 + 39\,649\,632\,\mathrm{C}^2 + 438\,468\,672\,C + 2976\,768)$$

  Since $C = 24$ we thus find $d = 21296876 = 2^2.31.41.59.71$ as obtains for the Moonshine module $V^\natural$.
- Can considerable weaken the vacuum descendent condition on $\lambda^{(n)}$.
- Prove M simple?.
- Prove Moonshine Module unique?-Frenkel, Lepowsky, Meurmann conjecture.