Singularity in Quantum Mechanics
and
the Calogero Model

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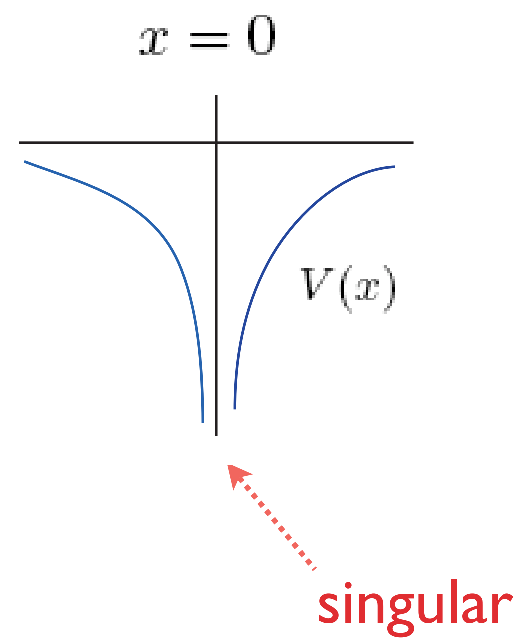
1. Singularity in quantum mechanics



Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

demand: H be self-adjoint = probability conservation

→ connection condition at the singularity

to find connection conditions ...

total probability (norm)

$$\|\psi(x,t)\|^2 = \langle \psi, \psi \rangle = 1 \qquad \langle \phi, \psi \rangle = \int_{\mathbb{R}\setminus\{0\}} dx\, \phi^*(x)\psi(x)$$

inner-product

Schrödinger equation $$i\hbar \frac{\partial}{\partial t}\psi = H\,\psi$$

$$\frac{\partial}{\partial t}\|\psi\|^2 = -\frac{1}{i\hbar}\left(\langle H\psi, \psi\rangle - \langle \psi, H\psi\rangle\right)$$

$$= -\int_{\mathbb{R}\setminus\{0\}} dx\, \frac{d}{dx} j(x) = j(+0) - j(-0) = 0$$

probability conservation

probability current $$j(x) = -\frac{i\hbar}{2m}\left((\psi^*)'\psi - \psi^*\psi'\right)(x)$$

$V(x)$ divergent $\Rightarrow$ $\psi(x), \psi'(x)$ divergent at $x = 0$

Wronskian is well-defined

$$W[\phi, \psi](x) = \phi(x)\psi'(x) - \psi(x)\phi'(x)$$

if $\psi, \phi, H\psi, H\phi$ are square integrable near $x = 0$

inner-product is finite $\langle \psi, \phi \rangle < +\infty$

$$\Rightarrow \quad \langle \phi, H\psi \rangle - \langle H\phi, \psi \rangle = \frac{\hbar^2}{2m} \left(W[\phi^*, \psi]_{+0} - W[\phi^*, \psi]_{-0} \right) < +\infty$$

Wronskians are finite separately

probability current

$$
\begin{aligned}
j(x) &\propto (\psi^*)'\psi - \psi^*\psi' = W[\psi^*, \psi] \\
&= \begin{vmatrix} \psi^* & \psi^{*\prime} \\ \psi & \psi' \end{vmatrix} = \begin{vmatrix} \psi^* & \psi^{*\prime} \\ \psi & \psi' \end{vmatrix} \begin{vmatrix} {\varphi_1}' & {\varphi_2}' \\ -\varphi_1 & -\varphi_2 \end{vmatrix} \\
&= \begin{vmatrix} \psi^*{\varphi_1}' - \psi^{*\prime}\varphi_1 & \psi^*{\varphi_2}' - \psi^{*\prime}\varphi_2 \\ \psi{\varphi_1}' - \psi'\varphi_1 & \psi{\varphi_2}' - \psi'\varphi_2 \end{vmatrix} \\
&= W[\psi^*, \varphi_1]W[\psi, \varphi_2] - W[\psi^*, \varphi_2]W[\psi, \varphi_1]
\end{aligned}
$$

with the help of

reference modes

square integrable near $x = 0$

$$
H\varphi_i(x) = E\,\varphi_i(x), \qquad W[\varphi_1, \varphi_2](x) = 1
$$

arbitrary

probability current

$$
\begin{aligned}
j(x) \propto (\psi^*)'\psi - \psi^*\psi' &= W[\psi^*, \psi] \\
&= \begin{vmatrix} \psi^* & \psi^{*\prime} \\ \psi & \psi' \end{vmatrix} = \begin{vmatrix} \psi^* & \psi^{*\prime} \\ \psi & \psi' \end{vmatrix} \begin{vmatrix} \varphi_1{}' & \varphi_2{}' \\ -\varphi_1 & -\varphi_2 \end{vmatrix} \\
&= \begin{vmatrix} \psi^*\varphi_1{}' - \psi^{*\prime}\varphi_1 & \psi^*\varphi_2{}' - \psi^{*\prime}\varphi_2 \\ \psi\varphi_1{}' - \psi'\varphi_1 & \psi\varphi_2{}' - \psi'\varphi_2 \end{vmatrix} \\
&= W[\psi^*, \varphi_1]W[\psi, \varphi_2] - W[\psi^*, \varphi_2]W[\psi, \varphi_1]
\end{aligned}
$$

with the help of

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arbitrary

probability current

$$
\begin{aligned}
j(x) &\propto (\psi^*)'\psi - \psi^*\psi' = W[\psi^*,\psi] \\
&= \begin{vmatrix} \psi^* & \psi^{*\prime} \\ \psi & \psi' \end{vmatrix} = \begin{vmatrix} \psi^* & \psi^{*\prime} \\ \psi & \psi' \end{vmatrix} \begin{vmatrix} {\varphi_1}' & {\varphi_2}' \\ -\varphi_1 & -\varphi_2 \end{vmatrix} \\
&= \begin{vmatrix} \psi^*{\varphi_1}' - \psi^{*\prime}\varphi_1 & \psi^*{\varphi_2}' - \psi^{*\prime}\varphi_2 \\ \psi{\varphi_1}' - \psi'\varphi_1 & \psi{\varphi_2}' - \psi'\varphi_2 \end{vmatrix} \\
&= W[\psi^*,\varphi_1]W[\psi,\varphi_2] - W[\psi^*,\varphi_2]W[\psi,\varphi_1]
\end{aligned}
$$

with the help of

reference modes

square integrable near $x = 0$

$$
H\varphi_i(x) = E\,\varphi_i(x), \qquad W[\varphi_1,\varphi_2](x) = 1
$$

arbitrary

probability current

ill-defined

$$j(x) \propto (\psi^*)'\psi - \psi^*\psi' = W[\psi^*, \psi]$$

$$= \begin{vmatrix} \psi^* & \psi^{*\prime} \\ \psi & \psi' \end{vmatrix} = \begin{vmatrix} \psi^* & \psi^{*\prime} \\ \psi & \psi' \end{vmatrix} \begin{vmatrix} \varphi_1' & \varphi_2' \\ -\varphi_1 & -\varphi_2 \end{vmatrix}$$

$$= \begin{vmatrix} \psi^*\varphi_1' - \psi^{*\prime}\varphi_1 & \psi^*\varphi_2' - \psi^{*\prime}\varphi_2 \\ \psi\varphi_1' - \psi'\varphi_1 & \psi\varphi_2' - \psi'\varphi_2 \end{vmatrix}$$

$$= W[\psi^*, \varphi_1] W[\psi, \varphi_2] - W[\psi^*, \varphi_2] W[\psi, \varphi_1]$$

well-defined

with the help of

reference modes

square integrable near $x = 0$

$$H\varphi_i(x) = E\,\varphi_i(x), \qquad W[\varphi_1, \varphi_2](x) = 1$$

arbitrary

probability conservation

$$j(-0) = j(+0) \iff \Psi'^{\dagger}\Psi = \Psi^{\dagger}\Psi' \iff |\Psi - iL_0\Psi'| = |\Psi + iL_0\Psi'|$$

boundary vectors

$$\Psi = \begin{pmatrix} W[\psi,\varphi_1]_{+0} \\ W[\psi,\varphi_1]_{-0} \end{pmatrix}, \qquad \Psi' = \begin{pmatrix} W[\psi,\varphi_2]_{+0} \\ -W[\psi,\varphi_2]_{-0} \end{pmatrix}$$

connection condition

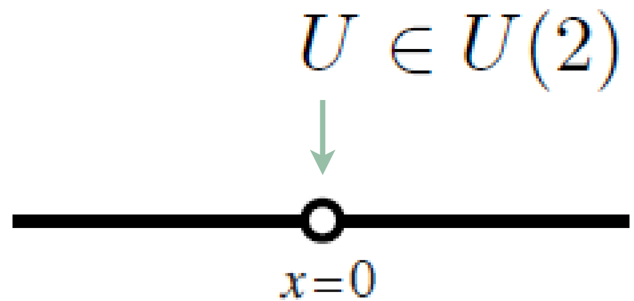
$$(U - I)\Psi + iL_0(U + I)\Psi' = 0$$

scale constant

characteristic matrix

$$U = e^{i\xi}\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} = e^{i\xi}\begin{pmatrix} \alpha_R + i\alpha_I & \beta_R + i\beta_I \\ -\beta_R + i\beta_I & \alpha_R - i\alpha_I \end{pmatrix} \in U(2)$$

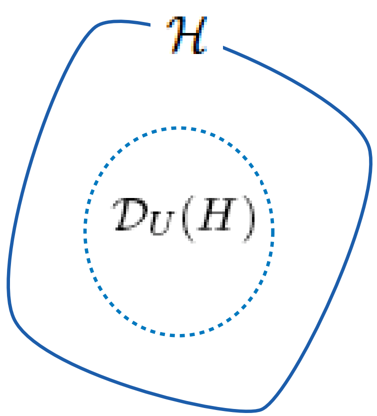
there exists a U(2) family of 'distinct' singularities



cf. theory of self-adjoint extension (inequivalent quantizations)

self-adjoint domains $\mathcal{D}_U(H) \subset \mathcal{H}$ of H

form a $U(2)$ family (from deficiency indices)



remark: self-adjoint extensions depend on how $V(x)$ diverges

boundary vectors

$$\Psi = \begin{pmatrix} W[\psi, \varphi_1]_{+0} \\ W[\psi, \varphi_1]_{-0} \end{pmatrix}, \qquad \Psi' = \begin{pmatrix} W[\psi, \varphi_2]_{+0} \\ -W[\psi, \varphi_2]_{-0} \end{pmatrix}$$

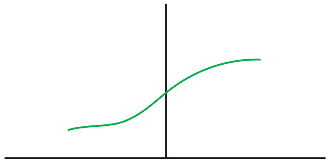
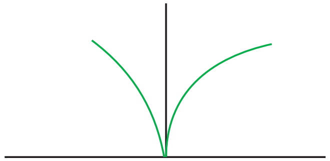
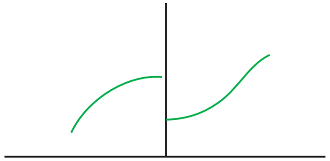
for regular $V(x)$ for $x \neq 0$

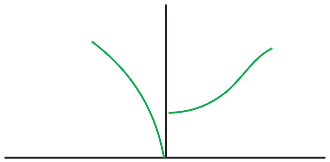
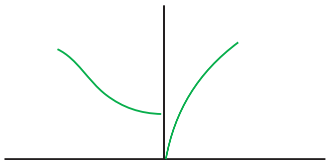
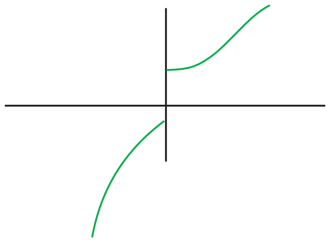
we may choose the reference modes s.t.

$$\varphi_1(\pm 0) = 0, \qquad \varphi_1'(\pm 0) = 1, \qquad \varphi_2(\pm 0) = -1, \qquad \varphi_2'(\pm 0) = 0$$

$\Rightarrow$ $$\Psi \Rightarrow \begin{pmatrix} \psi(+0) \\ \psi(-0) \end{pmatrix}, \qquad \Psi' \Rightarrow \begin{pmatrix} \psi'(+0) \\ -\psi'(-0) \end{pmatrix}$$

connection condition expressed
by linear combinations of $\psi(\pm 0)$ and $\psi'(\pm 0)$

	characteristic matrix	connection condition	
free	$U=\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\psi(+0)=\psi(-0)$ $\psi'(+0)=\psi'(-0)$	
Dirichlet	$U=-I$	$\psi(+0)=\psi(-0)=0$	
Neumann	$U=I$	$\psi'(+0)=\psi'(-0)=0$	

	characteristic matrix	connection condition	
chiral	$U = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\psi'(+0) = 0$ $\psi(-0) = 0$	
anti-chiral	$U = -\sigma_3$	$\psi(+0) = 0$ $\psi'(-0) = 0$	
Hadamard	$U = \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$\psi(+0) - (1+\sqrt{2})\psi(-0) = 0$ $\psi'(+0) + (1-\sqrt{2})\psi'(-0) = 0$	

δ-potential

$$V(x) = c\,\delta(x)$$

connection condition

$$-\frac{\hbar^2}{2m}\left[\psi'(+0) - \psi'(-0)\right] + c\,\psi(+0) = 0$$

$$\psi(+0) = \psi(-0)$$

characteristic matrix

$$U = \pm i e^{i\phi} \begin{pmatrix} \cos\phi & i\sin\phi \\ i\sin\phi & \cos\phi \end{pmatrix}$$

$$\phi = -\arctan\left(\frac{mL_0}{\hbar^2}c\right)$$

has scale constant L_0

ϵ-potential
(δ'-potential)

'opposite' to δ with discontinuity in ψ

connection condition

$$\psi(+0) - \psi(-0) + c\,\psi'(+0) = 0$$

$$\psi'(+0) = \psi'(-0)$$

characteristic matrix

$$U = e^{i\phi}\begin{pmatrix} \cos\phi & -i\sin\phi \\ -i\sin\phi & \cos\phi \end{pmatrix}$$

$$\phi = \operatorname{arccot}\left(\frac{c}{2L_0}\right)$$

has scale constant L_0

control of the property of quantum singularities yields a variety of phenomena/applications such as

- spectral anholonomy (Berry phase)
- strong-weak coupling duality
- N = 2 or 4 supersymmetry
- qubit (q-abacus)
- emergence of pressure

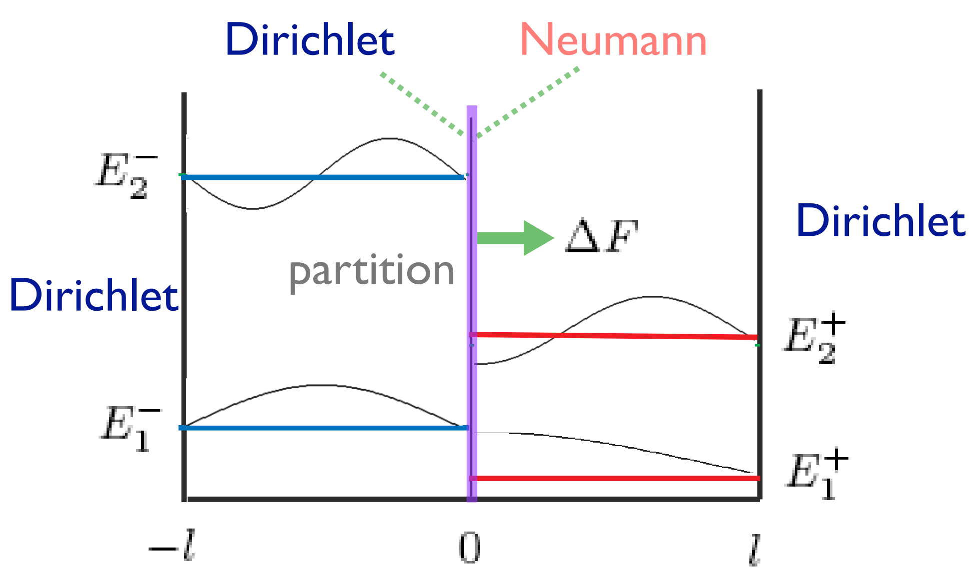
  → partition in quantum well

- inequivalent quantizations

  → N = 3 Calogero model

2. Quantum pressure and statistics

chiral connection

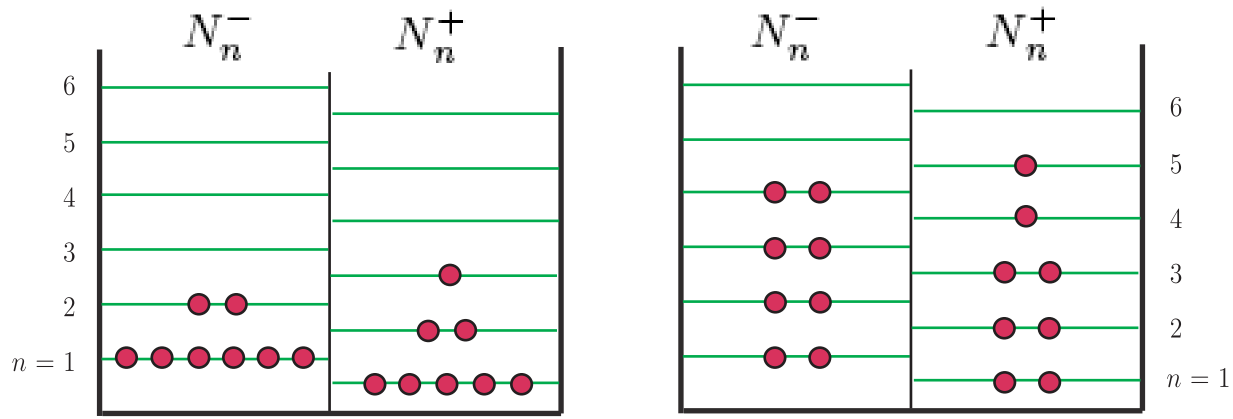


pressure by **distinct** boundary conditions on the left and right

boundary conditions $\psi(-0) = 0, \quad \psi'(+0) = 0, \quad \psi(\pm l) = 0$

energy levels $E_n^+ = \left(n - \frac{1}{2}\right)^2 \mathcal{E}, \quad E_n^- = n^2 \mathcal{E}, \quad \mathcal{E} = \frac{\hbar^2}{2m}\left(\frac{\pi}{l}\right)^2$

bosons fermions



net pressure

particle number at level n

$$\Delta F = F^- - F^+, \qquad F^{\pm} = -\sum_n \frac{\partial E_n^{\pm}}{\partial l} N_n^{\pm} \qquad N = \sum_n N_n^{\pm}$$

dimensionless force/temp.

$$f = \frac{l}{2\mathcal{E}} F \qquad t = \frac{k}{\mathcal{E}} T$$

$t = 2.3 \times T$ for

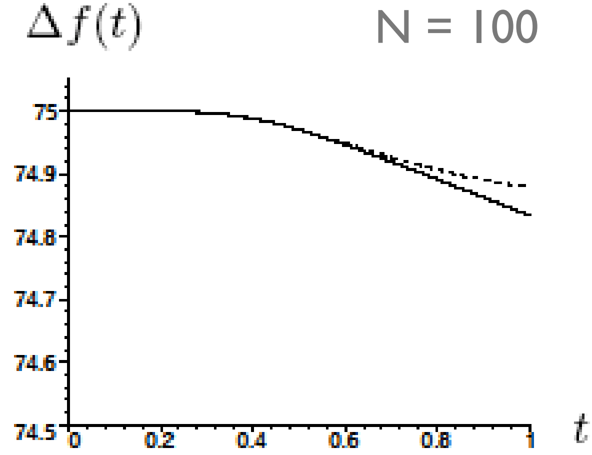
$m = m_e = 9.1 \times 10^{-31}\, kg$

$l = 100\, nm$

(i) bosonic case

$$N_n^{\pm} = \frac{1}{e^{\alpha^{\pm} + \beta E_n^{\pm}} - 1} \qquad \beta = \frac{1}{kT} = \frac{1}{\mathcal{E} t}$$

N = 100



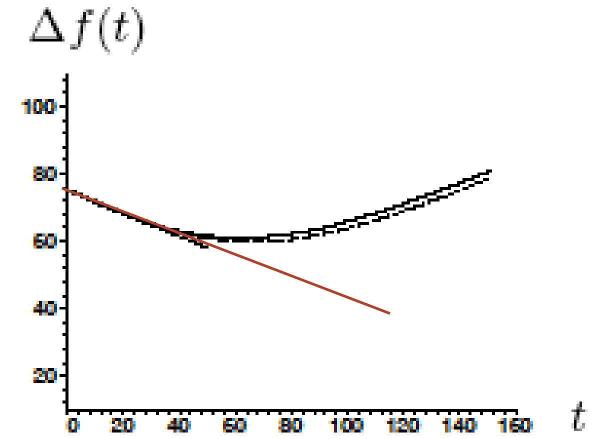
extreme low temp.

$\alpha^{\pm} + \beta E_1^{\pm} \ll 1$ and $t < 1$:

$$\Delta f(t) \approx \frac{3}{4} N + (3\, e^{-3/t} - 2\, e^{-2/t})$$

low temp. $1 < t < N/3$:

$$\Delta f(t) \approx \frac{3}{4} N - \frac{t}{(e-1)^2}$$



high temp. $t \gg 1$:

$$\Delta f(t) \approx \frac{N}{2} \left(\frac{t}{\pi} \right)^{1/2} - \frac{N}{\pi} \left[(\sqrt{2} - 1) N - \frac{1}{2} \right]$$

(ii) fermionic case

$$N_n^{\pm} = \frac{1}{e^{\alpha^{\pm} + \beta E_n^{\pm}} + 1}$$

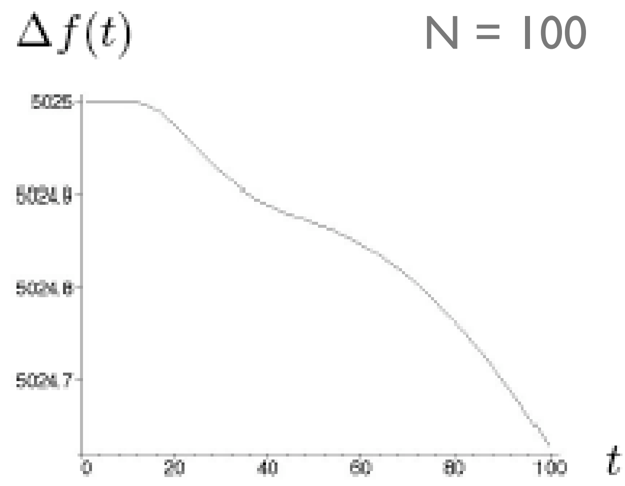
extreme low temp.

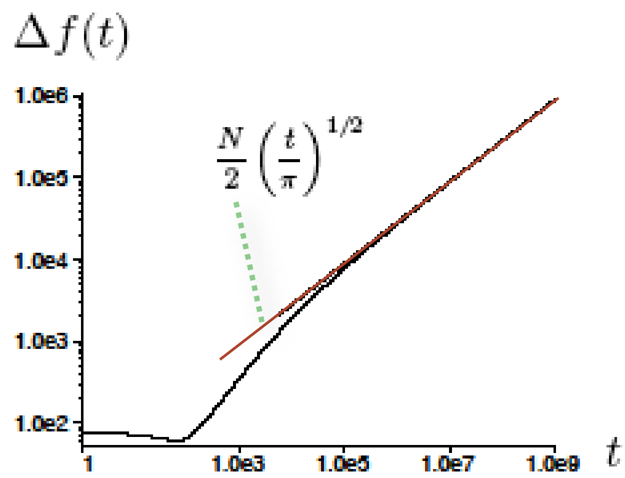
$$\alpha + \beta E_N \ll -1, \quad \alpha + \beta E_{N+1} \gg 1:$$

$$\Delta f(t) \approx \frac{N}{2}\left(N + \frac{1}{2}\right) + 2N e^{-N/t}\left(e^{-1/(2t)} - 1\right)$$

high temp. $t \gg 1$:

$$\Delta f(t) \approx \frac{N}{2}\left(\frac{t}{\pi}\right)^{1/2} + \frac{N}{\pi}\left[(\sqrt{2} - 1)N + \frac{1}{2}\right]$$





characteristic temperature/statistics dependence

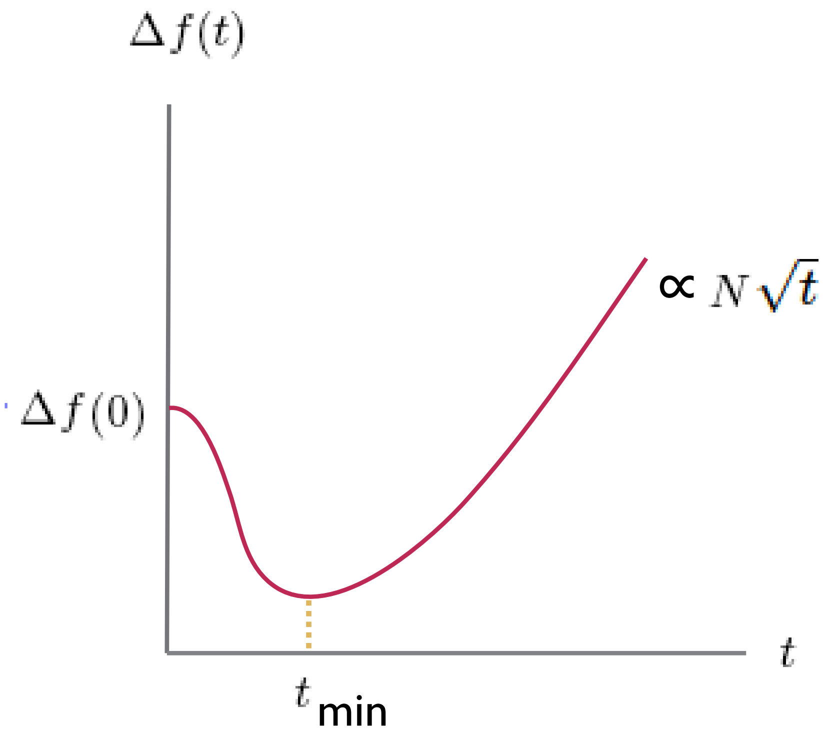
1) unique minimum t_{min}

2) zero temp. limit

$$\Delta f_{\text{boson}}(0) = \mathcal{O}(N)$$

$$\Delta f_{\text{fermion}}(0) = \mathcal{O}(N^2)$$

scales



3) high temp. behaviour

$$\Delta f(t) \approx \frac{N}{2}\left(\frac{t}{\pi}\right)^{1/2}$$

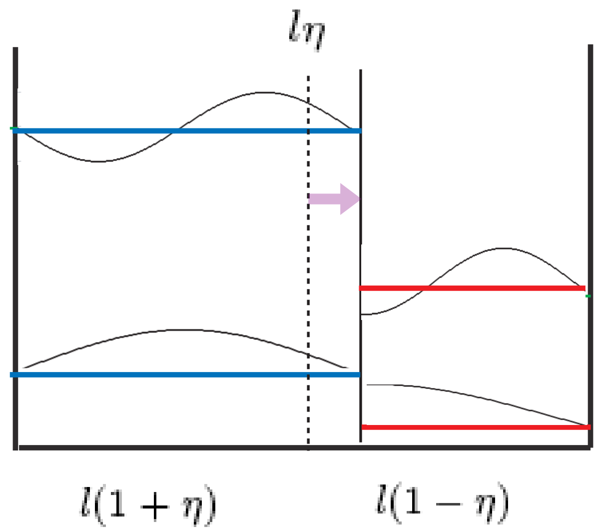
for both bosons and fermions

scales

$\mathcal{O}(N)$ for bosons

$\mathcal{O}(N^2)$ for fermions

measurement of pressure by partition shift



stability condition

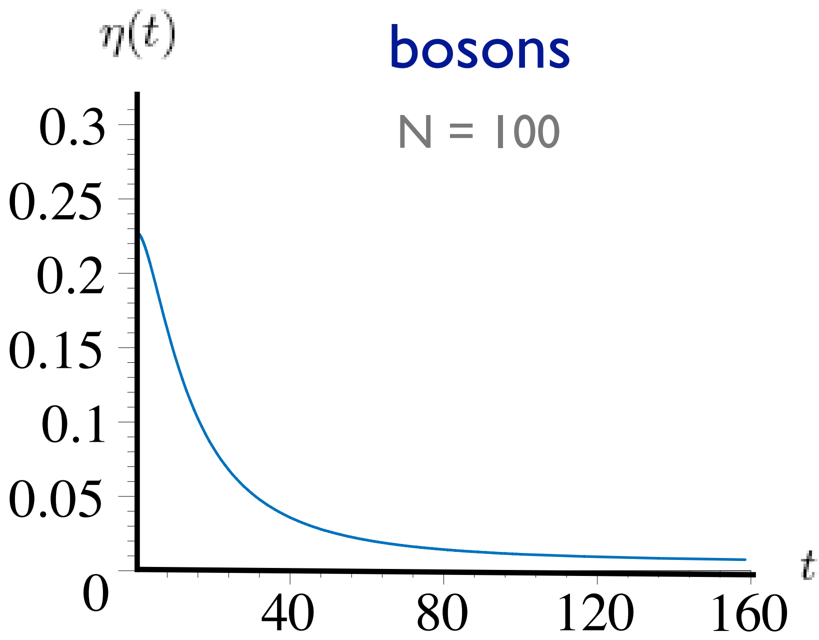
$$0 = \Delta F(t) = \left(\frac{\hbar\pi}{m}\right)^2 \frac{1}{l^3}\left[\frac{f^-(t)}{(1+\eta)^3} - \frac{f^+(t)}{(1-\eta)^3}\right]$$

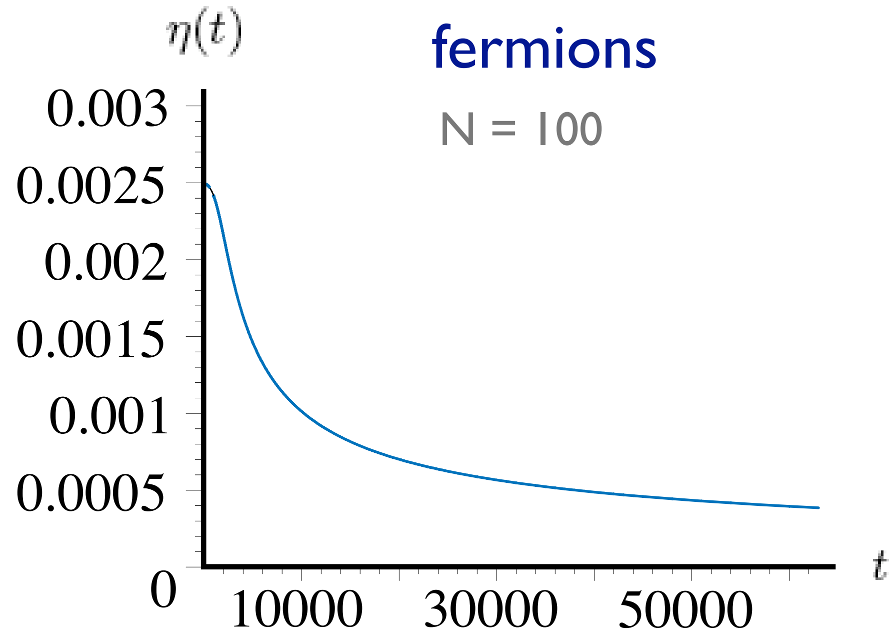
$$\eta(t) = \frac{r-1}{r+1}$$

$$r = \left(\frac{f^-}{f^+}\right)^{1/3}$$

at zero temp.

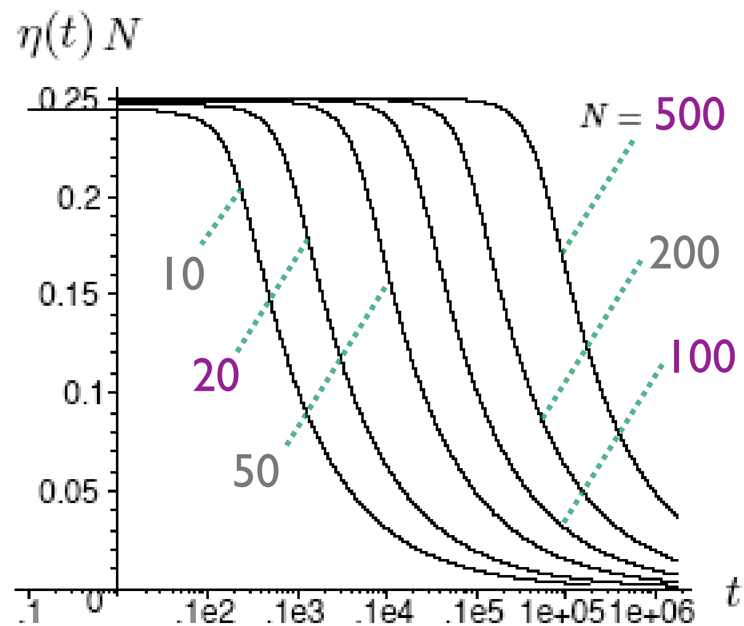
$$\eta_{\text{boson}}(0) \simeq 0.227 \qquad \eta_{\text{fermion}}(0) \simeq \frac{1}{4}\frac{1}{N} = \frac{1}{2}\frac{1}{N_{\text{tot}}}$$

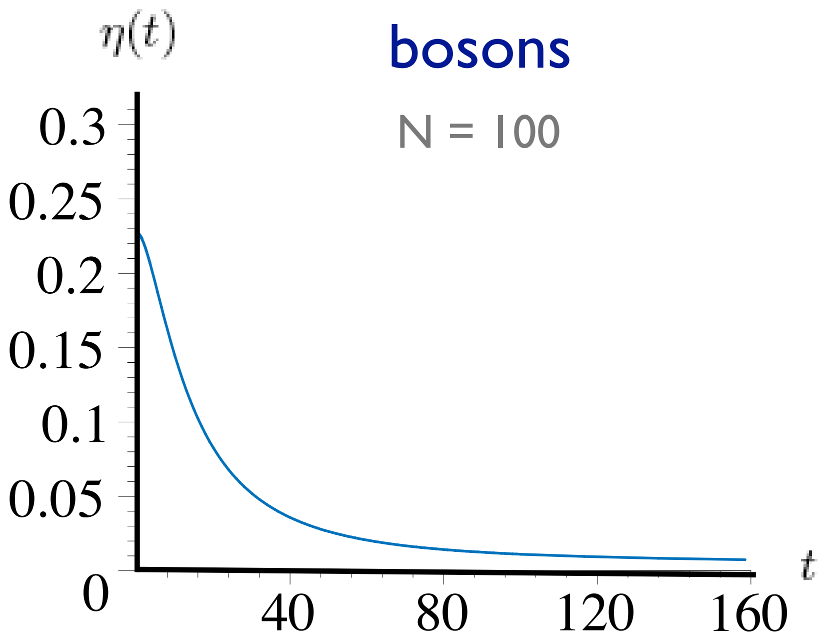


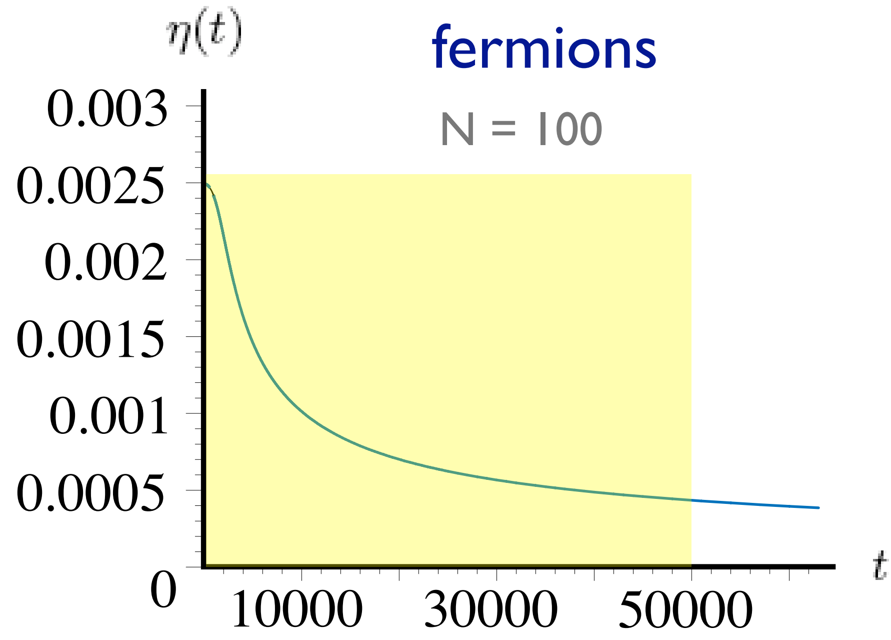


bosons vs fermions

- temp. dep. is similar
- order is 1/N for fermions
- scaling law for N







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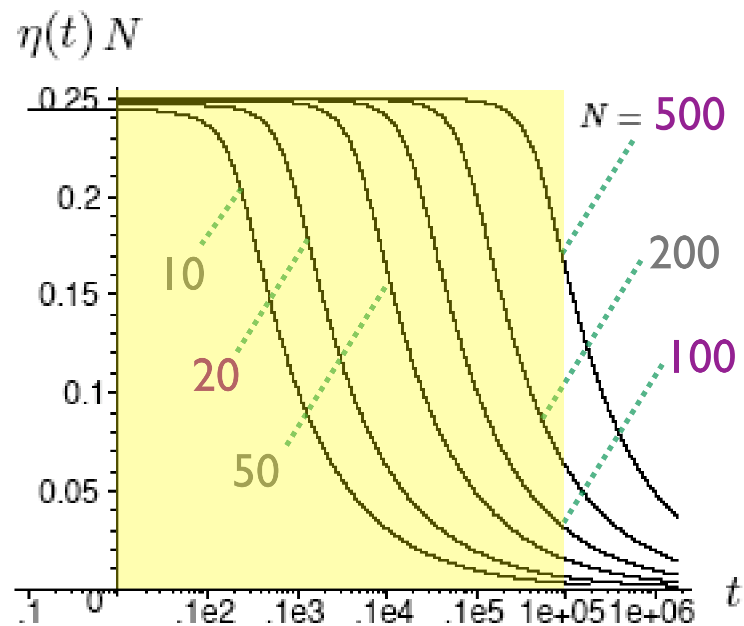


3. N = 3 Calogero Model

N particle Calogero

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \sum_{i=2}^{N}\sum_{j=1}^{i-1}\{\frac{1}{4}m\omega^2(x_i - x_j)^2 + g(x_i - x_j)^{-2}\}$$

singular

separation of variables

$$H = H_0 + H_{rel}$$

centre of mass — relative

$$H_{rel} = H_r + r^{-2}H_\Omega$$

radial
$$H_r = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} - \frac{\hbar^2}{2m}\frac{N-2}{r}\frac{d}{dr} + \frac{1}{4}Nm\omega^2r^2$$

angular
$$H_\Omega = -\frac{\hbar^2}{2m}\Delta_\Omega + g\sum_{i=2}^{N}\sum_{j=1}^{i-1}[r/(x_i - x_j)]^2$$

solution for the relative part

$$H_\Omega \eta_\lambda = \lambda \eta_\lambda$$

$$H_{r,\lambda} R_{E,\lambda} = E R_{E,\lambda} \quad \text{with} \quad H_{r,\lambda} = H_r + \lambda r^{-2}$$

N = 3: polar coordinates (r, ϕ)

$$x_1 - x_2 = r\sqrt{2}\sin\phi$$
$$x_2 - x_3 = r\sqrt{2}\sin(\phi + \frac{2}{3}\pi)$$
$$x_3 - x_1 = r\sqrt{2}\sin(\phi + \frac{4}{3}\pi).$$

angular $$M := H_\Omega = -\frac{d^2}{d\phi^2} + \frac{g}{2}\frac{9}{\sin^2 3\phi}$$ ← singular

radial $$H_{r,\lambda} = -\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{3}{8}\omega^2 r^2 + \frac{\lambda}{r^2}$$ ← singular

$$\hbar = 2m = 1$$

our strategy

$M := H_\Omega$ → find self-adjoint extensions

$H_{r,\lambda}$ → find self-adjoint extensions

range of coupling constant

$-\frac{1}{2} < g < \frac{3}{2}$ → admits nontrivial self-adjoint extensions

parametrize:

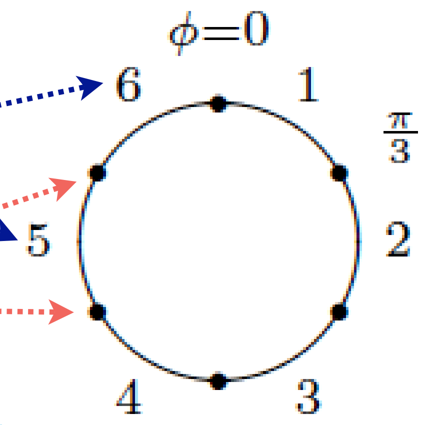
$$g = 2\nu(\nu - 1) \quad \text{with} \quad \frac{1}{2} < \nu < \frac{3}{2}, \quad (\nu \neq 1)$$

angular part

$$M := H_\Omega = -\frac{d^2}{d\phi^2} + \frac{g}{2}\frac{9}{\sin^2 3\phi}$$

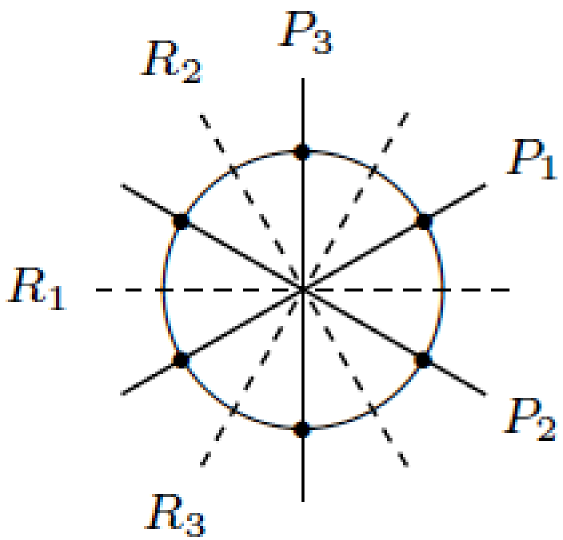
sectors

6 singularities



$$\mathcal{S} = \{\frac{k\pi}{3} \,|\, k = 0, 1, 2, 3, 4, 5\}$$

reflections generate symmetry group D_6



$$P_3 : \phi \mapsto -\phi, \qquad R_3 : \phi \mapsto \frac{\pi}{3} - \phi \qquad (\mathrm{mod}\ 2\pi)$$

rotation $\mathcal{R}_{\frac{\pi}{3}} = R_3 \circ P_3 : \phi \mapsto \phi + \frac{\pi}{3}$

'exchange-S_3' $P_n = (\mathcal{R}_{\frac{\pi}{3}})^n \circ P_3 \circ (\mathcal{R}_{\frac{\pi}{3}})^{-n}$

'mirror-S_3' $R_n = (\mathcal{R}_{\frac{\pi}{3}})^n \circ R_3 \circ (\mathcal{R}_{\frac{\pi}{3}})^{-n}$ for $n = 1, 2.$

require: D_6 symmetry for connection connections

→ D_6 invariant quantizations

reference modes

$$\varphi_i^0 \text{ for } i = 1, 2 \qquad W[\varphi_1^0, \varphi_2^0] := \varphi_1^0 \frac{d\varphi_2^0}{d\phi} - \frac{d\varphi_1^0}{d\phi}\varphi_2^0 = 1$$

$$\varphi_k^{R_i\theta}(\phi) = (-1)^k \varphi_k^\theta(R_i\phi) \qquad \forall k = 1, 2, \quad i = 1, 2, 3, \quad \theta \in \mathcal{S}$$

boundary vectors

$$B_\theta(\psi) := \begin{bmatrix} W[\psi, \varphi_1^\theta]_{\theta+} \\ W[\psi, \varphi_1^\theta]_{\theta-} \end{bmatrix}, \quad B'_\theta(\psi) := \begin{bmatrix} W[\psi, \varphi_2^\theta]_{\theta+} \\ -W[\psi, \varphi_2^\theta]_{\theta-} \end{bmatrix} \quad \text{for} \quad \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3},$$

$$B_\theta(\psi) := \begin{bmatrix} W[\psi, \varphi_1^\theta]_{\theta-} \\ W[\psi, \varphi_1^\theta]_{\theta+} \end{bmatrix}, \quad B'_\theta(\psi) := \begin{bmatrix} -W[\psi, \varphi_2^\theta]_{\theta-} \\ W[\psi, \varphi_2^\theta]_{\theta+} \end{bmatrix} \quad \text{for} \quad \theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}.$$

connection connections

$$(U_\theta - \mathbb{1}_2)B_\theta(\psi) + \mathrm{i}(U_\theta + \mathbb{1}_2)B'_\theta(\psi) = 0 \qquad \forall\theta \in \mathcal{S},$$

$U_\theta = U$ *for all* $\theta \in \mathcal{S}$ and $U = \sigma_1 U \sigma_1$ → D_6

→ $U = e^{\mathrm{i}\alpha I} e^{\mathrm{i}\beta\sigma_1} = e^{\mathrm{i}\alpha} \begin{pmatrix} \cos\beta & \mathrm{i}\sin\beta \\ \mathrm{i}\sin\beta & \cos\beta \end{pmatrix}$

angular eigenstates

basic solutions $M\eta^k_{\pm,\mu} = 9\mu^2\eta^k_{\pm,\mu}$ $\hat{R}_3\eta^1_{\pm,\mu} = \pm\eta^1_{\pm,\mu}$

angular eigenvalue

parity

basic solutions for sector 1

$$\eta^1_{+,\mu}(\phi) = \begin{cases} b_2(\mu)v_{1,\mu}(\phi) - b_1(\mu)v_{2,\mu}(\phi) & \text{if } 0 < \phi \le \frac{\pi}{6} \bmod 2\pi \\ b_2(\mu)v_{1,\mu}(\frac{\pi}{3}-\phi) - b_1(\mu)v_{2,\mu}(\frac{\pi}{3}-\phi) & \text{if } \frac{\pi}{6} \le \phi < \frac{\pi}{3} \bmod 2\pi \\ 0 & \text{otherwise} \end{cases}$$

$$\eta^1_{-,\mu}(\phi) = \begin{cases} a_2(\mu)v_{1,\mu}(\phi) - a_1(\mu)v_{2,\mu}(\phi) & \text{if } 0 < \phi \le \frac{\pi}{6} \bmod 2\pi \\ -a_2(\mu)v_{1,\mu}(\frac{\pi}{3}-\phi) + a_1(\mu)v_{2,\mu}(\frac{\pi}{3}-\phi) & \text{if } \frac{\pi}{6} \le \phi < \frac{\pi}{3} \bmod 2\pi \\ 0 & \text{otherwise} \end{cases}$$

where

hypergeometric function

$$v_{1,\mu}(\phi) := |\sin 3\phi|^{\nu} F\left(\frac{\nu-\mu}{2}, \frac{\nu+\mu}{2}, \nu+\frac{1}{2}; \sin^2 3\phi\right)$$

$$v_{2,\mu}(\phi) := |\sin 3\phi|^{1-\nu} F\left(\frac{1-\nu-\mu}{2}, \frac{1-\nu+\mu}{2}, -\nu+\frac{3}{2}; \sin^2 3\phi\right)$$

$$a_1(\mu) = \frac{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{\nu+1+\mu}{2})\Gamma(\frac{\nu+1-\mu}{2})}, \qquad a_2(\mu) = \frac{\Gamma(-\nu+\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{-\nu+2+\mu}{2})\Gamma(\frac{-\nu+2-\mu}{2})},$$

$$b_1(\mu) = \frac{6\,\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{\nu+\mu}{2})\Gamma(\frac{\nu-\mu}{2})}, \qquad b_2(\mu) = \frac{6\,\Gamma(-\nu+\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{-\nu+1+\mu}{2})\Gamma(\frac{-\nu+1-\mu}{2})}.$$

other sectors

$$\eta^k_{\pm,\mu}(\phi) = \eta^1_{\pm,\mu}(\phi - (k-1)\frac{\pi}{3}), \quad \text{for} \quad k = 2, \ldots, 6.$$

general solutions

$$\eta_\mu(\phi) = \sum_{k=1}^{6} \left(C^k_+ \eta^k_{+,\mu}(\phi) + C^k_- \eta^k_{-,\mu}(\phi) \right)$$

coefficients $C^k_\pm$. to be determined from connection conditions with boundary vectors

$$B_0(\eta_\mu) = (3(2\nu-1))^{\frac{1}{2}} \begin{bmatrix} -C^1_+ b_1(\mu) - C^1_- a_1(\mu) \\ -C^6_+ b_1(\mu) + C^6_- a_1(\mu) \end{bmatrix},$$

$$B'_0(\eta_\mu) = (3(2\nu-1))^{\frac{1}{2}} \begin{bmatrix} C^1_+ b_2(\mu) + C^1_- a_2(\mu) \\ C^6_+ b_2(\mu) - C^6_- a_2(\mu) \end{bmatrix}.$$

representations of D_6

$$12 = 1 + 1 + 1 + 1 + 2^2 + 2^2$$

1 dim. 2 dim.

character table of D_6

conjugacy class	$\{e\}$	$\{R_i\}$	$\{P_i\}$	$\{\mathcal{R}^{\pm 1}_{\pi/3}\}$	$\{\mathcal{R}^{\pm 2}_{\pi/3}\}$	$\{\mathcal{R}^{3}_{\pi/3}\}$
χ^{++}	1	1	1	1	1	1
χ^{-+}	1	-1	1	-1	1	-1
χ^{+-}	1	1	-1	-1	1	-1
χ^{--}	1	-1	-1	1	1	1
$\chi^{(2)}$	2	0	0	1	-1	-2
$\tilde{\chi}^{(2)}$	2	0	0	-1	-1	2

radial part

$\lambda = (3\mu)^2$

$$\mathcal{H}_{r,\lambda} := \sqrt{r} \circ H_{r,\lambda} \circ \frac{1}{\sqrt{r}} = -\frac{d^2}{dr^2} + \frac{3}{8}\omega^2 r^2 + \frac{\lambda - \frac{1}{4}}{r^2}$$

angular eigenvalue

$$\mathcal{H}_{r,\lambda}\rho = E\rho$$

$\lambda \geq 1$ .... unique self-adjoint extension

$\lambda < 1$ .... U(1) self-adjoint extensions

boundary condition $\dfrac{W[\rho, \varphi_1]_{0+}}{W[\rho, \varphi_2]_{0+}} = \kappa(\lambda)$ at $r = 0$

U(1) parameter

basic solution

$$\rho_E = \frac{\Gamma(1-\sqrt{\lambda})}{\Gamma(-\xi-\sqrt{\lambda})}\rho_{E,1} - \frac{\Gamma(1+\sqrt{\lambda})}{\Gamma(-\xi)}\rho_{E,2}$$

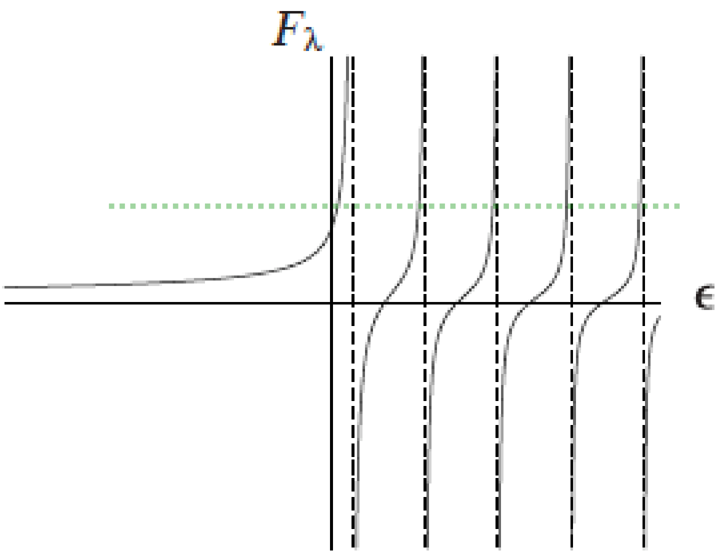
$$\rho_{E,1}(r) = \sigma^{\frac{1}{2}(\frac{1}{2}+\sqrt{\lambda})} e^{-\frac{1}{2}\sigma} \Phi(-\xi, \sqrt{\lambda}+1, \sigma),$$
$$\rho_{E,2}(r) = \sigma^{\frac{1}{2}(\frac{1}{2}-\sqrt{\lambda})} e^{-\frac{1}{2}\sigma} \Phi(-\xi-\sqrt{\lambda}, 1-\sqrt{\lambda}, \sigma).$$

confluent hypergeometric function

spectral condition

$$F_\lambda(\epsilon) := \frac{\Gamma(-\epsilon + \frac{1-\sqrt{\lambda}}{2})}{\Gamma(-\epsilon + \frac{1+\sqrt{\lambda}}{2})} = -\frac{\Gamma(-\sqrt{\lambda})}{\Gamma(\sqrt{\lambda})}\kappa(\lambda)$$

$$\epsilon := \frac{E}{4c}, \qquad c := \sqrt{\frac{3}{8}}\omega,$$



if $\kappa(\lambda) = 0$ → Calogero's choice

$$\rho_{m,\lambda}(r) = r^{\frac{1}{2}+\sqrt{\lambda}} e^{-\frac{1}{2}cr^2} L_m^{\sqrt{\lambda}}(cr^2).$$

generalized Laguerre polynomial

$$E_{m,\lambda} = 2c(2m + 1 + \sqrt{\lambda}), \qquad m = 0, 1, 2, \ldots,$$

explicitly solvable cases

1) Dirichlet case $U = -1_2$

$$\eta^A_\mu(\phi) = \sum_{k=1}^{6} C^k_- \eta^k_{-,\mu}(\phi), \qquad \mu = 2n + 1 + \nu, \qquad n = 0, 1, 2, \ldots$$

$$\eta^B_\mu(\phi) = \sum_{k=1}^{6} C^k_+ \eta^k_{+,\mu}(\phi), \qquad \mu = 2n + \nu,$$

multiplicity 6

total eigenstates

$$\Psi^A_{mn}(r,\phi) = R_{m,\lambda}(r)\, \eta^A_\mu(\phi), \qquad E^A_{mn} = 2c\,(2m + 1 + 3(2n + 1 + \nu))\,,$$

$$\Psi^B_{mn}(r,\phi) = R_{m,\lambda}(r)\, \eta^B_\mu(\phi), \qquad E^B_{mn} = 2c\,(2m + 1 + 3(2n + \nu))\,,$$

$$R_{m,\lambda}(r) = r^{-\frac{1}{2}} \rho_{m,\lambda}(r)$$

restriction to boson/fermion sectors → Calogero's solution

2) free case $U = \sigma_1$

1 dim. irrep. angular states

$$\begin{aligned}
\eta_\mu^{A+}(\phi) &= -c(\phi)\, a_1(\mu) v_{2,\mu}(\phi), & \mu &= 2n+1+(1-\nu), \\
\eta_\mu^{A-}(\phi) &= t(\phi)\, a_2(\mu) v_{1,\mu}(\phi), & \mu &= 2n+1+\nu, \\
\eta_\mu^{B+}(\phi) &= -b_1(\mu) v_{2,\mu}(\phi), & \mu &= |2n+(1-\nu)|, \\
\eta_\mu^{B-}(\phi) &= s(\phi)\, b_2(\mu) v_{1,\mu}(\phi). & \mu &= 2n+\nu.
\end{aligned}$$

total eigenstates

$$\begin{aligned}
\Psi_{mn}^{++}(r,\phi) &= R_{m,\lambda}(r)\, \eta_\mu^{B+}(\phi), & E_{mn}^{++} &= 2c\,(2m+1+3|2n+(1-\nu)|)\,, \\
\Psi_{mn}^{-+}(r,\phi) &= R_{m,\lambda}(r)\, \eta_\mu^{A+}(\phi), & E_{mn}^{-+} &= 2c\,(2m+1+3(2n+1+(1-\nu)))\,, \\
\Psi_{mn}^{+-}(r,\phi) &= R_{m,\lambda}(r)\, \eta_\mu^{B-}(\phi), & E_{mn}^{+-} &= 2c\,(2m+1+3(2n+\nu))\,, \\
\Psi_{mn}^{--}(r,\phi) &= R_{m,\lambda}(r)\, \eta_\mu^{A-}(\phi), & E_{mn}^{--} &= 2c\,(2m+1+3(2n+1+\nu))\,,
\end{aligned}$$

2 dim. irrep. angular states

$$\eta_{\mu,\tau}(\phi) = -\frac{\mathrm{i}q(\mu)}{\Im(\tau)} v_{1,\mu}(\phi) + v_{2,\mu}(\phi)$$

$$q(\mu) = \frac{3\cos^2 \pi\nu}{2\pi^2} 2^{-2\nu}\,\Gamma(-\nu + \frac{1}{2})\,\Gamma(-\nu + \frac{3}{2})\,\Gamma(\nu + \mu)\,\Gamma(\nu - \mu)$$

total eigenstates

$$\Psi^{(2)+}_{mn,\tau}(r,\phi) = R_{m,\lambda}(r)\,\eta^{(2)+}_{\mu,\tau}(\phi), \qquad E^{(2)+}_{mn} = 2c\,(2m + 1 + 3(2n + (1 - \Delta(\nu))))\,,$$

$$\tilde{\Psi}^{(2)+}_{mn,\tau}(r,\phi) = R_{m,\lambda}(r)\,\tilde{\eta}^{(2)+}_{\mu,\tau}(\phi), \qquad \tilde{E}^{(2)+}_{mn} = 2c\,(2m + 1 + 3(2n + 1 + (1 - \Delta(\nu)))\,,$$

$$\tilde{\Psi}^{(2)-}_{mn,\tau}(r,\phi) = R_{m,\lambda}(r)\,\tilde{\eta}^{(2)-}_{\mu,\tau}(\phi), \qquad \tilde{E}^{(2)-}_{mn} = 2c\,(2m + 1 + 3(2n + \Delta(\nu)))\,,$$

$$\Psi^{(2)-}_{mn,\tau}(r,\phi) = R_{m,\lambda}(r)\,\eta^{(2)-}_{\mu,\tau}(\phi), \qquad E^{(2)-}_{mn} = 2c\,(2m + 1 + 3(2n + 1 + \Delta(\nu)))\,.$$

$$\Delta(\nu) := \frac{1}{\pi}\arccos\left(\frac{1}{2}\cos\pi\nu\right)$$

harmonic oscillator limit $\nu \longrightarrow 1$

$$\eta_k^{\pm}(\phi) := e^{\pm ik\phi}, \qquad k = 0, 1, 2, \ldots.$$

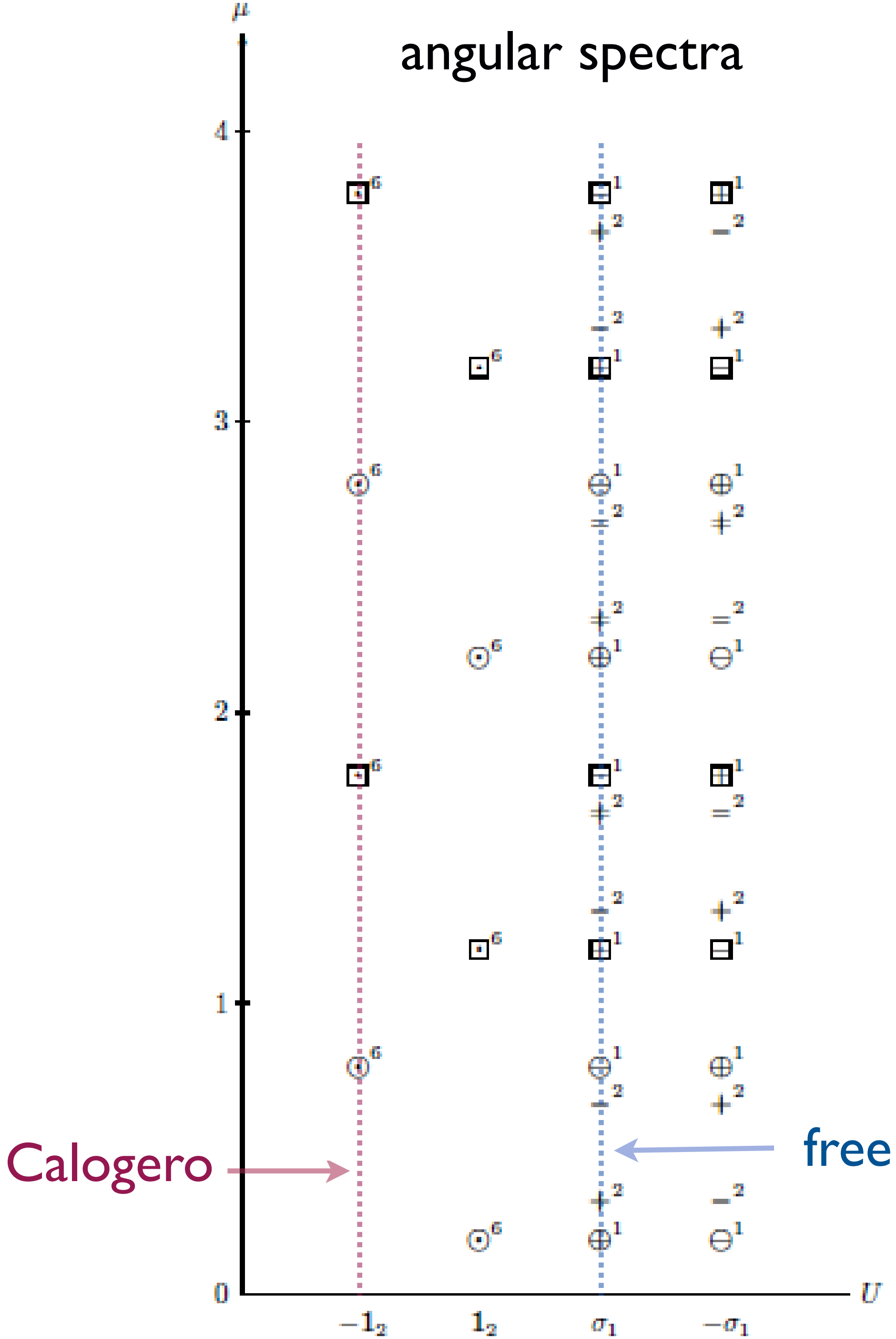
$$R_{m,\lambda}(r) = r^k e^{-\frac{1}{2}cr^2} L_m^k(cr^2)$$

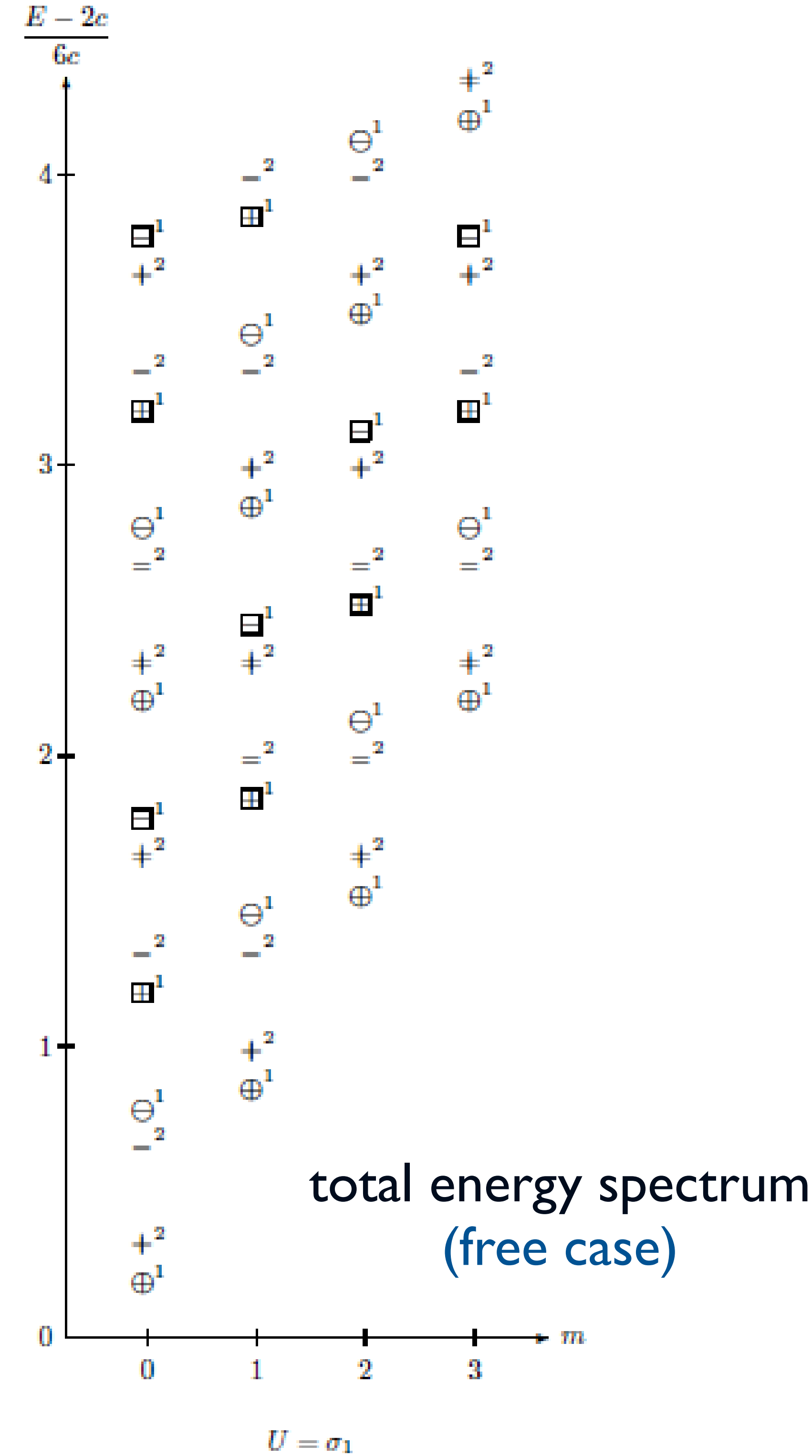
$$\Psi_{mk}^{\pm}(r,\phi) = R_{m,\lambda}(r)\, \eta_k^{\pm}(\phi), \qquad E_{mk}^{\pm} = 2c\,(2m + 1 + k)$$

reduces to the standard 2 dim. harmonic oscillator

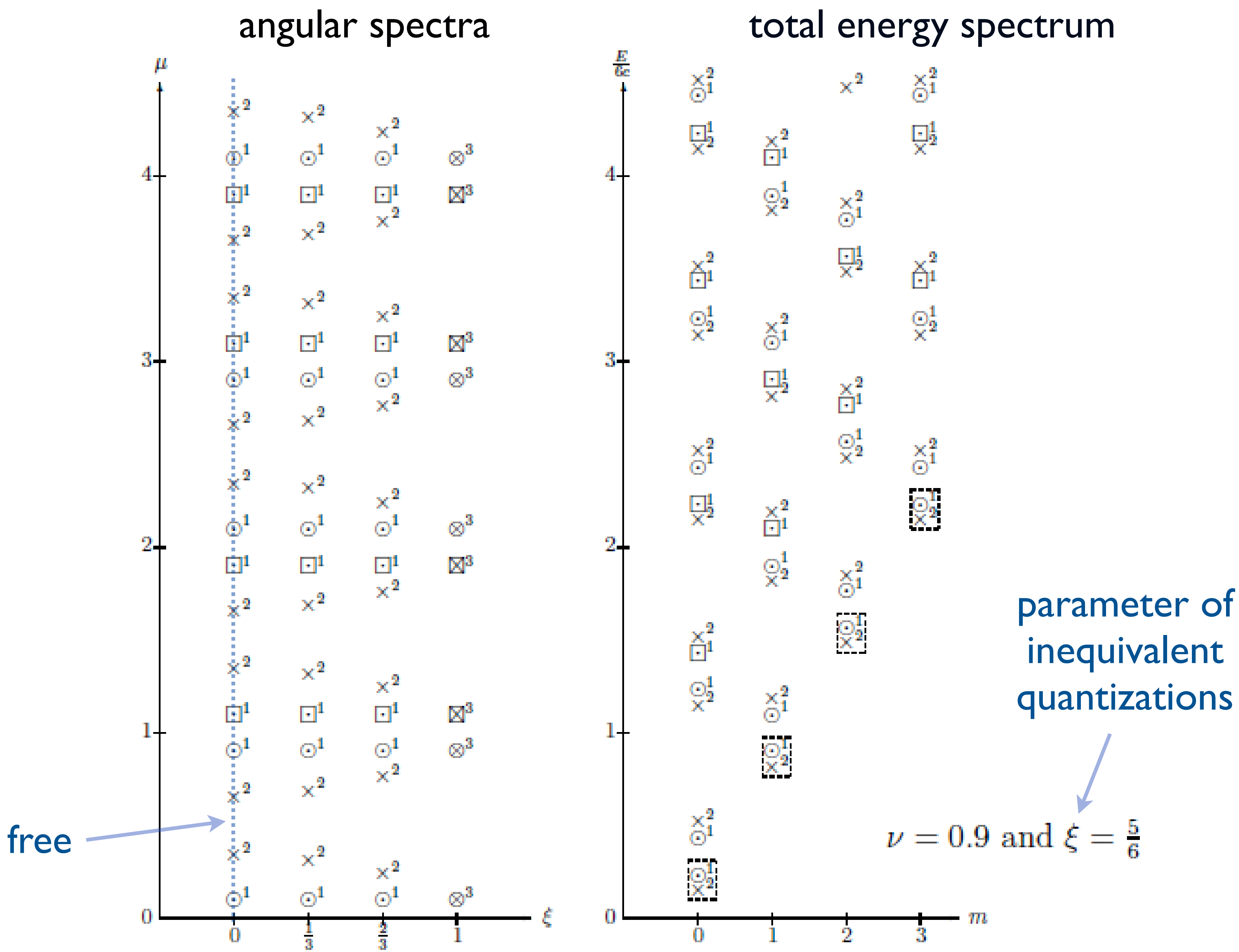
smooth limit (unlike other cases)

→ distingiuished quantization


angular spectra
μ
0
1
2
3
4
Calogero
free
U
-1_2
1_2
σ_1
$-\sigma_1$


$\frac{E-2c}{6c}$
0
1
2
3
4
total energy spectrum
(free case)
m
0
1
2
3
$U=\sigma_1$

cf.) Mirror-S_3 and scale invariant quantizations



Summary

- U(2) family of different singularities (self-adjoint extensions) for each singularity on a line

- Resultant quantum systems exhibit distinct physical properties (e.g., energy spectra or pressure) depending on the characteristics of the singularity

- These properties may also depend on the statistics and the number of the particles in a particular manner (scaling laws in quantum well)

- Calogero model admits a variety of inequivalent quantizations with distinct spectra including Calogero's original one as a special case