Non-local finite size effects in the dimer model

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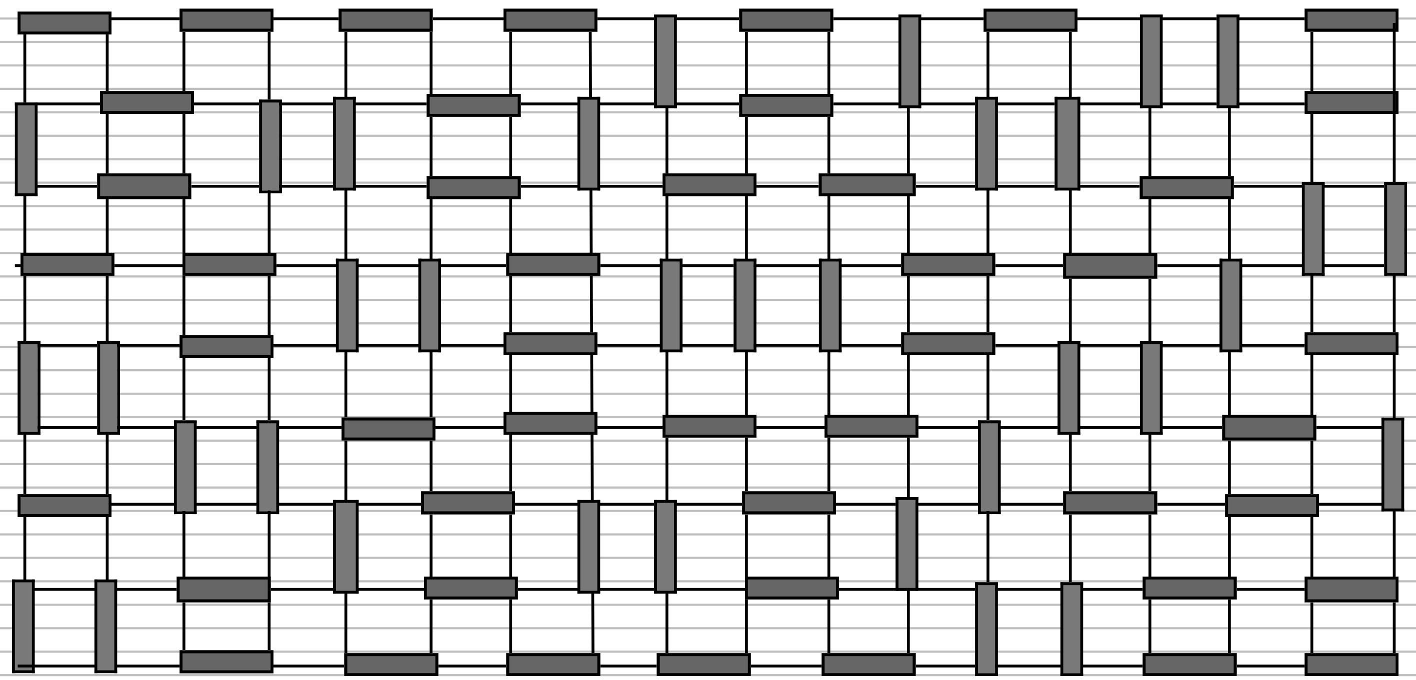
Basics

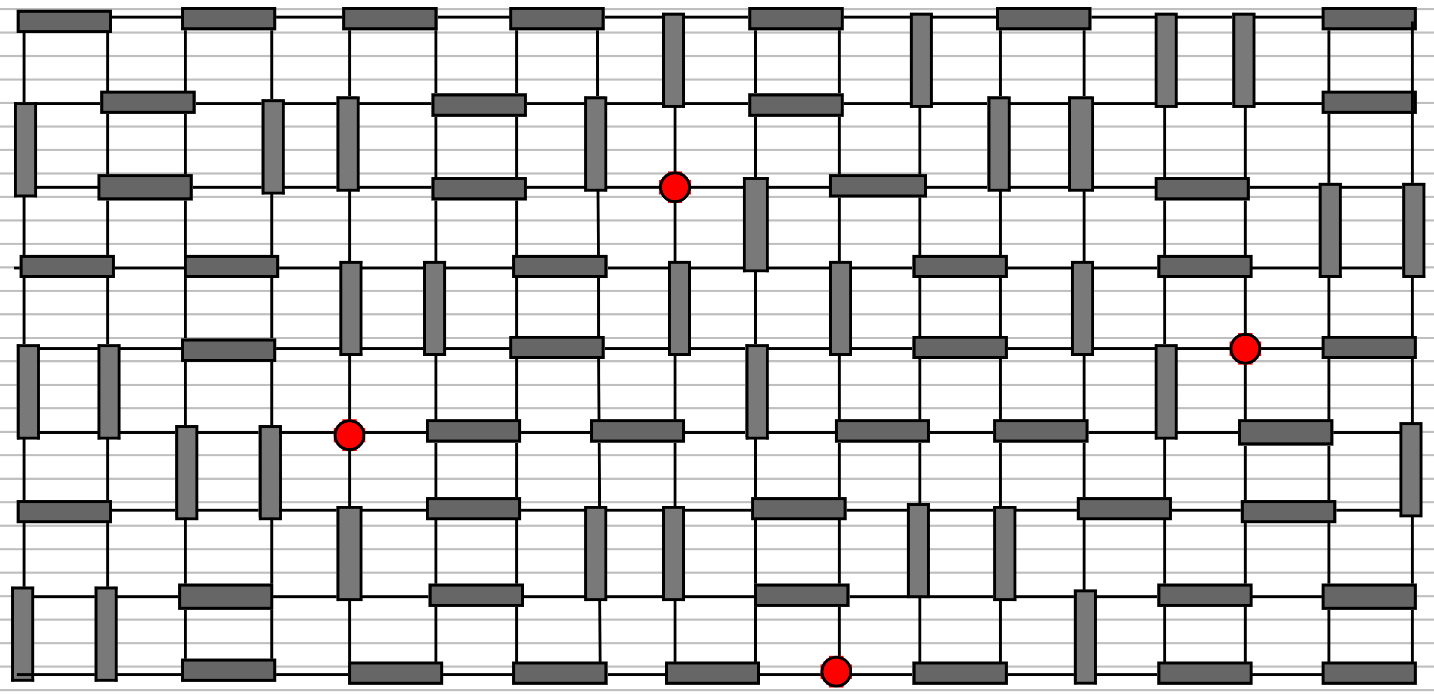
Take a grid with N sites and consider all arrangements of $N/2$ dimers $=$ dominoes so that all sites are covered.



9 x 18

Basics

More generally, we may introduce vacancies = monomers, i.e. sites which cannot be covered.



Basics

Corresponding to these arrangements of monomers-dimers, we introduce a partition function (for square lattice)

$$Z(x, y|z_1, \ldots, z_m) = \sum_{\text{cover.}} x^{n_h} y^{n_v}.$$

Counts the number of dimer coverings in presence of m vacancies located at positions $z_1, \ldots, z_m$ in bulk or on boundaries.
Weights x and y assigned to horizontal and vertical dimers.

Case $m = 0$ fairly well-understood for many lattices
$m > 0$ more difficult ($m = 2$ vacancies in bulk ✓)

Basics

Questions include:

- phase transitions ?
- correlations of vacancies/monomers ? of bond occupation ?
- finite size corrections ?
- CFT description ?

Here : finite size corrections for infinite strip and cylinder, on square lattice with $x = y = 1$ (critical but no phase transition), and no vacancies

$\longrightarrow$ central charge c, boundary conditions of underlying CFT

History

Pioneers: Kasteleyn, Fisher, Temperley, Ferdinand, Wu

Two vacancies: Fisher & Stephenson, correlation $\sim \frac{1}{\sqrt{r}}$

Finite size: Fisher, Ferdinand, Hartwig, Brankov, Wu, Kong

Spanning trees: Temperley, Priezzhev, Kenyon, Propp, Wilson

General

On a rectangular grid $M \times N$ and free boundary conditions, partition function is simply

$$Z(M,N) = \#\ \text{dimer coverings}$$

It grows exponentially fast with volume:

$2 \times 2 \ : \ Z = 2$

$4 \times 4 \ : \ Z = 36$

$8 \times 8 \ : \ Z = 12\,988\,816$

$10 \times 10 \ : \ Z = 258\,584\,046\,368$

$9 \times 18 \ : \ Z = 4.6528 \times 10^{18}$

General

Dimer model is in the class of free fermion theories, giving rise to exact determinantal formula

$$Z(M,N) = \prod_{k=1}^{M}\prod_{\ell=1}^{N}\left|2\cos\frac{\pi k}{M+1} + 2i\cos\frac{\pi \ell}{N+1}\right|^{1/2}$$

It follows that

$$Z(M,N) \sim (e^{\mathrm{G}/\pi})^{MN} = (1.79162)^{MN/2}$$

and the free energy per unit vertical length (portion $1 \times N$) is

$$F(N) = -\lim_{M\to\infty}\frac{1}{M}\log Z(M,N) = -\frac{\mathrm{G}}{\pi}N + \dots$$

(Fisher, '61)

Finite size corrections

The full expression for the free energy contains subdominant terms, in this case odd powers of N^{-1} (plus constant). One finds

$$N \text{ odd}: \; F(N) = -\frac{\mathrm{G}}{\pi}N - \frac{\mathrm{G}}{\pi} + \frac{1}{2}\log(1+\sqrt{2}) + \frac{\pi}{12N} + \ldots$$

$$N \text{ even}: \; F(N) = -\frac{\mathrm{G}}{\pi}N - \frac{\mathrm{G}}{\pi} + \frac{1}{2}\log(1+\sqrt{2}) - \frac{\pi}{24N} + \ldots$$

(Fisher '61, Ferdinand '67, Ivashkevich, Izmailian & Hu '02)

The interesting term is the third one, proportional to $1/N$ ...

Finite size corrections

The general form of the free energy is

$$F(N) = f_{\text{bulk}} N + 2 f_{\text{surf}} + \frac{A}{N} + \ldots$$

with f_{bulk} = bulk free energy per site
f_{surf} = excess of free energy per boundary site

The constant A is universal, b.c. dependent and related to effective central charge (Blöte, Cardy, Nightingale + Affleck)

$$A = -\frac{\pi}{24} c_{\text{eff}} = \pi (h_{\min} - \frac{c}{24})$$

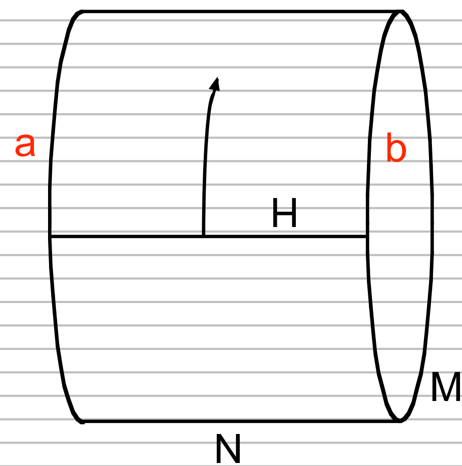
$h_{\min}$ = smallest conformal weight in Hilbert space with prescribed b.c.'s.

Finite size corrections

This follows from universal part of partition function (bulk and surface contributions subtracted off)

$$Z_{a|b}(M,N) = \mathrm{Tr} e^{-MH_{a|b}} = \mathrm{Tr} e^{-\frac{\pi M}{N}(L_0 - \frac{c}{24})}$$
$$\sim e^{-\frac{\pi M}{N}(h_{\min} - \frac{c}{24})}$$

$$-\lim_{M\to\infty} \frac{1}{M} \log Z_{a|b} = \frac{\pi}{N}(h_{\min} - \frac{c}{24})$$



Paradox

Compare now $A = \pi(h_{\text{min}} - \frac{c}{24})$ with dimer model data

$$N \text{ odd}: \quad A = \frac{\pi}{12}$$

$$N \text{ even}: \quad A = -\frac{\pi}{24}$$

Looks paradoxical : either same h_{min} but then c depends on parity of N, or same central charge but then h_{min} is different in both cases, even though the b.c. are the same ... !!!

Number of authors take $h_{\text{min}} = 0$ (reasonable) and claim that $c = -2$ for N odd and $c = 1$ for N even.

Similar situation on cylinder: same argument leads to $c = -\frac{1}{2}$ for N odd and $c = 1$ for N even.

Boundary conditions

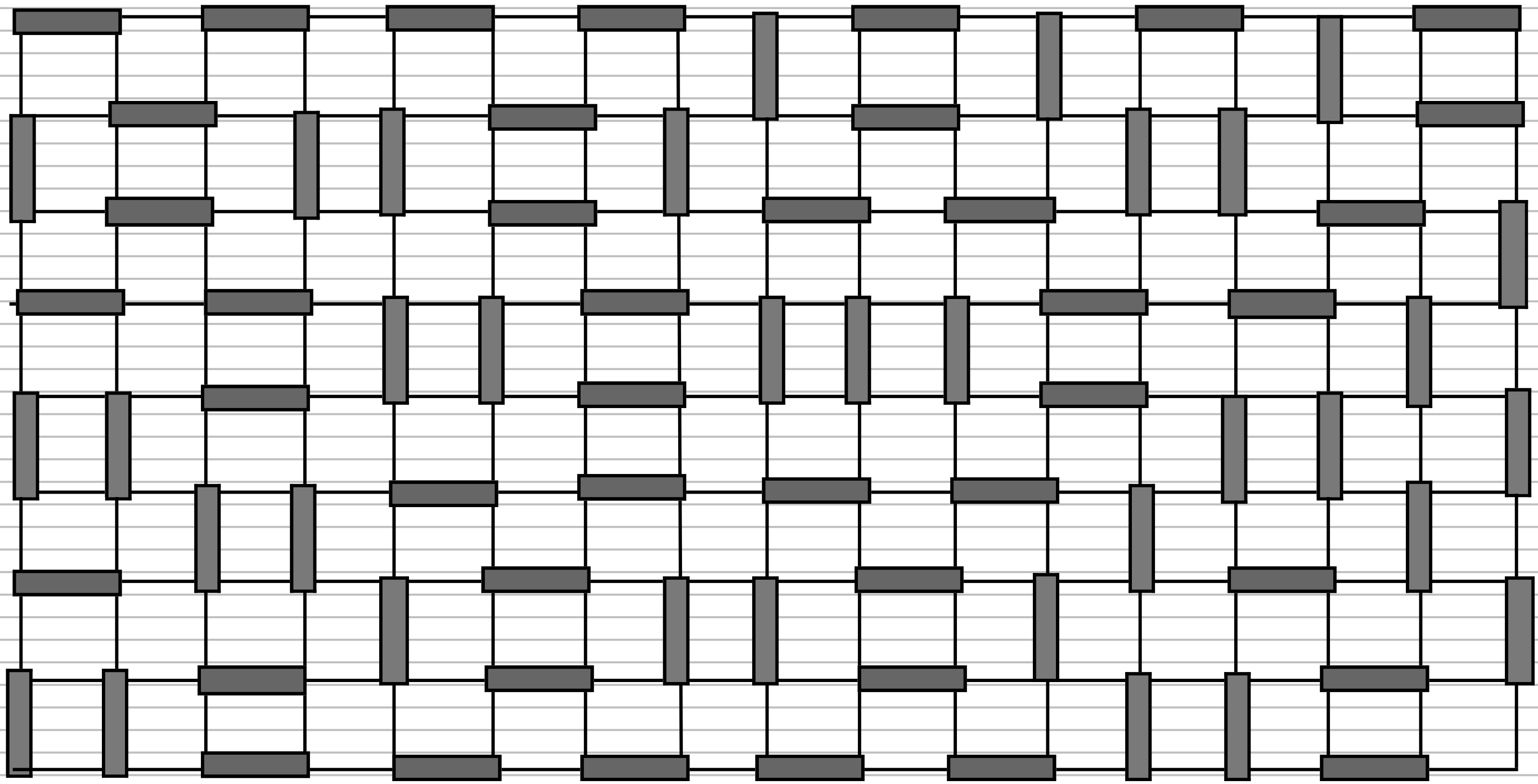
In trying to understand the paradox, looks more reasonable to keep c fixed (minimal amount of locality ...), but then explain why the value of $h_{\min}$ changes ...

In fact, $h_{\min}$ changes because the boundary conditions are modified by a change of parity of N.

To best see it, go from dimer configurations to another description, in terms of spanning trees.

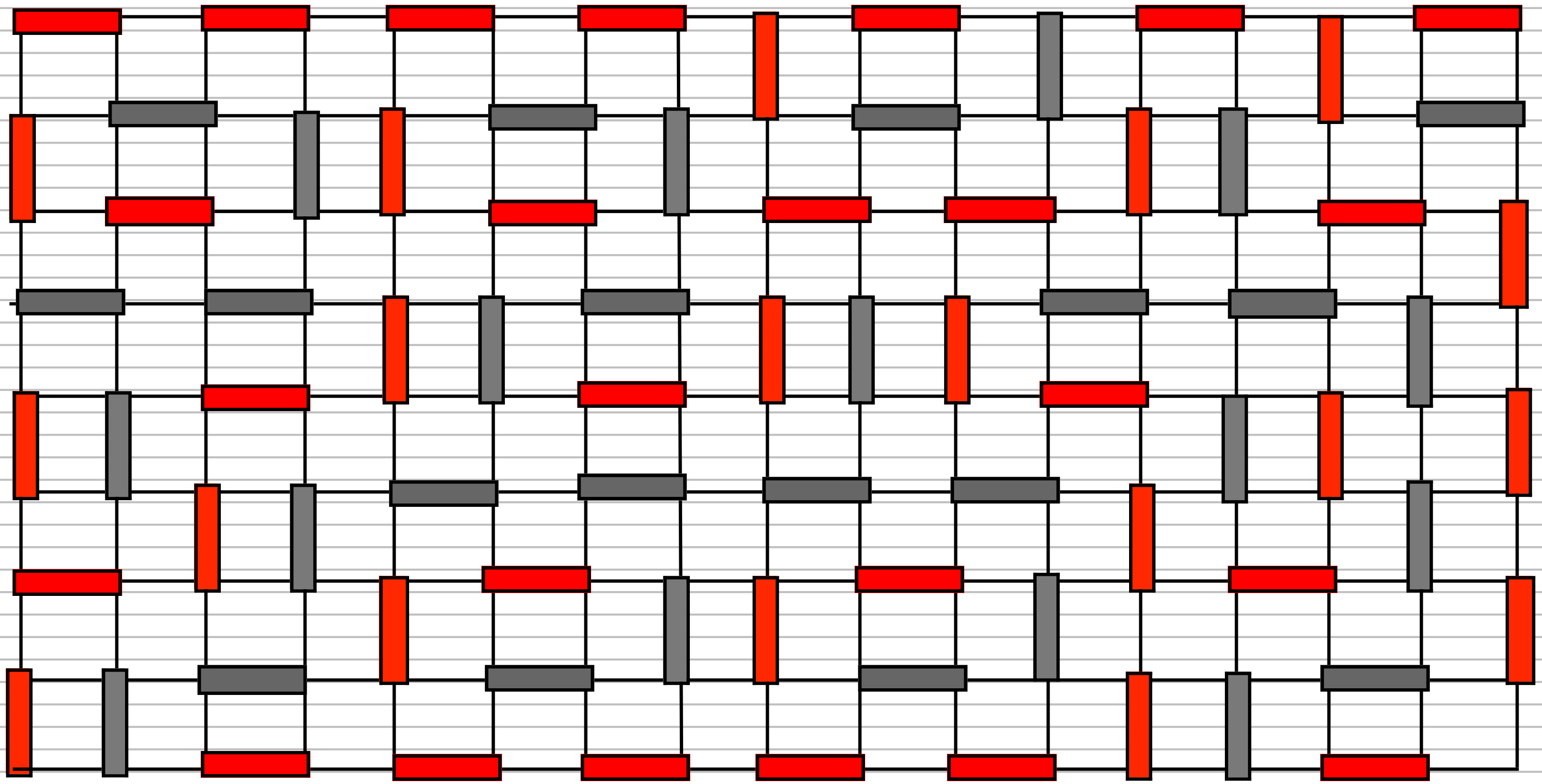
(First discovered for odd-odd lattices by Temperley '74)

N odd



9 x 17

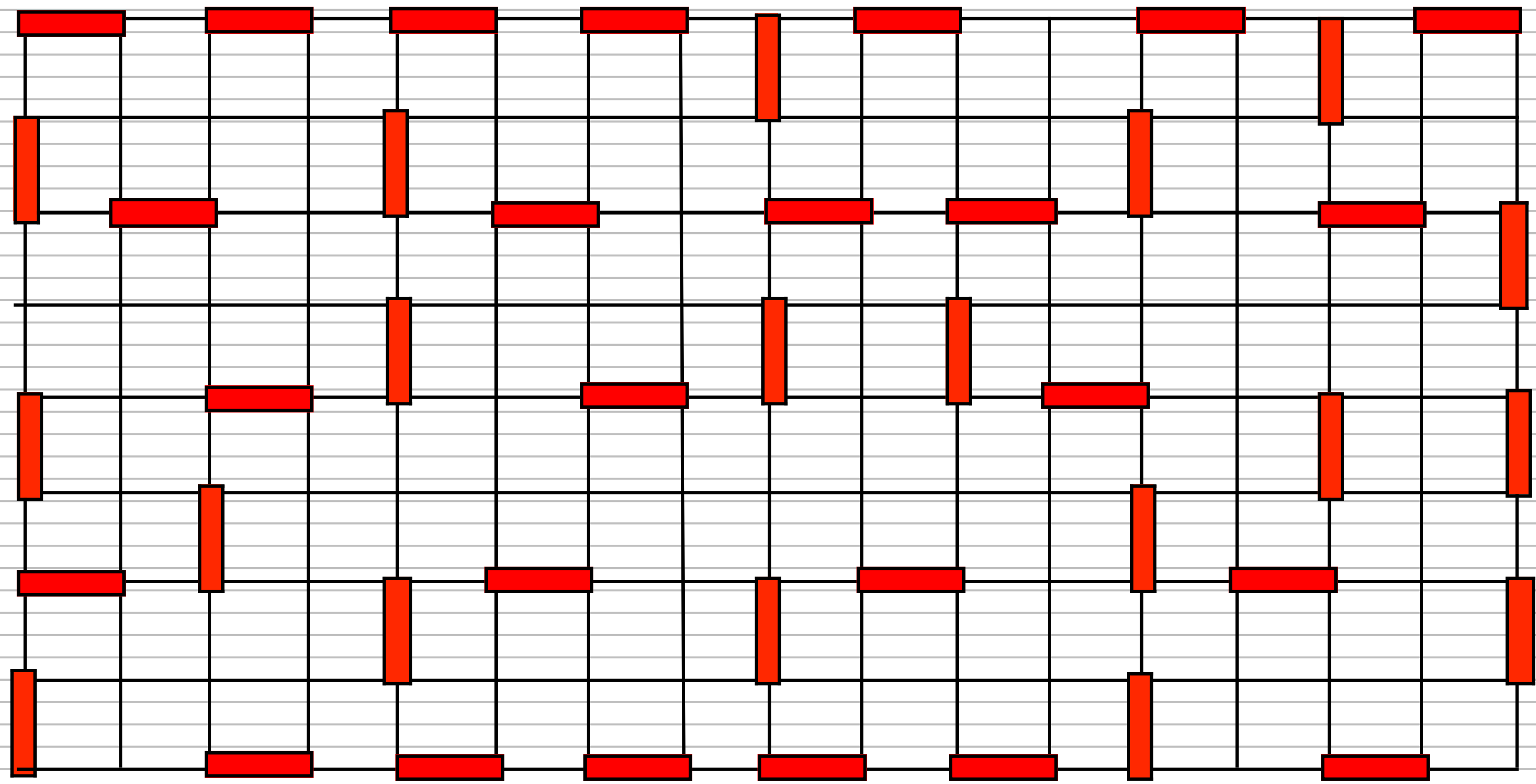
N odd



9 x 17

Red dimers touch odd-odd sublattice

N odd



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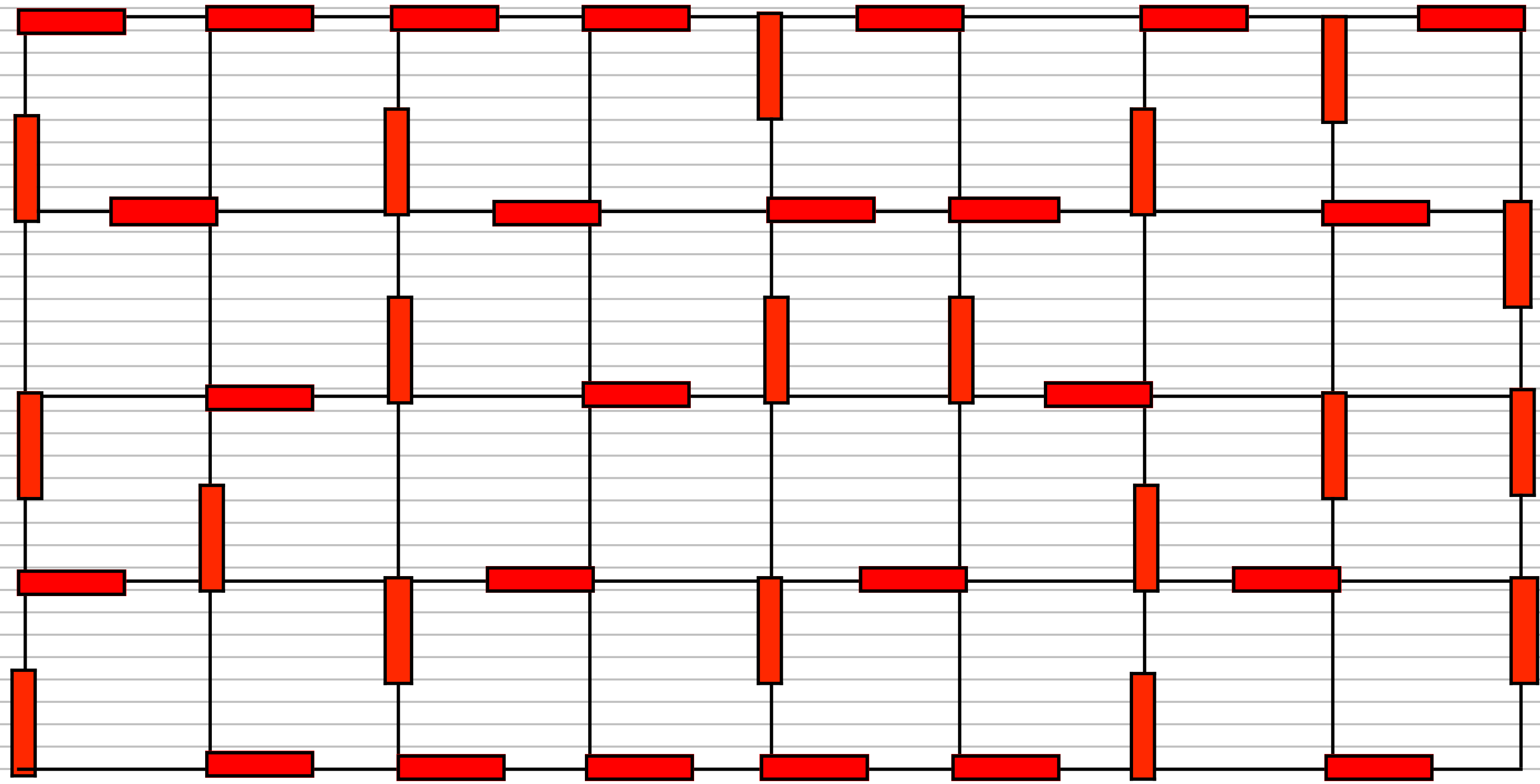
N odd

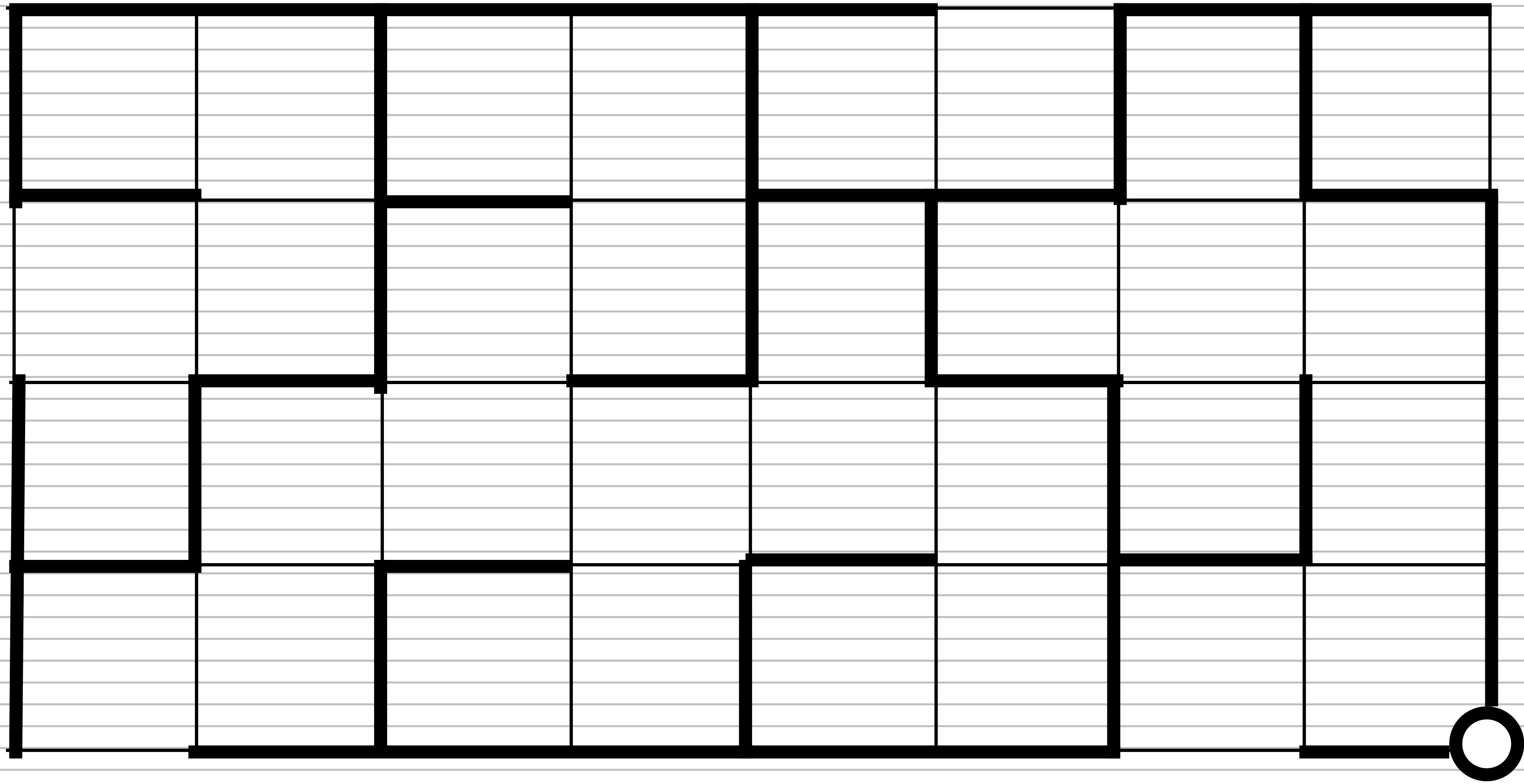


5 x 9

N odd

5 x 9

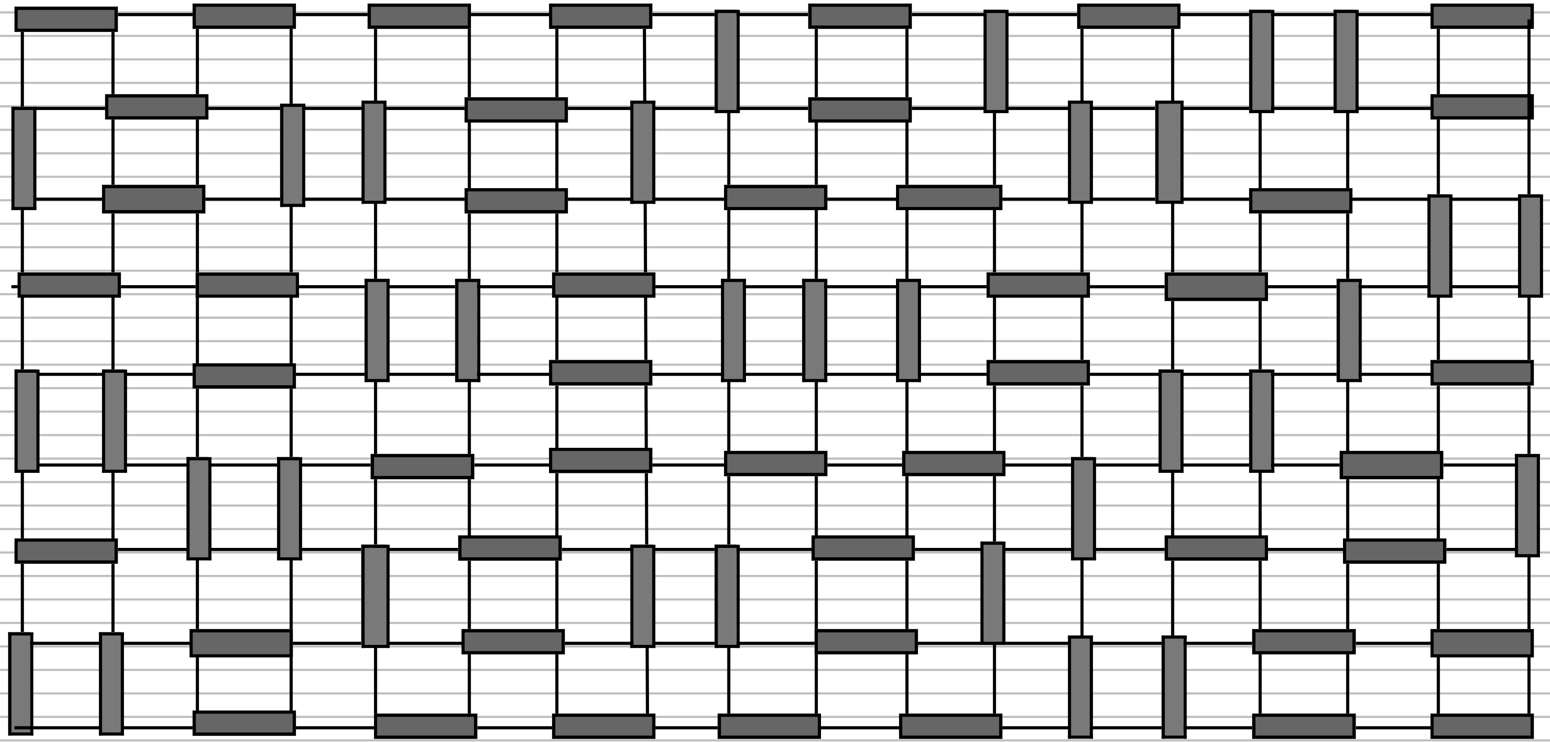
N odd



5 x 9

Spanning tree (covering graph with no loop) rooted at one site → boundary condition is closed on all four sides.

N even



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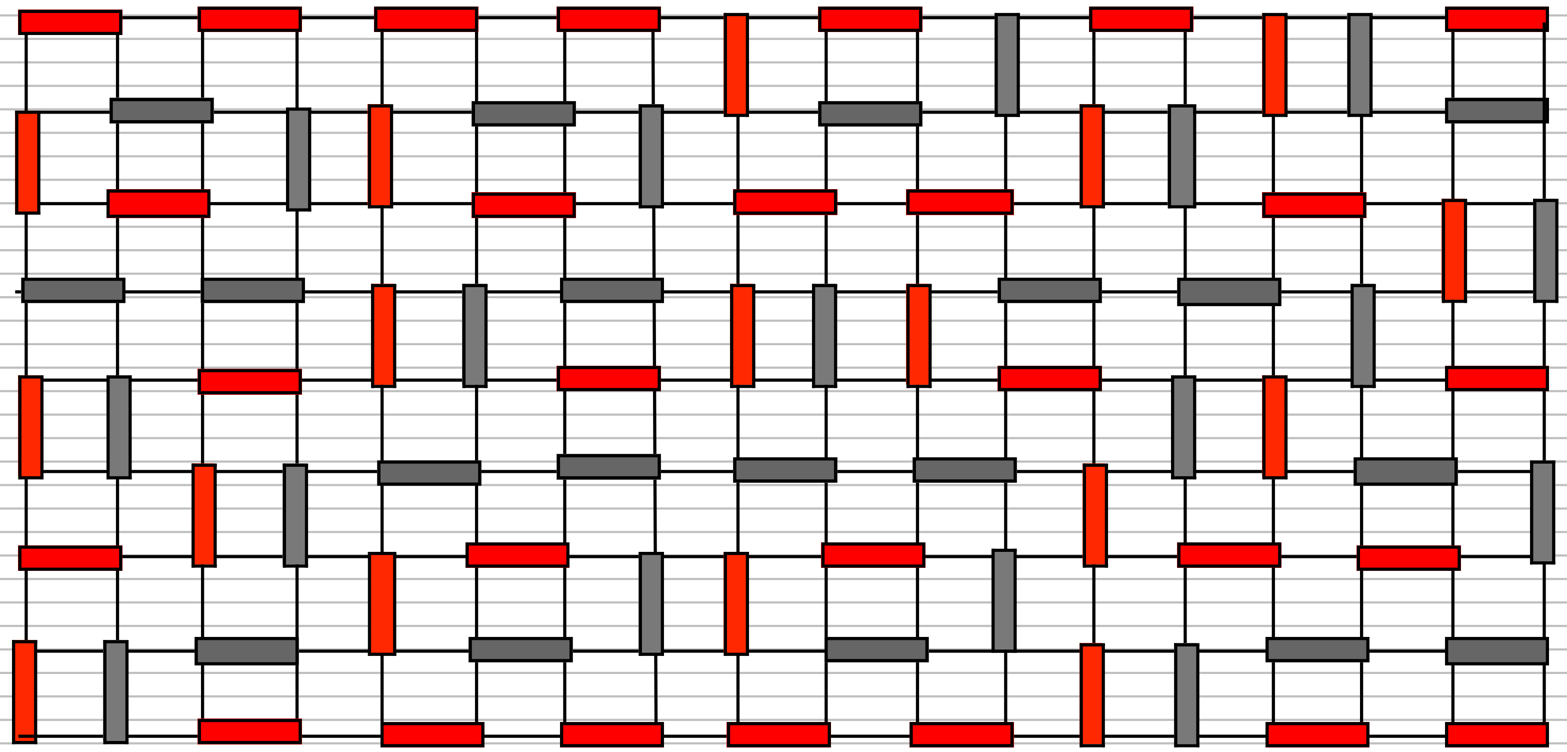
N even



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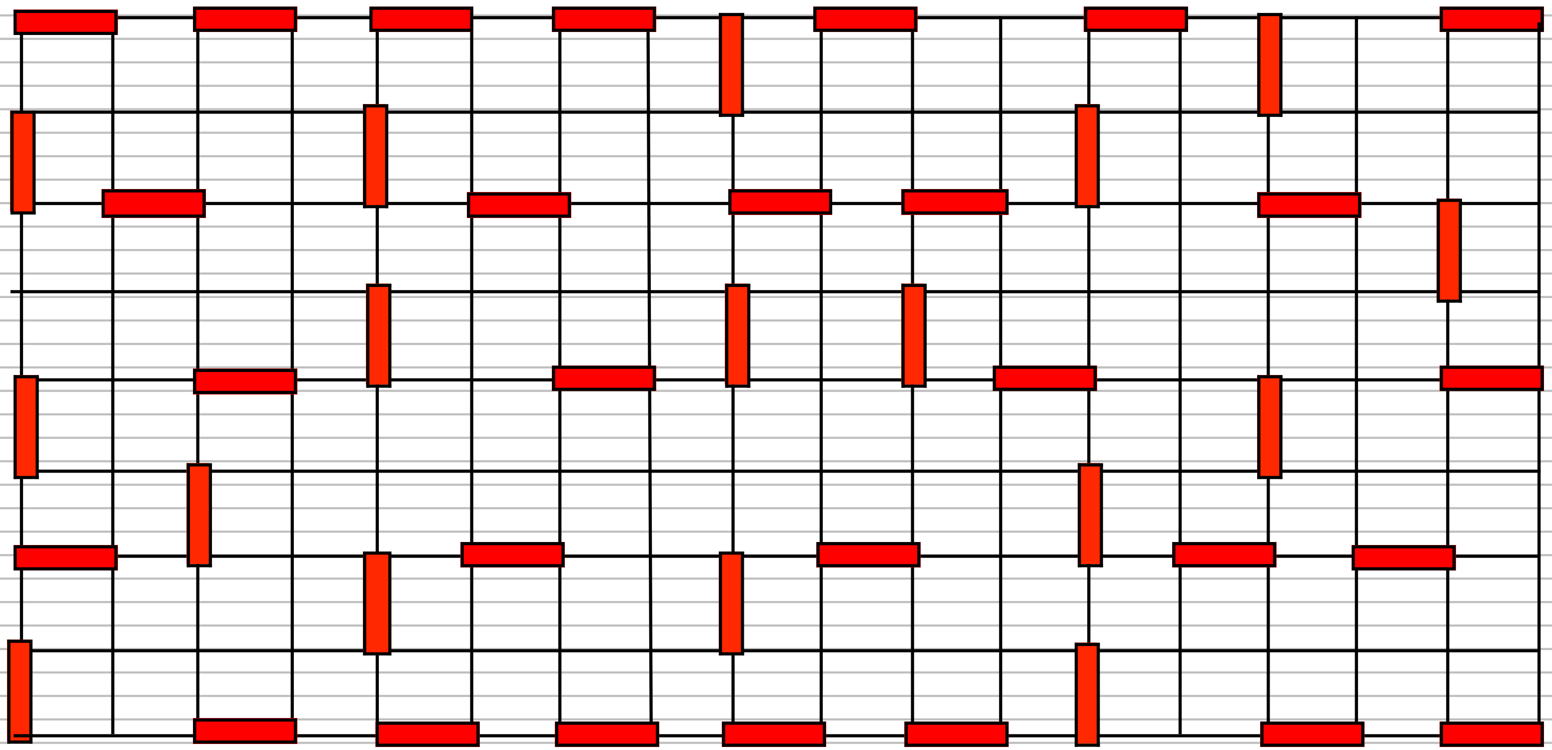

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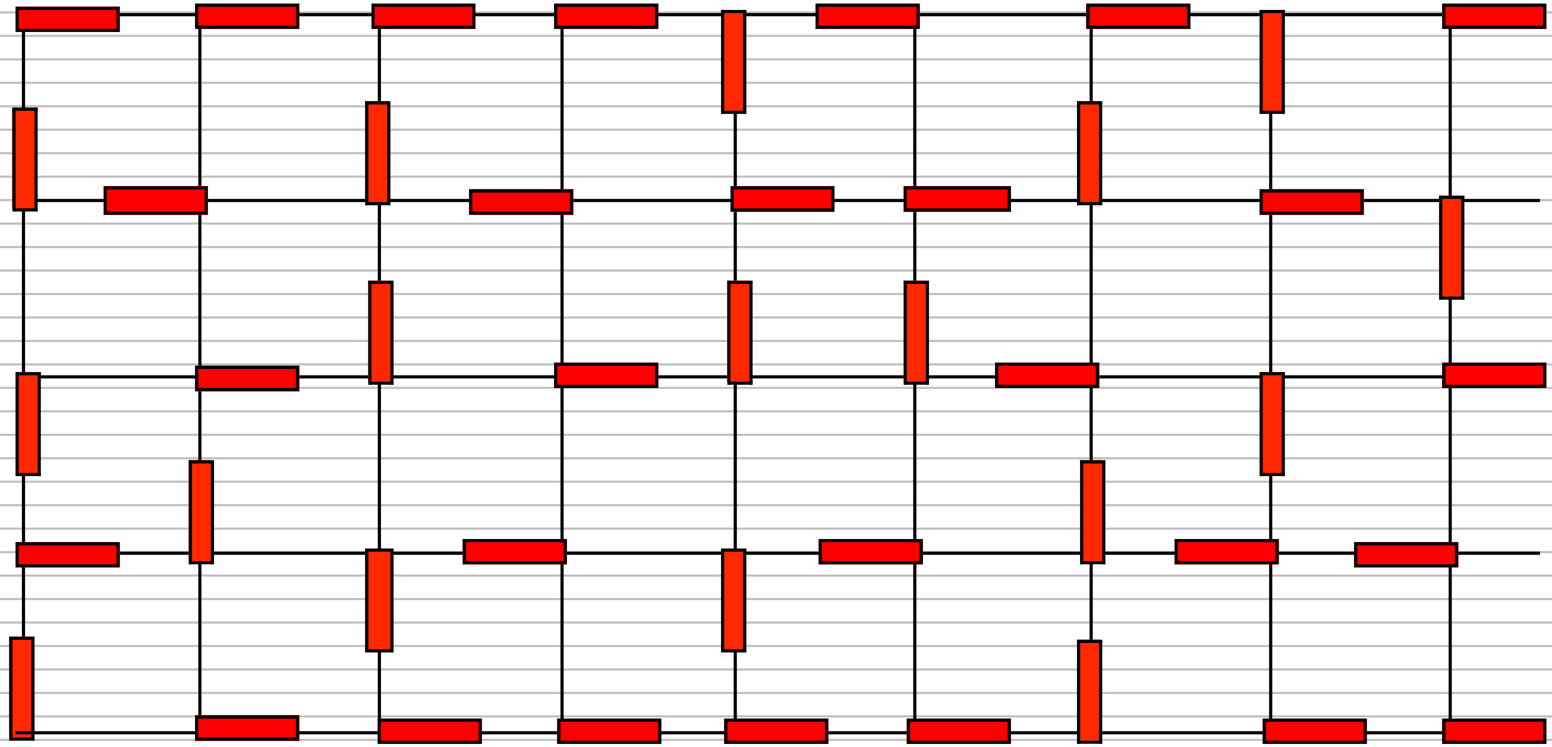

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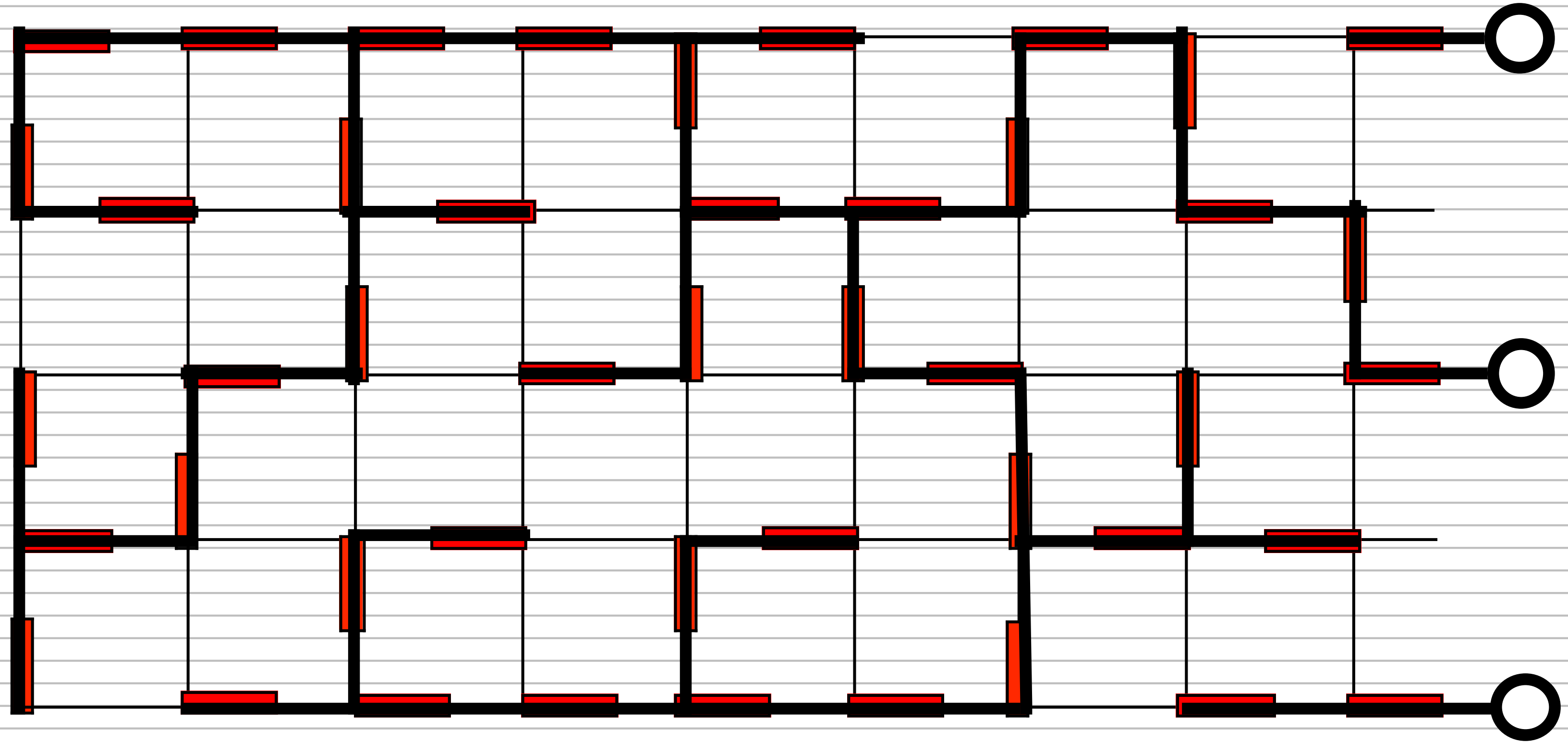

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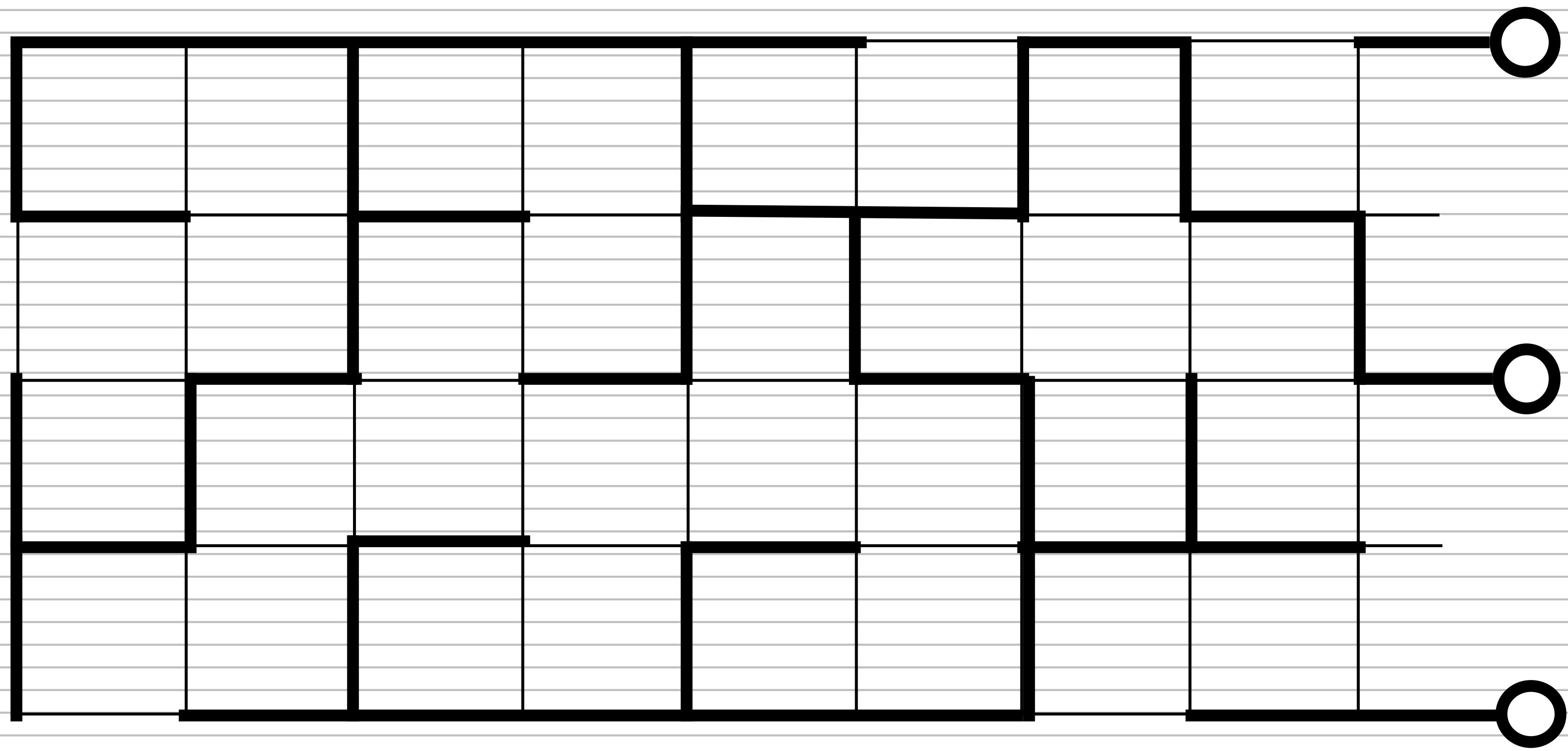
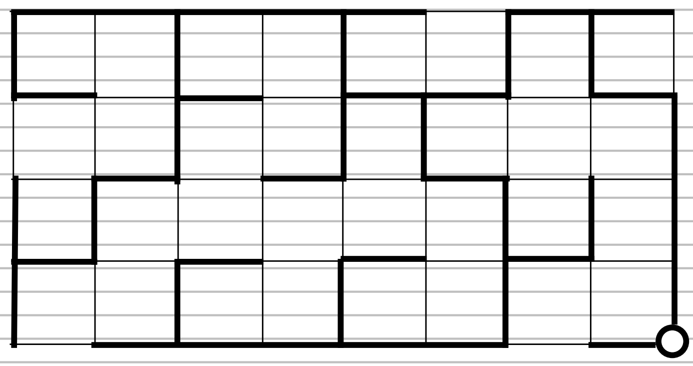


5 x 9

Spanning tree rooted at possibly every site of right side $\longrightarrow$ boundary condition is closed on three sides and open on right side.

Change of b.c.

Width N odd



closed (Neu) on left and right sides

$$\longrightarrow h_{\min}^{\mathrm{N,N}} = 0$$

$$A = \frac{\pi}{12} = \pi(h_{\min} - \frac{c}{24})$$

$$\boxed{c = -2}$$

Width N even

closed on left, open (Dir) on right

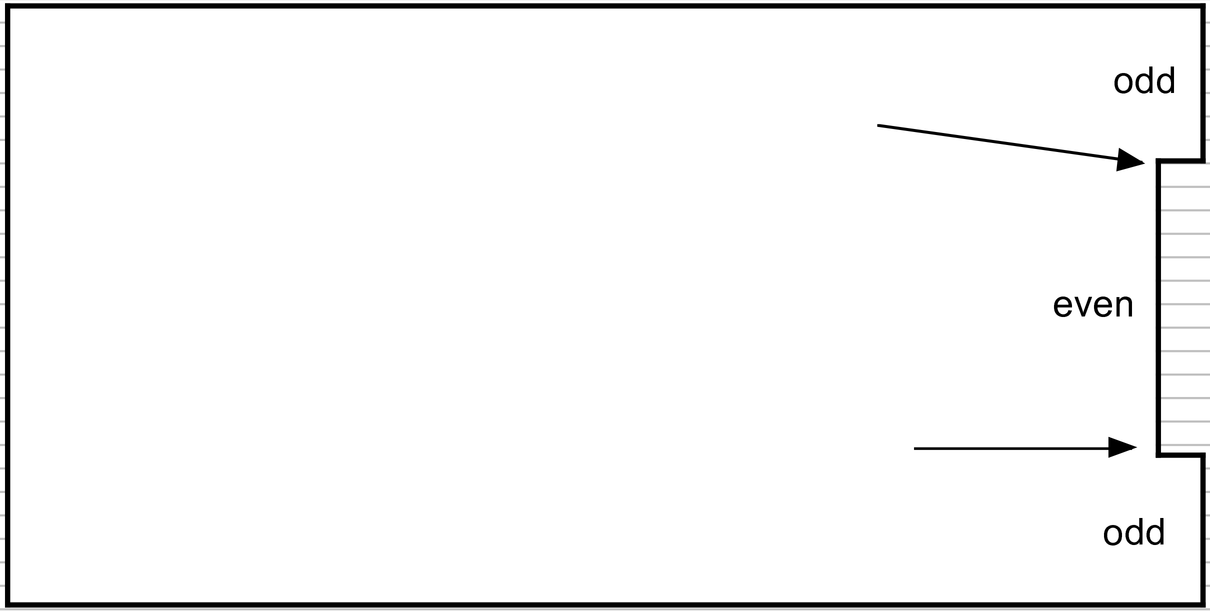
$$\longrightarrow h_{\min}^{\mathrm{N,D}} = -\tfrac{1}{8}$$

$$A = -\frac{\pi}{24} = \pi(h_{\min} - \frac{c}{24})$$

$$\boxed{c = -2}$$

B.c. changing field

Value $h_{\mathrm{min}} = -\frac{1}{8}$ confirmed by studying effect of change of parity:



Removal of one layer of sites on distance x is implemented by the insertion of b.c.c.f. $\mu(0)$ and $\mu(x)$, with μ is boundary primary of weight $-1/8$.

Cylinder

Similar, slightly more complex, situation on cylinder with perimeter N.

N odd: $A = \frac{\pi}{12}$, same value as for strip.

Correspondence with spanning trees holds, but a (defect) line of roots is to be inserted longitudinally $\rightarrow$ unwraps the cylinder to a strip (cfr Ferdinand).

N even: $A = -\frac{\pi}{6}$, four times the value on strip.

Spanning trees now grow on loops wrapped around the cylinder $\rightarrow$ proper boundary condition is antiperiodic $\rightarrow$ $h_{\text{min}} = -\frac{1}{8}$. Agrees with

$$A = 4\pi(h_{\text{min}} - \frac{c}{24}), \qquad \text{and} \quad c = -2.$$

CFT is OK

We get a consistent picture for finite size corrections in dimer model, leading to

$c = -2$ (well-known value for spanning trees/sandpile model)

provided proper attention is paid to boundary conditions, better analyzed in terms of spanning trees.

Based on:
N. Izmailian, V. Priezzhev, P.R. & C.K. Hu
Logarithmic CFT and Boundary Effects in the Dimer Model
Phys. Rev. Lett. 95, 260602 (Dec. 05)