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Q-Deformations and Integrable Motions on Manifolds with Curvature

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- AIM : to show the intimate relation between algebraic notions and quantities (namely $q-$Poisson coalgebras) and geometric ones (integrable motions on 2D manifolds with constant and non-constant curvature)
- TOOLS: Hopf-algebra structure of $Non-Standard$ $q-$deformations

PLAN OF THE LECTURE

1. Hamiltonians with co-algebra symmetry
2. Non-Standard deformations
3. Integrable Hamiltonians and non-constant curvature
4. Super-integrable Hamiltonians and constant curvature
5. More degrees of freedom. Separation of variables
6. From Classical to Quantum

I. Poisson Coalgebra $(sl(2,\mathbb{C}),\Delta)$

$$sl(2,\mathbb{C}) := \{J_3, J_+, J_-\}$$

$$\{J_3, J_\pm\} = \pm 2 J_\pm$$

$$\{J_+, J_-\} = 4 J_3$$

- Δ : co-associative Poisson Homomorphism:

$$\Delta : (sl(2,\mathbb{C}) \longrightarrow (sl(2,\mathbb{C}) \oplus (sl(2,\mathbb{C}))$$

$$\Delta(J_k) = J_k \oplus I + I \oplus J_k$$

- One particle symplectic realization:

$$J_-^{(1)} = q_1^2 \quad J_+^{(1)} = p_1^2 + b_1/q_1^2 \quad J_3^{(1)} = q_1 p_1$$

- Casimir function

$$\mathcal{C}^{(1)} = J_- J_+ + J_3^2 = b_1$$

- From 1- to 2- (and to many-) particle symplectic realization through Δ

$$J_-^{(2)} = q_1^2 + q_2^2 \quad J_+^{(2)} = p_1^2 + p_2^2 + b_1/q_1^2 + b_2/q_2^2$$

$$J_3^{(2)} = q_1 p_1 + q_2 p_2$$

- Fundamental property:

  Any smooth function $\mathcal{H}^{(2)} = \mathcal{H}(J_-^{(2)}, J_+^{(2)}, J_3^{(2)})$ (*) defines a completely integrable two-particle system, as it is equipped with the extra-integral of motion $\mathcal{C}^{(2)}$,reading:

$$\mathcal{C}^{(2)} = \Delta(\mathcal{C}) =$$

$$(q_1 p_2 - q_2 p_1)^2 + (\frac{b_1}{q_1^2} + \frac{b_2}{q_2^2})(q_1^2 + q_2^2)$$

- Hence, integrability of any Hamiltonian (*) is merely a consequence of coalgebra symmetry

It is worth to notice that, moreover, there are exceptional hamiltonians of type (*) which are Superintegrable (SI), namely, a further integral of motion exists:

$$\{\mathcal{H}^{(2)}, \mathcal{I}^{(2)}\} = 0$$

Example: if we consider a generic hamiltonian of the form:

$$\mathcal{H} = \frac{1}{2} J_+ \mathcal{F}(J_-)$$

for linear $\mathcal{F}$ we get a super-integrable system.

II. Integrable Systems through Non-Standard Deformations of $(sl(2,\mathbb{C}),\Delta)$

- Deformed PB:

$$\{J_3, J_+\} = 2J_+ \cosh zJ_- \qquad \{J_3, J_-\} = -2\,\frac{\sinh zJ_-}{z} \qquad \{J_-, J_+\} = 4J_3$$

- Casimir function

$$\mathcal{C}_z = \frac{\sinh zJ_-}{z} J_+ - J_3^2$$

- Deformed Coproduct

$$\Delta_z(J_-) = J_- \otimes 1 + 1 \otimes J_- \qquad \Delta_z(J_i) = J_i \otimes \mathrm{e}^{zJ_-} + \mathrm{e}^{-zJ_-} \otimes J_i \quad i = +, 3$$

z: real deformation parameter

- One and two particle symplectic realization

One-particle:

$$J_- = q_1^2 \qquad J_3 = \frac{\sinh zq_1^2}{zq_1^2} q_1 p_1$$
$$J_+ = \frac{\sinh zq_1^2}{zq_1^2} p_1^2$$

Two-particle:

$$J_- = q_1^2 + q_2^2 \qquad J_3 = \frac{\sinh zq_1^2}{zq_1^2}\, q_1 p_1\, \mathrm{e}^{zq_2^2} + \frac{\sinh zq_2^2}{zq_2^2}\, q_2 p_2\, \mathrm{e}^{-zq_1^2}$$
$$J_+ = \left(\frac{\sinh zq_1^2}{zq_1^2}\, p_1^2 + \frac{zb_1}{\sinh zq_1^2} \right) \mathrm{e}^{zq_2^2} + \left(\frac{\sinh zq_2^2}{zq_2^2}\, p_2^2 + \frac{zb_2}{\sinh zq_2^2} \right) \mathrm{e}^{-zq_1^2}$$

Two-particle Casimir:

$$\mathcal{C}_z = \frac{\sinh zq_1^2}{zq_1^2} \frac{\sinh zq_2^2}{zq_2^2} (q_1 p_2 - q_2 p_1)^2\, \mathrm{e}^{-zq_1^2} \mathrm{e}^{zq_2^2} + (b_1 \mathrm{e}^{2zq_2^2} + b_2 \mathrm{e}^{-2zq_1^2})$$
$$+ \left(b_1 \frac{\sinh zq_2^2}{\sinh zq_1^2} + b_2 \frac{\sinh zq_1^2}{\sinh zq_2^2} \right) \mathrm{e}^{-zq_1^2} \mathrm{e}^{zq_2^2}.$$

Most general integrable deformation of the free motion in $\mathbb{E}^2$ ($\mathcal{H} = \frac{1}{2}(p_1^2 + p_2^2)$) :

$$\mathcal{H} = \frac{1}{2} J_+ f(z J_-)$$

Simplest choice: $f(x) = 1$: however, not superintegrable!

Superintegrable hamiltonian:

$$\mathcal{H}_z^S = \frac{1}{2} J_+ \exp(z J_-)$$

i.e : $f(x) = \exp(x)$

Extra-integral:

$$\mathcal{I}_z = \frac{\sinh z q_1^2}{z q_1^2} p_1^2 \exp(q_1^2) = J_+^{(1)} \exp(z J_-^{(1)})$$

$\mathcal{H}_z^S, \mathcal{I}_z, \mathcal{C}_z$: functionally independent

Natural interpretation:

Hamiltonians of the form $J_+ f(zJ_-)$ are deformed kinetic energies:

$$\mathcal{H}_z = \mathcal{T}_z(q_i, p_i)$$

We will show:

1. $\mathcal{H}_z^I$: geodesic motion in 2D Riemannian space or 1+1 rel. space-time, with curvature depending both on z and on the point (q,p);
2. $\mathcal{H}_z^S$: geodesic motion … with curvature depending just on z

III. Integrable Deformations and Non-Constant Curvature

Let $\mathcal{H}_z^I(q_i, p_i) \quad \to \mathcal{T}_z^I(q_i, \dot{q}_i) \quad$ (Legendre Transformation):

$$\mathcal{T}_z^I(q_i, \dot{q}_i) = \frac{1}{2}\left(\frac{(\dot{q}_1)^2 \exp(-z(q_2)^2}{s_z(q_1^2)} + \frac{(\dot{q}_2)^2 \exp(z(q_1)^2}{s_z(q_2^2)}\right)$$

$$s_z(x) := \frac{\sin(zx)}{zx}$$

yields a geodesic flow on a 2D space.

- Metric:

$$ds^2 \equiv \frac{\exp(-z(q_2)^2}{s_z(q_1^2)} dq_1^2 + \frac{\exp(z(q_1)^2}{s_z(q_2^2)} dq_2^2 := g_{11}(q) dq_1^2 + g_{22}(q) dq_2^2$$

- Gaussian curvature:

$$K = -\frac{1}{(g_{11}g_{22})^{\frac{1}{2}}}\{\frac{\partial}{\partial q_1}(g_{11}^{-\frac{1}{2}}\frac{\partial}{\partial q_1}g_{22}^{\frac{1}{2}}) + \frac{\partial}{\partial q_2}(g_{22}^{-\frac{1}{2}}\frac{\partial}{\partial q_2}g_{11}^{\frac{1}{2}})\} = \; -z\sinh[z(q_1^2+q_2^2)]$$

K: negative and nonconstant!

Notice: To give a nonconstant curvature, the exponentials appearing in the deformed coproducts are essential

Geometry is better seen through a change of variables.

$$\cosh(\lambda_1 \rho) = \exp z(q_1^2 + q_2^2) \quad (\rho > 0)$$

$$\sin^2(\lambda_2 \theta) = \frac{\exp(2zq_1^2) - 1}{\exp z(q_1^2 + q_2^2) - 1}$$

Remarks

- We have set $z = \lambda_1^2$ and we have introduced a new real parameter λ_2, related with the *signature* of the metric.
- The new variable $\cosh(\lambda_1 \rho)$ is a collective variable, function of $\Delta(J_-)$; its role will be further specified later.
- The zero-deformation limit (improperly called the "classical limit") $z \to 0$ is in fact the flat limit $K \to 0$. In this limit $\rho \to 2(q_1^2 + q_2^2)$, $\sin^2(\lambda_2 \theta) \to \frac{q_1^2}{q_1^2 + q_2^2}$

Metric in the new variables:

$$ds^2 = \frac{1}{\cosh(\rho)}(d\rho^2 + \frac{\lambda_2^2}{\lambda_1^2}\sinh^2(\lambda_1\rho)d\theta^2) = \frac{1}{\cosh(\rho)}ds_0^2$$

$$ds_0^2 : \mathrm{so-called} \quad \mathrm{CK\ (Cayley-Klein)\ metric.}$$

$$K = K(\rho) = -\frac{1}{2}\lambda_1^2\frac{\sinh^2(\lambda_1\rho)}{\cosh(\lambda_1\rho)}$$

$$z \in \mathbb{R}^+ : K < 0; \qquad z \in \mathbb{R}^- : K \mathrm{periodic}$$

Kinetic energy and Hamiltonian:

$$\mathcal{T}_z^I(q,\dot{q}) = \frac{1}{2}\frac{1}{\cosh(\lambda_1\rho)}((\dot{\rho})^2 + \frac{\lambda_2^2}{\lambda_1^2}\sinh^2(\lambda_1\rho)(\dot{\theta})^2))$$

$$\mathcal{H}_z^I(q,p) = \frac{1}{2}\cosh(\lambda_1\rho)(p_\rho^2 + \frac{\lambda_1^2}{\lambda_2^2}\sinh^{-2}(\lambda_1\rho)(p_\theta)^2)$$

Moreover, as $(p_\theta)^2 = \mathcal{C}_z^I$, the usual reduction to the radial coordinate can be performed.

Specializations:

- $\lambda_2 \in \mathbb{R}$: $z \in \mathbb{R}^+ : \text{ def. Hyperbolic} - \text{space}; \quad z \in \mathbb{R}^- : \text{ def. sphere}$
- $\lambda_2 \in i\mathbb{R}$: $z \in \mathbb{R}^+ : \text{ def. DS} - \text{space}; \quad z \in \mathbb{R}^- : \text{ def. ADS} - \text{space}$

IV. Super-Integrable Deformations and Constant Curvature

- We start from the Superintegrable Hamiltonian:

$$\mathcal{H}_z^S = \frac{1}{2} J_+ \exp(z J_-)$$

- Legendre Transform → the two-body "free" Lagrangian (Kinetic energy):

$$\mathcal{T}_z^S(q, \dot{q}) = \frac{1}{2}\left(\frac{\exp(-z(q_1^2 + 2q_2^2))}{s_z(q_1^2)}(\dot{q}_1)^2 + \frac{\exp(-z q_2^2)}{s_z(q_2^2)}(\dot{q}_2)^2\right)$$

- Associated metric:

$$ds^2 = \left(\frac{\exp(-z(q_1^2 + 2q_2^2))}{s_z(q_1^2)}(\dot{q}_1)^2 + \frac{\exp(-z q_2^2)}{s_z(q_2^2)}(\dot{q}_2)^2\right)$$

- Gaussian curvature:

$$K(q, z) = z \quad = \quad \text{const.}$$

- Change of variables (as before):

$$ds^2 = \frac{1}{\cosh^2(\lambda_1 \rho)}(d\rho^2 + \frac{\lambda_2^2}{\lambda_1^2}\sinh^2(\lambda_1 \rho) d\theta^2) =$$

$$= \frac{1}{\cosh^2(\lambda_1 \rho)} ds_0^2$$

- New radial variable:

$$r = \int_0^\rho \frac{dx}{\cosh(\lambda_1 x)}$$

whence:

$$\tan(\lambda_1 r) = \sinh(\lambda_1 \rho)$$

$$\cos(\lambda_1 r) = \frac{1}{\cosh(\lambda_1 \rho)}$$

Finally:

$$\mathcal{T}_z^S = \frac{1}{2}(\dot{r})^2 + \frac{\lambda_2^2}{\lambda_1^2}\sin^2(\lambda_1 r)(\dot{\theta})^2)$$
$$\mathcal{H}_z^S = \frac{1}{2}(p_r)^2 + \frac{\lambda_1^2}{\lambda_2^2 \sin^2(\lambda_1 r)}(p_\theta)^2)$$

Integrals of motion:

$$\mathcal{C}_z^S = p_\theta^2; \quad \mathcal{I}_z^S = (sin(\lambda_2\theta)p_r + \frac{\lambda_1}{\lambda_2}\frac{\cos(\lambda_2\theta)}{\tan(\lambda_1 r)}p_\theta)^2$$

Comment:

The change of variable $\rho \to r$ through $dr = d\rho(\cosh(\lambda_1\rho))^{-\frac{1}{2}}$ is of course admissible even in the non-superintegrable case; however, with negligible advantage.

- Question : *Are there* other choices for the Hamiltonian yielding constant curvature?
- Answer: Yes, *there are !* However, I cannot say at the moment whether they all yield superintegrable systems.

In fact, let:

$$\mathcal{H}_z^S = \frac{1}{2} J_+ f(z J_-)$$

and ask for $K(\rho, z)$ be costant. It turns out:

$$K(x,z)/z = f' \cosh x + (f'' - f - (f')^2/f) \sinh x =$$

$$= f[g \cosh x + (g' - 1) \sinh x]; \quad g := f'/f$$

$$K' = 0 \equiv 2y \cosh x + y' \sinh x = 0 : \quad y := 2g' + g^2 - 1$$

yielding: $y = \frac{A}{\sinh^2(x)}$;

Solving for g, we get for $F := f^{\frac{1}{2}}$:

$$F'' = \frac{1}{4}(1 + \frac{A}{\sinh^2 x})F$$

whose general solution is ($A := \lambda(\lambda - 1)$):

$$F = (\sinh x)^{\lambda}[C_1 \sinh(x/2)^{(1-2\lambda)} + C_2 \cosh(x/2)^{1-2\lambda)}]$$

V. Many-Body Case; preliminary results

Co-algebra symmetry $\rightarrow$ N-body integrable version.

Example:N-body version of the simplest Hamiltonian:

$$\mathcal{H}_z^{I(N)} = \frac{1}{2}\sum_{j=1}^{N} s_z(q_j^2)p_j^2 \exp(z\sum_{k\neq j}\mathrm{sgn}(k-j)q_k^2)$$

Again we get a "free" Lagrangian:

$$\mathcal{T}_z^{I(N)} = \frac{1}{2}\sum_{i=1}^{N} \frac{(\dot{q}_i)^2 \exp(-z\sum_{k\neq j}\mathrm{sgn}(j-k)q_k^2)}{s_z(q_i^2)}$$

with the obvious corresponding metric.

The following coordinates are the most suitable to understand the nature of the problem, and to enforce separation (here I put for simplicity $\lambda_1 = 1$,$\lambda_2 = 0$):

$$\xi_0 = \cosh^2(\rho) := \Pi_{i=1}^{N} \exp 2zq_i^2$$

$$\xi_k = \sinh^2(\rho)\Pi_{j=1}^{k-1}\sinh^2\theta_j\cosh^2\theta_k = \Pi_{i=1}^{N-k}\exp(2zq_i^2)(\exp(2zq_{N-k+1}^2 - 1)\ \ (k = 1,\ldots,N-1)$$

$$\xi_N = \sinh^2(\rho)\Pi_{j=1}^{N}\sinh^2\theta_j$$

$$\xi_0^2 - \sum_{k=1}^{N}\xi_k^2 = 1.$$

• Geodesic flow in $(\rho, \vec{\theta})$ variables:

The Hamiltonian reads

$$\mathcal{H}_z^{I(N)} = \cosh(\rho)[p_\rho^2 + \frac{1}{\sinh^2(\rho)}\sum_{j=1}^{N}(\Pi_{k=1}^{j-1}\frac{1}{\sin^2(\theta_k)})p_{\theta_j}^2]$$

and the Integrals of motion are:

$$\mathcal{I}_j = \Delta^{(j)}\mathcal{C} = (\Pi_{k=1}^{j-1}\frac{1}{\sin^2(\theta_k)})p_{\theta_j}^2$$

So we are left with a one-dimensional problem.

- Main advantage (and limitation) of dynamical systems with co-algebra symmetry: for any N, you end up with a typical mean field dynamics: The system has a cluster structure: each cluster, whose dynamical variables are given by the partial coproducts of the (q-deformed) Lie algebra generators, moves as a single particle in a field generated self-consistently by the individual constituents. The coupling between the clusters and the mean field is parametrized by the appropriate partial Casimirs.

- The models can be extended to incorporate the interaction with an external central field, preserving integrability. It is enough modifying the Hamiltonian by adding an arbitrary function of J_-. In this way, we have constructed Hamiltonian describing an integrable deformation of Harmonic or Kepler motion on a curved background, reducing to the usual one as $z \to 0$.

VI. Towards Quantization

The Poisson brackets relations are replaced by the following CRs:

- Deformed CRs:

$$[J_3, J_+\,]_- = [J_+ \cosh z, J_-]_+ \qquad [J_3, J_-]_- = -2\frac{\sinh zJ_-}{z} \qquad [J_-, J_+]_- = 4J_3$$

- Casimir operator

$$\mathcal{C}_z = \frac{1}{2}[\frac{\sinh zJ_-}{z}, J_+]_+ - J_3^2$$

- Realization.

  As the coproduct map has no ordering ambiguities, also in the quantum case the basic information is encoded in the one-dimensional case. We use the coordinate $x = \lambda_1 \rho$ and get:

$$\hat{J}_- = \lambda_1^{-2} \log\cosh x$$
$$\hat{J}_3 = \frac{1}{2}[\partial_x, \sinh(x)]_+$$
$$\hat{J}_+ = \lambda_1^2(\partial_x h(x)\partial_x + \frac{1}{h(x)}) \quad h(x) = 2\cosh x$$

- Notice the additional term $\frac{1}{h(x)}$ with respect to the classical case.

As an example we consider just the quantum analog of the geodesic motion with nonconstant curvature, thus taking $\hat{J}_+$ as the Hamiltonian operator. After a further (trivial) gauge transformation, we arrive at the equation (μ: "spectral parameter")

$$\Box \quad \psi_{xx} = (\mu \mathrm{sech} x + \frac{1}{4})\psi$$

1. $\Box$ is a special case of Heun differential equation with parameters:

$$a = -1; \quad \mu = q: \quad \gamma = 0; \quad \delta = 1; \alpha, \beta = \pm 1/2$$

2. Extra-dimensions result in the addition of appropriate centrifugal terms, controlled by the partial Casimirs.

- Examples

1. E_1: Geodesic motion on constant curvature surfaces
2. E_2: Deformed Harmonic motion on constant curvature surfaces.
3. E_3: Geodesic motion on nonconstant curvature surfaces

EXAMPLE I

Let:

$$\mathcal{H} = J_+ \exp(zJ_-) = \frac{\exp(2zq^2 - 1)}{2zq^2} p^2$$

Define

$$a_i = J_{3,i} \exp(zJ_{-,i}); \quad b_i = J_{+,i} \exp(zJ_{-,i}); \quad c_i = J_{-,i}$$

$$\mathcal{C}_{z,i} = \exp(-2zc_i)(a_i^2 + b_i \frac{\exp(2zc_i) - 1}{2z})$$

We don't work with single particle variables, but first use:

$$
\begin{aligned}
& a = \Delta^{(2)}(a_1) = \Delta^{(2)}(J_{3,1}) \exp(z\Delta^{(2)} J_{-,1}) = \\
& a_1 + \exp(2zc_1)a_2 \\
& b = \Delta^{(2)}(b_1) = b_1 + \exp(2zc_1)b_2 \; := \; \mathcal{H}_2 \\
& c = \Delta^{(2)}(c_1) = c_1 + c_2
\end{aligned}
$$

Then, turn to a_1, b_1, c_1

Remark: Geometric variables: $\cosh(\lambda_1\rho) = \exp(2zc)$ $\quad \sin^2(\lambda_2\theta) = \frac{\exp(2zc_1)-1}{\exp(2zc)-1}$

According with the previous outlined strategy, we start by solving the simplest equation, involving collecting variables, then solve for single-particle dynamics

Evolution equations for collective variables:

$$
\begin{aligned}
& \dot{a} = 2b + 4a^2 = E + 4a^2 \\
& \dot{b} = 0 \\
& \dot{c} = 4a
\end{aligned}
$$

There are two cases, according to the sign of zE.

1. $2zE = \gamma^2 > 0, \quad \gamma \in \mathbb{R}$; then:

$$a = \frac{E}{\gamma}\tanh(2\gamma(t - t_0))$$
$$\cosh(\lambda_1 \rho) = \exp(2zc) = \cosh(2\gamma(t - t_0))$$

   Notice: The "radius" ρ grows linearly in time.

2. $2zE = -\gamma^2 < 0, \quad \gamma \in \mathbb{R}$. Hyperbolic functions are replaced by trigonometric ones. However, having to do with free motion, the energy E has to be taken as positive. So it is z that changes sign, and consequently the variable ρ, which again evolves linearly in time, has to be viewed as an angle.

The one-body variables obey the system of nonlinear equations:

$$\dot{a}_1 = 2b_1 + 4za_1^2 + 4b_2 \exp(2c_1 z)\frac{\exp(2c_1 z) - 1}{2z}$$
$$\dot{b}_1 = 8za_1 \exp(2zc_1)b_2$$
$$\dot{c}_1 = 4a_1$$

which can be explicitly solved in terms of trigonometric/hyperbolic functions.

You may proceed as follows:

- From the second and the third equation, you get:

$$\exp(2zc_1) = \frac{E - b_1}{b_2} \quad b_2 : \text{constant of the motion}$$

- Then, you use the one-body Casimir $\delta^{(1)}$, such that:

$$exp(2zc_1) = \frac{za_1^2 + b_1}{z\delta^{(1)} + b_1}$$

  to eliminate a_1 in favor of b_1, c_1, finally getting the evolution equation for $\gamma_1 := \exp(2zc_1)$:

$$\dot{\gamma}_1 = 8\gamma_1 \sqrt{zb_2(\gamma_1 - \gamma_+)(\gamma_1 - \gamma_-)}$$

  the parameters $\gamma_\pm$ being given in terms of the constants $b_2, \delta^{(1)}, E$.

- For $zE < 0, \gamma_\pm \in \mathbb{R}$ the solution is given in terms of trigonometric functions and reads:

$$\gamma_1 = \frac{\gamma_+ \gamma_-}{\gamma_+ \cos^2(\sqrt{zE}(t - t_0)) + \gamma_- \sin^2(\sqrt{zE}(t - t_0))}$$

EXAMPLE II

Let

$$\mathcal{H} = \exp(zJ_-)(J_+\omega^2\frac{\sinh(zJ_-)}{z})$$

It describes the motion of a particle in the field given by a $q-$deformed harmonic oscillator, on a surface with constant curvature.

The dynamical variables and the Casimir are defined as before. The equations for the collective variables are easily written down in terms of variables $a, b, \gamma := \exp(2zc) = \cosh(\lambda_1 \rho)$.

$$\begin{aligned}
\dot{a} &= 2b + 4za^2 + \omega^2 \exp(2zc)(\frac{\exp(2zc) - 1}{z}) \\
\dot{b} &= -4\omega^2 a\gamma \\
\dot{\gamma} &= 8za\gamma
\end{aligned}$$

Thanks to the integrals of motion $\mathcal{H}, \mathcal{C}_z$ one finally gets a first order evolution equation for γ ($\longrightarrow$ for ρ):

$$\dot{\gamma} = 8z\gamma\sqrt{\omega^2(\gamma - \gamma_+)(\gamma_- - \gamma)}$$

For suitable values of $\mathcal{H}, \mathcal{C}_z$ the motion for γ is periodic, expressed in terms of trigonometric functions, just as that for the one-body variables derived in the previous example, and confined in the interval $[\gamma_-, \gamma_+]$.

Following again the same path, one now considers evolution equations for single-particle variables:

$$\dot{a}_1 = 2b_1 + 4za_1^2 + 2\exp(2zc)\frac{1-2\exp(2c_1 z)}{2z}(2zb_2\exp(-2zc_2+\omega^2)$$
$$\dot{b}_1 = 8za_1\exp(2zc)\frac{\omega^2 + b_2\exp(-2zc_2)}{2z}$$
$$\dot{c}_1 = 4a_1 \tag{0.1}$$

As for the geodesic case, the constants of the motion $\mathcal{H}, \mathcal{C}_z, \delta^{(1)}, \delta^{(2)}$ yield finally first order equations for the above degrees of freedom. The simplest one involves $\exp(2zc_1) = \gamma_1$and reads:

$$\dot{\gamma}_1 = 8z\gamma_1\sqrt{k(\xi_+ - \gamma_1)(\xi_- - \gamma_1)}$$

which is again solvable in terms of trigonometric/hyperbolic functions.

EXAMPLE III

As for the geodesic motion on surfaces with noncostant curvature, just a few preliminary remarks.

Recall that in *polar* variables $\rho, p_\rho, \theta, p_\theta$ for the so-called deformed ADS space-time the Hamiltonian reads:

$$\mathcal{H} = \cos\rho(p_\rho^2 + \frac{p_\theta^2}{\sin^2(\rho)})$$

The corresponding evolution equation for the collective variable $\cos\rho$ is obtained by inverting the integral:

$$t = \int^{\cos\rho} \frac{dy}{\sqrt{y(E(1-y^2) - p_\theta^2 y)}}$$

In suitable rescaled variables ($a = \cot(p_\theta^2/|E|)$, one get for $\cos\rho$ a periodic motion, with the following period:

$$T = (|E|)^{-\frac{1}{2}} \int_0^a \frac{dx}{\sqrt{x(x-a)(x+a^{-1})}}$$