Casimir force between planes as a boundary finite size effect

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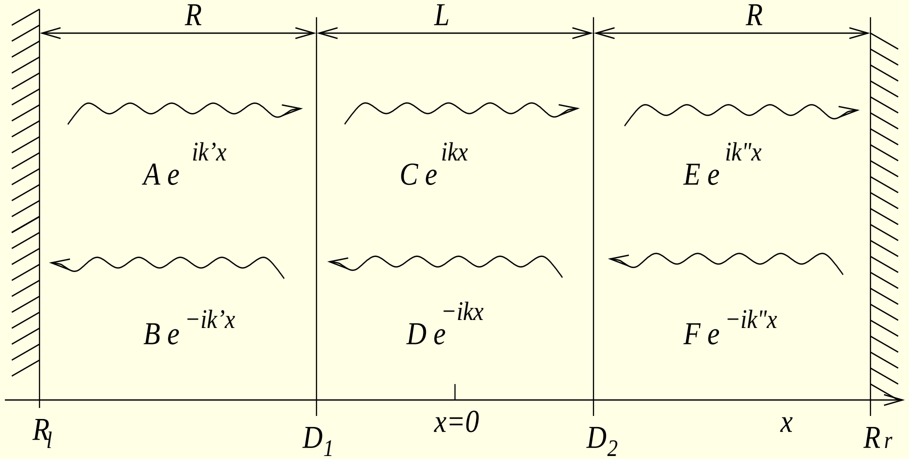


Plan of the talk:

Motivation and review of Casimir effect

Boundary state formalism

Derivation of Casimir energy

Applications

- scalar field with Robin boundary condition
- dielectric slabs separated by vacuum slot
- massless fermion with “bag boundary condition”

Conclusions

Casimir energy / Casimir force

Casimir energy / Casimir force

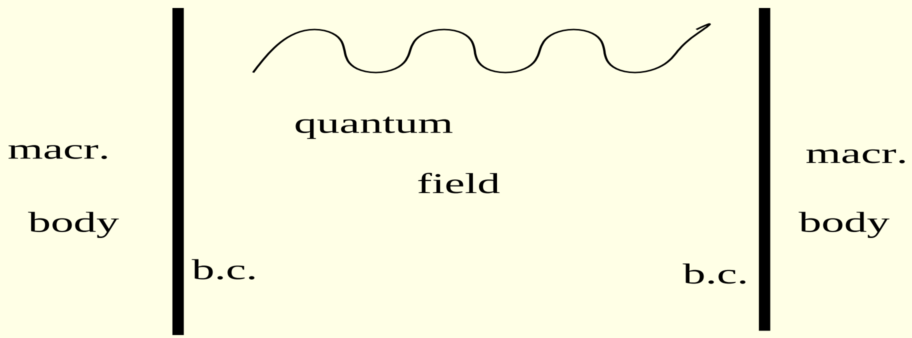
manifestation of zero point fluctuations

volume dependence of ground state “Casimir “ energy

Casimir energy / Casimir force

manifestation of zero point fluctuations

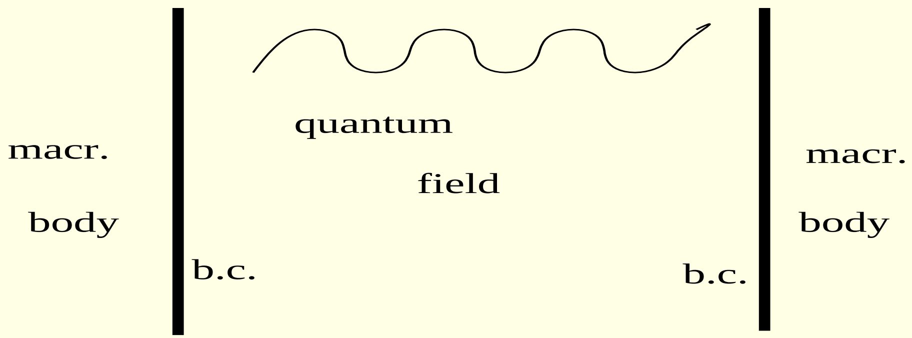
volume dependence of ground state "Casimir " energy



Casimir energy / Casimir force

manifestation of zero point fluctuations

volume dependence of ground state “Casimir “ energy



aim: dependence on boundary conditions in planar geometry

finite size effects (FSE) in boundary QFT

QFT in periodic box Lüscher FSE in terms of S matrix

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BPT   2 dim. IBQFT on strip L FSE   in terms of R matrix

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- depending only on R

- can be summed up for free bulk

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R from   "one boundary"   geometry

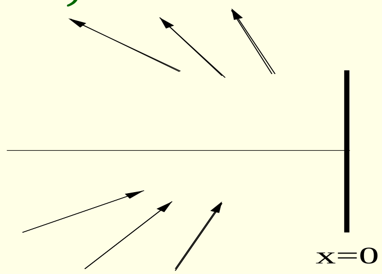
Boundary state formalism

$\Phi(x,\vec{y},t)$ in $D+1$ dim. $\mathcal{L}=\frac{1}{2}(\partial_t\Phi)^2-\frac{1}{2}(\partial_x\Phi)^2-\frac{1}{2}\left(\vec{\partial}\Phi\right)^2-V(\Phi)$

restricted to $x<0$ by $V_B(\Phi(0,\vec{y},t))$ at $x=0$

there are TWO Hamiltonian descriptions

t time, boundary in x $\mathcal{H}_B=\left\{|k_\perp,\vec{k}_\parallel;k'_\perp,\vec{k'}_\parallel;\dots\rangle_B\right\}$



$|\text{in}\rangle$ $t\to-\infty$ $\forall k_\perp>0$
$|\text{out}\rangle$ $t\to\infty$ $\forall k_\perp<0$

reflection matrix $R_{\alpha\beta}={}_B^{out}\langle\alpha|\beta\rangle_B^{in}$

one particle elastic $k_\perp=m_{\text{eff}}(\vec{k}_{||})\sinh\theta$ $m_{\text{eff}}(\vec{k}_{||})=\sqrt{m^2+\vec{k}_{||}^2}$

$$R(\theta,m_{\text{eff}}(\vec{k}_{||}))\,(2\pi)^D\,\delta\,(\theta-\theta')\,\delta^{(D-1)}\left(\vec{k}_\parallel-\vec{k}'_\parallel\right)={}_B^{out}\langle k'_\perp,\vec{k}'_{||}|k_\perp,\vec{k}_{||}\rangle_B^{in}$$

boundary reduction formulae (Bajnok, Böhm, Takács) R, $R_{\alpha\beta}\leftrightarrow$ corr. fns.

$$H_B=\int_{-\infty}^0 dx\int d\vec{y}\left[\tfrac{1}{2}\Pi^2+\tfrac{1}{2}(\partial_x\Phi)^2+\tfrac{1}{2}(\vec{\partial}\Phi)^2+V(\Phi)+\delta(x)V_B(\Phi)\right]$$

second description in crossed channel $\tau = ix$ time $\xi = -it$ space

$$\mathcal{H} = \left\{ |k_1, \vec{k}_1; k_2, \vec{k}_2; \ldots; k_n, \vec{k}_n\rangle \right\}$$ $\in$ bulk theory

bulk Hamiltonian

boundary condition in time as initial state

$$G(\tau_i, \xi_i, \vec{y}_i) = \langle 0| \, \Phi(\tau_1, \xi_1, \vec{y}_1) \ldots \Phi(\tau_N, \xi_N, \vec{y}_N) \, |B\rangle$$

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boundary state $|B\rangle \in \mathcal{H}$ from equality of corr. functions in the two pictures

$$|B\rangle = \left\{ 1 + K_1 A^{\dagger}_{in}(0,0) + \int_0^\infty \frac{d\theta}{2\pi} \int \frac{d^{D-1}\vec{k}_{||}}{(2\pi)^{D-1}} K_2 \left(\theta, m_{\mathrm{eff}}(\vec{k}_{||})\right) A^{\dagger}_{in}(-\theta, -\vec{k}_{||}) A^{\dagger}_{in}(\theta, \vec{k}_{||}) + \ldots \right\} |0\rangle$$

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$K_1 \neq 0$ iff ${}_B\langle 0| \, \Phi\left(t, x, \vec{y}\right) |0\rangle_B \neq 0$ keep $K_1 = 0$

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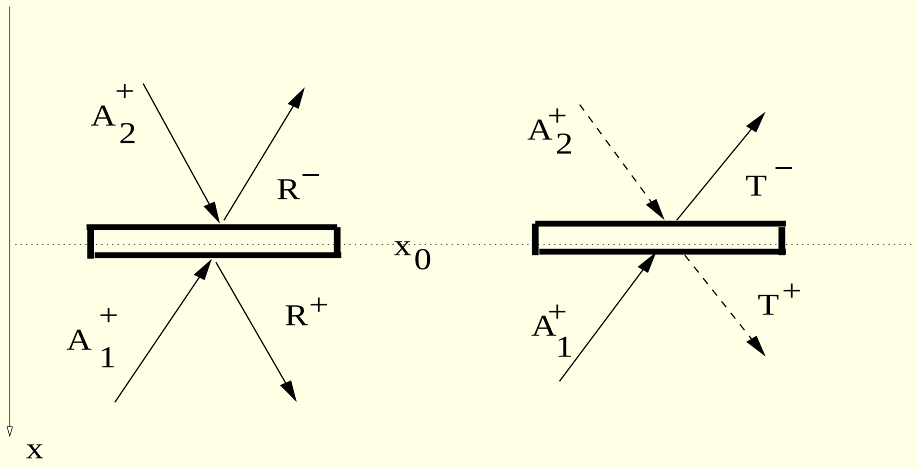
can be summed up for free bulk and elastic reflection

$$|B\rangle = \exp\left(\int_0^\infty \frac{d\theta}{2\pi} \int \frac{d^{D-1}\vec{k}_{||}}{(2\pi)^{D-1}} K_2(\theta, m_{\text{eff}}(\vec{k}_{||})) A^\dagger_{in}(-\theta, -\vec{k}_{||}) A^\dagger_{in}(\theta, \vec{k}_{||}) \right) |0\rangle$$

Casimir effect: boundary conditions with transmissions "defect"



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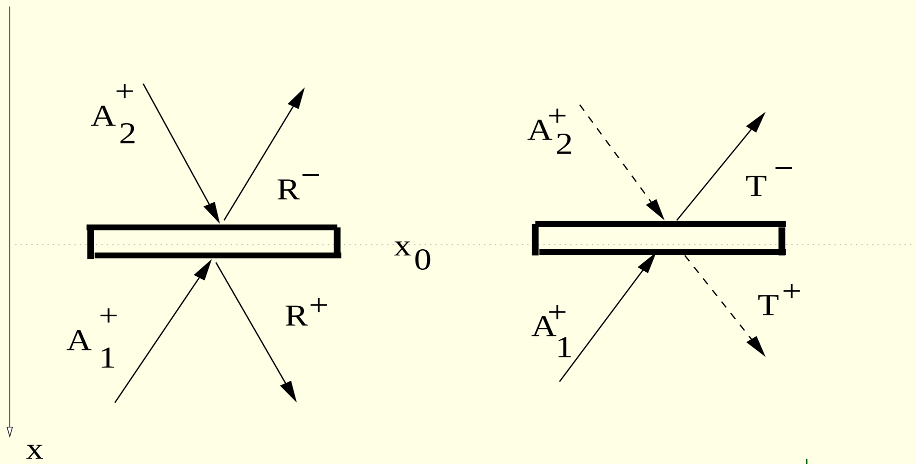

folding trick (Bajnok, George) maps to boundary theory $R^{\pm}$ reflections $T^{\pm}$ transmissions

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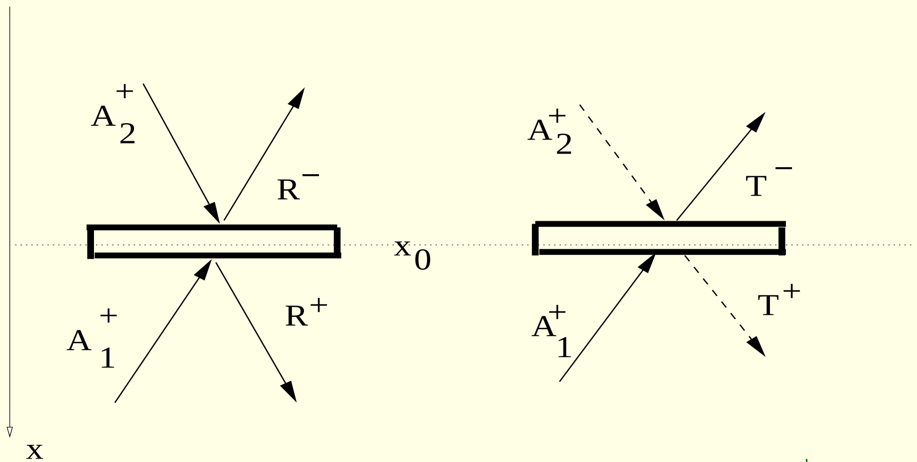

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defect operator $\tilde{\theta} = \frac{i\pi}{2} - \theta$ $\tilde{m} = m_{\text{eff}}(\vec{k}_{||})$

$$D = 1 + \int_{-\infty}^{\infty} \frac{d\theta}{4\pi} \int \frac{d^{D-1}\vec{k}_{||}}{(2\pi)^{D-1}} \big[R^+(\tilde{\theta},\tilde{m}) A_1^\dagger(-\theta,-\vec{k}_{||}) A_1^\dagger(\theta,\vec{k}_{||}) + T^+(\tilde{\theta},\tilde{m}) A_1^\dagger(-\theta,-\vec{k}_{||}) A_2(-\theta,-\vec{k}_{||}) + T^-(\tilde{\theta},\tilde{m}) A_1(\theta,\vec{k}_{||}) A_2^\dagger(\theta,\vec{k}_{||}) + R^-(\tilde{\theta},\tilde{m}) A_2(\theta,\vec{k}_{||}) A_2(-\theta,-\vec{k}_{||}) \big] + \dots$$

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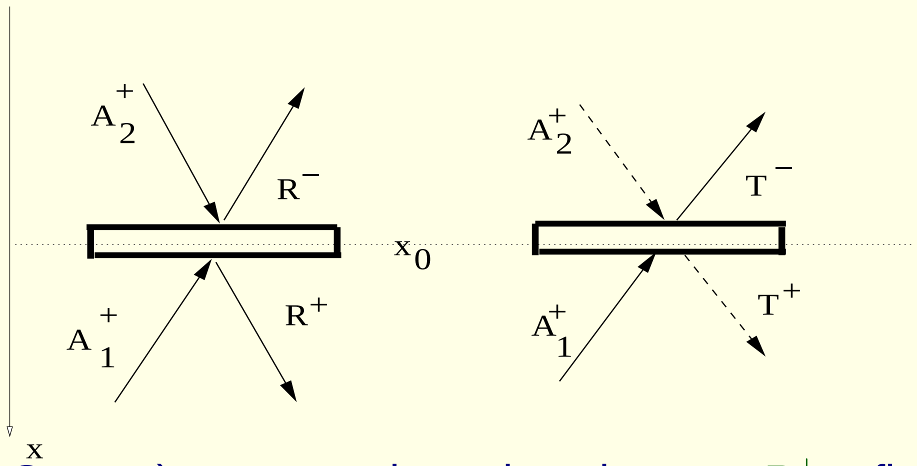

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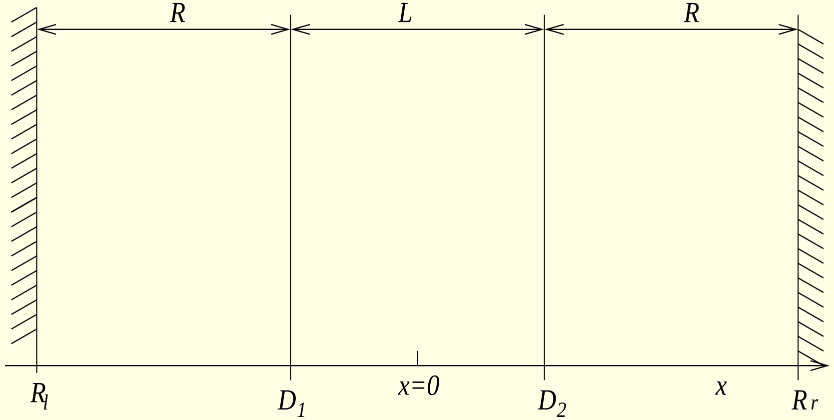
can be summed up for free bulk and elastic reflections/transmissions

Derivation of Casimir energy



$$D_i = \begin{pmatrix} R_i^+(\theta, m_{\text{eff}}(\vec{k}_{||})) & T_i^-(\theta, m_{\text{eff}}(\vec{k}_{||})) \\ T_i^+(\theta, m_{\text{eff}}(\vec{k}_{||})) & R_i^-(\theta, m_{\text{eff}}(\vec{k}_{||})) \end{pmatrix}$$

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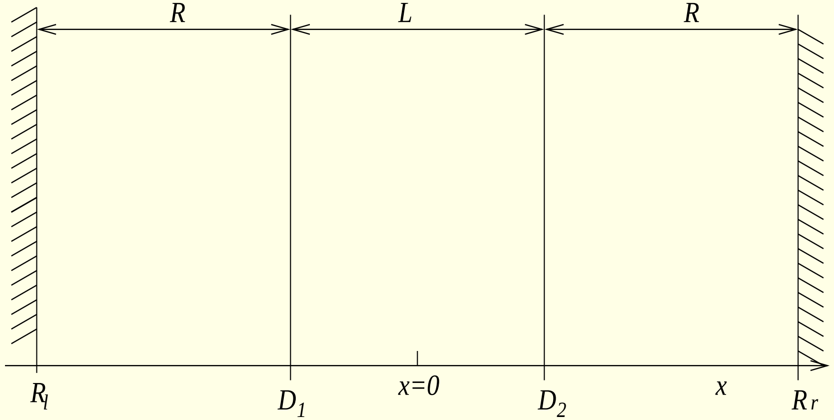

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H_B Hamiltonian $\mathcal{H}_B$ crossed channel H_x^i ($i = 1, ..3$) Hamiltonian $\mathcal{H}$

infinite dimensions compactified perimeter T

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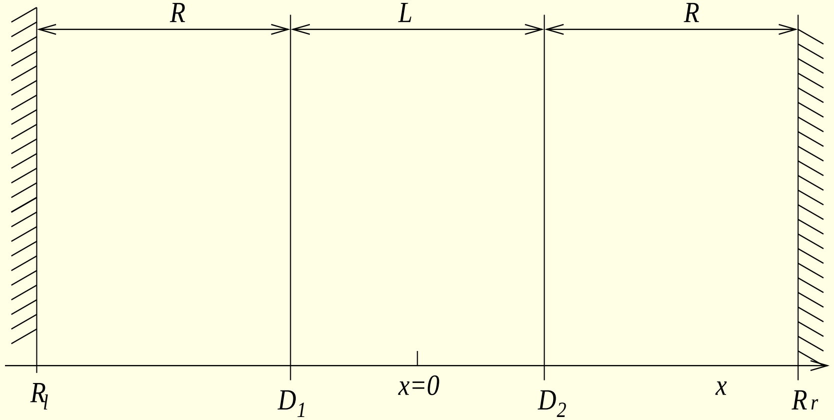

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$$Z_R(L,T) = \mathrm{Tr}_{\mathcal{H}_B} \mathrm{e}^{-TH_B} = \langle B_l | \, \mathrm{e}^{-RH_x^{(1)}} D_1 \mathrm{e}^{-LH_x^{(2)}} D_2 \mathrm{e}^{-RH_x^{(3)}} \, | B_r \rangle$$

$$E(L) = -\lim_{T\to\infty} \frac{1}{T^D} \log Z_\infty(L,T)$$

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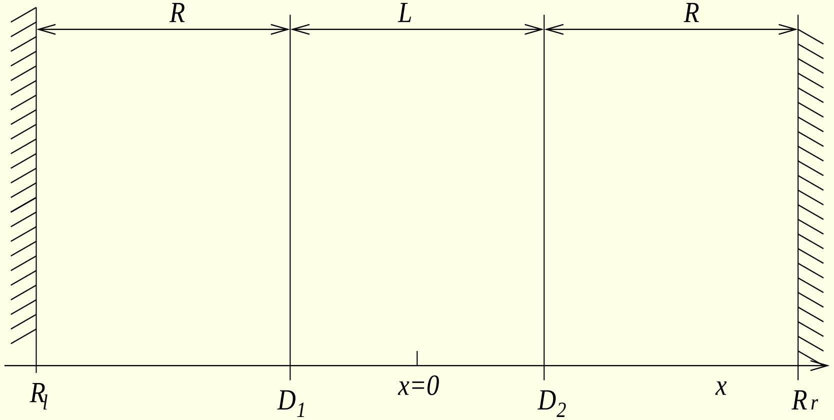

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$$Z_\infty(L,T) = \sum_n \langle B_l | \, 0 \rangle \, \langle 0 | \, D_1 \, | n \rangle \, \langle n | \, D_2 \, | 0 \rangle \, \langle 0 \, | B_r \rangle \, \text{e}^{-LE_n} = \sum_n \langle 0 | \, D_1 \, | n \rangle \, \langle n | \, D_2 \, | 0 \rangle \, \text{e}^{-LE_n}$$

the result

$$E(L) \;=\; -\int_{-\infty}^{\infty}\frac{d\theta}{4\pi}\,\cosh\theta\int\frac{d^{D-1}\vec{k}_{||}}{(2\pi)^{D-1}}m_{\text{eff}}(\vec{k}_{||})R_1^-(\tfrac{i\pi}{2}+\theta,m_{\text{eff}}(\vec{k}_{||}))R_2^+(\tfrac{i\pi}{2}-\theta,m_{\text{eff}}(\vec{k}_{||}))\mathrm{e}^{-2m_{\text{eff}}(\vec{k}_{||})\cosh\theta L}+O(e^{-3mL})$$

- multi particle terms suppressed e^{-mLN}
- bulk, boundary interactions in $R^{\pm}_{1,2}$ (BYBE for $2d$ IBQFT)
- “infrared” viewpoint large volume expression
- universality can be summed up for free bulk and elastic reflection TBA like

$$E(L)=\pm\int_{-\infty}^{\infty}\frac{d\theta}{4\pi}\,\cosh\theta\int\frac{d^{D-1}\vec{k}_{||}}{(2\pi)^{D-1}}m_{\text{eff}}(\vec{k}_{||})\log[1\mp R_1^-(\tfrac{i\pi}{2}+\theta,m_{\text{eff}}(\vec{k}_{||}))R_2^+(\tfrac{i\pi}{2}-\theta,m_{\text{eff}}(\vec{k}_{||}))\mathrm{e}^{-2m_{\text{eff}}(\vec{k}_{||})\cosh\theta L}]$$

upper/lower boson/fermion for $L\gg m^{-1}$ gives back large volume expr.

Applications

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massive scalar field $V(\Phi) = \frac{m^2}{2}\Phi^2$ with Robin boundary conditions

$$\partial_x \Phi - c_1 \Phi|_{x=0} = 0; \qquad \partial_x \Phi + c_2 \Phi|_{x=L} = 0; \qquad c_1, c_2 \geq 0,$$

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the reflection amplitudes on the planes

$$R_j(\theta, m_{\mathrm{eff}}(\vec{k}_{||})) = \frac{m_{\mathrm{eff}}(\vec{k}_{||}) \sinh\theta - ic_j}{m_{\mathrm{eff}}(\vec{k}_{||}) \sinh\theta + ic_j}; \quad j = 1, 2.$$

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$$E(L) = \frac{1}{(4\pi)^{D/2}\Gamma(D/2)} \int_0^\infty dq q^{D-1} \log\left(1 - \frac{\sqrt{m^2+q^2} - c_1}{\sqrt{m^2+q^2} + c_1} \frac{\sqrt{m^2+q^2} - c_2}{\sqrt{m^2+q^2} + c_2} e^{-2L\sqrt{m^2+q^2}}\right)$$

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$c_j \to 0$ Neumann $c_j \to \infty$ Dirichlet $\longrightarrow$ Ambjorn-Wolfram

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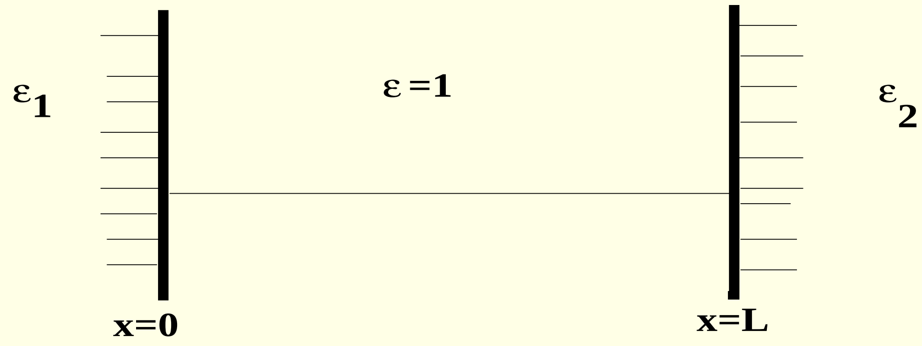
$m \to 0$ limit $\longrightarrow$ Albuquerque-Cavalcanti

parallel dielectric slabs



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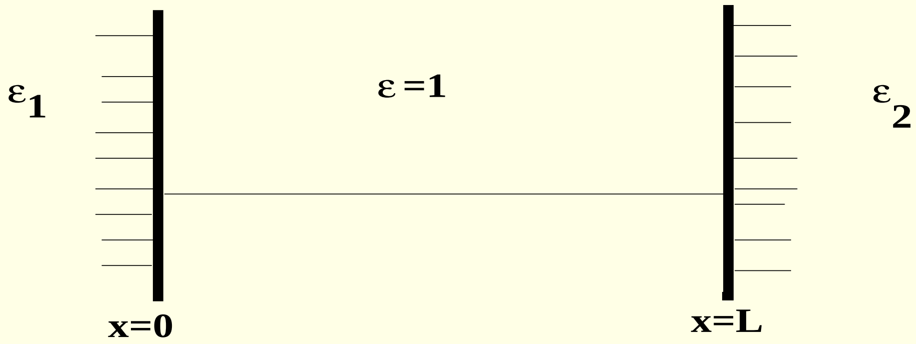
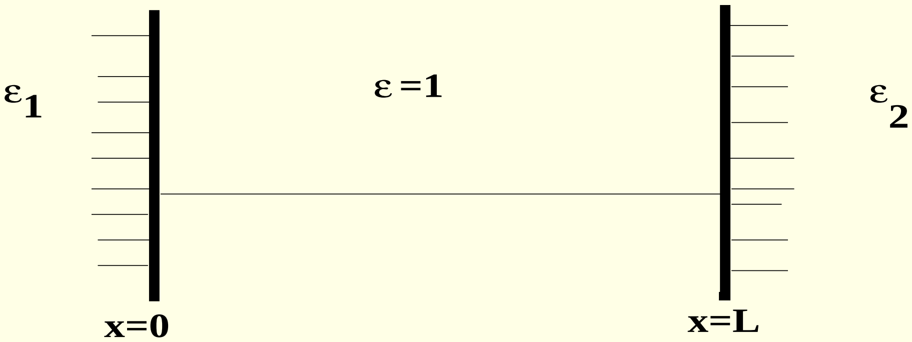


$R^{\pm}_{1,2}(i\frac{\pi}{2}+\theta)$ reflection amplitudes of electromagnetic waves (Jackson)

parallel dielectric slabs



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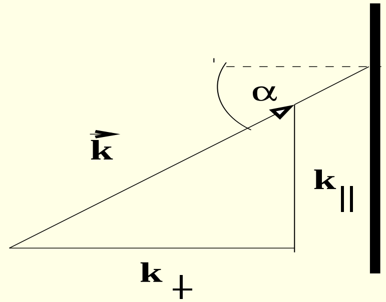
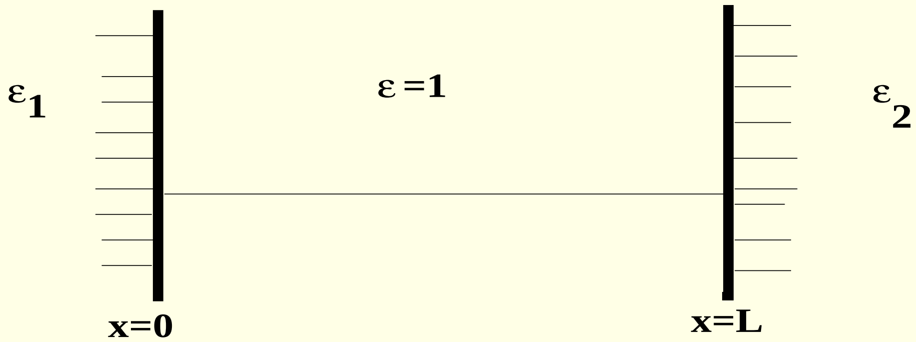


$\vec{E}$ can be parallel and perpendicular

parallel dielectric slabs



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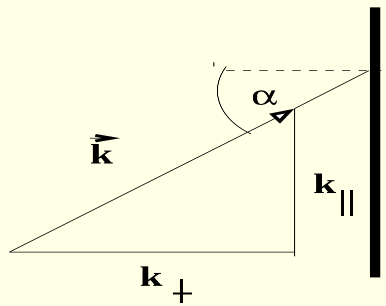
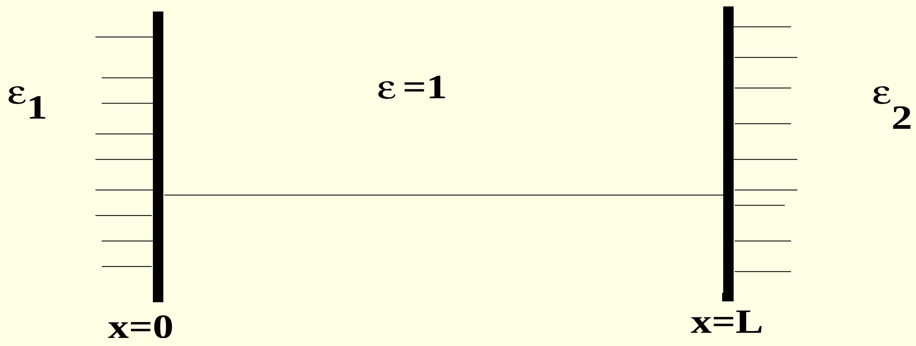


$\vec{E}$ can be parallel and perpendicular

$$R^{(i)}_{\text{par}}(\omega,\vec{k}_{||}) = \frac{\epsilon_i\sqrt{\omega^2-\vec{k}^2_{||}} - \sqrt{\epsilon_i\omega^2-\vec{k}^2_{||}}}{\epsilon_i\sqrt{\omega^2-\vec{k}^2_{||}} + \sqrt{\epsilon_i\omega^2-\vec{k}^2_{||}}}, \quad R^{(i)}_{\text{perp}}(\omega,\vec{k}_{||}) = \frac{\sqrt{\omega^2-\vec{k}^2_{||}} - \sqrt{\epsilon_i\omega^2-\vec{k}^2_{||}}}{\sqrt{\omega^2-\vec{k}^2_{||}} + \sqrt{\epsilon_i\omega^2-\vec{k}^2_{||}}},$$

$$\epsilon_i = \epsilon_i(\omega) \qquad \mu_i = 1$$

parallel dielectric slabs



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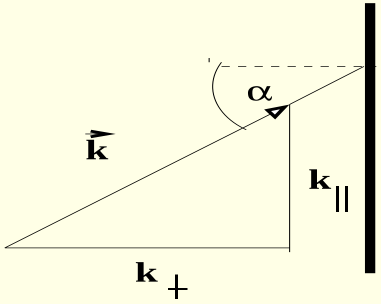

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continuing $\theta \to \theta + i\frac{\pi}{2}$ means $\omega \to i m_{\text{eff}}(\vec{k}_{||})\,\sinh\theta = i\zeta$

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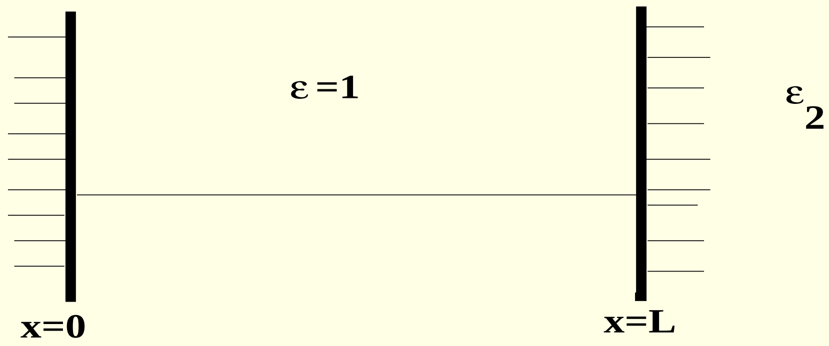

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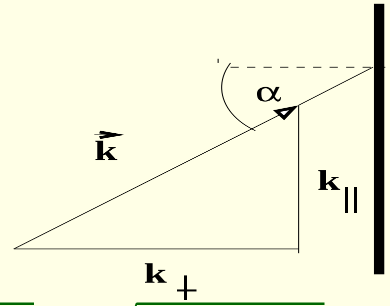

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Casimir force $\mathcal{F}=-\frac{\partial E}{\partial L}$ agrees with Lifshitz et al.

Conclusions

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