Symplectic Hecke correspondence in Hitchin systems

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1) A.Levin, A.Zotov, M.O.,
Hitchin systems– symplectic Hecke correspondence and two-dimensional version,
Comm.Math.Phys **236** (2003) 93-133

2) A.Zotov, M.O.,
Isomonodromic problems on elliptic curve, rigid tops and reflection equations,
Two talks at RIMS, Kyoto, November 2004.

2) A.Levin, A.Zotov, M.O.,
Painleve VI, Rigid Tops and Reflection Equation
math.QA/0508058

Anton Kapustin, Edward Witten,
"Electric-Magnetic Duality And The Geometric Langlands Program,"
hep-th/0604151

Anton Kapustin,
"Wilson-'t Hooft operators in four-dimensional gauge theories and S-duality,"
hep-th/0501015

HOW MONOPOLES PROVIDE AN EQUIVALENCE OF INTEGRABLE SYSTEMS

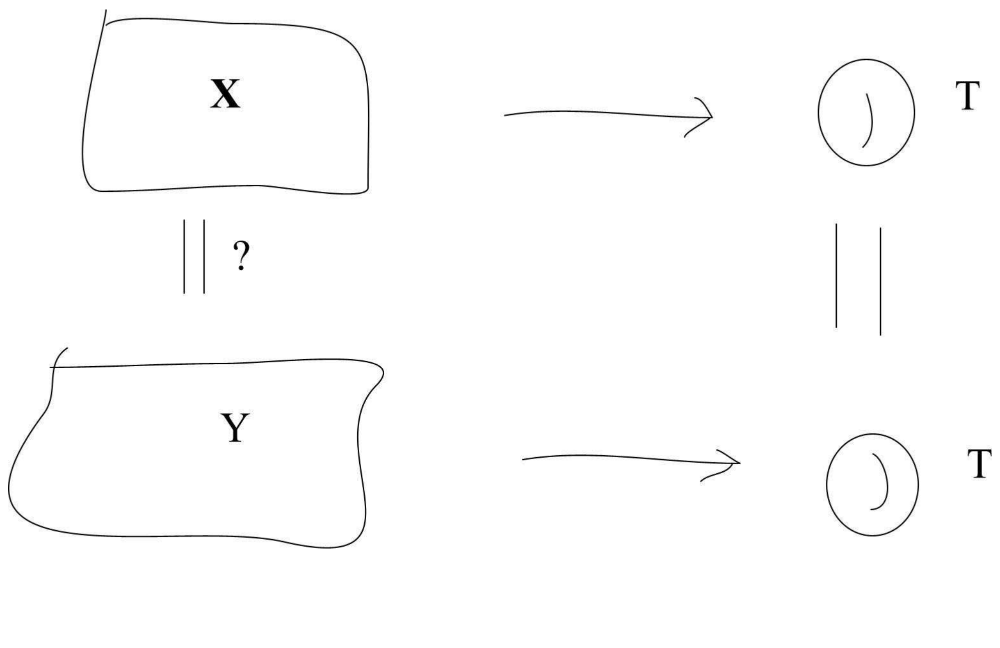
Let $\mathbf{X}$ and $\mathbf{Y}$ be phase spaces of two completely integrable classical systems

$$\dim \mathbf{X} = \dim \mathbf{Y} = 2n$$

Then there exists a symplectic transform to the action-angle variables

$$\mathbf{X} \to T^*\mathbf{T}\,, \quad \mathbf{Y} \to T^*\mathbf{T}\,,$$

$\mathbf{T}$ – the Liouville torus, $\dim \mathbf{T} = n$.



EXAMPLES

1. Elliptic Calogero-Moser system $\Leftrightarrow$ Elliptic GL$(N, \mathbb{C})$ Top ;

2. Calogero-Moser field theory $\Leftrightarrow$ Landau-Lifshitz equation;

3. Painlevé VI $\Leftrightarrow$ Zhukovsky-Volterra gyrostat.

PLAN

1. Two examples of Hitchin systems - Elliptic Calogero-Moser (ECMS) systems and Elliptic Tops (ET).

2. What is the Hitchin systems?

3. Higgs bundles description of ECMS and ET.

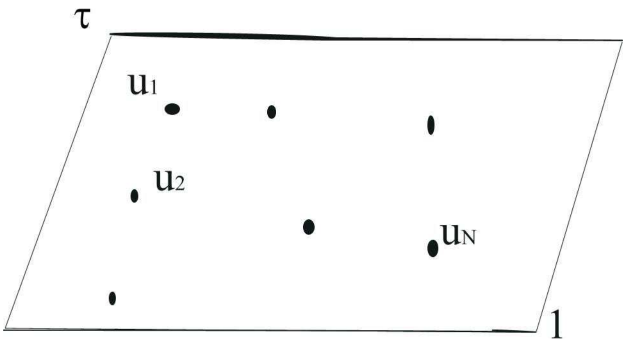
4. Symplectic Hecke correspondence - general approach.

5. Symplectic Hecke correspondence ECMS $\to$ ET.

N-body Elliptic Calogero-Moser System (ECM)

Σ_τ - elliptic curve $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$.

Phase space $\mathcal{R}^{ECM} = (\mathbf{v}, \mathbf{u})$:
$\mathbf{u} = (u_1, \ldots, u_N)$, $u_j \in \Sigma_\tau$ - coordinates of particles,
$\mathbf{v} = (v_1, \ldots, v_N)$, $(v_j \in \mathbb{C})$ - momentum vector,
Poisson brackets $\{v_j, u_k\} = \delta_{jk}$.



Hamiltonian:

$$H^{CM} = \frac{1}{2}|\mathbf{v}|^2 + \nu^2 \sum_{j<k} \wp(u_j - u_k)\,,$$

$\wp(z)$ - Weierschtrass function,
$\wp(z+1) = \wp(z+\tau) = \wp(z)$, $\wp(z) \sim \frac{1}{z^2}$, $z \to 0$,
ν^2 - coupling constant.

Elliptic Top on $\mathrm{GL}(N,\mathbb{C})$ **(ET)**

Basis in $\mathrm{gl}(N,\mathbb{C})$

$$\mathbb{Z}_N^{(2)} = (\mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z})\,, \quad \mathbf{e}_N(x) = \exp\frac{2\pi\imath}{N}x$$

$$T_a = \frac{N}{2\pi i}\mathbf{e}_N(\frac{a_1a_2}{2})Q^{a_1}\Lambda^{a_2}\,,\ a = (a_1,a_2) \in \mathbb{Z}_N^{(2)}$$

$$Q_N = \mathrm{diag}(1, \mathbf{e}_N(1), \ldots, \mathbf{e}_N(N-1))\,,$$

$$\Lambda_N = \sum_{j=1,N,\ (mod\ N)} E_{j,j+1}$$

$$[T_\alpha, T_\beta] = \frac{N}{\pi}\sin\frac{\pi}{N}(\alpha\times\beta)T_{\alpha+\beta}\,,$$

Poisson brackets on Lie coalgebra $\mathfrak{g}^* = \mathrm{gl}(N,\mathbb{C})^*$

$\mathbf{S} = \sum_{\alpha\in\mathbb{Z}_N^{(2)}\setminus(0,0)} S_\alpha T_\alpha \in \mathfrak{g}^*$

$$\{S_\alpha, S_\beta\} = \frac{N}{\pi}\sin\frac{\pi}{N}(\alpha\times\beta)S_{\alpha+\beta}\,.$$

Phase space of ET

$$\mathcal{R}^{ET} = \{\mathbf{S} \in \mathfrak{g}^* \,|\, \mathbf{S} = gS_0g^{-1}\,,\ g \in \mathrm{GL}(N,\mathbb{C})\}$$

$\mathcal{R}^{ET} \sim \mathcal{O}$ - coadjoint orbit.

Euler-Arnold Hamiltonian

$$H^{ET} = -\frac{1}{2}\mathrm{tr}(\mathbf{S}\cdot\mathbf{J}(\mathbf{S}))$$

$\mathbf{J}(\mathbf{S})\ :\ S_\alpha \to \wp_\alpha S_\alpha$ - *inverse inertia tensor*

$$\wp_\alpha = \wp\left(\frac{\alpha_1 + \alpha_2\tau}{N}\right)$$

$\alpha \in \tilde{\mathbb{Z}}_N^{(2)} = \mathbb{Z}_N^{(2)} \setminus (0,0)$

Equations of motion

$$\partial_t\mathbf{S} = \{H^{ET},\mathbf{S}\} = [\mathbf{J}(\mathbf{S}),\mathbf{S}]\,,$$

$$\partial_t S_\alpha = \frac{N}{\pi}\sum_{\gamma\in\tilde{\mathbb{Z}}_N^{(2)}} S_\gamma S_{\alpha-\gamma}\wp_\gamma \sin\frac{\pi}{N}(\alpha\times\gamma)\,.$$

Symplectic Hecke Correspondence
ECM $\Leftrightarrow$ ET

Assume that the coadjoint orbit is degenerate

$$\mathbf{S}_0 = \nu \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \ddots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \dots & 1 & 0 \end{pmatrix}$$

Then $\dim \mathcal{R}^{ET} = 2N = \dim \mathcal{R}^{CM}$

There exists the SH transformation

$$\boxed{\mathbf{S} = \mathbf{S}(\mathbf{v}, \mathbf{u}, \nu)}$$

of the ECM system

$$\partial_t u_m = -\nu^2 \sum_{j \neq m} \partial^2_{u_m} \wp(u_j - u_m)$$

to the ET equation

$$\partial_t S_\alpha = \frac{N}{\pi} \sum_{\gamma \in \tilde{\mathbb{Z}}^{(2)}_N} S_\gamma S_{\alpha-\gamma} \wp_\gamma \sin \frac{\pi}{N}(\alpha \times \gamma)\,.$$

WHAT IS THE HITCHIN SYSTEMS?

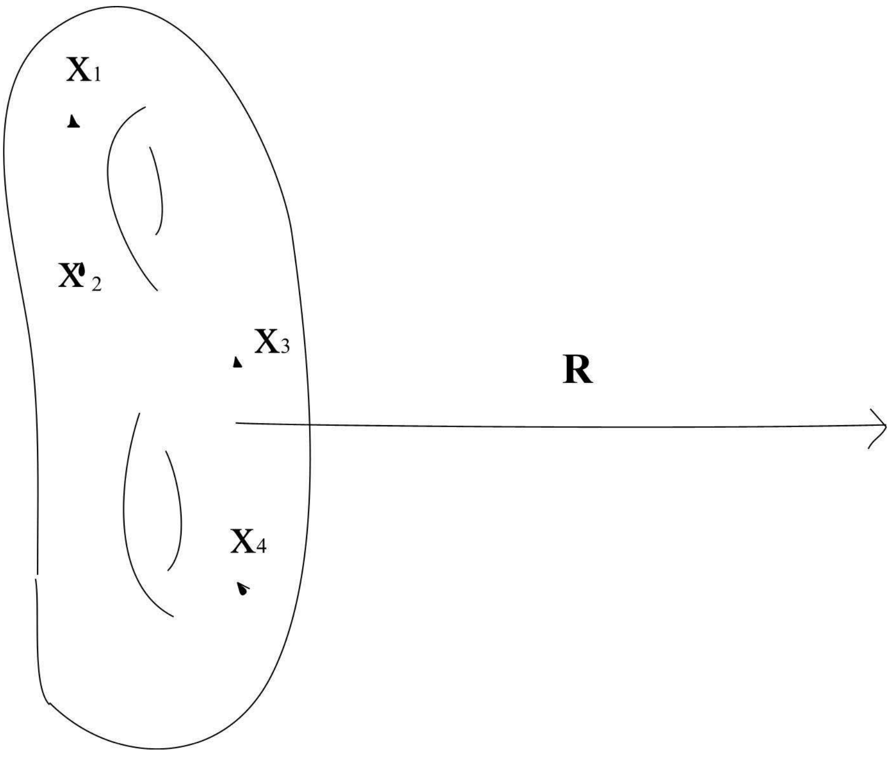
TOPOLOGICAL GL$(N, \mathbb{C})$ FIELD THEORY

SPACE-TIME:

$\mathbb{R} \times C_{g,n}$,

$\Sigma_{g,n}$ – Riemann curve of genus g with n marked points $(x_1, \ldots, x_n)$.

$(z, \bar{z})$ - local complex coordinates.



Fields $\mathcal{R} = (\bar{A}, \Phi, \mathbf{S}^a)$:

1) Vector fields: $(\bar{\partial} + \bar{A}) \otimes d\bar{z}$,

$\bar{A}(z, \bar{z}) \, : \, \Sigma_{g,n} \to \mathrm{gl}(N, \mathbb{C})$,

2) Higgs fields $\Phi(z, \bar{z}) \otimes dz$, $\Phi \, : \, \Sigma_{g,n} \to \mathrm{gl}(N, \mathbb{C})$

3) Spin variables, attributed to the marked points $\mathbf{S}^a$, $a = 1, \ldots, n$,

$\mathbf{S}^a = g^{-1} \mathbf{S}^a(0) g$, $\mathbf{S}^a \in \mathcal{O}^a$- coadjoint $\mathrm{GL}(N, \mathbb{C})$-orbit.

The background charge (the Chern class):

$$\boxed{c_1 = \int_{C_{g,n}} \partial \bar{A}}$$

$\mathcal{R}(c_1)$ – Higgs bundles with the quasi-parabolic structures at the marked points

Poisson brackets:

Let T_α, $\alpha = (\alpha_1, \alpha_2)$ be a basis in $\mathrm{gl}(N, \mathbb{C})$, $\alpha_{1,2} = 0, 1, \ldots, N-1$, $[T\alpha, T_\beta] = C^\gamma_{\alpha,\beta} T_\gamma$.

1) Canonical brackets:

$$\bar{A} = \sum_\alpha \bar{A}_\alpha T_\alpha \,, \quad \Phi = \sum_\beta \Phi_\beta T_\beta \,.$$

$$\{\Phi_\alpha, \bar{A}_\beta\} = \langle T_\alpha T_\beta \rangle \,, \quad (\langle \quad \rangle = trace \ in \ ad)$$

2) Linear (Lie) brackets:

$\mathbf{S}^a = \sum_\alpha S^a_\alpha T_\alpha$

$$\{S^a_\alpha, S^b_\beta\} = \delta^{a,b} C^\gamma_{\alpha,\beta} S^a_\gamma$$

HITCHIN HAMILTONIANS

Let $\Omega^{(1-j,1)}(\Sigma_{g,n})$ $(j=2,\ldots,N)$ be the space of differentials of type $\left(\frac{\partial}{\partial z}\right)^{j-1}\otimes d\bar{z}$, vanishing at the marked points.

ν_{j,k_j} - basis in $\Omega^{(1-j,1)}$, $k_j=1,\ldots,n_j$,
$n_j=(2j-1)(g-1)+jn$

$$H_{j,k}=\frac{1}{j}\int_{C_{g,n}}\nu_{j,k}\langle\Phi^j\rangle\,.$$

$$\boxed{\{H_{(j_1,k_1)},H_{(j_2,k_2)}\}=0}$$

Number of integrals

$$d_{N,g,n}=\sum_{j=1}^{N-1}n_j=N^2(g-1)+1+\frac{1}{2}N(N-1)n\,.$$

Equations of motion:

$$\frac{\partial}{\partial t_{j,k}}\Phi = \{\nabla H_{j,k}, \Phi\} = 0\,,$$

$$\frac{\partial}{\partial t_{j,k}}\bar{A} = \nu_{j,k}\Phi^{j-1}\,,$$

$$\frac{\partial}{\partial t_{j,k}}\mathbf{S}^{a} = 0$$

ACTION

$$\mathcal{S} = \sum_{j=2}^{N} \sum_{k=1}^{n_j} \int_{\mathbb{R}_{j,k}} \int_{C_{g,n}} \Big(\langle \Phi \partial_{j,k} \bar{A} \rangle +$$

$$+ \sum_{a=1}^{n} \langle \mathbf{S}^a g_a^{-1} \partial_{j,k} g_a \rangle - H_{j,k} \Big) \, dt_{j,k} \,,$$

$$\partial_{j,k} = \frac{\partial}{\partial t_{j,k}} \,.$$

$$H_{j,k} = \frac{1}{j} \int_{\Sigma_{g,n}} \nu_{j,k} \langle \Phi^j \rangle \,.$$

GAUGE SYMMETRIES

Gauge group

$$\mathcal{G} = \{\text{smooth maps} : \Sigma_{g,n} \to \mathrm{GL}(N, \mathbb{C})\}$$

The action is invariant with respect to the gauge action

$$\bar{A} \to f^{-1}\bar{\partial} f + f^{-1}\bar{A} f \,,$$

$$\Phi \to f^{-1}\Phi f \,,$$

$$\mathbf{S}^a \to (f^a)^{-1}\mathbf{S}^a f^a \,, \quad f^a = f(z,\bar{z})|_{z=x_a} \,.$$

The Gauss law (the Hitchin equation)

$$\bar{\partial}\Phi + [\bar{A}, \Phi] = \sum_{a=1}^{n} \mathbf{S}^a \delta(z - x_a, \bar{z} - \bar{x}_a) \,.$$

Physical degrees of freedom - reduced phase space

$\mathcal{R}^{red} = \mathcal{R}(\bar{A}, \Phi, \mathbf{S}^a)/(\text{Gauss law}) + (\text{gauge fixing})$

$\mathcal{R}^{red}$ –the moduli space of Higgs bundles

$$\mathcal{R}^{red} \sim \mathcal{R}//\mathcal{G}$$

Dimension of the reduced phase space

$$\dim \mathcal{R}^{red} = 2(g-1+\frac{1}{2}n)N^2 - Nn + 2$$

Integrability:

• Number of integrals

$$\boxed{d_{N,g,n} = \sum_{j=1}^{N-1} n_j = N^2(g-1) + 1 + \tfrac{1}{2}N(N-1)n\,.}$$

•• Involutivity

$$\boxed{\{H_{j,k}, H_{j',k'}\}\,.}$$

Equations of motion on reduced phase space

Let us fix a gauge

$$\bar{A}_0 = (f^{-1}\bar{\partial}f)[\bar{A}] + f^{-1}[\bar{A}]\bar{A}f[\bar{A}]\,.$$

Then

$$L = f^{-1}[\bar{A}]\Phi f[\bar{A}]$$

$$\bar{\partial}L + [\bar{A}_0, L] = \sum_{a=1}^{n} \mathbf{S}^a \delta(x_a, \bar{x}_a)$$

Eqs. on $\mathcal{R}^{red}$:

$$\boxed{\partial_{j,k}L = [L, M_{j,k}]} \text{ – Lax equation}$$

$$\bar{\partial}M_{j,k} + [M_{j,k}, \bar{A}_0] = L^{j-1}\mu_{j,k} - \partial_{j,k}\bar{A}_0$$

<u>Spectral curve</u>

$$\mathcal{C}\ :\ \ f(\lambda, z) = 0\,,\ \ f(\lambda, z) = \det(\lambda - L(z)).$$

The main goal is a description of the map

(Symplectic Hecke Correspondence)

$$\boxed{\Xi\,:\,\mathcal{R}^{red}(c_1)\;\rightarrow\;\mathcal{R}^{red}(c_1+1)\,.}$$

$$\dim\mathcal{R}^{red}=2(g-1+\frac{1}{2}n)N^2-Nn+2\,.$$

$\Xi\,:\,L(c_1)\rightarrow L(c_1+1).$

EXAMPLES

1.Elliptic Calogero-Moser systems (CMS) $(c_1 = 0)$

$C_{1,1} \sim \Sigma_\tau$ - elliptic curve $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with a marked point $z = 0$.

$$\bar{A}(z+1) = \bar{A}(z)\,,\ \Phi(z+1) = \Phi(z)\,,$$

$$\bar{A}(z+\tau) = \bar{A}(z)\,,\quad \Phi(z+\tau) = \Phi(z)\mathbf{e}(-\mathbf{u})\,.$$

The degenerate orbit $\mathcal{O}$ at the marked point $z = 0$, $\mathbf{S} \in \mathcal{O}$.

Reduced phase space $\mathcal{R}^{CM}$:

1) The gauge fixing: $\bar{A} \to \bar{A}_0 = \text{diag}(u_1, \ldots, u_N)$, $(\sum u_j = 0)$,
$\mathbf{u}$ - moduli of the bundle degree zero ($c_1 = 0$).
$\mathcal{O} \to \nu \mathbf{S}_0$

2)The solution of the Gauss law:

$$\boxed{\Phi \to L^{CM} = V + X},$$

$$V = \text{diag}(v_1, \ldots, v_N), \quad \sum v_j = 0,$$

$$X_{jk} = \nu \phi(u_j - u_k, z),$$

$$\phi(u, z) = \frac{\theta(u+z)\theta'(0)}{\theta(u)\theta(z)},$$

$$\theta(z) = q^{\frac{1}{8}} \sum_{n \in \mathbf{Z}} (-1)^n \exp 2\pi\imath(\tfrac{1}{2}n(n+1)\tau + nz),$$

$$\mathcal{R}^{CM} = \{\mathbf{v}, \mathbf{u}\}, \quad \dim \mathcal{R}^{CM} = 2(N-1)$$

2. Elliptic Top (ET)$(c_1 = 1)$

$C_{1,1} \sim \Sigma_\tau$ - elliptic curve $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with a marked point $z = 0$.

$$\bar{A}(z+1) = Q_N \bar{A}(z) Q_N^{-1} \,,$$

$$\Phi(z+1) = Q_N \Phi(z) Q_N^{-1} \,,$$

$$\bar{A}(z+\tau) = \Lambda_N \bar{A}(z) \Lambda_N^{-1} \,,$$

$$\Phi(z+\tau) = \Lambda_N \Phi(z) \Lambda_N^{-1} \,,$$

The degenerate orbit $\mathcal{O}$ at the marked point $z = 0$.

Reduced phase space $\mathcal{R}^{ET}$:

1)The gauge fixing: $\bar{A} \to \bar{A}_0 = 0$.
The coadjoint orbit:

$$\mathcal{R}^{red} = \{\mathcal{O} = \mathbf{S} = g\mathbf{S}_0 g^{-1}\}\,,$$

$\mathbf{S} = \sum_{\alpha \in \mathbb{Z}_N^{(2)} \setminus (0,0)} S_\alpha T_\alpha \in \mathfrak{g}^*$
2)The Gauss law solution:

$$\boxed{\Phi \to L^{ET} = \sum_{\alpha \in \mathbb{Z}_N^{(2)} \setminus (0,0)} S_\alpha \varphi_\alpha(z) T_\alpha}\,,$$

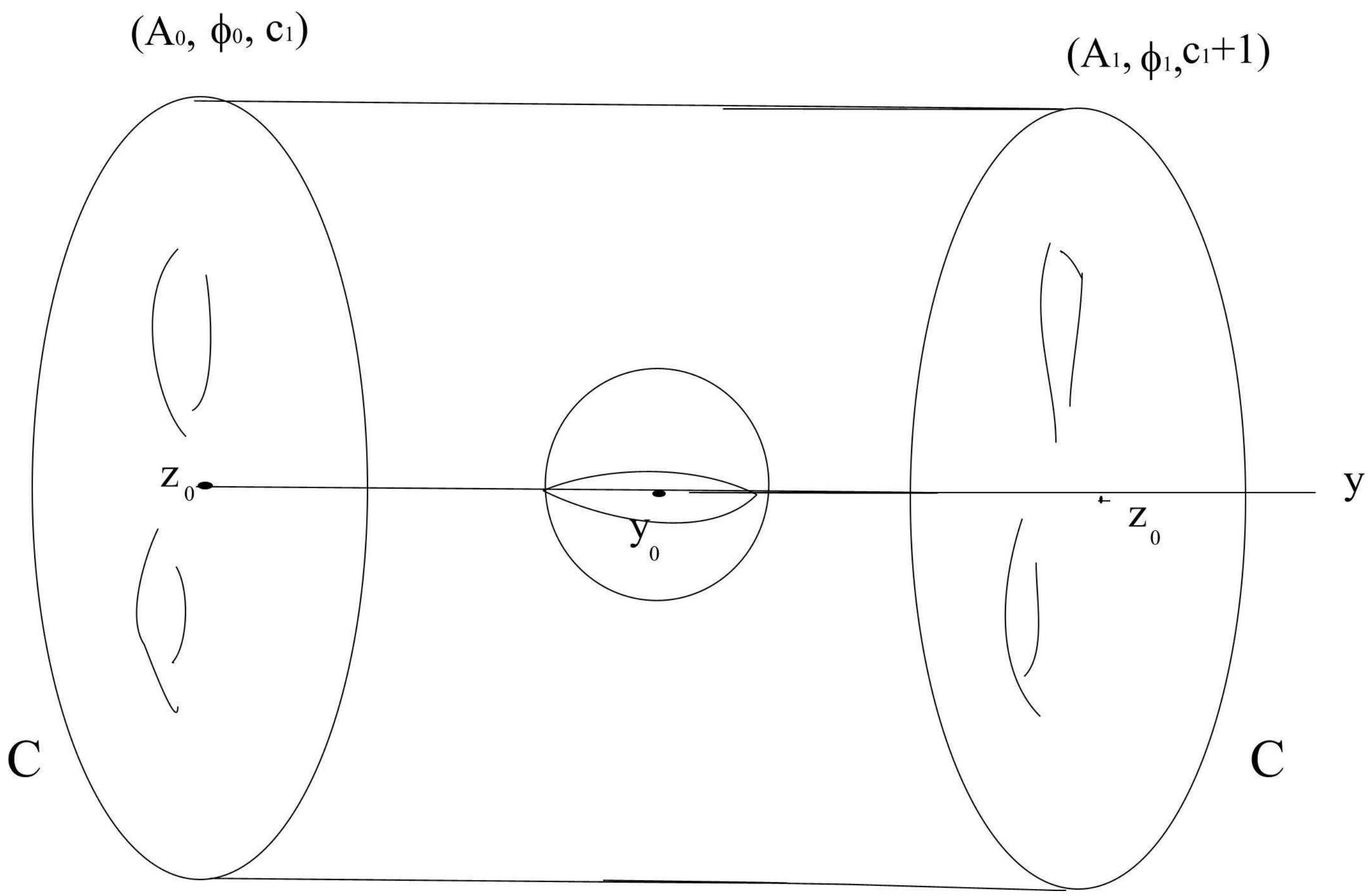
$\varphi_\alpha(z) = \mathbf{e}_N(\alpha_2 z)\phi(\frac{\alpha_1 + \alpha_2 \tau}{N}, z)$

$$\dim \mathcal{R}^{ET} = \dim \mathcal{R}^{CMS} = 2N - 2\,.$$

HECKE CORRESPONDENCE

$M = C \times \mathbb{R}\,,\quad M = \{p = (z, \bar{z}, y)\}$
Monopole at $p_0 = (z_0, y_0)$

$$\partial \bar{A} \sim \partial_y \frac{B}{|p - p_0|}\,,\;\; B = \mathrm{diag}(1, 0, \ldots, 0)\,.$$



The monopole increase the Chern class:
$c_1 \to c_1 + 1$

Symplectomorphisms of CMS and ET

There exists the upper modification Ξ^+ such that

$$L^{ET} = \Xi^+ L^{ECM} (\Xi^+)^{-1}$$

• *Quasi-periodicity*:

$$\Xi^+(z+1,\tau) = -Q \times \Xi^+(z,\tau)\,,$$

$$\Xi^+(z+\tau,\tau) = \tilde{\Lambda}(z,\tau) \times \Xi^+(z,\tau) \times \mathrm{diag}(\mathbf{e}_N(u_j))$$

•• *An eigen-vector* $\mathbf{r} = (r_1, \dots, r_N)$

$$\mathbf{p}\mathbf{r} = p_i^0 \mathbf{r}\,, \quad \mathbf{p} \in \tilde{\mathcal{O}}$$

such that $\Xi^+ \mathbf{r} = 0$.

$$\Xi^+(z,\mathbf{u},\mathbf{r}_i) = \tilde{\tilde{\Xi}}(z)\mathrm{diag}\left(\frac{(-1)^l}{r_l}\prod_{j<k;j,k\neq l}\theta(u_k-u_j)\right)$$

$$\tilde{\Xi}_{ij}(z, \mathbf{u}; \tau) = \theta\left[\begin{array}{c} \frac{i}{N} - \frac{1}{2} \\ \frac{N}{2} \end{array}\right](z - Nu_j, N\tau)$$

$$\theta\left[\begin{array}{c} a \\ b \end{array}\right](z, \tau) =$$

$$\sum_{j \in \mathbb{Z}} \exp 2\pi\imath \left((j + a)^2 \frac{\tau}{2} + (j + a)(z + b) \right) .$$

EXAMPLE ($N = 2$)

Two-body ECM:

$$\boxed{\partial_t u = v\,,\ \ \partial_t v = -\nu^2 \partial_u \wp(2u)}$$

Euler top on SL$(2, \mathbb{C})$

$$\vec{S} = (S_1, S_2, S_3) \in \mathbb{C}^3\,,\ \ S_1^2 + S_2^2 + S_3^2 \sim \nu^2$$

$$\boxed{\partial_t \vec{S} = \vec{S} \times (\vec{J} \cdot \vec{S})}$$

$\vec{J} \cdot \vec{S} = (J_1 S_1, J_2 S_2, J_3 S_3)$,
$J_1 = \wp(\frac{\tau}{2}),\ J_2 = \wp(\frac{1+\tau}{2}),\ J_3 = \wp(\frac{1}{2})$

Symplectic Hecke Correspondence

$$(v, u) \to \vec{S}$$

$$\{v, u\} = 1\,,\ \ \{S_\alpha, S_\beta\} = \frac{2}{\pi} \epsilon_{\alpha\beta\gamma} S_\gamma\,.$$

Hecke correspondence

$$S_1 = -v\frac{\theta_{10}(0)}{\theta'_{11}(0)}\frac{\theta_{10}(2u)}{\theta_{11}(2u)} -$$

$$\nu\frac{\theta^2_{10}(0)}{\theta_{00}(0)\theta_{01}(0)}\frac{\theta_{00}(2u)\theta_{01}(2u)}{\theta^2_{11}(2u)},$$

$$S_2 = -v\frac{\theta_{00}(0)}{\imath\theta'_{11}(0)}\frac{\theta_{00}(2u)}{\theta_{11}(2u)} -$$

$$\nu\frac{\theta^2_{00}(0)}{\imath\theta_{10}(0)\theta_{01}(0)}\frac{\theta_{10}(2u)\theta_{01}(2u)}{\theta^2_{11}(2u)},$$

$$S_3 = -v\frac{\theta_{01}(0)}{\theta'_{11}(0)}\frac{\theta_{01}(2u)}{\theta_{11}(2u)} -$$

$$\nu\frac{\theta^2_{01}(0)}{\theta_{00}(0)\theta_{10}(0)}\frac{\theta_{00}(2u)\theta_{10}(2u)}{\theta^2_{11}(2u)},$$

$$S_1^2 + S_2^2 + S_3^2 \sim \nu^2$$

Open problem:

Classical S-duality:
The spin variables at the marked points $\Longleftrightarrow$ the Hecke transformations

$$\mathbf{S}^a \Longleftrightarrow \Xi^a \,.$$