SUPERSYMMETRY AT FINITE TEMPERATURE

and

THE ENERGY-MOMENTUM TENSOR

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Spontaneous Breaking of Scale Invariance and the Energy Momentum Tensor

Under general conditions, scale invariance imply also invariance under the conformal group $SO(d+1,1)$

The scale S_μ and conformal current $K_{\mu\nu}$ are:

$$
\begin{aligned}
S^\mu &= x_\lambda \tilde{T}^{\mu\lambda} \\
K^{\mu\nu} &= x^2 \tilde{T}^{\mu\nu} - 2x^\mu x^\lambda \tilde{T}^\nu_\lambda .
\end{aligned}
$$

$\tilde{T}_{\mu\nu}$ is the "improved" energy momentum tensor
.

The divergences $\partial^\mu S_\mu$ of the scale and conformal $\partial^\mu K_{\mu\nu}$ currents are proportional to the trace of the energy momentum tensor $\tilde{T}_\mu{}^\mu$
Two mechanisms for breaking scale invariance:

(a) Explicit breaking of scale invariance, anomaly in the trace of the energy momentum tensor.

(b) Spontaneous breaking of scale invariance (e.g. Nambu–Jona-Lasinio in the BCS theory)

In conventional quantum field theories the two mechanisms occur simultaneously, no massless Nambu Goldston boson (Dilaton).

For example, the scale current is conserved at tree level for a scale invariant scalar potential

$$V(\phi) = g\phi^n$$

will check

$$\partial^\mu S_\mu = \tilde{T}^\nu{}_\nu = ? = 0$$

where, in general

$$\tilde{T}^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \mathcal{S}(\phi)$$

$\mathcal{S}(\phi)$ is the action on a Riemannian manifold

$$\begin{aligned}
&\mathcal{S}(\phi) = \\
&\quad \int \sqrt{-g} \mathrm{d}^d x \tfrac{1}{2} \{ g^{\alpha\beta}(x) \nabla_\alpha \phi(x) \nabla_\beta \phi(x) \\
&\quad - V(\phi) - \xi R(x) \phi^2(x) \}
\end{aligned}$$

Varying with respect to $h_{\mu\nu}$ near flat space $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ one than obtains

$$\begin{aligned}
&\tilde{T}^{\mu\nu} \\
&= \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \left(\tfrac{1}{2} (\partial\phi)^2 - V(\phi) \right) \\
&\qquad + \xi (\eta^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu) \phi^2
\end{aligned}$$

Using the equation of motion the trace is given by

$$\begin{aligned}\partial_\mu S^\mu &= \tilde{T}^\nu{}_\nu \\ &= (\partial\phi)^2\,[1 - d/2 + 2\xi(d-1)] \\ &+ \phi\partial^2\phi\,[-d/n + 2\xi(d-1)]\,.\end{aligned}$$

So, in d dimensions

$$\xi = \frac{1}{4}\left(\frac{d-2}{d-1}\right) \quad \text{and} \quad n = \frac{2d}{d-2}\,.$$

(ξ is independent of the potential, e.g. $\xi = \frac{1}{6}$ in $d = 4$).

Supersymmetric models in the large N limit

$O(N)$ invariant supersymmetric action (d=3):

$$\mathcal{S} = \int \mathrm{d}^3x \, \mathrm{d}^2\theta \left[\tfrac{1}{2}\bar{\mathrm{D}}\Phi \cdot \mathrm{D}\Phi + NU(\Phi^2/N)\right]$$

$O(N)$ vector: $\Phi(\theta, x) = \varphi + \bar{\theta}\psi + \frac{1}{2}\bar{\theta}\theta F$

Components - generic super-potential:

$$\mathcal{S} = \int \mathrm{d}^3x \, \tfrac{1}{2}[-\bar{\psi}\not{\partial}\psi + (\partial_\mu\varphi)^2 - U'(\varphi^2/N)\bar{\psi}\psi$$
$$- 2U''(\varphi^2/N)(\bar{\psi}\varphi)(\varphi\psi)/N + \varphi^2 U'^2(\varphi^2/N)]$$

Will show that under certain conditions:

(1) A supersymmetric ground state with $m_\psi = m_\phi \neq 0$ exists even in a renormalized scale invariant theory.

(2) At a critical value of the coupling constant, though $m_\psi = m_\phi \neq 0$ there is no explicit breaking of scale invariance $< \partial^\mu S_\mu > \sim < \tilde{T}^\nu_{\ \nu} > = \ 0$.

(3) At a certain strength of the attractive force between $O(N)$ bosons and fermions, massless Goldston-Bosons ($O(N)$ singlet Dilaton and Dilatino) are created as bound states.

(4) The relation of **(3)** to the **double scaling limit** in O(N) matrix and vector theories and stringy nature of quantum field theory.

(5) Finite temperature effects on **(1)-(4)** and unusual phase transitions in the supersymmetric model in d=3.

Large N methods
for supersymmetric actions

Introduce two new superfields:

$$L(\theta,x)=M+\bar\theta\ell+\tfrac12\bar\theta\theta\lambda$$
$$R(\theta,x)=\rho+\bar\theta\sigma+\tfrac12\bar\theta\theta s.$$

Add an extra term to the action:

$$\mathcal{S}(\Phi,L,R)=\int \mathrm{d}^3x\,\mathrm{d}^2\theta\{\tfrac12\bar{\mathrm{D}}\Phi\cdot\mathrm{D}\Phi+NU(R)+L(\theta)[\Phi^2(\theta)-NR(\theta)]\}$$

Integrate out $N-1$ components of Φ, ($\Phi_1\equiv\phi$).

$$\mathcal{Z}=\int[\mathrm{d}\phi][\mathrm{d}R][\mathrm{d}L]\,\mathrm{e}^{-\mathcal{S}_N(\phi,R,L)}$$

$$\mathcal{S}_N=\int \mathrm{d}^3x\,\mathrm{d}^2\theta[\tfrac12\bar{\mathrm{D}}\phi\mathrm{D}\phi+NU(R)+L(\phi^2-NR)]+\tfrac12(N-1)\mathrm{Str}\,\ln[-\bar{\mathrm{D}}\mathrm{D}+2L].$$

Note: action $\sim N$ and thus three saddle point equations

Action density: $\mathcal{E} = \mathcal{S}_N/\text{volume}$:

$$\begin{aligned}\mathcal{E}/N =& \tfrac{1}{2} M^2 \varphi^2/N \\ &+ \frac{1}{24\pi}(m-|M|)^2(m+2|M|)\end{aligned}$$

m is the boson mass, M is the fermion mass.

$\mathcal{E}$ is positive for all saddle points and has an absolute minimum at $m_\varphi \equiv m = |M| = m_\psi$ (a supersymmetric ground state).

Φ^4 super-potential in $d = 3$: phase structure

$$U(\Phi^2/N) = (\mu/N)\Phi^2 + \tfrac{1}{2}(u/N^2)\Phi^4$$

Gap equations (saddle point equations) reduce to

$$M = \mu - \mu_c + u\frac{\varphi^2}{N} - \frac{u}{4\pi}|M|\,, \quad M\varphi = 0$$

Note the special case:
$\mu - \mu_c \equiv \mu_R = 0$ in the $O(N)$ symmetric phase ($\varphi = 0$).

The gap equation is:

$$M = -\frac{u}{4\pi}|M|$$

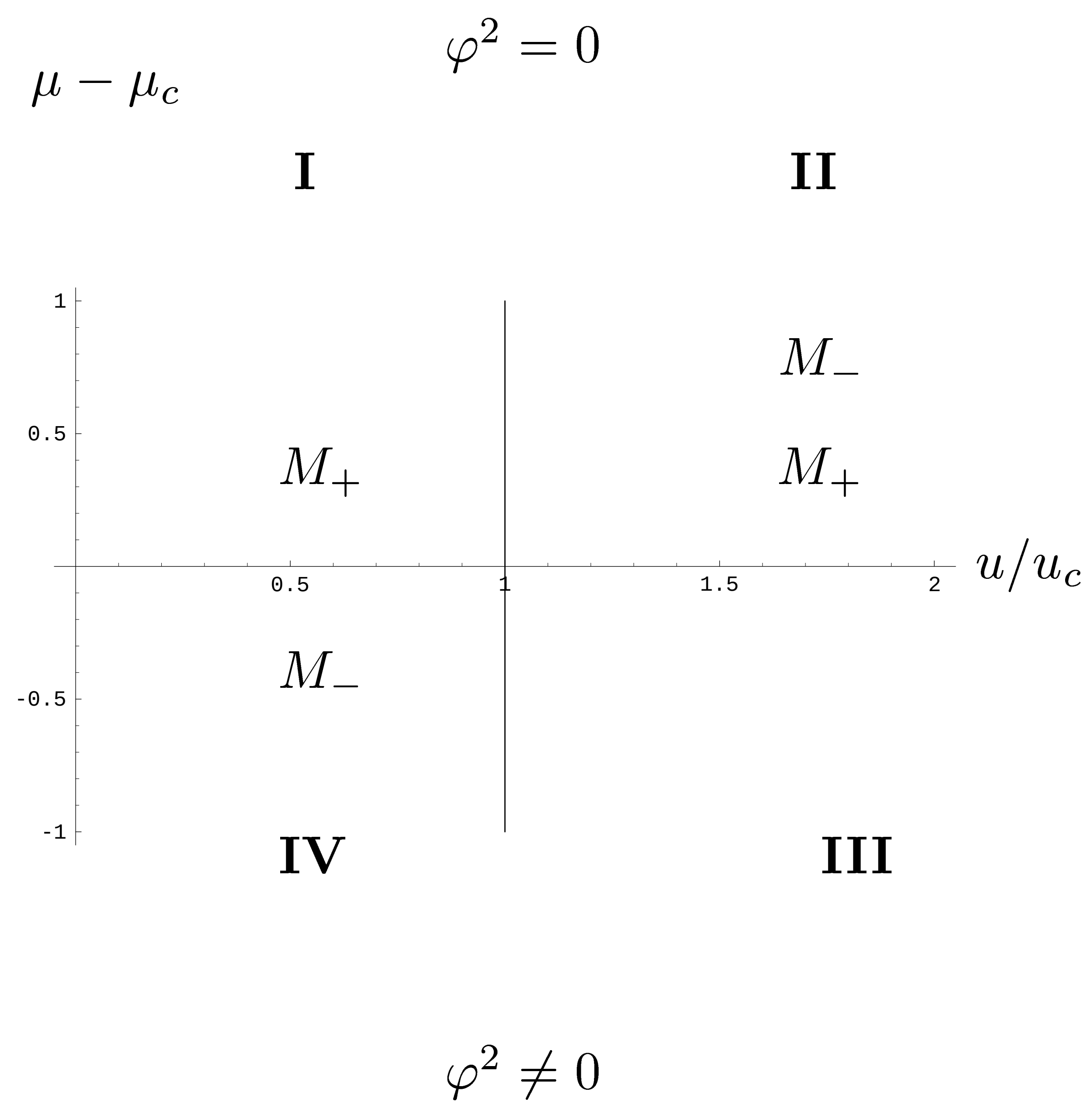


Fig. 1 Summary of the phases of the model in the $\{\mu - \mu_c, u\}$ plane. Here $m_\varphi = m_\psi = |M_\pm| = (\mu - \mu_c)/(u/u_c \pm 1)$. The lines $u = u_c$ and $\mu - \mu_c = 0$ are lines of first and second order phase transitions.

The action density $\mathcal{E}(m,\varphi)$:

$$\frac{1}{N}\mathcal{E}(m,\varphi) = \tfrac{1}{2}M^2(m,\varphi)\frac{\varphi^2}{N} + \frac{1}{24\pi}\left[m-|M(m,\varphi)|\right]^2 \times \left(m_\varphi + 2\,|M(m,\varphi)|\right)$$

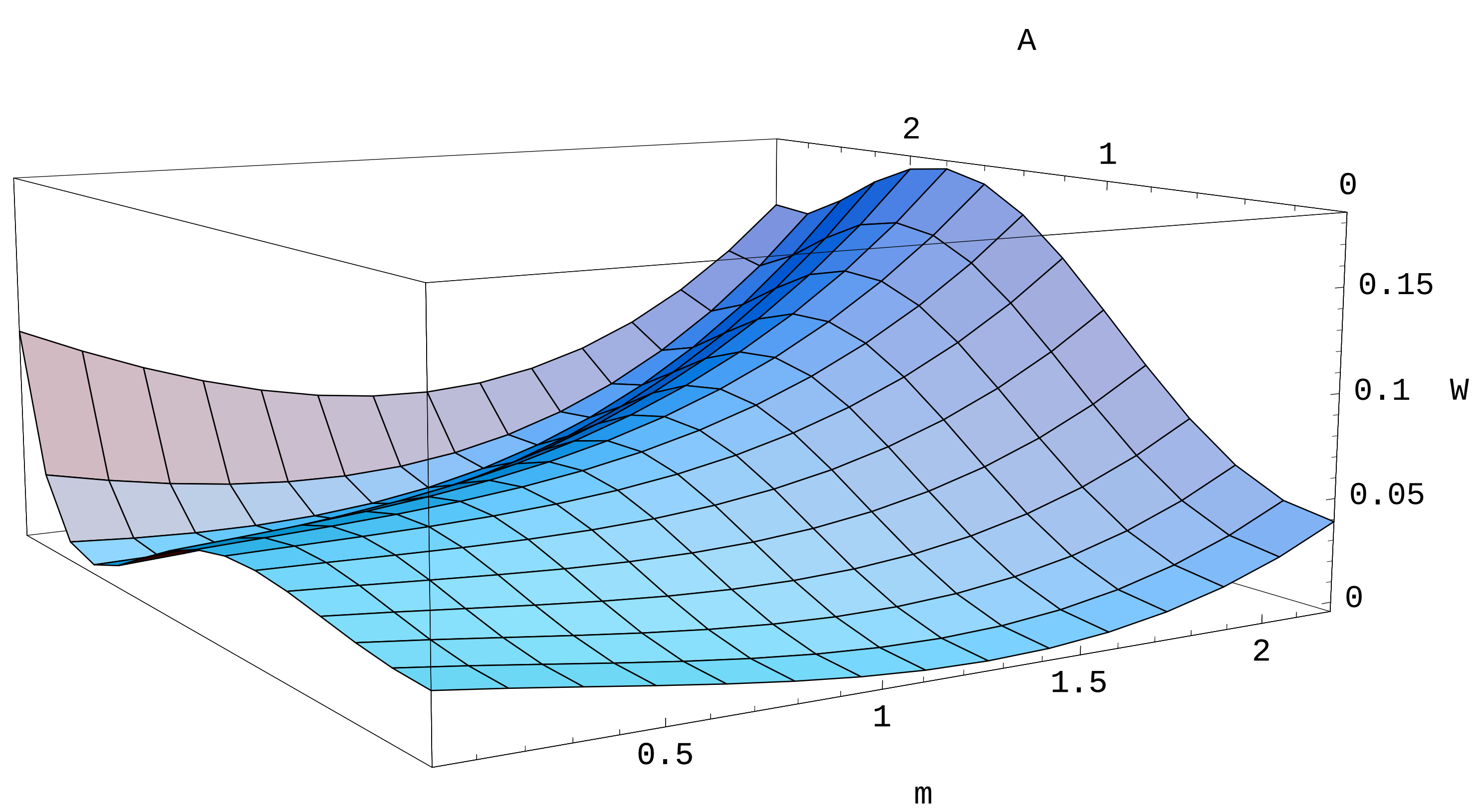


Fig. 2 **Region IV :** The action density $\frac{1}{N}\mathcal{E}(m, \varphi)$ as a function of the boson mass (m) and A, where $A^2 = \varphi^2/u_c$. Here $\mu - \mu_c = -1$, $u/u_c = 0.2$. As seen here there are two distinct degenerate phases. One is an ordered phase $(\varphi \neq 0)$ with a massless boson and fermion, the other is a symmetric phase $(\varphi = 0)$ with a massive $(m = |M_-|)$ boson and fermion.

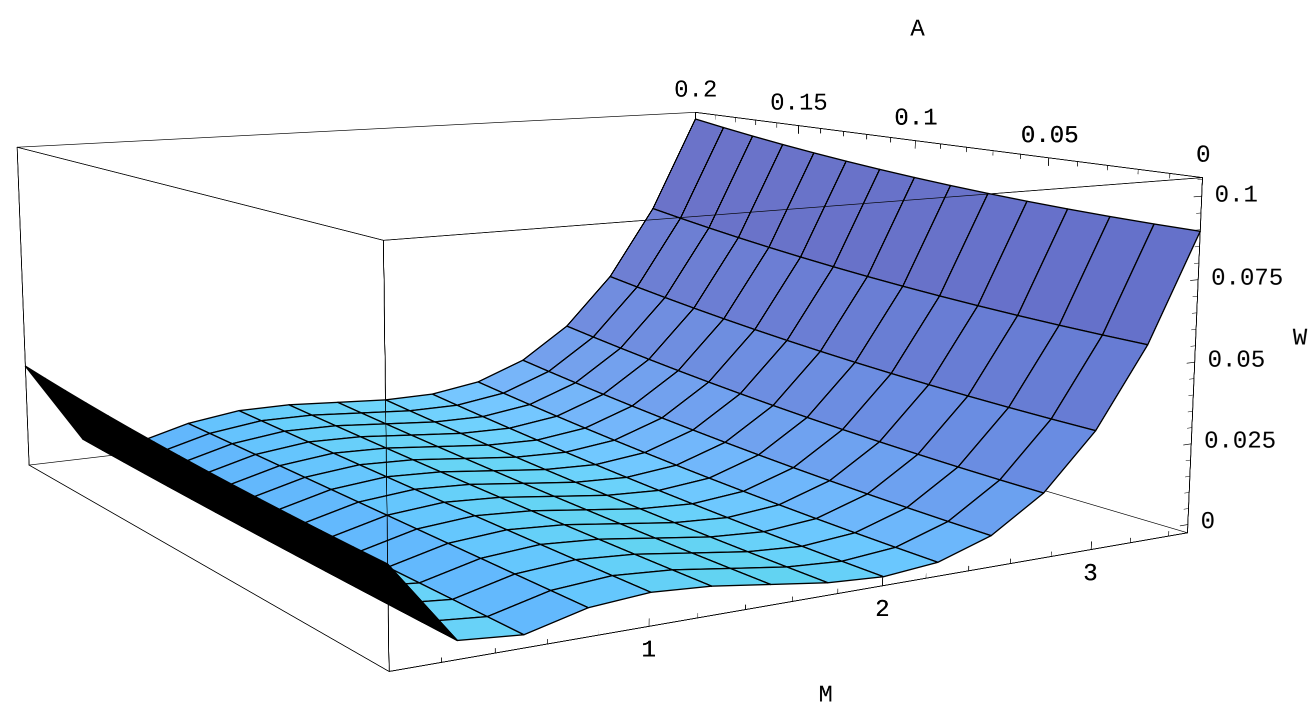


Fig. 3 **Region II:** The action density $\mathcal{E}(m, \varphi)$ as a function of the boson mass (m) and A, where $A^2 = \varphi^2/u_c$. Two degenerate, $O(N)$ symmetric phases exist with massive bosons (and massive fermions). Here $\mu - \mu_c = 1$ (sets the mass scale) $u/u_c = 1.5$.

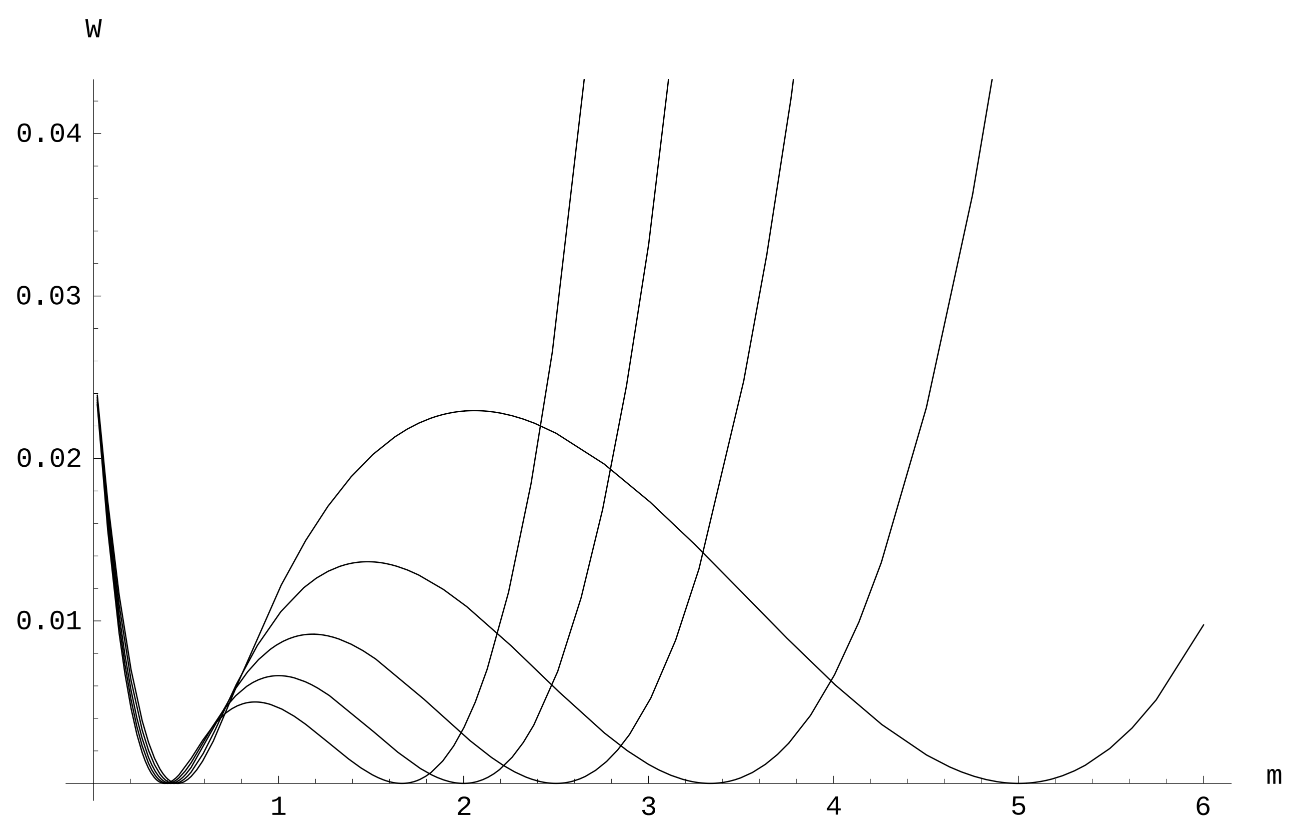


Fig. 4 The action density $\frac{1}{N}\mathcal{E}(m, \varphi = 0)$ in region **II** of Fig. 1. Here $\mu - \mu_c = 1$ (sets the mass scale) and u/u_c= varies between 1.6 and 1.2. There are two massive, degenerate $O(N)$ symmetric SUSY vacua with $\varphi = 0$ and possible large hierarchy of masses $|M_-/M_+| >> 1$ (as $u \to u_c$) with $m_\psi = m_\varphi = M_+ = (\mu - \mu_c)/(u/u_c + 1)$ and $m_\psi = m_\varphi = -M_- = (\mu - \mu_c)/(u/u_c - 1)$.

Remarks on the Double Scaling Limit

Geometry by dense Feynman graphs of the same topology ($1/N$ expansion).

Double scaling limit is enforced in

$$Z_N(g) = \int D\hat{\Phi}\; e^{\{-\beta \int d^d x Tr[\hat{\Phi}(x)\hat{K}\hat{\Phi}(x) + V(\hat{\Phi}(x))]\}}$$

$\hat{\Phi}(x)$ - NxN Hermitian matrix

V($\hat{\Phi}$) depends on coupling(s) constant(s) $\{g_i\}$.

Dynamically triangulated random surfaces **summed** on different topologies viewed as the manifold for string propagation ($d = 0$ Matrix models)

Nonperturbative treatment of string theory, when the double scaling limit is enforced ($N \to \infty$ and $g \to g_c$).

A major progress can be achieved if extended to $d > 1$ dimensions. One will then obtain a relation between d dimensional QFT and its possible representation as a stringy object.

$O(N)$ symmetric vector models represent discretized filamentary surfaces - randomly branched polymers in their double scaling limit.

Weak coupling limit, $\frac{1}{N} \sim \frac{1}{\beta} \to 0$ - a one-dimensional frozen Dyson gas, the planar graphs dominate.

Pauli repulsion between the eigenvalues - at strong coupling, at critical point $\{g_i\} = \{g_{iC}\}$ as $N \to \infty$ and $\frac{N}{\beta} \to 1$ (or $\{g_i\} \to \{g_{iC}\}$) non-planar graphs become important.

Genus (G) expansion of the free energy:

$$
\begin{aligned}
F = \ln Z_N \;&=\; a \;+\; b \ln \beta + \sum_{G,S} N^{2(1-G)} \;\; (\frac{N}{\beta})^S \;\; F_S \\
&\sim \sum_G (\frac{1}{N})^{2G-2} \mathcal{A}_G\{g_i\}
\end{aligned}
$$

Factorial growth of the positive $\mathcal{A}_G\{g_i\}$ with the genus G, thus the topological series is not Borel summable and a nonperturbative approach is needed.

At a given topology, $\mathcal{A}_G\{g\}$ has a finite radius of convergence, when expanded in powers of the coupling constant g.

The critical exponents of this model are calculable since the model belongs to the same universality class of two-dimensional, conformally invariant matter field in gravitational background.

Special situation: $\quad u = u_c = 4\pi$

$\langle LL \rangle$ propagator, and massless fermion and boson $O(N)$ singlet bound states

The $\langle LL \rangle$ action

$$-\frac{N}{2u}\int \mathrm{d}^3x\,\mathrm{d}^2\theta(L-\mu)^2 \\ +\tfrac{1}{2}(N-1)\mathrm{Str}\,\ln\left(-\bar{\mathrm{D}}\mathrm{D}+2L\right)$$

$$\Delta_L^{-1} = -\frac{N}{4\pi|M|}\{1 \\ +[M+|M|(u_c/u)]\delta^2(\theta'-\theta)\}\,\mathrm{e}^{i\bar{\theta}\not{p}\theta'}$$

Corresponds to a bound state super-particle of mass $2M(1-u_c/u)$. At the special point $u=u_c$ the mass vanishes.

Namely, massless boson and fermion, O(N) singlets associated with the spontaneous breakdown of scale invariance. Dilaton and Dilatino masses

$$m_{D_\psi} = m_{D_\phi} = 2M(1-u/u_c) \to 0$$

as $u \to u_c$

E.g. The $\psi \cdot \varphi$ scattering amplitude

$T_{\psi\cdot\varphi,\psi\cdot\varphi}(p^2)$, in the limit $p^2 \to 0$ satisfies:

$$
\begin{aligned}
T_{\psi\varphi,\psi\varphi}(p^2) &\sim \frac{-i2u}{N}\left[1+\frac{u}{4\pi}\frac{m_\psi}{|m_\psi|}-\frac{u}{8\pi}\frac{\not{p}}{|m_\psi|}\right]^{-1} \\
&\to i\frac{16\pi}{N}\frac{|m_\psi|}{\not{p}}
\end{aligned}
$$

Namely, a masless $O(N)$ singlet, fermion-boson bound state Dilatino for $m_\psi < 0$ and $u \to u_c$
If we slightly deviate from the critical coupling u_c, dilatino acquires a mass given by

$$m_{D_\psi} = 2\left(1-\tfrac{u_c}{u}\right)|m_\psi|$$

Similarly, in the boson-boson scattering amplitude $T_{\varphi\cdot\varphi,\varphi\cdot\varphi}$ or fermion-fermion $T_{\psi\cdot\psi,\psi\cdot\psi}$ or fermion-fermion to boson-boson scattering amplitude $T_{\psi\cdot\psi,\varphi\cdot\varphi}$ one finds the Dilaton pole at

$$m^2_{D_\varphi} = 4\left(1-\tfrac{u_c}{u}\right)^2 m^2_\varphi$$

A supersymmetric non-linear σ-model at large N

Consider the supersymmetric n.l. σ-model in d dimensions, $2 \le d \le 3$.

$$\mathcal{S}(\Phi) = \frac{1}{2\kappa} \int \mathrm{d}^d x \, \mathrm{d}^2\theta \, \bar{\mathrm{D}}\Phi \cdot \mathrm{D}\Phi$$

Dimension $d = 3$.

$$\begin{aligned}\mathcal{E}/N = {} & \frac{1}{2N\kappa} M^2 \varphi^2 \\ & + \frac{1}{24\pi}(m - |M|)^2 (m + 2|M|)\end{aligned}$$

Dimension $d = 2$.

$$\frac{\mathcal{E}}{N} = \frac{1}{8\pi} \left[m^2 - M^2 + 2M^2 \ln(M/m) \right].$$

Contin. *Dimension* $d = 2$.

$$M = m = \Lambda\, \mathrm{e}^{-2\pi/\kappa} \,.$$

$O(N)$ supersymmetric model at finite temperature

$$\mathcal{Z} = \int [\mathrm{d}\phi][\mathrm{d}R][\mathrm{d}L]\, \mathrm{e}^{-\mathcal{S}_N(\phi,R,L)}$$

$$\begin{aligned}\mathcal{S}_N = \int \mathrm{d}^3x\, \mathrm{d}^2\theta [\tfrac{1}{2}\bar{\mathrm{D}}\phi \mathrm{D}\phi + NU(R) \\ + L(\phi^2 - NR] \\ + \tfrac{1}{2}(N-1)\mathrm{Str}\, \ln\left[-\bar{\mathrm{D}}\mathrm{D} + 2L\right]\end{aligned}$$

$$\begin{aligned}\Delta(k,\theta,\theta) = \frac{1}{k^2 + M_T^2 + \lambda} \\ + \bar{\theta}\theta M_T \left(\frac{1}{k^2 + M_T^2 + \lambda} - \frac{1}{k^2 + M_T^2}\right)\end{aligned}$$

M_T the expectation value of $M(x)$ at finite temperature T.

$$
\begin{aligned}
&G_2(m_T, T) \\
&\qquad = \frac{T}{(2\pi)^{d-1}} \sum_{n \in \mathcal{Z}} \int^{\Lambda} \frac{\mathrm{d}^{d-1}k}{(2\pi n T)^2 + k^2 + m_T^2} \\
&\qquad = \int^{\Lambda} \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \frac{1}{\omega(k)} \left(\frac{1}{2} + \frac{1}{\mathrm{e}^{\beta\omega(k)} - 1} \right)
\end{aligned}
$$

Fermions:

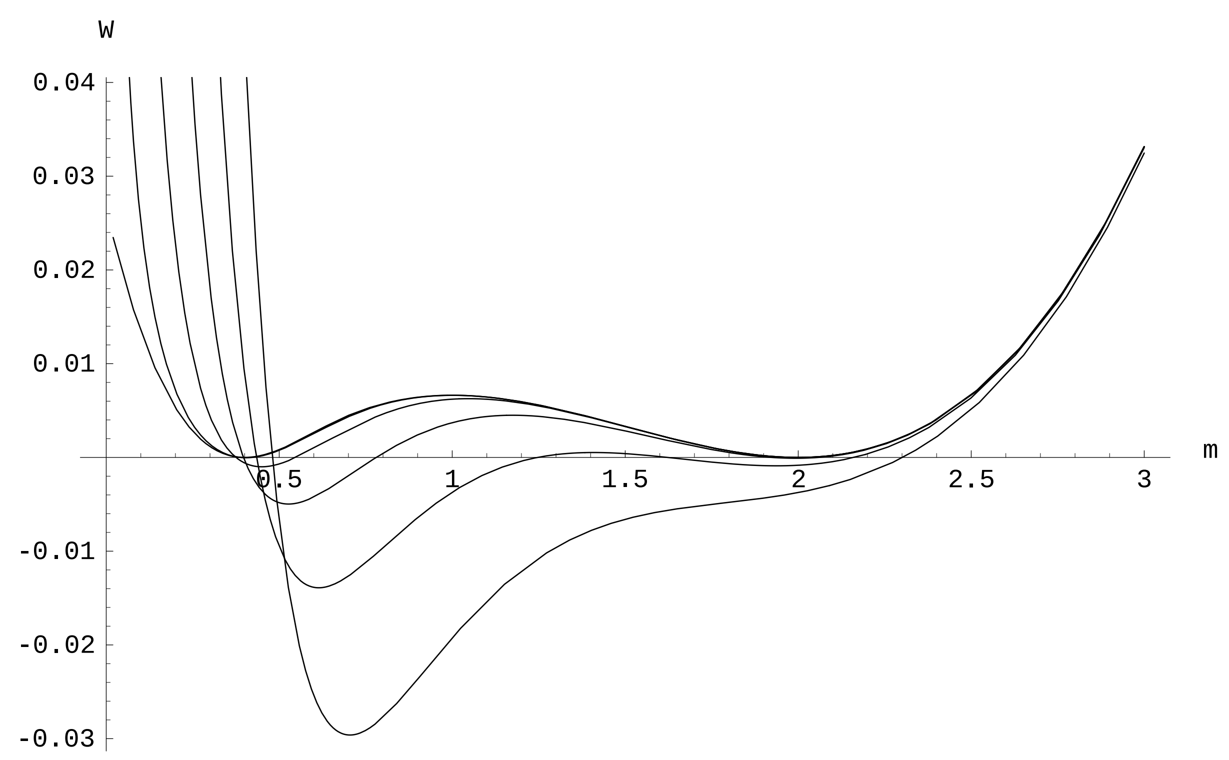
$$
\begin{aligned}
&\mathcal{G}_2(M_T, T) \\
&\qquad = \int^{\Lambda} \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \frac{1}{\omega(k)} \left(\frac{1}{2} - \frac{1}{\mathrm{e}^{\omega(k)/T} + 1} \right)
\end{aligned}
$$

with $\omega(k) = \sqrt{k^2 + M_T^2}$.

$$
\begin{aligned}
\frac{1}{N}\mathcal{F} = \quad & -\frac{F^2}{2N} + M_T\frac{F\varphi}{N} \\
& + \lambda\frac{\varphi^2}{2N} + \tfrac{1}{2}s(U'(\rho) - M_T) \\
& - \frac{1}{12\pi}\left(m_T^3 - |M_T|^3\right) \\
& + \tfrac{1}{2}\lambda(\rho_c - \rho) \\
+ T\int \frac{\mathrm{d}^2k}{(2\pi)^2} & \left\{\ln[1 - \mathrm{e}^{-\beta\omega_\varphi}] - \ln[1 + \mathrm{e}^{-\beta\omega_\psi}]\right\}
\end{aligned}
$$

Peculiar transitions occur in this system

Region **(II)** $\mu - \mu_c \geq 0$ and $u \geq u_c$: (Fig. 4).



Ground state free energy at $\varphi = 0$ as a function of the boson mass (m) at different temperatures. Here $\mu - \mu_c = 1$(sets the mass scale) , $u/u_c = 1.5$ and T varies between $T = 0 - -0.5$. At $T = 0$ two degenerate phases with a light $m = m_+$ and heavier $m = m_- > m_+$ boson (and fermion). Light mass phase is affected as temperature increases.

Region **(IV)** $\mu \leq 0$ and $u \leq u_c$: Fig. 2).

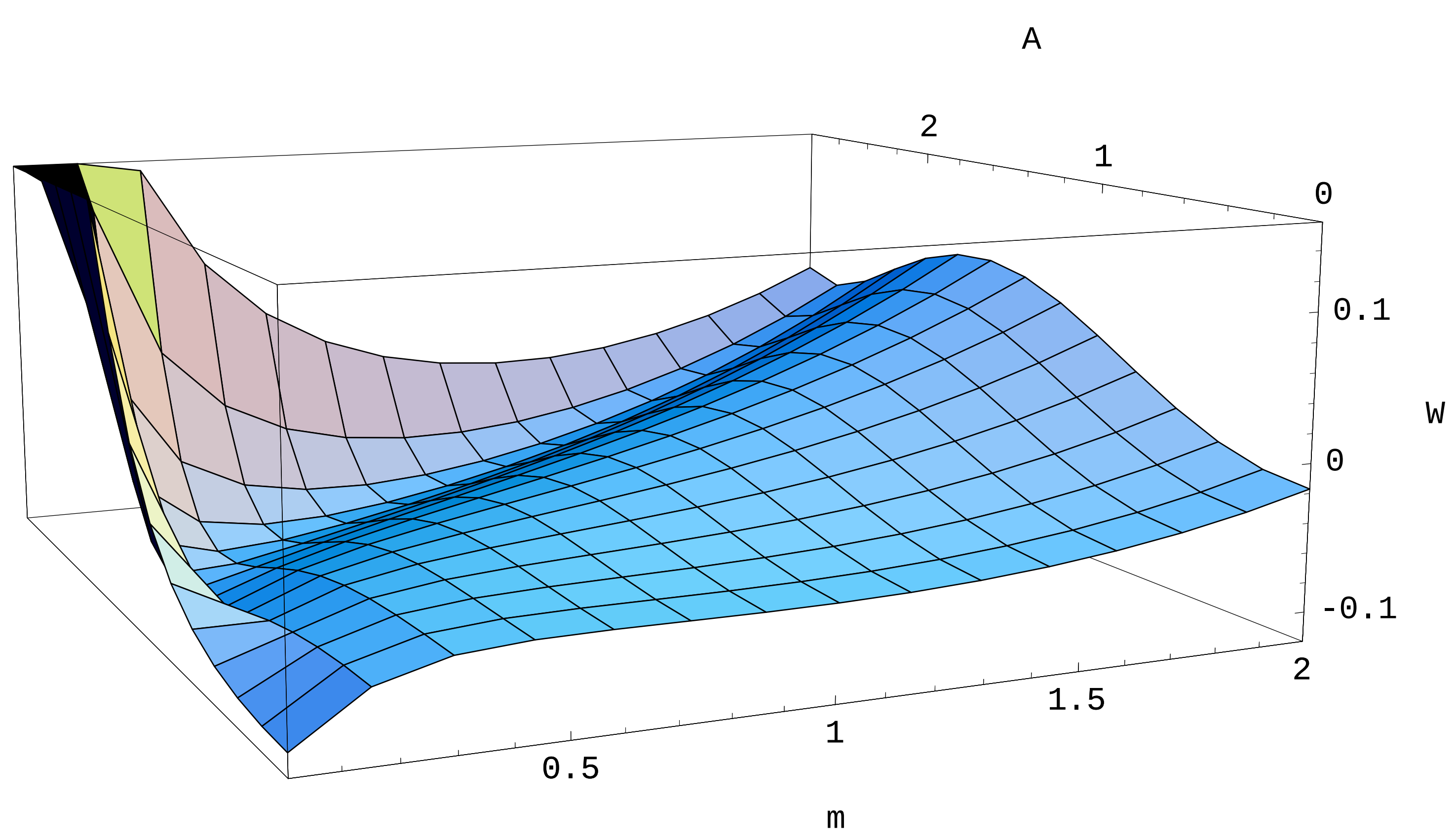


Fig. 5 Same as Fig. 2 but the temperature has been increased from $T = 0$ (in Fig. 2) to $T = 0.7$ (here). $\frac{1}{N}\mathcal{F}(m \equiv m_\varphi, \varphi, T)$ as a function of the boson mass (m) and A, where $A^2 = \varphi^2/u_c$. Here $\mu = -1$ and $u/u_c = 0.2$. A non-degenerate $O(N)$ symmetric ground state ($\varphi = 0$) appears with a very small boson mass (the non-zero mass is not seen here due to the limited resolution of the plot).

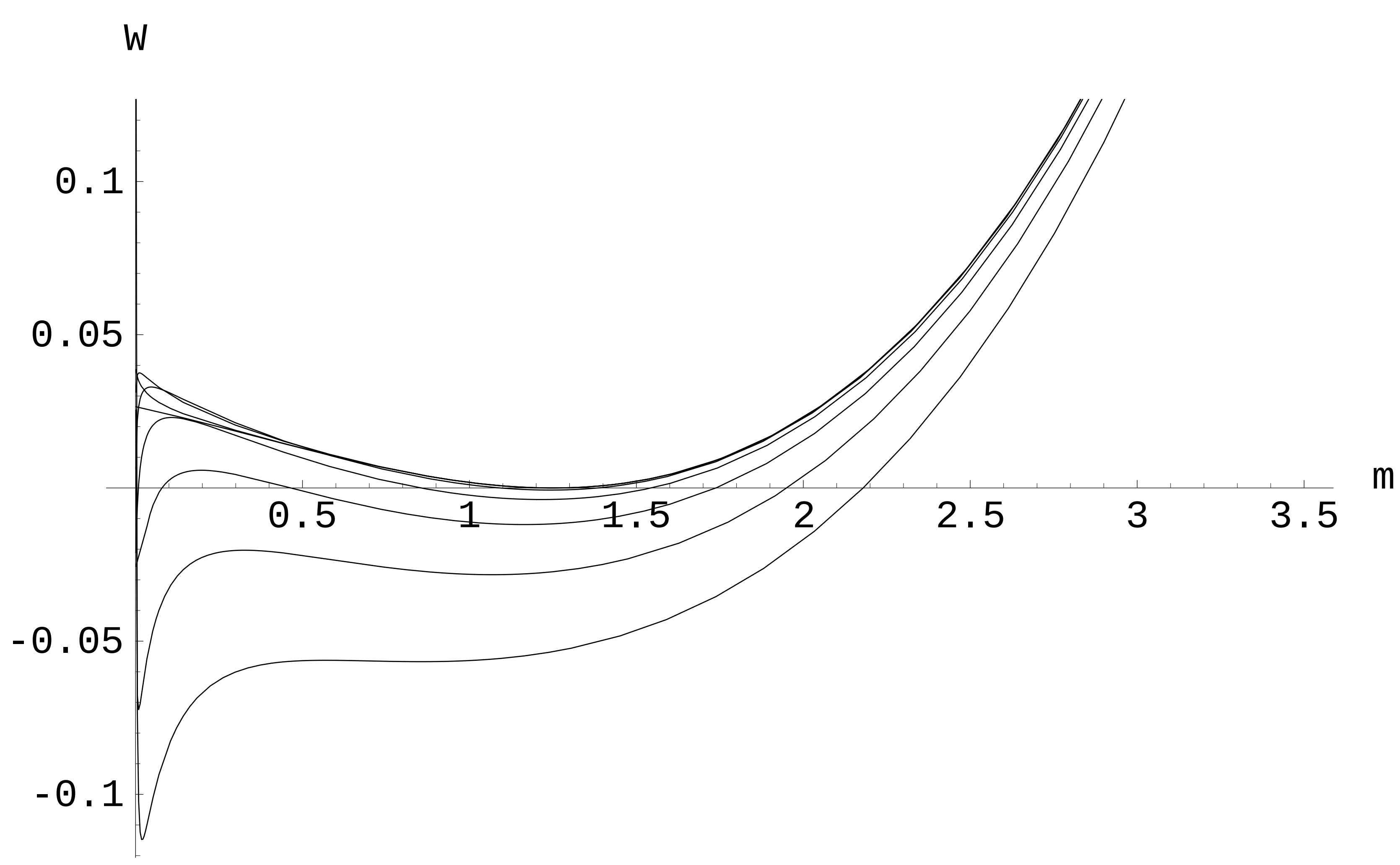


Fig. 6 This figure displays the effect of increasing the temperature from $T = 0$ in Fig. 2 to T that varies between $T = 0 - 0.7$ (Fig. 5 has $T = 0.7$). The ground state free energy $\frac{1}{N}\mathcal{F}(m \equiv m_\varphi, \varphi, T)$ at $\varphi = 0$ is plotted as a function of the boson mass (m) at different temperatures. Here $\mu = -1$, $u/u_c = 0.2$. At $T = 0$ there are two degenerate phases: An $O(N)$-symmetric phase, shown here, with a massive ($m = m_-$) boson and fermion and an ordered phase ($\varphi \neq 0$) with massless particles (both phases are shown in Fig. 2). At finite temperatures the $O(N)$ symmetry is restored (see Fig. 5) and a small mass ground state appears, the heavy mass state decays into the small mass ground state as seen here.

Supersymmetric $O(N)$ non-linear σ-model at finite temperature

$$\mathcal{Z} = \int [\mathrm{d}\Phi][\mathrm{d}L]\, \mathrm{e}^{-\mathcal{S}(\Phi,L)}$$

$$\mathcal{S}(\Phi,L) = \frac{1}{2\kappa}\int \mathrm{d}^d x\, \mathrm{d}^2\theta\, \bar{\mathrm{D}}\Phi\cdot\mathrm{D}\Phi + L(\Phi^2 - N)$$

The free energy is given by

$$\frac{1}{N}\mathcal{F} = \tfrac{1}{2}(M_T^2 - m_T^2)(\frac{1}{\kappa} - \frac{1}{\kappa_c}) - \frac{1}{12\pi}(m_T^3 - |M_T|^3) + T\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\{\ln[1-\mathrm{e}^{-\beta\omega_\varphi}] - \ln[1+\mathrm{e}^{-\beta\omega_\psi}]\}$$

(for $\varphi = 0$):

$$\frac{1}{N}\mathcal{F} = \frac{1}{24\pi}(m_T - |M_T|)^2(m_T + 2|M_T|)$$

$$+\frac{T}{4\pi}(m_T^2 - M_T^2)\ln(1-\mathrm{e}^{-m_T/T})$$

$$+T\int\frac{\mathrm{d}^2k}{(2\pi)^2}\{\ln(1-\mathrm{e}^{-\beta\omega_\varphi}) - \ln(1+\mathrm{e}^{-\beta\omega_\psi})\}$$

notation

$$X(\kappa, T) = \exp\left[\frac{2\pi}{T}\left(\frac{1}{\kappa_c} - \frac{1}{\kappa}\right)\right].$$

An interesting non-analytic behaviour:

$$\begin{cases} M_T = 0 & \text{for } X < 2, \\ M_T = 2T\ln\left[\frac{1}{2}(X+\sqrt{X^2-4})\right] & \\ & \text{for } X > 2 \end{cases}$$

The boson thermal mass m_T:

$$m_T = 2T\ln\left[\tfrac{1}{2}(X+\sqrt{X^2+4})\right].$$

For $\kappa < \kappa_c$ and $T \to 0$ we find the asymptotic behaviour

$$m_T \sim TX(\kappa, T),$$

Dimension $d = 2$. At high temperature

$$\frac{T}{m_T} \sim \frac{1}{\pi}\ln(m_T/m) \sim \frac{1}{\pi}\ln(T/m),$$

$$
\begin{aligned}
\langle p_2^a | T^{\mu\nu} | p_1^a \rangle = {} & p_1^\mu p_2^\nu + p_2^\mu p_1^\nu \\
& - \eta^{\mu\nu} p_1 p_2 - \eta^{\mu\nu} m^2 \\
& + \tfrac{1}{4} \left(q^\mu q^\nu - \eta^{\mu\nu} q^2 \right) \times \\
\times & \left[1 - 8 \int_0^1 dx x(1-x) \left[1 + \tfrac{x(1-x)q^2}{m^2} \right]^{-\frac{1}{2}} \right] \times \\
\times & \left[1 - \int_0^1 dx \left[1 + \tfrac{x(1-x)q^2}{m^2} \right]^{-\frac{1}{2}} \right]^{-1}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\langle p_2^a | T^{\mu\nu} | p_1^a \rangle = {} & p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - \eta^{\mu\nu} p_1 p_2 - \eta^{\mu\nu} m^2 \\
& + \left(q^\mu q^\nu - \eta^{\mu\nu} q^2 \right) \left(\tfrac{1}{4} - \tfrac{m^2}{q^2} \right)
\end{aligned}
$$

The trace of the energy-momentum tensor:

$$
\begin{aligned}
T_\mu^\mu &= 2p_1p_2 - 3p_1p_2 - 3m^2 \\
&+ \left(q^2 - 3q^2\right)\left(\tfrac{1}{4} - \tfrac{m^2}{q^2}\right) \\
&= -p_1p_2 - m^2 - \tfrac{q^2}{2} \\
&= -\tfrac{1}{2}((p_1 - p_2)^2 + 2p_1p_2) - m^2 = 0
\end{aligned}
$$

Used (here, Euclidean space) $p_1^2 = p_2^2 = -m^2$.

Energy momentum tensor in one particle fermionic state

$$\int d^3x e^{iqx} \left\langle p^a_{2_\psi} \left| T_{\mu\nu}(x) \right| p^a_{1_\psi} \right\rangle =$$
$$-\frac{1}{4}\overline{u}(p_2)\left[(p_{1\nu}\gamma_\mu + p_{1\mu}\gamma_\nu) + (p_{2\nu}\gamma_\mu + p_{2\mu}\gamma_\nu) + 2\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) m\right] u(p_1)$$

This expression is traceless.

Energy momentum tensor for the SUSY Lagrangian in 3D

$$L = \tfrac{1}{2}[\nabla_\alpha \varphi \nabla^\alpha \varphi - \mu_0^2 \varphi^2] + \tfrac{1}{2}\overline{\psi}(i\gamma^\alpha \nabla_\alpha - \mu_0)\psi$$
$$- (u/N)\mu_0(\varphi^2)^2 - \tfrac{(u/N)^2}{2}(\varphi^2)^3$$
$$- \tfrac{(u/N)}{2}\varphi^2(\overline{\psi}\psi) - \xi R(x)\varphi^2$$

$\nabla_\alpha = V_\alpha{}^\mu \frac{\partial}{\partial x^\mu}$ for scalars

where $V_\alpha{}^\mu$ is a tetrad, and

$\nabla_\alpha = V_\alpha{}^\mu \frac{\partial}{\partial x^\mu} + \frac{1}{2}\sigma^{\beta\gamma} V_\beta{}^\nu V_\alpha{}^\mu V_{\gamma\nu;\mu}$ for fermions.

The covariant derivative is $V_{\gamma\nu;\mu} = \frac{\partial V_{\gamma\nu}}{\partial x^\mu} - \Gamma^\lambda_{\nu\mu} V_{\gamma\nu}$ and $\sigma^{\alpha\beta} = \frac{1}{4}\left[\gamma^\alpha, \gamma^\beta\right]$ are the generators of the Lorentz group representation for spin $\frac{1}{2}$.

The action is given in tetrad formalism in $d = 3$:

$$\delta S_{matter} = \int d^3x \sqrt{-g} U^{\alpha}{}_{\mu} \delta V_{\alpha}{}^{\mu}$$

$$T_{\mu\nu} = V_{\alpha\mu} U^{\alpha}{}_{\nu}$$

where $\qquad U^{\alpha}{}_{\mu} = \frac{1}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta V_{\alpha}{}^{\mu}}$
and thus:

$$T_{\mu\nu}(x) = \frac{V_{\alpha\mu}(x)}{\det[V(x)]} \frac{\delta S_{matter}}{\delta V_{\alpha}{}^{\nu}}$$

The SUSY energy-momentum tensor in 3D ($\xi = \frac{1}{8}$ in 3D) is then given by:

$$\begin{aligned} T_{\mu\nu} &= \partial_\mu \varphi \partial_\nu \varphi + \tfrac{i}{4} (\overline{\psi} \gamma_\mu \partial_\nu \psi + \overline{\psi} \gamma_\nu \partial_\mu \psi) \\ &- \eta_{\mu\nu} [\tfrac{1}{2} \partial_\alpha \varphi(x) \partial^\alpha \varphi(x) - \tfrac{\mu_0^2}{2} \varphi^2 \\ &\qquad - (u/N) \mu_0 \left(\varphi^2\right)^2 - \tfrac{(u/N)^2}{2} \left(\varphi^2\right)^3] \\ &- \eta_{\mu\nu} \left(\tfrac{1}{2} \overline{\psi} i \partial\!\!\!/ \psi - \tfrac{\mu_0}{2} \overline{\psi} \psi - \tfrac{(u/N)}{2} \varphi^2 \left(\overline{\psi} \psi\right) \right) \\ &- \tfrac{1}{8} \left(\partial^2_{\mu\nu} \varphi^2 - \eta_{\mu\nu} \partial^2 \varphi^2 \right) \end{aligned}$$

In a covariant, symmetrized form in a general non-flat background, the energy momentum tensor is then given by:

$$
\begin{aligned}
T_{\mu\nu}^{SUSY} &= \tfrac{1}{8}\overline{\psi} i\left(\gamma_\mu \overrightarrow{\nabla}_\nu + \gamma_\nu \overrightarrow{\nabla}_\mu\right)\psi - \tfrac{1}{8}\overline{\psi} i\left(\gamma_\mu \overleftarrow{\nabla}_\nu + \gamma_\nu \overleftarrow{\nabla}_\mu\right)\psi \\
&+ \nabla_\mu\varphi\nabla_\nu\varphi \\
&- g_{\mu\nu}\left(\tfrac{1}{2}\overline{\psi} i\not{\nabla}\psi - \tfrac{\mu_0}{2}\overline{\psi}\psi - \tfrac{u/N}{2}\varphi^2\left(\overline{\psi}\psi\right)\right) \\
&- g_{\mu\nu}\left(\tfrac{1}{2}\nabla_\rho\varphi\nabla^\rho\varphi - \tfrac{1}{2}\mu_0^2\varphi^2 - (u/N)\mu_0\left(\varphi^2\right)^2 - \tfrac{(u/N)^2}{2}\left(\varphi^2\right)^3\right) \\
&+ \xi\left(\tfrac{g_{\mu\nu}}{2}R - R_{\mu\nu}\right)\varphi^2 + \xi\left[g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu\right]\varphi^2
\end{aligned}
$$

Which reduces to the former expression in the case of flat space.

Energy momentum tensor in one particle fermionic state

$$\int d^3x e^{iqx} \left\langle p^a_{2_\psi} \left| T_{\mu\nu}\left(x\right) \right| p^a_{1_\psi} \right\rangle =$$
$$-\frac{1}{4}\overline{u}\left(p_2\right)\left[\left(p_{1\nu}\gamma_\mu + p_{1\mu}\gamma_\nu\right) + \left(p_{2\nu}\gamma_\mu + p_{2\mu}\gamma_\nu\right) + 2\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)m\right]u\left(p_1\right)$$

This expression is traceless.

Scalar-Fermion thermal mass difference at finite temperature

$$m_A^2 - m_\psi^2 = u[\tfrac{m_\psi}{2\pi}(|m_\psi| - m_A) + \tfrac{m_\psi}{\beta\pi}\ln(\tfrac{1+e^{-\beta|m_\psi|}}{1-e^{-\beta m_A}})]$$

Clearly $\quad m_\varphi^2 \neq m_\psi^2 \quad$ at $\quad T \neq 0$

Dilatino mass:

$$M_\psi^D \approx 2\left(1 + \tfrac{u}{u_c}\tfrac{m_\psi}{|m_\psi|}\right) + \tfrac{u}{u_c}\tfrac{\delta}{m_\psi}$$

δ is the boson-fermion thermal mass difference.

$$\frac{1}{N}\left<T_{11}\right>_T = \frac{1}{N}\left<T_{22}\right>_T = -\frac{(m_\varphi - |m_\psi|)^2(m_\varphi + 2|m_\psi|)}{24\pi} + \frac{m_\psi^2 - m_\varphi^2}{4\pi\beta}\ln(1-e^{-\beta m_\varphi}) + \frac{1}{2\pi\beta^3}\int_{\beta|m_\psi|}^{\beta m_\varphi} y\ln(1-e^{-y})dy$$

$$\frac{1}{N}\left<T_{00}\right>_T = -\frac{(m_\varphi - |m_\psi|)^2(m_\varphi + 2|m_\psi|)}{24\pi} + \frac{m_\psi^2 + m_\varphi^2}{4\pi\beta}\ln(1-e^{-\beta m_\varphi}) - \frac{1}{\pi\beta^3}\int_{\beta|m_\psi|}^{\beta m_\varphi} y\ln(1-e^{-y})dy - \frac{m_\psi^2}{2\pi\beta}\ln(1+e^{-\beta|m_\psi|})$$

and thus the trace of the energy momentum tensor is:

$$\frac{1}{N}\left< T_\mu{}^\mu\right>_T = -\frac{(m_\varphi - |m_\psi|)^2(m_\varphi + 2|m_\psi|)}{8\pi} + \frac{3m_\psi^2}{4\pi\beta}\ln(1-e^{-\beta m_\varphi}) - \frac{m_\varphi^2}{4\pi\beta}\ln(1-e^{-\beta m_\varphi}) - \frac{m_\psi^2}{2\pi\beta}\ln(1+e^{-\beta|m_\psi|})$$

Using the gap equations this simplifies to:

$$\left< T_\mu{}^\mu\right>_T = N(m_\psi^2 - m_\varphi^2)\frac{\mu_R}{2u}$$

Taking into account the gap equations, we get the following expression for the thermal expectation value of the energy-momentum trace in our model

$$\left\langle T_\mu{}^\mu \right\rangle_T = N(m_\psi^2 - m_\varphi^2)\tfrac{\mu_R}{2u}$$

As have been shown, supersymmetry is softly broken when the temperature is turned on. Thus scale invariance is guaranteed by $\mu_R = 0$ requirement.