Elliptic Quantum Groups

and

Elliptic Lattice Models

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1) K, in preparation,

2) Kojima, K and Weston, *Nucl.Phys.* B720(2005)348.

3) Kojima and K, *J.Math.Phys.* 45(2004)3146.

4) Kojima and K, *Comm.Math.Phys.* 239(2003)405.

5) Jimbo, K, Odake and Shiraishi, *Comm.Math.Phys.* 199(1999)605,

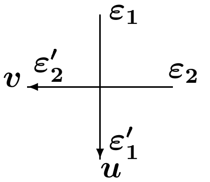
6) Jimbo, K, Odake and Shiraishi, *TransformationGroups*, 4(1999)303.

7) K, *Comm.Math.Phys.* 195(1998)373.

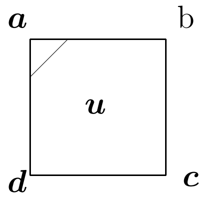
1. Introduction

• 2-dim. Exactly Solvable Lattice Models

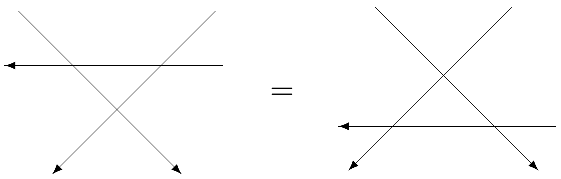
Vertex models

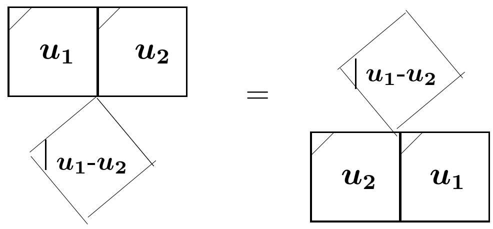
$R(u-v)^{\varepsilon_1\varepsilon_2}_{\varepsilon'_1\varepsilon'_2} =$ 

Face models

$W\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| u\right) =$ 

Yang-Baxter Eq. :





Elliptic solutions ass. with Affine Lie Algebra :

- Baxter '73 $\widehat{\mathfrak{sl}}_2$
- Belavin '81 $\widehat{\mathfrak{sl}}_N$
- Andrews-Baxter-Forrester '84 $\widehat{\mathfrak{sl}}_2$
- Jimbo-Miwa-Okado '87, '88
  $A^{(1)}_N,\ B^{(1)}_N,\ C^{(1)}_N,\ D^{(1)}_N$
- Kuniba '91; Kuniba-Suzuki '91
  $A^{(2)}_{2N},\ A^{(2)}_{2N+1};\ G^{(1)}_2$

• Success of $U_q(\mathfrak{g})$ in Trig. Vertex Models

- Derivation of $R(z)$ (Jimbo'86)
- Vertex operators : (Frenkel-Reshetikhin'92)

$$\Phi(z) : V(\lambda) \longrightarrow V(\mu) \otimes V_z$$

- Correlation functions (Jimbo-Miwa et.al.'92)

$$\sim \mathrm{tr}_{V(\lambda)} q^d\ \Phi(z_1)\cdots\Phi(z_n)$$

- **Elliptic Quantum Groups**

Foda-Iohara-Jimbo-Miwa-Kedem-Yan '94

Felder '95, K '98, Fronsdal '97, JKOS '99

Quasi-Hopf deformation:

$$
\begin{array}{lll}
 & E(r) & \mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N) \quad (p = q^{2r}, r \in \mathbb{C}) \\
 & \nearrow & \\
U_q(\mathfrak{g}) & & \\
 & \searrow & \\
 & F(\lambda) & \mathcal{B}_{q,\lambda}(\mathfrak{g}) \quad (\mathfrak{g} : \text{aff.Lie alg.}, \ \lambda \in \mathfrak{h})
\end{array}
$$

<u>EX.</u> **The Face Type Elliptic Quantum Group $\mathcal{B}_{q,\lambda}(\mathfrak{g})$**

$F(\lambda) \in U_q \otimes U_q \ (\lambda \in \mathfrak{h})$, invertible, satisfying

$$\Rightarrow \quad F(\lambda)(\Delta \otimes \mathrm{id})F(\lambda) = F^{(23)}(\lambda + h^{(1)})(\mathrm{id} \otimes \Delta)F(\lambda)$$

$$(U_q(\mathfrak{g}), \Delta, \varepsilon, S, \mathcal{R}) \ \rightarrow \ \mathcal{B}_{q,\lambda}(\mathfrak{g}) = (U_q(\mathfrak{g}), \Delta_\lambda, \varepsilon, S, \mathcal{R}(\lambda), ..)$$

$$\Delta_\lambda(a) = F(\lambda)\Delta(a)F(\lambda)^{-1} \qquad \forall a \in U_q(\mathfrak{g}),$$

$$\mathcal{R}(\lambda) = F^{(21)}(\lambda)\mathcal{R}F^{-1}(\lambda), \quad \text{etc.}$$

$$
\begin{aligned}
&\mathcal{R}^{(12)}(\lambda + h^{(3)})\mathcal{R}^{(13)}(\lambda)\mathcal{R}^{(23)}(\lambda + h^{(1)}) \\
&\qquad = \mathcal{R}^{(23)}(\lambda)\mathcal{R}^{(13)}(\lambda + h^{(2)})\mathcal{R}^{(12)}(\lambda)
\end{aligned}
$$

The Face Type Twistor $F(\lambda)$

(Jimbo-K-Odake-Shiraishi '99)

- $\{h_l\}$: basis of $\mathfrak{h}$, $\{h^l\}$: dual basis, $\mathfrak{h} \cong \mathfrak{h}^*$
- $\rho \in \mathfrak{h}^*$ s.t. $(\rho|\alpha_i) = \frac{1}{2}(\alpha_i|\alpha_i)$
- $\mathcal{R}$: the universal R matrix of $U_q(\mathfrak{g})$
- $\varphi_\lambda = \mathrm{Ad}(q^{2(\lambda-\rho)+\sum h_l h^l})$

Theorem

$$F(\lambda) = \prod_{m\geq 1}^{\curvearrowleft} ((\varphi_\lambda)^m \otimes \mathrm{id})(q^T \mathcal{R})^{-1}, \quad T = \sum h_l \otimes h^l$$

satisfies the shifted cocycle condition.

Hence

$$(U_q(\mathfrak{g}), \Delta, \varepsilon, S, \mathcal{R}) \xrightarrow{F(\lambda)} \mathcal{B}_{q,\lambda}(\mathfrak{g}) = (U_q(\mathfrak{g}), \Delta_\lambda, \varepsilon, S, \mathcal{R}(\lambda))$$

Similarly,

$$(U_q(\widehat{\mathfrak{sl}}_N), \Delta, \varepsilon, S, \mathcal{R}) \xrightarrow{E(r)} \mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N) = (U_q(\widehat{\mathfrak{sl}}_N), \Delta_r, \varepsilon, S, \mathcal{R}(r))$$

However,

No a priori reason that $\mathcal{B}_{q,\lambda}(\mathfrak{g})$ and $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$ are Elliptic !

$\widehat{\mathfrak{sl}}_N$ case : Frønsdal '97, JKOS '99, Kojima-K '03

$A_2^{(2)}$ case : Kojima-K '04

We will show

- For any affine Lie algebra $\mathfrak{g}$ and any finite dim. rep. (π_V, V) and (π_W, W) of $\mathcal{B}_{q,\lambda}(\mathfrak{g})$,

$$(\pi_V \otimes \pi_W)\mathcal{R}(z,-\lambda) = \sum_{i,i'j,j'} C_{VW} \left(\begin{array}{ccc|c} \lambda & v_i & \mu & \\ w_{j'} & & w_j & z \\ \mu' & v_{i'} & \nu & \end{array} \right) E_{ii'} \otimes E_{jj'},$$

and the matrix elements of C_{VW} are elliptic.

$\longrightarrow$ § II

- In the vector representation (π_V, V), $(\pi_V \otimes \pi_V)\mathcal{R}(z,\lambda)$ coincides with the face weight obtained by Jimbo-Miwa-Okado '87, '88.

(conjectured by Frenkel and Reshetikhin '92)

$\longrightarrow$ § III

II. $\mathcal{R}(\lambda)$ and Connection Problem for the q-KZ Eq.

$U_q(\mathfrak{g})$ ($\mathfrak{g}$: affine Lie alg.)

- M_λ : irr. Verma module with h.w. λ $\left(\begin{array}{l}\text{antidominant,}\\ \text{generic}\end{array}\right)$
- $(\pi_V, V), (\pi_W, W)$: finite dim.rep.,
- $V_z = V \otimes \mathbb{C}[z, z^{-1}]$, W_z : evaluation rep.
- $\Psi^\mu_\lambda(z) = z^{\Delta_\mu - \Delta_\lambda}\widetilde{\Psi}^\mu_\lambda(z)$, $\quad \Delta_\lambda = \frac{(\lambda|\lambda+2\rho)}{2(k+h^\vee)}$,

  $\widetilde{\Psi}^\mu_\lambda(z)$: $M_\lambda \to V_z \otimes M_\mu$, $\quad \widetilde{\Psi}^\mu_\lambda(z)x = \Delta(x)\widetilde{\Psi}^\mu_\lambda(z)$.
- $M_\lambda \xrightarrow{\Psi^\mu_\lambda(z_1)} V_{z_1}\otimes M_\mu \xrightarrow{\mathrm{id}\otimes\Psi^\nu_\mu(z_2)} V_{z_1}\otimes W_{z_2}\otimes M_\nu$

$$J_{VW}(z_1, z_2) = \left\langle \mathrm{id}\otimes\mathrm{id}\otimes u^*_\nu, \left(\mathrm{id}\otimes\Psi^\nu_\mu(z_2)\right)\Psi^\mu_\lambda(z_1)u_\lambda\right\rangle,$$
$$\left(\overset{\text{abbreviate}}{=} \left\langle \Psi^\nu_\mu(z_2)\Psi^\mu_\lambda(z_1)\right\rangle\right)$$

Theorem (q-KZ Eq.) (Frenkel-Reshetikhin '92)

$$J_{VW}(q^{2(k+h^\vee)}z_1, z_2) = (q^{-\bar{\nu}-\bar{\lambda}-2\bar{\rho}}\otimes\mathrm{id})R_{VW}(z_1/z_2)J_{VW}(z_1, z_2)$$

k : the level of λ

Solution: $J_{VW}(z_1, z_2) = G(z_2/z_1)J(z_2/z_1)$,

- $G(z)$ is meromorphic and determined uniquely.
- $J(z_2/z_1)$ is unique, analytic in $|z_1| > |z_2|$ and analytically continued to $|z_1| < |z_2|$.

Theorem (Frenkel-Reshetikhin '92)

(1) $$\Psi_{\mu}^{\nu,w}(z_2)\Psi_{\lambda}^{\mu,v}(z_1)$$

$$= \sum_{i',j',\mu'} P\Psi_{\mu'}^{\nu,v_{j'}}(z_1)\Psi_{\lambda}^{\mu',w_{i'}}(z_2)C_{VW}\left(\begin{array}{ccc|c} \lambda & v & \mu & \\ w_{i'} & & w & \frac{z_1}{z_2} \\ \mu' & v_{j'} & \nu & \end{array}\right),$$

where $P(v\otimes w) = w\otimes v$,

(2) C_{VW} satisfies the face type YBE, the unitarity and the crossing unitarity relations.

(3) C_{VW} is expressed as a ratio of elliptic theta functions.

The VO is determined uniquely by the leading vector v

s.t. $\widetilde{\Psi}_{\lambda}^{\mu}(z)u_\lambda = v\otimes u_\mu + \sum_{i,n} v_i z^{-n}\otimes u_{i,n}, \quad (v_i\in V,\ u_{i,n}\in M_\mu)$

$v\in V_\lambda^\mu = \{v\in V \,|\, \mathrm{wt}(v) = cl(\lambda-\mu)\}$, $\qquad \mathrm{wt}(u_{i,n}) < u_\mu$.

We write this VO as $\Psi_{\lambda}^{\mu,v}(z)$.

$J_{VW}(z_1, z_2)$ and $F_{VW}(z, \lambda)$

Lemma

$$\left\langle \widetilde{\Psi}^{\nu,w}_{\mu}(z_2)\widetilde{\Psi}^{\mu,v}_{\lambda}(z_1)\right\rangle = F_{VW}\left(\frac{z_1}{z_2}, -\lambda\right)^{-1} v \otimes w$$

Idea : Etingof-Varchenko '99 for $U_q(\bar{\mathfrak{g}})$ ($\bar{\mathfrak{g}}$: simple Lie alg.)

Sketch of proof.

- The difference eq. for $F(\lambda)$ (JKOS '99):

$$F_{VW}(pz, \lambda) = (\bar{\varphi}_\lambda \otimes \mathrm{id})(F_{VW}(z, \lambda)) \cdot R_0^{VW}(pz)$$

$$F(z, \lambda) = \prod_{m \geq 1}^{\curvearrowleft} ((\varphi_\lambda)^m \otimes \mathrm{id})\mathcal{R}_0^{-1}(z), \quad \mathcal{R}_0 = q^T \mathcal{R}$$

$$\lambda - \rho = rd + \bar{\lambda} - \bar{\rho}, \qquad p = q^{2r}$$

$$\varphi_\lambda = \mathrm{Ad}((pq^{2c})^d) \circ \bar{\varphi}_\lambda, \quad \bar{\varphi}_\lambda = \mathrm{Ad}(q^{2(\bar{\lambda}-\bar{\rho})+\sum \bar{h}_i \bar{h}^i})$$

Cf. Arnaudon-Buffenoir-Ragoucy-Roche '98

for $U_q(\bar{\mathfrak{g}})$ ($\bar{\mathfrak{g}}$: simple Lie alg.)

- Under the identification $p = q^{-2(k+h^\vee)}$, the eq. for $F_{VW}(z_1/z_2, -\lambda)^{-1}$ on $v \otimes w$ coincides with the q-KZ eq. for $\left\langle \widetilde{\Psi}^{\nu,w}_{\mu}(z_2)\widetilde{\Psi}^{\mu,v}_{\lambda}(z_1)\right\rangle$.

- The solution is unique up to an overall constant factor.

Theorem

$$R_{VW}(z,-\lambda) = \sum_{\substack{i,i',j,j' \\ \mathrm{wt}(v_i)+\mathrm{wt}(w_j) \\ =\mathrm{wt}(v_{i'})+\mathrm{wt}(w_{j'})}} C_{VW}\left(\begin{array}{ccc|c} \lambda & v_i & \mu & \\ w_{j'} & & w_j & z \\ \mu' & v_{i'} & \nu & \end{array}\right) E_{jj'} \otimes E_{ii'}$$

where $v_i \in V_\lambda^\mu,\ v_{i'} \in V_{\mu'}^\nu,\ w_j \in W_\mu^\nu,\ w_{j'} \in W_\lambda^{\mu'}$.

Proof. Set $z = z_1/z_2$.

$$R_{VW}\,(z,\text{-}\lambda)\,(v_i \otimes w_j)$$

$$= F_{WV}^{(21)}(z^{-1},\text{-}\lambda) R_{VW}(z) F_{VW}(z,\text{-}\lambda)^{-1}(v_i \otimes w_j)$$

$$= F_{WV}^{(21)}(z^{-1},\text{-}\lambda) R_{VW}(z) \left\langle \widetilde{\Psi}_\mu^{\nu,w_j}(z_2) \widetilde{\Psi}_\lambda^{\mu,v_j}(z_1) \right\rangle$$

$$= F_{WV}^{(21)}(z^{-1},\text{-}\lambda) \sum_{\mu'} P \left\langle \widetilde{\Psi}_{\mu'}^{\nu,v_{i'}}(z_1) \widetilde{\Psi}_\lambda^{\mu,w_{j'}}(z_2) \right\rangle C_{VW}\left(\begin{array}{ccc|c} \lambda & v_i & \mu & \\ w_{j'} & & w_j & z \\ \mu' & v_{i'} & \nu & \end{array}\right)$$

$$= P F_{WV}(z^{-1},\text{-}\lambda) \sum_{\mu'} F_{WV}\left(\frac{z_2}{z_1},\text{-}\lambda\right)^{-1} (w_{j'} \otimes v_{i'}) C_{VW}\left(\begin{array}{ccc|c} \lambda & v_i & \mu & \\ w_{j'} & & w_j & z \\ \mu' & v_{i'} & \nu & \end{array}\right)$$

$$= \sum_{\mu'} C_{VW}\left(\begin{array}{ccc|c} \lambda & v_i & \mu & \\ w_{j'} & & w_j & z \\ \mu' & v_{i'} & \nu & \end{array}\right) (v_{i'} \otimes w_{j'}),$$

Note $F_{WV}^{(21)}(z^{-1},\text{-}\lambda) = P F_{WV}(z^{-1},\text{-}\lambda) P,\ P^2 = \mathrm{id}$.

We will show

- For any affine Lie algebra $\mathfrak{g}$ and any finite dim. rep. (π_V, V) and (π_W, W) of $\mathcal{B}_{q,\lambda}(\mathfrak{g})$,

$$(\pi_V \otimes \pi_W)\mathcal{R}(z, -\lambda) = \sum_{i,i'j,j'} C_{VW} \left(\begin{matrix} \lambda & v_i & \mu \\ w_{j'} & & w_j \\ \mu' & v_{i'} & \nu \end{matrix} \middle| \, z \right) E_{ii'} \otimes E_{jj'},$$

and the matrix elements of C_{VW} are elliptic.

$\longrightarrow$ § II

- In the vector representation (π_V, V), $(\pi_V \otimes \pi_V)\mathcal{R}(z, \lambda)$ coincides with the face weight obtained by Jimbo-Miwa-Okado '87, '88.

(conjectured by Frenkel and Reshetikhin '92)

$\longrightarrow$ § III

III. Vector Representation $\mathfrak{g} = A_N^{(1)}, B_N^{(1)}, C_N^{(1)}, D_N^{(1)}$

Jimbo-Miwa-Okado's sol. of the face type YBE

- $$J = \{1,2,\cdots,N+1\} \text{ for } A_N^{(1)}$$
  $$= \{1,2,\cdots,N,(0,)-N,\cdots,-2,-1\} \text{ for } B_N^{(1)}, C_N^{(1)}, D_N^{(1)}$$
- $\hat{\mu}$ $(\mu \in J)$: weights $\in$ the vec. rep. of $\bar{\mathfrak{g}}$
- $a_\mu = (a+\rho|\hat{\mu}), \quad a_{\mu\nu} = a_\mu - a_\nu \qquad (a \in \mathfrak{h}^*)$
- $[u] = \vartheta_1\left(\frac{u}{r}\middle|\tau\right), \quad r(\neq 0) \in \mathbb{C}$

$$W\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| u\right) = \kappa(u)\bar{W}\left(\begin{matrix} a & b \\ c & d \end{matrix}\middle| u\right),$$

$$(I) \quad \bar{W}\left(\begin{matrix} a & a+\hat{\mu} \\ a+\hat{\mu} & a+2\hat{\mu} \end{matrix}\middle| u\right) = 1 \quad (\mu \neq 0),$$

$$\bar{W}\left(\begin{matrix} a & a+\hat{\mu} \\ a+\hat{\mu} & a+\hat{\mu}+\hat{\nu} \end{matrix}\middle| u\right) = \frac{[1][a_{\mu\nu}-u]}{[1+u][a_{\mu\nu}]} \quad (\mu \neq \nu),$$

$$\bar{W}\left(\begin{matrix} a & a+\hat{\nu} \\ a+\hat{\mu} & a+\hat{\mu}+\hat{\nu} \end{matrix}\middle| u\right) = \frac{[u]\sqrt{[a_{\mu\nu}+1][a_{\mu\nu}-1]}}{[1+u][a_{\mu\nu}]} \quad (\mu \neq \nu),$$

$$(II) \quad \bar{W}\left(\begin{matrix} a & a+\hat{\nu} \\ a+\hat{\mu} & a \end{matrix}\middle| u\right) = \frac{[u][1][a_{\mu,-\nu}+1+\eta-u]}{[\eta-u][1+u][a_{\mu,-\nu}+1]}\sqrt{G_{a\mu}G_{a\nu}} \quad (\mu \neq \nu),$$

$$\bar{W}\left(\begin{matrix} a & a+\hat{\mu} \\ a+\hat{\mu} & a \end{matrix}\middle| u\right) = \frac{[\eta+u][1][a_{\mu,-\mu}+1+2\eta-u]}{[\eta-u][1+u][a_{\mu,-\mu}+1+2\eta]} - \frac{[u][1][a_{\mu,-\mu}+1+\eta-u]}{[\eta-u][1+u][a_{\mu,-\mu}+1+2\eta]}\sum_{\kappa\neq\mu}\frac{[a_{\mu,-\kappa}+1+2\eta]}{[a_{\mu,-\kappa}+1]}G_{a\mu}$$

η: the crossing parameter

$G_{a\mu} = \frac{G_{a+\hat{\mu}}}{G_a}$, $\quad G_a$: the principally specialized char. for $\mathfrak{g}^\vee$

A Connection Matrix for the difference eq. for $F(z,\lambda)$ in the vec. rep.

$$F(pz,\lambda) = (\bar{\varphi}_\lambda \otimes \mathrm{id})(F(z,\lambda)) \cdot q^T R(pz)$$

$$\bar{\varphi}_\lambda = \mathrm{Ad}(q^{2(\bar{\lambda}-\bar{\rho})+\sum \bar{h}_i \bar{h}^i}), \quad T = \sum \bar{h}_i \otimes \bar{h}^i$$

- The R-matrix of $U_q(\mathfrak{g})$ (Jimbo '86, Bazhanov '87) :

$$R(z) = \rho(z) \Bigg\{ \sum_{\substack{i\in J \\ i\neq 0}} E_{i,i}\otimes E_{i,i} + b(z) \sum_{\substack{i,j \\ i\neq \pm j}} E_{i,i}\otimes E_{j,j} + \sum_{\substack{i\prec j \\ i\neq -j}} \Big(c(z) E_{i,j}\otimes E_{j,i} + z c(z) E_{j,i}\otimes E_{i,j} \Big) + \frac{1}{(1-q^2 z)(1-\xi z)} \sum_{i,j} a_{ij}(z) E_{i,j}\otimes E_{-i,-j} \Bigg\},$$

$$b(z) = \frac{q(1-z)}{1-q^2 z}, \quad c(z) = \frac{1-q^2}{1-q^2 z}, \quad \xi = q^{h^\vee}, \ldots$$

- Noting $F(\lambda) = \prod_{m\geq 1}^{\curvearrowleft} ((\varphi_\lambda)^m \otimes \mathrm{id})(q^T \mathcal{R})^{-1}$,

$$F(z,\lambda) = f(z) \Bigg\{ \sum_{\substack{i\in J \\ i\neq 0}} E_{i,i}\otimes E_{i,i} + \sum_{\substack{i,j \\ i\neq \pm j}} X_{ij}^{ij}(z) E_{i,i}\otimes E_{j,j} + \sum_{\substack{i\prec j \\ i\neq -j}} \Big(X_{ij}^{ji}(z) E_{i,j}\otimes E_{j,i} + X_{ji}^{ij}(z) E_{j,i}\otimes E_{i,j} \Big) + \sum_{i,j} X_{i,-i}^{j,-j}(z) E_{i,j}\otimes E_{-i,-j} \Bigg\}.$$

- The difference equation for $F(z,\lambda)$:

1×1 blocks:

$$\begin{aligned} f(pz) &= q^{\frac{N}{N+1}}\rho(pz)f(z) \qquad \text{for } A_N,\\ &= q\rho(pz)f(z) \qquad \text{for } B_N, C_N, D_N. \end{aligned}$$

2×2 blocks:

$$\begin{pmatrix} X^{ij}_{ij}(pz) & X^{ji}_{ij}(pz)\\ X^{ij}_{ji}(pz) & X^{ji}_{ji}(pz)\end{pmatrix}$$
$$= q\begin{pmatrix} X^{ij}_{ij}(z) & q^{-2(a_i-a_j)}X^{ji}_{ij}(z)\\ q^{2(a_i-a_j)}X^{ij}_{ji}(z) & X^{ji}_{ji}(z)\end{pmatrix}\begin{pmatrix} b(pz) & c(pz)\\ pzc(pz) & b(pz)\end{pmatrix},$$
$$(i,j\in J,\ i\prec j,\ i\neq -j).$$

$|J|\times|J|$ block:

$$X^{j,-j}_{i,-i}(pz)=\sum_{k\in J}\frac{q^{-2(a_i-a_k+1)}a_{kj}(pz)}{(1-pq^2z)(1-p\xi z)}X^{k,-k}_{i,-i}(z) \quad (i,j\in J).$$

- The 2×2 block consists of two indep. 2nd order difference eqs. Their connection formulae determine the counterpart of (I) of W.

- The part (II) of W is determined uniquely from (I) by the YBE, the unitarity and the crossing unitarity relations.

The 2nd order difference equation

$$(p^c - p^{a+b+1}z)X(p^2z) - \{(p+p^c) - (p^a+p^b)pz\}X(pz) + p(1-z)X(x) = 0$$

Two independent solutions

$${}_2\phi_1\begin{pmatrix} p^a \ p^b \\ p^c \end{pmatrix}; p, z \Bigg) \quad \text{and} \quad z^{1-c}{}_2\phi_1\begin{pmatrix} p^{a-c+1} \ p^{b-c+1} \\ p^{2-c} \end{pmatrix}; p, z \Bigg)$$

The connection formula (Mimachi '89):

$$\begin{aligned}
&{}_2\phi_1\left(\begin{matrix} p^a \ p^b \\ p^c \end{matrix}; p, \frac{1}{z}\right) \\
&= \frac{\Gamma_p(c)\Gamma_p(b-a)\Theta_p(p^{1-a}z)}{\Gamma_p(b)\Gamma_p(c-a)\Theta_p(pz)}{}_2\phi_1\left(\begin{matrix} p^a \ p^{a-c+1} \\ p^{a-b+1} \end{matrix}; p, p^{c-a-b+1}z\right) \\
&+\frac{\Gamma_p(c)\Gamma_p(a-b)\Theta_p(p^{1-b}z)}{\Gamma_p(a)\Gamma_p(c-b)\Theta_p(pz)}{}_2\phi_1\left(\begin{matrix} p^b \ p^{b-c+1} \\ p^{b-a+1} \end{matrix}; p, p^{c-a-b+1}z\right)
\end{aligned}$$

$$\Theta_p(z) = (z;p)_\infty (p/z;p)_\infty (p;p)_\infty,$$

$$(z;p)_\infty = \prod_{n=0}^{\infty}(1-zp^n)$$

Summary

- $(\pi_V \otimes \pi_W)\mathcal{R}(z,\lambda)$ is elliptic !

Application to Elliptic Lattice Models

- Face type: $\mathcal{B}_{q,\lambda}(\mathfrak{g})$

– The elliptic Drinfeld currents: $U_{q,p}(\mathfrak{g})$ (K '98, JKOS '99)

◎ Free field realization of the space of states

$\mathcal{H}_{a,m}$ ($\cong$ irr. q-$W(\bar{\mathfrak{g}})$ module) and VOs $\Phi(z), \Psi(z)$: $\widehat{\mathfrak{sl}}_N$

◎ Correlation func. (or Form factor) (Lukyanov-Pugai '96; Asai et al. '96; Kojima-K '03)

· $\mathrm{tr}_{\mathcal{H}_{a,m}}(q^{H_{a,m}}\Phi(z_1)\cdots\Psi(w_1)\cdots)$

$\rightsquigarrow$ Continuum limit ?

· Affine Toda (with imaginary coupling ?)

- Vertex type: $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_N)$

– No satisfactory realization.

– But the Vertex-Face correspondence allows us to realize the Vertex model in terms of the face model

◎ Correlation func. (or Form factor)

$\widehat{\mathfrak{sl}}_2$ case: Lashkevich-Pugai '98, Kojima-K-Weston '05

$\rightsquigarrow$ Continuum limit ?

· Sine-Gordon, Super Sine-Gordon, Fractional Super Sine-Gordon, $\cdots$