GAUGING THE DEFORMED WZW MODEL

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LIE GROUPS IN A DUAL LANGUAGE

A Lie group B can be viewed as the commutative Hopf algebra $Fun(B)$ equipped with the coproduct $\Delta : Fun(B) \to Fun(B) \otimes Fun(B)$, the counit $\varepsilon : Fun(B) \to \mathbf{R}$ and the antipode $S : Fun(B) \to Fun(B)$. We have

$$S(x)(b) = x(b^{-1}), \quad x \in Fun(B), b \in B,$$

$$\varepsilon(x) = x(e_B),$$

where e_B is the unit element of B. The coproduct is often written as

$$\Delta x = \sum_\alpha x'_\alpha \otimes x''_\alpha \equiv x' \otimes x'',$$

where x's are in $Fun(B)$ and

$$x'(b_1)x''(b_2) = x(b_1 b_2), \quad b_1, b_2 \in B.$$

The Lie algebra $Lie(B)$ is the space of ε-derivations of $Fun(B)$:

$$Lie(B) = \{t : Fun(B) \to \mathbf{R}, t(xy) = \varepsilon(x)t(y) + t(x)\varepsilon(y)\},$$

where $x, y \in Fun(B)$. The commutator $[,.]$ of $Lie(B)$ is defined as

$$[t_1, t_2](x) = t_1(x')t_2(x'') - t_1(x'')t_2(x').$$

POISSON-LIE GROUPS

Let $(M, \{.,.\}_M)$ and $(N, \{.,.\}_N)$ be Poisson manifolds. The direct product manifold $M \times N$ can be naturally equipped with the so-called product Poisson bracket $\{.,.\}_{M\times N}$ which is fully determined by two conditions:

1) Both $Fun(M) \otimes 1$ and $1 \otimes Fun(N)$ are Lie subalgebras of $Fun(M \times N)$;

2)$Fun(M) \otimes 1$ commutes with $1 \otimes Fun(N)$ in $Fun(M \times N)$.

A smooth map $\varphi : M \to N$ is called Poisson, if it preserves the Poisson brackets, i.e. if

$$\{\varphi^* f_1, \varphi^* f_2\}_M = \varphi^* \{f_1, f_2\}_N, \quad f_1, f_2 \in Fun(N).$$

A Lie group B equipped with a Poisson bracket $\{.,.\}_B$ is called the Poisson-Lie group if the group multiplication $B \times B \to B$ is the Poisson map. Equivalently, in the dual language, this means

$$\Delta\{x, y\}_B = \{x', y'\}_B \otimes x'' y'' + x' y' \otimes \{x'', y''\}_B, \quad x, y \in Fun(B).$$

THE DRINFELD DOUBLE

Let D be an even-dimensional Lie group equipped with a maximally Lorentzian biinvariant metric. If $Lie(D) = Lie(G) \dot{+} Lie(B)$, where G and B are null subgroups, D is called the Drinfeld double of G or the Drinfeld double of B.

Although the word "Poisson" does not appear in the definition of D, the structure of the Drinfeld double naturally permits to construct many examples of Poisson-Lie groups and of Poisson manifolds. Indeed, let t_i form a basis of $Lie(B)$. We can choose the dual basis T^i of $Lie(G)$ such that

$$(t_i, T^j)_D = \delta_i^j,$$

where the non-degenerate Ad-invariant inner product $(.,.)_D$ in $Lie(D)$ is given by the metric tensor at the unit element of D. Then the following expression defines the Poisson-Lie bracket on D:

$$\{f_1, f_2\}_D = \bigtriangledown^L_{T^i} f_1 \bigtriangledown^L_{t_i} f_2 - \bigtriangledown^R_{T^i} f_1 \bigtriangledown^R_{t_i} f_2, \quad f_1, f_2 \in Fun(D).$$

Note the definition of the objects $\bigtriangledown^L, \bigtriangledown^R$, e.g.:

$$\bigtriangledown^L_{T^i} f = T^i(f')f'', \quad \bigtriangledown^R_{t_i} f = t_i(f'')f'.$$

Because of the property of ε-derivation of t_i, T^i, the $\bigtriangledown^L, \bigtriangledown^R$ are differential operators verifying the Leibniz rule.

POISSON-LIE SUBGROUPS

Let $G, \{.,.\}_G$ be a Poisson-Lie group and H its subgroup. Denote I_H the ideal of $Fun(G)$ consisting of functions vanishing on H. If I_H is the Poisson ideal, i.e. if

$$\{I_H, Fun(G)\}_G \subset I_H,$$

the Poisson bracket $\{.,.\}_G$ naturally descends to a Poisson bracket $\{.,\}_H$ on $Fun(G)/I_H \equiv Fun(H)$. It turns out that $\{.,.\}_H$ is in fact the Poisson-Lie bracket on H.

Example: Both null (or isotropic) subgroups G and B are the Poisson-Lie subgroups of the Drinfeld double $(D, \{.,.\}_D)$. Actually, the Poisson-Lie groups G and B equipped with the respective induced Poisson-Lie brackets $\{.,.\}_G$ and $\{,.\}_B$ are called mutually dual to each other.

Note: Modulo a subtle issue of factoring by a discrete subgroup, the Drinfeld double $(D, \{.,.\}_D)$ can be uniquely reconstructed from $(G, \{.,.\}_G)$. Thus, given a Poisson-Lie group $(G, \{.,.\}_G)$, we can find its (unique) double $(D, \{.,.\}_D)$ and, hence, its (unique) dual Poisson-Lie group $(B, \{.,.\}_B)$. If H is the Poisson-Lie subgroup of G, its dual Poisson-Lie group C is the factor group B/N, where N is a normal subgroup of B.

POISSON-LIE SYMMETRY

<u>G-definition</u>: Let (M, ω_M) be a symplectic manifold and denote $\{.,.\}_M$ the Poisson bracket obtained by the inversion of the symplectic form ω_M. Let $(G, \{.,.\}_G)$ be a Poisson-Lie group acting on M. We say that M is Poisson-Lie symmetric if the action map $G \times M \to M$ is Poisson.

<u>B-definition</u>: Let (M, ω_M) be a symplectic manifold and denote $\{.,.\}_M$ the Poisson bracket obtained by the inversion of the symplectic form ω_M. Let $(B, \{.,.\}_B)$ be a Poisson-Lie group and $\mu : M \to B$ be a smooth map. To every function $x \in Fun(B)$ we can associate a vector field $w_x \in Vect(M)$ acting on functions on M as follows:

$$w_x f = \{f, \mu^* x'\}_M \mu^* S(x'').$$

We say that μ realizes the Poisson-Lie symmetry of M if the map $w : Fun(B) \to Vect(M)$ is homomorphism of Lie algebras.

The B-definition (based on $\{.,.\}_B$) is not quite equivalent to the G-definition (based on $\{.,.\}_G$) due to global topological effects. For instance, starting from the B-definition, one can show that the image of the map w in $Vect(M)$ is isomorphic to $Lie(G)$, but we cannot conclude that the $Lie(G)$-action on M can be lifted to the G-action. On the other hand, starting from the G-definition, the global topology of M may prevent the existence of the (moment) map μ.

SYMPLECTIC REDUCTION

GENERALITIES

The symplectic reduction is the method of construction of a new symplectic manifold R starting from the old one M. It works as follows: First we note that $Fun(M)$ is the Poisson algebra, i.e. the Lie algebra compatible with the commutative point-wise multiplication in $Fun(M)$. By the compatibility is meant the Leibniz rule:

$$\{f, gh\}_M = \{f, g\}_M h + \{f, h\}_M g, \quad f, g, h \in Fun(M).$$

Let J be a multiplicative ideal of the algebra $Fun(M)$ which is also the Poisson subalgebra of $Fun(M)$, i.e. $\{J, J\}_M \subset J$. We can now construct a new Poisson algebra $\tilde{A}$ defined as follows

$$\tilde{A} = \{f \in Fun(M);\ \{f, J\}_M \in J\}.$$

By construction, J is not only the multiplicative ideal of $\tilde{A}$ but it is also the Poisson ideal, i.e. $\{\tilde{A}, J\}_M \subset J$. Obviously, the factor algebra $A_R \equiv \tilde{A}/J$ inherits the Poisson bracket from $\tilde{A}$ hence it becomes itself the Poisson algebra. In "good" cases, the algebra A_R can be identified with a Poisson algebra $Fun(R)$ of functions on a (so called reduced) symplectic manifold R.

SYMPLECTIC REDUCTION
GAUGING THE POISSON-LIE SYMMETRY

The symplectic reduction is often put in relation with the Poisson-Lie actions of Lie groups on the symplectic manifold M and with the corresponding moment maps $\mu : M \to B$. In this context, the symplectic reduction is often referred to as gauging the Poisson-Lie symmetry.

The fact that the group multiplication $B \times B \to B$ is the Poisson map implies that the kernel of the counit $Ker(\varepsilon)$ is the Poisson subalgebra of $(Fun(B), \{.,.\}_B)$. Suppose that the moment map μ is also Poisson, the pullback $\mu^*(Ker(\varepsilon))$ is therefore the Poisson subalgebra of $(Fun(M), \{.,.\}_M)$. Thus the role of the ideal J from the general definition of the symplectic reduction is played by the ideal of $Fun(M)$ generated by $\mu^*(Ker(\varepsilon))$.

Let us suppose that the set P of points of M mapped by μ to the unit element e_B of B forms a smooth submanifold of M. It turns out that the action of the symmetry group G (locally induced by the moment map μ) leaves P invariant. Let us moreover suppose that the G-action on P is free. The basis P/G of this G-fibration can be then identified with the reduced symplectic manifold R.

If the moment map μ is not Poisson, the Poisson-Lie symmetry cannot be gauged and it is therefore called anomalous.

TWISTED HEISENBERG DOUBLE

DEFINITION

Consider a metric preserving outer automorphism κ of the Drinfeld double D and suppose that D is κ-decomposable, i.e. for every element $K \in D$ it exists a unique $g \in G$ and a unique $b \in B$ such that $K = \kappa(b)g^{-1}$ and a unique $\tilde{g} \in G$ and a unique $\tilde{b} \in B$ such that $K = \kappa(\tilde{g})\tilde{b}^{-1}$.

Denote $\Lambda_{L,R} : D \to B$, $\Xi_{L,R} \to G$ the maps defined by the decompositions above, i.e.

$$\Lambda_L(K) = b, \quad \Lambda_R(K) = \tilde{b}, \quad \Xi_R(K) = g, \quad \Xi_L(K) = \tilde{g}.$$

THEOREM: Let D be a decomposable Drinfeld double and $T^i \in Lie(G)$ the dual basis of $t_i \in Lie(B)$. Then

1) The (basis independent) expression

$$\{f_1, f_2\}_H \equiv \nabla^R_{T^i} f_1 \nabla^R_{t_i} f_2 - \nabla^L_{\kappa(t_i)} f_1 \nabla^L_{\kappa(T^i)} f_2, \quad f_1, f_2 \in Fun(D)$$

is Poisson bracket defining a symplectic structure on D.

2) The twisted left action of G on D: $g \triangleright K = \kappa(g)K$ is the Poisson-Lie symmetry whose moment map is Λ_L.

3) The right action of G on D: $g \triangleright K = Kg^{-1}$ is the Poisson-Lie symmetry whose moment map is Λ_R.

DEFINITION: The pair $(D, \{.,.\}_H)$ is called the twisted Heisenberg double.

TWISTED HEISENBERG DOUBLE

VECTOR GAUGING

Let H be a Poisson-Lie subgroup of G and $C = \rho(B)$ its dual Poisson-Lie group. Suppose that $\kappa(B) = B$ and consider two actions $H \times D \to D$:

$$(1) \qquad h \triangleright K = \kappa[h]K\Xi_R(\kappa[h\Lambda_L(K)]), \quad h \in H,\ K \in D,$$

$$(2) \quad h \triangleright K = \kappa[\Xi_L^{-1}(\Lambda_R^{-1}(K)h^{-1})]Kh^{-1}, \quad h \in H,\ K \in D.$$

It is easy to verify that, in both cases, it holds:

$$(h_1 h_2) \triangleright K = h_1 \triangleright (h_2 \triangleright K).$$

THEOREM: Both actions above are Poisson-Lie symmetries of $(D, \{.,.\}_H)$. Their moment maps $\mu_{1,2} : D \to C$ are non-anomalous and they are given, respectively, by

$$\mu_1(K) = \rho\Big(\kappa[\Lambda_L(K)]\Lambda_R(K)\Big), \quad \mu_2(K) = \rho\Big(\kappa^{-1}[\Lambda_R(K)]\Lambda_L(K)\Big).$$

The theorem implies that the actions (1) and (2) can be gauged. The corresponding reduced symplectic manifold can be called the gauged (twisted) Heisenberg double. Note also a special case when B is Abelian group. The actions (1) and (2) then coincide and they are both given by a much simpler formula:

$$h \triangleright K = \kappa[h]Kh^{-1}.$$

WZW MODEL

The phase space of the standard WZW model is a particular twisted Heisenberg double D. The group structure on D reads

$$(\chi, g).(\tilde{\chi}, \tilde{g}) = (\chi + Ad_g\tilde{\chi}, g\tilde{g}),$$
$$(\chi, g)^{-1} = (-Ad_{g^{-1}}\chi, g^{-1}),$$

where g is an element of a loop group LG and χ an element of $Lie(LG)$.

The Lie algebra $Lie(D)$ consists of pairs of elements of $Lie(LG)$ with the following commutator

$$[\phi \oplus \alpha, \psi \oplus \beta] = ([\phi, \beta] + [\alpha, \psi], [\alpha, \beta]).$$

The bi-invariant metric on D comes from Ad-invariant bilinear form $(.,.)_{\mathcal{D}}$ on $Lie(D)$

$$(\phi \oplus \alpha, \psi \oplus \beta)_{\mathcal{D}} = (\phi|\beta) + (\psi|\alpha),$$

where

$$(\alpha|\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\sigma \mathrm{Tr}(\alpha(\sigma)\beta(\sigma)),$$

The metric preserving automorphism κ of the group D reads

$$\kappa(\chi, g) = (\chi + k\partial_\sigma g g^{-1}, g),$$

where k is an (integer) parameter. The null Poisson-Lie subgroups are

$$G = \{(\chi, g) \in D; \chi = 0\},$$
$$B = \{(\chi, g) \in D; g = e\}.$$

GAUGED WZW MODEL

Every subgroup LH of LG is automatically Poisson-Lie subgroup because B is Abelian. The dual Poisson-Lie group C to LH can be identified with $Lie(LH)$ whose (Abelian) group structure is given by the addition of vectors. The actions (1) and (2) then coincide and they are both given by a simple formula:

$$h \triangleright K = \kappa[h]Kh^{-1}, \quad h \in LH.$$

The moment maps μ_1 and μ_2 also coincide:

$$\mu_{1,2}(g,\chi) = P_H(J_L(g,\chi) + J_R(g,\chi)),$$

where P_H is the orthogonal projector on $Lie(LH)$ and the standard Kac-Moody currents are given by:

$$J_L(g,\chi) = \chi, \quad J_R(g,\chi) = -Ad_{g^{-1}}\chi + kg^{-1}\partial_\sigma g.$$

Fix two elements α, β of $Lie(LH)$ and calculate:

$$\{(J_L|\alpha),(J_L|\beta)\}_H = (J_L|[\alpha,\beta]) + k(\alpha,\partial_\sigma\beta),$$

$$\{(J_R|\alpha),(J_R|\beta)\}_H = (J_R|[\alpha,\beta]) - k(\alpha,\partial_\sigma\beta),$$

$$\{(\mu_1|\alpha),\mu_1|\beta)\}_H = (\mu_1|[\alpha,\beta]).$$

We observe that the Poisson brackets of the moment map μ_1 are indeed non-anomalous, therefore the moment map μ_1 can serve as the basis for the symplectic reduction. The reduced symplectic structure is that of the gauged WZW model.

u-DEFORMED WZW MODEL

The structure of the twisted Heisenberg double D of the u-deformed WZW model is the same as that of the standard WZW model except for the definition of the null subgroup B. Let $\mathcal{T}$ be the Cartan subalgebra of $Lie(G)$ and denote $P_{\mathcal{T}}$ the projector from $Lie(G)$ to $\mathcal{T}$, orthogonal with respect to the scalar product $(.|.)$. Let $U : \mathcal{T} \to \mathcal{T}$ be a linear operator, skew-symmetric with respect to $(.|.)$. Define $u = U \circ P_{\mathcal{T}}$. Then

$$B = \{(\chi, g) \in D; g = e^{u(\chi)}\}.$$

The non-Abelian modification of B results in the modification of the symplectic structure. In particular, the u-deformed symplectic form becomes

$$\omega_u = \frac{1}{2}(dJ_L \wedge|dgg^{-1}) - \frac{1}{2}(dJ_R \wedge|g^{-1}dg) + \frac{1}{2}(u(dJ_L)\wedge|dJ_L) + \frac{1}{2}(u(dJ_R)\wedge|dJ_R).$$

Thus e.g. the brackets of the Kac-Moody currents change correspondingly:

$$\{J_L^{\alpha,m}, J_L^{\beta,n}\}_H = c^{\alpha\beta} J_L^{\alpha+\beta,m+n} - <\alpha, U(H^\mu)><\beta, H^\mu> J_L^{\alpha,m} J_L^{\beta,n},$$

where H^μ form an orthonormal basis of $\mathcal{T}$, $c^{\alpha\beta}$ are the structure constants in $[E^\alpha, E^\beta] = c^{\alpha\beta} E^{\alpha+\beta}$ and

$$J_L^{\alpha,m} = (J_L|E^\alpha e^{im\sigma}).$$

Remind that the u-deformed WZW model is Poisson-Lie symmetric with respect to the twisted left and ordinary right action of LG.

GAUGED u-WZW MODEL

In the presence of the u-deformation, the Poisson-Lie bracket on LG does not vanish and a subgroup LS of LG is not necessarily Poisson-Lie subgroup. However, define a set

$$N = \{(\chi, g) \in D; g = e^{u(\chi)}, \chi \in (Lie(LS))^{\perp}\}.$$

It turns out that if u is such that N is a normal subgroup of B, then LS is the Poisson-Lie subgroup of LG and $B/N = C$ is its dual Poisson-Lie group. In what follows, we suppose that this is the case.

The actions (1) and (2) of LS on D do not coincide, nevertheless their gaugings produce the same gauged u-deformed WZW model. Thus, for concreteness, we make explicit only the action (1). It reads

$$s \triangleright (\chi, g) = (s\chi s^{-1} + k\partial_\sigma s s^{-1}, sgs_L^{-1}),$$

where

$$s_L = e^{-u(s\chi s^{-1} + \kappa\partial s s^{-1})} s e^{u(\chi)}, \;\; s \in LS.$$

It turns out, that (modulo the Cartan subalgebra current modes) the phase of the gauged u-WZW model can be obtained by imposing the constraints $P_S J_L = P_S J_R = 0$ on the non-gauged phase space. The reduced symplectic form is simply the pull-back of the non-reduced one to the submanifold determined by the constraints.