EXOTIC GALILEAN SYMMETRY AND NON-COMMUTATIVE MECHANICS, IN MATHEMATICAL & IN CONDENSED MATTER PHYSICS

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MATHEMATICAL PHYSICS

1972 Lévy-Leblond Group theory : 2-parameter "exotic" central extension of planar Galilei group. (No physical applications.)

1995-2000 Grigore, Brihaye, Duval, Lukierski : planar models with "exotic" Galilean symmetry

1995 Duval, Duval, PAH 2000 : for critical value of the magnetic field, only allowed motions follow Hall law ⇝ Laughlin wave functions of FQHE. Noncommutative mechanics.

CONDENSED MATTER PHYSICS

1995-2000 Niu et al. Berry term in dynamics of semiclassical Bloch electron

2003 Fang et al Anomalous Hall Effect (AHE) monopole in momentum space

2003 S.-C. Zhang Spin-Hall Effect

2004 Onoda et al Optical Hall Effect

EXOTIC MECHANICS IN THE PLANE

Lévy-Lebond 1972 : planar rotations commute $\rightsquigarrow$ "exotic" central extension of planar Galilei group. Extension parameters m, κ.

Boosts don't commute

$$[\mathcal{G}_1, \mathcal{G}_2] = i\kappa. \tag{1}$$

Duval, Grigore 1995 : mechanical model with exotic symmetry. Construction from first principles (∼ Souriau : coadjoint orbits). Free Hamiltonian structure

$$\Omega_0 = dp_i \wedge dx^i + \boxed{\tfrac{1}{2}\theta\, \varepsilon^{ij}\, dp_i \wedge dp_j}, \tag{2}$$

$$H_0 = \frac{\mathbf{p}^2}{2m}. \tag{3}$$

$\theta = \kappa/m^2$ noncommutative parameter :

$$\{x^1, x^2\} = \theta \equiv \frac{\kappa}{m^2}. \tag{4}$$

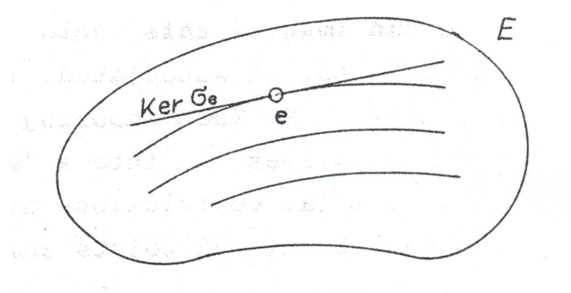
[Lukierski 1996 : acceleration-dependent model].

Souriau's framework : evolution space $(\mathbf{p}, \mathbf{q}, t)$ carries closed ($d\sigma = 0$) 2-form

$$\sigma = \Omega - dH \wedge dt. \tag{5}$$

Motion $\sim$ Ker σ. (2)-(3) $\Leftrightarrow$

$$\sigma_0 = dp_i \wedge dx^i + \tfrac{1}{2}\theta\, \varepsilon_{ij}\, dp^i \wedge dp^j - dH_0 \wedge dt \tag{6}$$



Motions : usual free. Exotic contribution only in conserved quantities

$$j = \mathbf{x} \times \mathbf{p} + \frac{\theta}{2}\mathbf{p}^2 \qquad \text{ang. mom.}$$
$$K_i = m x_i - p_i t + m\theta\epsilon_{ij}p_j \qquad \text{boosts} \tag{7}$$

separately conserved ($\sim$ SPIN !);

Coupling to gauge field Souriau 1970

$$\sigma \to \sigma_0 + eF \tag{8}$$

F electromagnetic field. $dF = 0$.

$$d\sigma = 0 \Leftrightarrow \text{JACOBI IDENTITY} \tag{9}$$

[Equivalent to rule $\mathbf{p} \to \mathbf{p} - e\mathbf{A}$ only e if $\theta = 0$]

Eqns of motion :

$$\begin{aligned} \boxed{m^*}\,\dot{x}_i &= p_i - \boxed{em\theta\,\varepsilon_{ij}E_j} \\ \dot{p}_i &= eE_i + eB\,\varepsilon_{ij}\dot{x}_j \end{aligned} \qquad (10)$$

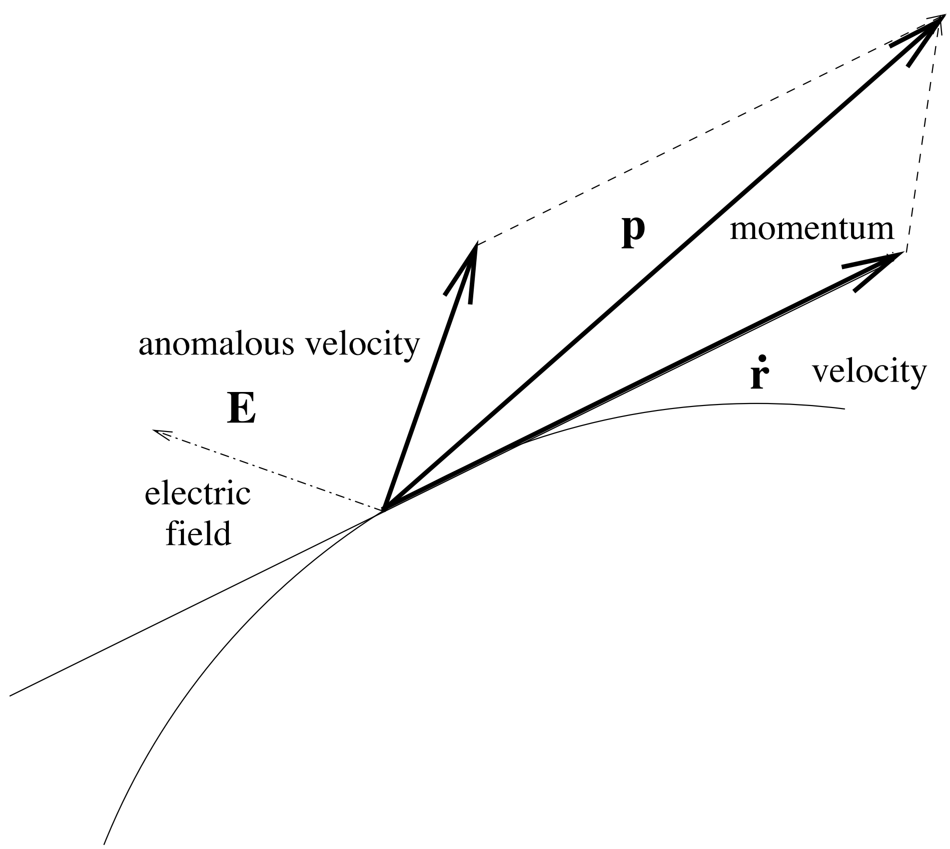
where

$$m^* = m(1 - e\theta B) \qquad (11)$$

field-dependent effective mass.

Velocity, $\dot{x}_i$, and momentum, p_i, not parallel !

Differ in anomalous velocity term $em\theta\,\varepsilon_{ij}E_j$



Evolution-space framework can be viewed as first-order variational calculus $\rightsquigarrow$ (10) comes from "exotic" phase-space Lagrangian

$$\int (\mathbf{p} - \mathbf{A}) \cdot \frac{d\mathbf{x}}{dt} - \left(\frac{p^2}{2m} + eV\right) + \frac{\theta}{2}\mathbf{p} \times \frac{d\mathbf{p}}{dt}. \qquad (12)$$

• If system regular,

$$m^* \neq 0 \;\Leftrightarrow\; \det \sigma \neq 0 \;\Leftrightarrow\; 1 - e\theta B \neq 0. \qquad (13)$$

$\Rightarrow$ also Hamiltonian, $\dot{\xi} = \{h, \xi^\alpha\}$ $[\xi = (p_i, x^j)]$
$h = \mathbf{p}^2/2m + eV$, Poisson brackets

$$\begin{aligned} \{x^1, x^2\} &= \frac{\theta}{1 - e\theta B}, \\ \{x^i, p_j\} &= \frac{\theta}{1 - e\theta B}\delta^i{}_j, \\ \{p_1, p_2\} &= \frac{\theta}{1 - e\theta B}eB. \end{aligned} \qquad (14)$$

Anomalous velocity from $\big\{V, x_i\big\} = \frac{\partial V}{\partial x_j}\big\{x_j, x_i\big\}$.

CRITICAL CASE

For vanishing effective mass $m^* = 0$ i.e. for

$$B = \frac{1}{e\theta} \tag{15}$$

exotic system becomes singular. Faddeev-Jackiw reduction yields 2-dimensional, simple system. Reduced model $\sim$ guiding center coordinates in Landau problem

$$Q_i = x_i + q_i = x_i - \frac{mE_i}{eB^2}. \tag{16}$$

Reduced Poisson bracket & energy

$$\{Q_1, Q_2\}_{red} = \frac{1}{eB}, \tag{17}$$

$$H_{red} = eV(Q_1, Q_2) + \theta^2 e^2 m \mathbf{E}^2/2. \tag{18}$$

(Peierls '1933, Dunne et al. '1991 electron in strong magnetic field.
Onsager'49 : same model to describe vortex dynamics in thin film of superfluid ^{4}He.

QHE explained by vortex motion !

• motions determined by (generalized) Hall law

$$\dot{Q} = \epsilon_{ij}\frac{E_j}{B}. \qquad (19)$$

"condensation into collective ground state"
(Laughlin '1983 for FQHE !)

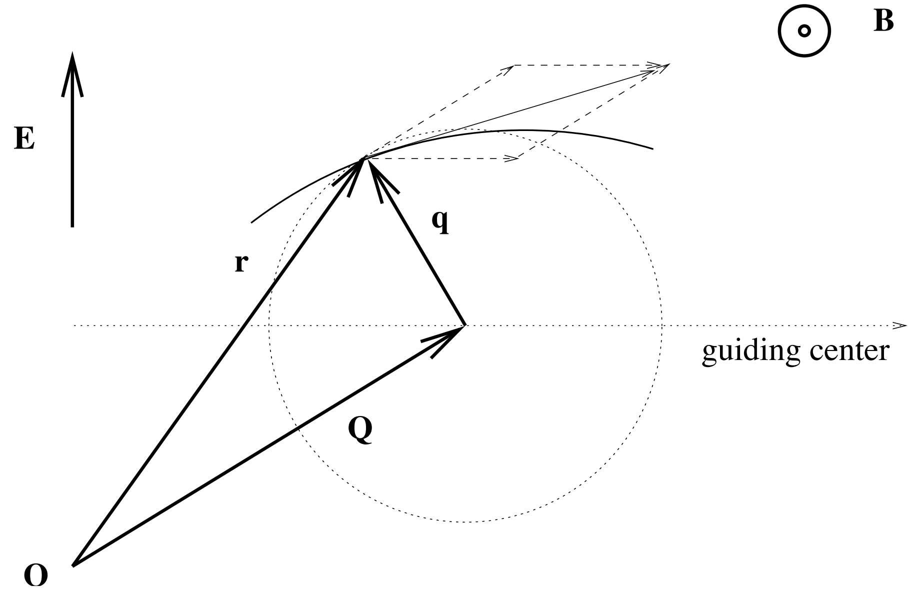
w_∞ symmetry of ground states of FQHE (1992)

• Quantization of reduced model yields
Laughlin wave functions:

$$\psi(z,\bar{z}) = e^{-z\bar{z}/4} f(z), \qquad (20)$$

where $z = \sqrt{eB}(Q_1 + iQ_2)$, $f(z)$ holomorphic.
Ground states Fractional Quantum Hall Effect.

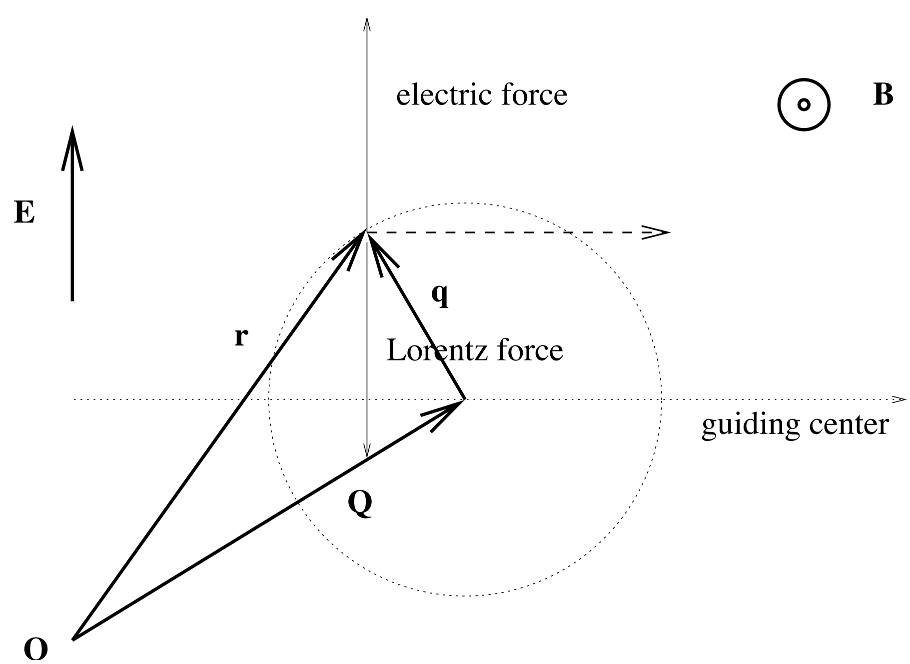
• constant fields $\mathbf{E} = const$, $B = const$.
cyclotronic motion around guiding center.



critical case $e\theta B = 1$: electric force canceled by Lorentz force.

$$e\dot{\mathbf{r}} \times \mathbf{B} = e\mathbf{E} \quad \Rightarrow \quad \dot{x}^i = \epsilon^{ij} \frac{E_j}{B}. \qquad (21)$$

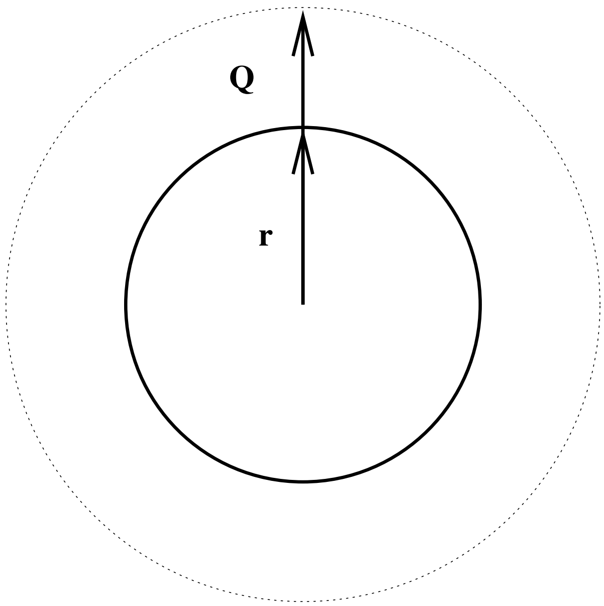
$\mathbf{q} = -m\mathbf{E}/2B^2 =$ const, $\dot{\mathbf{r}} = \dot{\mathbf{Q}}$ follows guiding center



• exotic oscillator $\mathbf{E} = -\omega^2 \mathbf{r}$ ($e = 1$). General motions whirling around elliptical orbits.

critical case $\theta B = 1$: $\mathbf{Q} = (1 + \theta^2 \omega^2)\mathbf{r}$. Circular motions with "Hall" angular velocity

$$\Omega = \frac{\omega^2 B}{B^2 + \omega^2} \tag{22}$$



NOT "electric force compensated by Lorentz" : non-newtonian dynamics ! $m\ddot{\mathbf{r}} =$ (force)+(terms)
Red. energy $\propto$ red. ang. momentum

$$H_{red} = \frac{\omega^2}{2}\Big(1 + \omega^2 \theta^2\Big) Q^2 \ \propto \ \mathrm{I}_{red} = \frac{B}{2} Q^2 \tag{23}$$

Spectrum :

$$E_n = \frac{\omega^2 \theta}{1 + \theta^2 \omega^2}\Big(\tfrac{1}{2} + n\Big) \qquad n = 0, 1, \ldots \tag{24}$$

SEMICLASSICAL BLOCH ELECTRON

Around SAME TIME, INDEPENDENTLY, similar theory in solid state physics.
1995-2000 Niu et al. Bloch electron in semiclassical model. Bloch wave function

$$\psi_{n,\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{n,\mathbf{k}}(\mathbf{r})$$

$u_{n,\mathbf{k}}(\mathbf{r})$ periodic. $\mathbf{k}$ crystal (quasi)momentum.

$$\mathcal{A}_j = i\Big\langle u_{n,\mathbf{k}}\Big|\frac{\partial u_{n,\mathbf{k}}}{\partial k_j}\Big\rangle \tag{25}$$

Berry connection. Berry curvature

$$\Theta_i(\mathbf{k}) = \epsilon_{ijl}\partial_{\mathbf{k}_j}\mathcal{A}_l(\mathbf{k}). \tag{26}$$

is (purely momentum-dependent). Yields extra term in equations of motion in n^{th} band

$$\dot{\mathbf{r}} = \frac{\partial \epsilon_n(\mathbf{k})}{\partial \mathbf{k}} - \boxed{\dot{\mathbf{k}} \times \vec{\Theta}(\mathbf{k}),} \tag{27}$$

$$\dot{\mathbf{k}} = -e\mathbf{E} - e\dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r}), \tag{28}$$

where $\mathbf{r}$ = electron's intracell position, $\epsilon_n(\mathbf{k})$ band energy.

Lagrangian for Eqns. (27-28)

$$\Big(k_i - eA_i(\mathbf{r},t)\Big)\dot{x}^i - \Big(\epsilon_n(\mathbf{k}) + eV(\mathbf{r},t)\Big) + \mathcal{A}^i(\mathbf{k})\dot{k}_i \tag{29}$$

- Also consistent with Hamiltonian structure

$$\{x^i, x^j\}^{Bloch} = \frac{\varepsilon^{ijk}\Theta_k}{1 + e\mathbf{B}\cdot\vec{\Theta}}, \tag{30}$$

$$\{x^i, k_j\}^{Bloch} = \frac{\delta^i{}_j + eB^i\Theta_j}{1 + e\mathbf{B}\cdot\vec{\Theta}}, \tag{31}$$

$$\{k_i, k_j\}^{Bloch} = -\frac{\varepsilon_{ijk}eB^k}{1 + e\mathbf{B}\cdot\vec{\Theta}} \tag{32}$$

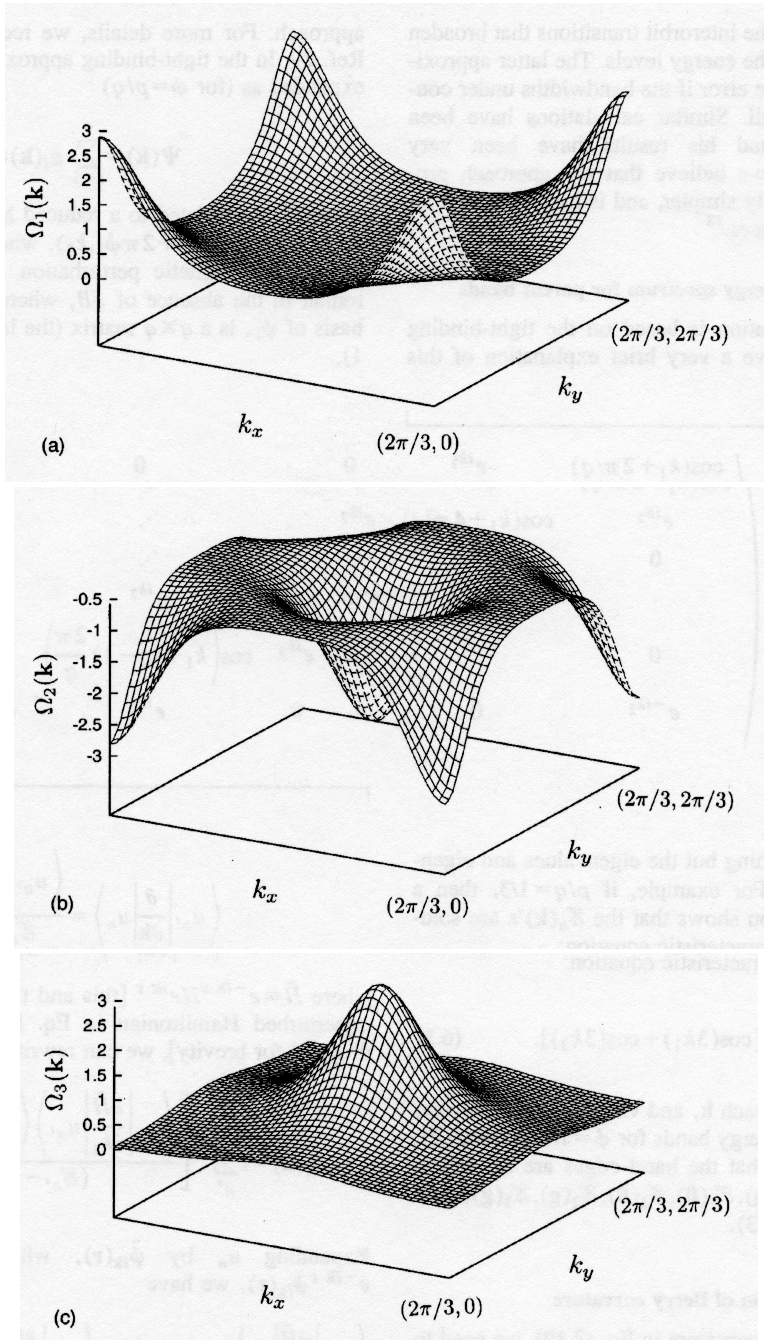
and Hamiltonian $h = \epsilon_n + eV$.

PB satisfies Jacobi identity !

- Restricted to plane, for $\epsilon_n(\mathbf{k}) = \mathbf{k}^2/2m$ and $\mathcal{A}_i = -(\theta/2)\epsilon_{ij}k_j$, eqns reduce to exotic equations (10). $\Theta_i = \theta\delta_{i3}$. Lagrangian (29) becomes "exotic" expression (12).

Critical case only for $\theta(\mathbf{k}) = \theta$ constant. Exotic galilean symmetry lost if θ not constant.

Chang, Niu PRB53,7010 (1006)



For a Bloch electron the Berry curvature is effectively $\mathbf{k}$-dependent

ANOMALOUS VELOCITY in (27)

$$\boxed{\dot{\mathbf{k}} \times \vec{\Theta}(\mathbf{k})} \qquad (33)$$

familiar in CONDENSED MATTER PHYSICS !

Instrumental in explaining:

- ANOMALOUS HALL Effect (AHE) with $B = 0$ in ferromagnetic semiconductors Karplus, Luttinger 1954, . . . Niu et al, Fang et al 2003 !

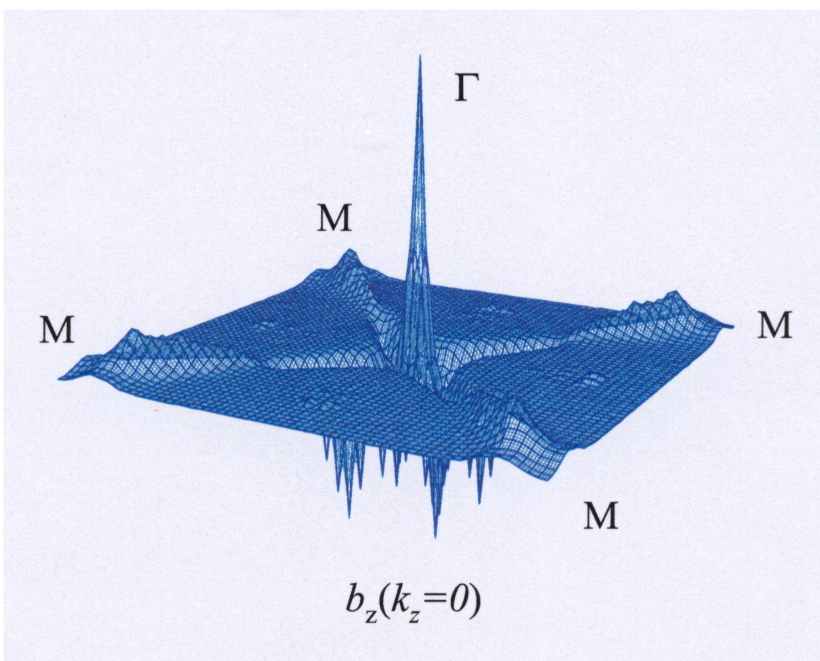
- SPIN-HALL effect ($\rightsquigarrow$ "spintronics") Zhang 2003 Sinova et al. 2004 spin-orbit interaction deviates trajectory.

AHE : MONOPOLE IN MOMENTUM SPACE

Bérard & Mohrbach 2003 $O(3)$ symmetry $\Rightarrow$ "$\mathbf{k}$-monopole"

$$\vec{\Theta} = \theta \frac{\mathbf{k}}{k^3}. \qquad (34)$$

2003 Fang et al. : AHE in ferromagnetic semiconductor $SrRuO_3$. Found numerically, consistently with experimental data, a k-monopole !



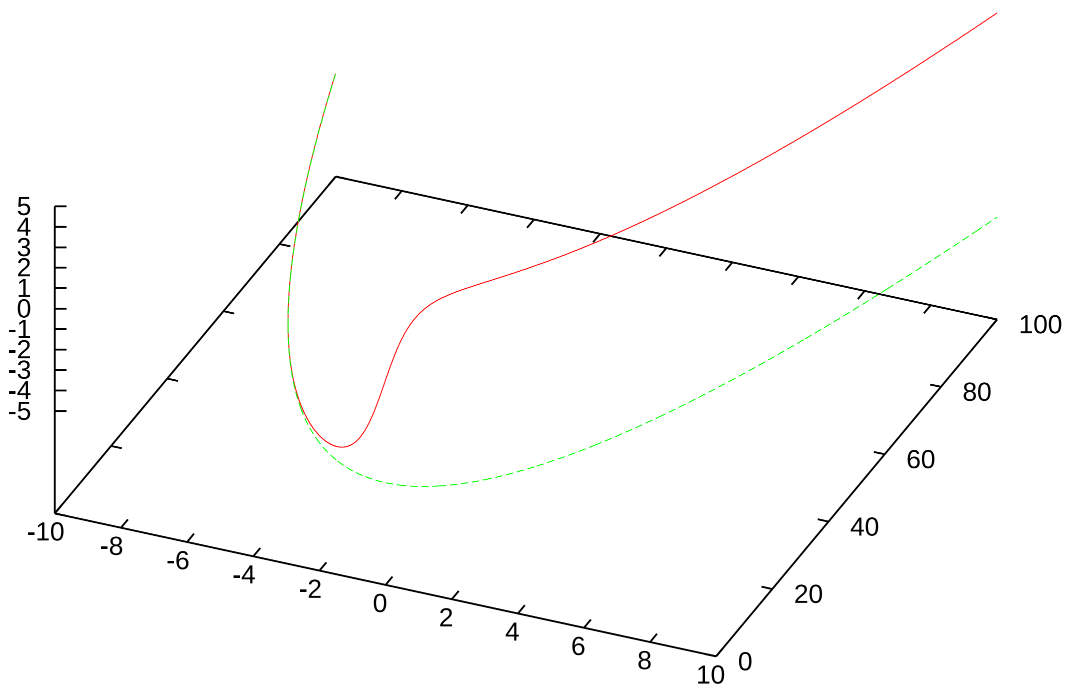
Eqns of motion for $B = 0$ e :

$$\dot{\mathbf{r}} = \frac{\partial \epsilon_n(\mathbf{k})}{\partial \mathbf{k}} - \boxed{\mathbf{E} \times \frac{\mathbf{k}}{k^3},} \qquad (35)$$

$$\dot{\mathbf{k}} = -e\mathbf{E}. \qquad (36)$$

The anomalous velocity deviates the trajectory

from the initial plane by $\Delta = \frac{2\theta}{k_0}$. $k_0 \neq 0$ minimum value of momentum.



Most of the shift comes when the momentum is small, i.e., when the particle passes close to the "monopole".

CONCLUSION

Similar structures arose, independently and around the same time (1995-2000) in Mathematical Physics ("exotic" galilean symmetry) and in Condensed Matter Physics (semiclassical Bloch electron).

Both theories contain the anomalous velocity term, first considered by Luttinger in 1954 and used to explain the Anomalous/Spin/Optical Hall effects.

This may well be the mathematical structure behind all Hall-type effects.