# Matrix Models, integrable systems and Riemann-Hibert methods

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## 1. Review: Hermitian 1-matrix models at finite $N$

**Partition function:**

$$\mathbf{Z}_n(V) := \int_{\mathcal{H}_N} dM\, e^{-N \mathrm{tr} V(M)}$$

The **potential** $V$ usually a real polynomial

$$V(x) := \sum_{a=1}^{d} \frac{t_a}{j} x^j,$$

e.g. **Gaussian Unitary Ensemble (GUE)**

$$V(H) = \alpha H^2$$

**Projection**: Diagonalize

$$H = U\mathrm{diag}(x_1, \dots x_N)U^\dagger$$

and integrate over $U(N)$ (*Weyl integration formula*)

$$\mathbf{Z}_N \propto \int dx_1 \dots \int dx_N \Delta^2(x_1, \dots, x_N) e^{-N\sum_{i=1}^N V(x_i)}$$

$$\Delta(x_1, \dots x_N) := \prod_{i<j}^N (x_i - x_j) \quad \text{(Vandermonde determinant)}$$

**Joint probability density** for eigenvalues

$$\begin{aligned} P_N(x_1, \dots, x_N) &= \frac{1}{\mathbf{Z}_N} \Delta^2(x_1, \dots, x_N) e^{-N\sum_{i=1}^N V(x_i)} \\ &= \frac{1}{\mathbf{Z}_N} e^{\left(\sum_{i\neq j} \ln(x_i - x_j) - N\sum_{i=1}^N V(x_i)\right)} \end{aligned}$$

**Orthogonal polynomials** (Heine integral formula)

$$\begin{aligned} p_N(x) &= \langle \det(x\mathbf{I} - M) \rangle \\ &= \frac{1}{Z_N} \int_\kappa \mathrm{d}x_1 \cdots \int dx_N \prod_{i=1}^N (x - x_i) \Delta^2(\underline{x}) \mathrm{e}^{-N\sum_{j=1}^N V(x_j)} \end{aligned}$$

$$\int p_n(x) p_m(x) e^{-NV(x)} dx = h_n \delta_{nm}$$

**All the statistical properties** of the spectrum are expressible in terms of these **orthogonal polynomials**.

**Partition function**:

$$\mathbf{Z}_N = N! \prod_{j=0}^{N-1} h_j$$

The $k$–**point correlation function**:

$$P_k^N(x_1, \ldots, x_k) = \det K_N(x_i, x_j)|_{1 \leq i,j, \leq k}$$

where

$$K_N(x, y) := \sum_{j=0}^{n-1} \frac{1}{h_j} p_j(x) p_j(y) e^{-\frac{N}{2}(V(x)+V(y))}$$

is the **Christoffel-Darboux kernel**.

The 1**-point function** (density of eigenvalues):

$$\rho_N(x) = P_1^N(x) = K_N(x, x)$$

The **Gap probability**:

$$E_J = \det(\mathbf{I} - \hat{K_N} \circ \chi_J), \quad J = \cup_{k=1}^m [a_{2k-1}, a_{2k}]$$

**Finite $N$ Spectral Curve**

The **recursion** and diffferential (**Freund**) relations for OP's

$$\begin{pmatrix} p_N(x) \\ p_{N+1}(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a_N & x+b_N \end{pmatrix} \begin{pmatrix} p_{N-1}(x) \\ p_N(x) \end{pmatrix}$$

$$\frac{d}{dx} \begin{pmatrix} p_{N-1}(x) \\ p_N(x) \end{pmatrix} = \mathbf{D}_N(x) \begin{pmatrix} p_{N-1}(x) \\ p_N(x) \end{pmatrix}$$

($\mathbf{D}(x) = 2 \times 2$ matrix valued polynomial of deg $= d$)

determine the **characteristic equation**

$$\det(y\mathbf{I} - \mathbf{D}_N(x)) = 0.$$

This defines the **finite $N$ spectral curve** (hyperelliptic), which determines the **moments** of the eigenvalues density

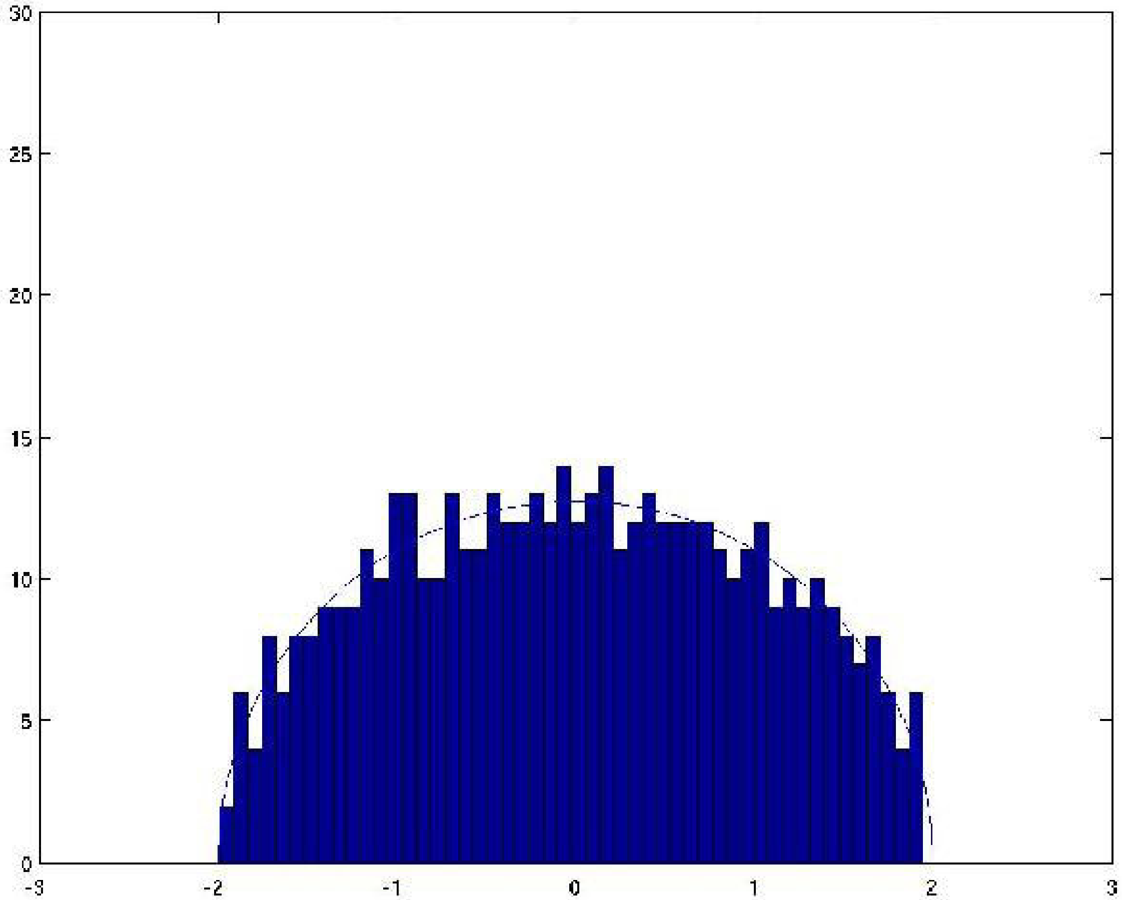
$$\int_{\mathbf{R}} x^j \rho_N(x) dx = \operatorname*{res}_{x=\infty} x^j y dx = -\frac{j}{N^2} \frac{\partial \ln \mathbf{Z}_N}{\partial t_j}$$

$N \to \infty$ (scaled) **continuum limit**

$$\lim_{N \to \infty} \rho_N(x) = \rho_{eq}(x)$$

For GUE: **"Wigner semi-circle law**:

$$\propto \sqrt{N^2 - x^2}$$



In general: minimize the **Free energy:**

$$\begin{aligned}\mathcal{F}_0 := & \quad -\lim_{N\to\infty}\frac{1}{N^2}\ln\mathcal{Z}_N \\ &= \min_{\rho(x)\geq 0}\Big[\int V(x)\rho(x)dx - \int\int \rho(x)\rho(x')\ln|x-x'|\Big]\end{aligned}$$

The **equilibrium density** $\rho_{\rm eq}$ in general is obtained from the variational equation $\delta\mathcal{F}_0 = 0$

$$2\mathcal{P}\int\frac{\rho_{\rm eq}(x)dx}{x-x'} = V'(x)$$

and is supported on a finite union of intervals $I = \cup_{a=1}^{k} I_a$.

It is related to the **resolvent** by

$$\omega(z) := \frac{1}{N}\lim_{n\to\infty}\left\langle \mathrm{tr}\frac{1}{M-z}\right\rangle = \int_I dx\frac{\rho_{\rm eq}(x)}{x-z}\ , \quad z\in\mathbb{C}\setminus I\ .$$

**Asymptotic spectral curve**:

The function $y = -\omega(x)$ satisfies an algebraic relation given by

$$y^2 = yV'(x) + R(x)$$

where $R(x)$ is a polynomial of degree less than $V'(x)$ , and the **moments** of $\rho_{\rm eq}$ are given by:

**Moments and spectral residue formulae:**

$$\int_I dx x^j\rho_{\rm eq}(x) = -\mathop{\rm res}_{z=\infty} z^j\omega(z)dz = j\partial_{t_J}\mathcal{F}_0$$

**Universality**: $k$-point correlation function ("bulk" region)

$$P_k(x_1, \ldots, x_k) \propto \det K_S(x_i, x_j)|_{1\leq i,j,\leq k}$$

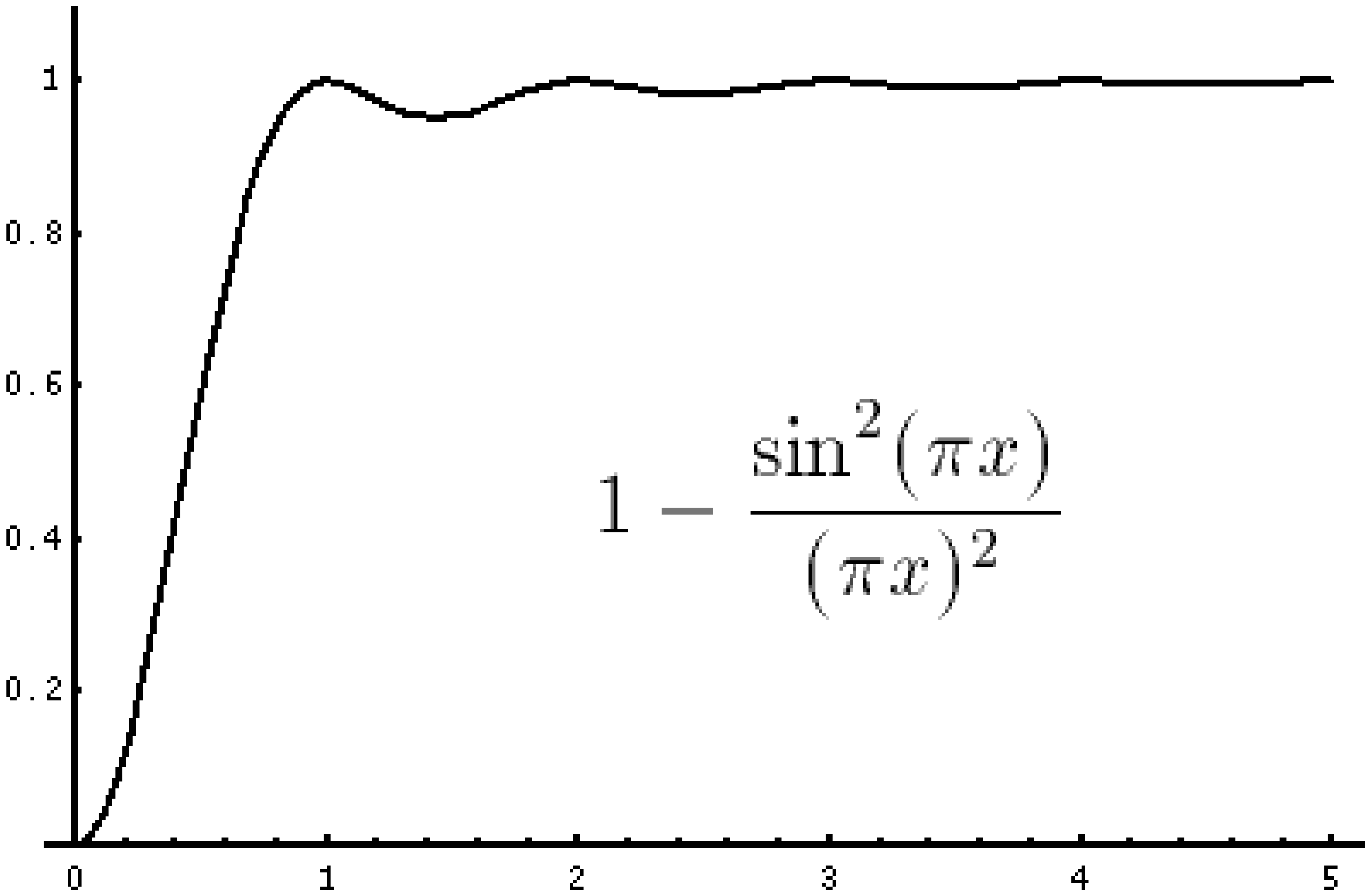
$$K_S(x_i, x_j) := \frac{\sin(\pi(x_i - x_j))}{\pi(x_i - x_j)} \qquad \text{(sine kernel)}$$

Proved via "Riemann-Hilbert" method:

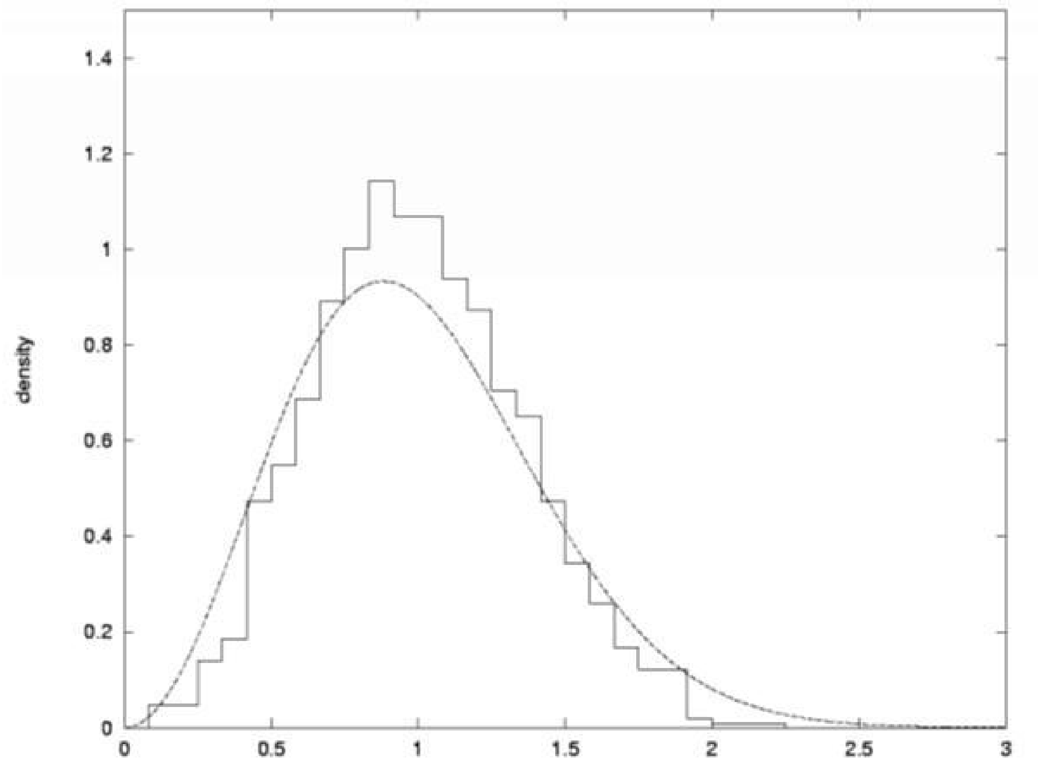
1) Scaled large $N$ asymptotics of OP's

2) "Nonlinear WKB method"

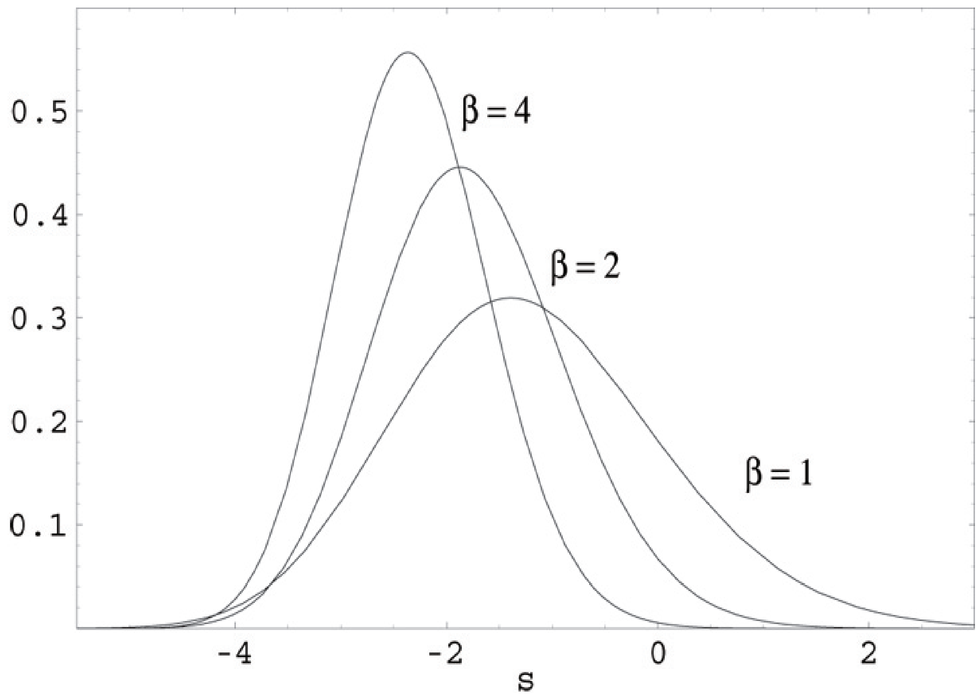
**Pair correlation function (level repulsion)**



**Bulk Spacing distributions** (Jimbo-Miwa $P_{\rm V}$):




**Edge Spacing distributions** (Tracy-Widom $P_{\rm II}$):



## 2. 2-matrix models

**Partition function:**

$$\mathbf{Z}_N(V_1, V_2) = \int\int d\mu(M_1, M_2)$$

$$\propto \int \prod_{i=1}^{N} \mathrm{d}x_i \mathrm{d}y_i \Delta(x)\Delta(y) \mathrm{e}^{-N\sum_{j=1}^{N}(V_1(x_j)+V_2(y_j)-x_j y_j)}$$

$$d\mu := \exp \mathrm{tr}\, (-V_1(M_1) - V_2(M_2) + M_1 M_2) dM_1 dM_2$$

$$V_1(x) = \sum_{a=1}^{d_1+1} \frac{u_a}{a} x^a \ , \qquad V_2(y) = \sum_{b=1}^{d_2+1} \frac{v_b}{b} y^b \ .$$

**Relation to bi-orthogonal polynomials**:

$$\pi_n(x) = x^n + \cdots, \qquad \sigma_n(y) = y^n + \cdots, \qquad n = 0, 1, \ldots$$

$$\int\int \mathrm{d}x\, \mathrm{d}y\ \pi_n(x)\sigma_m(y)\mathrm{e}^{-V_1(x)-V_2(y)+xy} = h_n \delta_{mn},$$

$$\mathbf{Z}_N = N! \prod_{j=0}^{N-1} h_j$$

**Fredholm Kernels:**

$$\overset{N}{K}_{12}(x, y) = \sum_{n=0}^{N-1} \frac{1}{h_n} \pi_n(x)\sigma_n(y)\mathrm{e}^{-V_1(x)-V_2(y)}$$

$$\overset{N}{K}_{11}(x, x') = \int \mathrm{d}y\ \underset{12}{\overset{N}{K}}(x, y)\, \mathrm{e}^{x'y}$$

$$\overset{N}{K}_{22}(y', y) = \int \mathrm{d}x\ \underset{12}{\overset{N}{K}}(x, y)\, \mathrm{e}^{xy'}$$

$$\overset{N}{K}_{21}(y', x') = \int\int \mathrm{d}x\, \mathrm{d}y \overset{N}{K}_{12}(x, y)\, \mathrm{e}^{xy'} \mathrm{e}^{x'y}$$

**Density of eigenvalues** :

$$\overset{N}{\rho}_1(x) = \frac{K^N{}_{11}(x,x)}{N}, \quad \overset{N}{\rho}_2(y) = \frac{K^N{}_{22}(y,y)}{N}$$

**Correlation functions:**

$$\overset{N}{\rho}_{11}(x,x') = \frac{1}{N^2}\det\begin{pmatrix} K^N{}_{11}(x,x) & K^N{}_{11}(x,x') \\ K^N{}_{11}(x',x) & K^N{}_{11}(x',x') \end{pmatrix}$$

$$\overset{N}{\rho}_{12}(x,y) = \frac{1}{N^2}\det\begin{pmatrix} K^N{}_{11}(x,x) & K^N{}_{12}(x,y) \\ K^N{}_{21}(y,x) - e^{xy} & (K^N{}_{22}(y,y) \end{pmatrix}$$

**Gap probablilities (spacing distributions):**

$$\overset{N}{p}{}^1_J = \det\left(\mathbf{I} - \overset{N}{\hat{\mathbf{K}}}_{11} \circ \chi_J\right), \quad \text{(Matrix } \mathrm{M}_1)$$

$$\overset{N}{p}{}^2_J = \det\left(\mathbf{I} - \overset{N}{\hat{\mathbf{K}}}_{22} \circ \chi_{\widetilde{J}}\right), \quad \text{(Matrix } \mathrm{M}_2)$$

$$E_{J,\widetilde{J}} = \det\left(\mathbf{I} - \overset{N}{\hat{\mathbf{K}}} \circ \mathrm{diag}(\chi_J, \chi_{\widetilde{J}})\right), \quad \text{(Matrix } \mathrm{M}_1)$$

where

$$\overset{N}{\hat{\mathbf{K}}} = \begin{pmatrix} \hat{\mathbf{K}}^N_{11} & \hat{\mathbf{K}}^N_{12} \\ \hat{\mathbf{K}}^N_{21} - \hat{\mathbf{E}} & \hat{\mathbf{K}}^N_{22} \end{pmatrix}$$

$$\hat{\mathbf{E}}(f)(x) := \int e^{xy} f(y) dy$$

(matrix Fredholm integral operator)

and $\chi_J, \chi_{\widetilde{J}}$ are the characteristic function of the sets $J, \widetilde{J}$.

**Support of biorthogonality measure**

$$\iint_{\kappa} dx dy f(x) g(y) e^{-V_1(x)-V_2(y)+xy}$$
$$:= \sum_j \sum_k \kappa_{jk} \int_{\Gamma_j} dx \int_{\hat{\Gamma}_k} dy f(x) g(y) e^{-V_1(x)-V_2(y)+xy}$$

where $\{\Gamma_j\}_{j=1,\ldots d_1}$, $\{\hat{\Gamma}_k\}_{k=1,\ldots d_2}$ are a basis of homologically independent curves in the complex $x$- and $y$- planes.

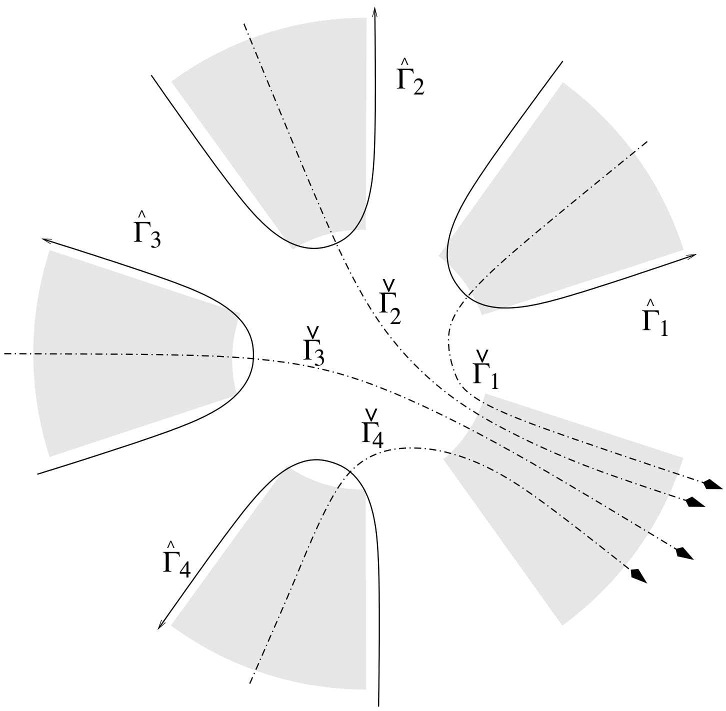



Figure 1: Wedge and antiwedge contours

**Wave vectors**

$$\underset{\infty}{\Psi}(x) := [\psi_0(x), ..., \psi_n(x), ...]^t$$
$$\underset{\infty}{\Phi}(y) := [\phi_0(y), ..., \phi_n(y), ...]^t$$
$$\psi_n(x) := \frac{1}{\sqrt{h_n}} \pi_n(x) \mathrm{e}^{-V_1(x)}$$
$$\phi_n(y) := \frac{1}{\sqrt{h_n}} \sigma_n(y) \mathrm{e}^{-V_2(y)}$$

**Recursions relations, differential equations**

$$x \underset{\infty}{\Psi}(x) = Q \underset{\infty}{\Psi}(x) \ , \qquad y \underset{\infty}{\Phi}(y) = P^t \underset{\infty}{\Phi}(y) \ ,$$
$$\partial_x \underset{\infty}{\Psi}(x) = -P \underset{\infty}{\Psi}(x) \ , \qquad \partial_y \underset{\infty}{\Phi}(y) = -Q^t \underset{\infty}{\Phi}(y)$$

$$Q := \begin{pmatrix} \alpha_0(0) & \gamma(0) & 0 & \cdots \\ \alpha_1(1) & \alpha_0(1) & \gamma(1) & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ \alpha_{d_2}(d_2) & \alpha_{d_2-1}(d_2) & \alpha_{d_2-2}(d_2) & \cdots \\ 0 & \alpha_{d_2}(d_2+1) & \alpha_{d_2-1}(d_2+1) & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

$$P := \begin{pmatrix} \beta_0(0) & \beta_1(1) & \beta_2(2) & \beta_3(3) & \cdots \\ \gamma(0) & \beta_0(1) & \beta_1(2) & \beta_2(3) & \cdots \\ 0 & \gamma(1) & \beta_0(2) & \beta_1(3) & \cdots \\ 0 & 0 & \gamma(2) & \beta_0(3) & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

**Fourier-Laplace transforms along wedge contours $\hat{\Gamma}_j$**

$$\widetilde{\phi}_m^{(j)}(x) := \int_{\hat{\Gamma}_j} \phi_m(y) e^{xy} dy$$

**Generalized Christoffel-Darboux identity**:

$$(z-x)\sum_{m=0}^{N-1} \widetilde{\phi}_m^{(j)}(x)\psi_m(z) = \sum_{a=0}^{d_2}\sum_{b=0}^{d_2} \widetilde{\phi}_{N-1+a}^{(j)}(x) \overset{N}{\mathbf{A}}_{ab} \psi_{N-d_2+b}(z),$$

**Christoffel-Darboux matrix $\overset{N}{\mathbf{A}}$**

$$\overset{N}{\mathbf{A}} := \begin{pmatrix} 0 & 0 & \cdots & 0 & -\gamma(N-1) \\ \alpha_{d_2}(N) & \alpha_{d_2-1}(N) & \cdots & \alpha_1(N) & 0 \\ 0 & \alpha_{d_2}(N+1) & \cdots & \alpha_2(N+1) & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_{d_2}(N+d_2-1) & 0 \end{pmatrix}.$$

## 3. The "direct" and "dual" fundamental systems

**"Second type" solutions** to recursions and differential equations

$$\psi_m^{(k)}(x) := \frac{1}{2\pi i}\int_{\check{\Gamma}_k} ds \iint_\kappa dzdw \frac{\psi_m(z)}{x-z}\frac{V_2'(s)-V_2'(w)}{s-w} \cdot e^{-V_2(w)+V_2(s)+zw-xs}, \quad 1 \le k \le d_2$$

**"Direct" fundamental system**

$$\underset{N}{\Psi}(x) = \left( \underset{N}{\Psi}^{(0)}(x)\ \underset{N}{\Psi}^{(1)}(x) \cdots\ \underset{N}{\Psi}^{(d_2)}(x) \right)$$

$$\underset{N}{\Psi}^{(k)}(x) := \begin{pmatrix} \psi_{N-d_2}^{(k)}(x) \\ \vdots \\ \psi_N^{(k)}(x) \end{pmatrix}$$

**"Dual fundamental system"**

$$\underset{N}{\widetilde{\Phi}}(x) := \begin{pmatrix} \underset{N}{\widetilde{\Phi}}^{(0)}(x) \\ \underset{N}{\widetilde{\Phi}}^{(1)}(x) \\ \vdots \\ \underset{N}{\widetilde{\Phi}}^{(d_2)}(x) \end{pmatrix}$$

with **row vectors**

$$\underset{N}{\widetilde{\Phi}}^{(k)}(x) := \left( \widetilde{\phi}_{N-1}^{(k)}(x) \cdots \widetilde{\phi}_{N-1+d_2}^{(k)}(x) \right)$$

$$\widetilde{\phi}_m^{(0)}(x) := e^{V_1(x)} \iint_\kappa dzdw \frac{\phi_m(w)}{x-z} e^{-V_1(z)+zw}, \quad m \in \mathbb{N}.$$

**Theorem 1** (Dual pairing) *For $N \geq d_2$, the matrices $\underset{N}{\widetilde{\Phi}}(x)$ and $\underset{N}{\Psi}(x)$ are bilinearly paired via the Christoffel–Darboux matrix*

$$\underset{N}{\widetilde{\Phi}}(x)\overset{N}{\mathbf{A}}\underset{N}{\Psi}(x) = \mathbf{I}$$

**Theorem 2.** (Recursion relations and differential equations) *The pair of matrices $\underset{N}{\Psi}(x)$ and $\underset{N}{\widetilde{\Phi}}(x)$ satisfy the recursion relations*

$$\underset{N+1}{\Psi}(x) = \underset{N}{\mathbf{a}}(x)\underset{N+1}{\Psi}(x) \quad \underset{N}{\widetilde{\Phi}}(x) = \underset{N+1}{\widetilde{\Phi}}(x)\underset{N}{\widetilde{\mathbf{a}}}(x)$$

*where*

$$\underset{N}{\mathbf{a}}(x) := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ \frac{-\alpha_{d_2}(N)}{\gamma(N)} & \cdots & \frac{-\alpha_1(N)}{\gamma(N)} & \cdots & \frac{(x-\alpha_0(N))}{\gamma(N)} \end{pmatrix}$$

$$\underset{N}{\widetilde{\mathbf{a}}}(x)\overset{N}{\mathbf{A}}\underset{N}{\mathbf{a}}(x) := \mathbf{I}$$

*and the differential equations*

$$\frac{\partial}{\partial x}\underset{N}{\Psi}(x) = -\underset{N}{\mathbf{D_1}}(x)\underset{N}{\Psi}(x),$$
$$\frac{\partial}{\partial x}\underset{N}{\widetilde{\Phi}}(x) = -\underset{N}{\widetilde{\Phi}}(x)\underset{N}{\widetilde{\mathbf{D}}_1}(x),$$

*where*

$$\underset{N}{\mathbf{D_1}}(x) := \begin{pmatrix} \beta_0(N-d_2) & \dots & \beta_{d_2-1}(N-1) & 0 \\ \gamma(N-d_2) & \ddots & \vdots & \vdots \\ 0 & \ddots & \beta_0(N-1) & 0 \\ 0 & \dots & \gamma(N-1) & V_1'(x) \end{pmatrix}$$

$$- \frac{\gamma(M)}{\alpha_{d_2}(M-1)} \begin{pmatrix} \alpha_{d_2-1}(N-1) & \dots & \alpha_0(N-1)-x & \gamma(N-1) \\ 0 & \dots & & 0 \\ \vdots & & & \vdots \\ 0 & \dots & & 0 \end{pmatrix}$$

$$- \begin{pmatrix} \nabla_Q V_1'(x)_{N-d_2,N-1} & \dots & \nabla_Q V_1'(x)_{N-d_2,N+d_2-1} \\ \vdots & & \vdots \\ \nabla_Q V_1'(x)_{N,N-1} & \dots & \nabla_Q V_1'(x)_{N,N+d_2-1} \end{pmatrix} \overset{N}{\mathbf{A}}$$

$$\underset{N}{\widetilde{\mathbf{D}}_1}(x) := \begin{pmatrix} V_1'(x) & 0 & \dots & 0 \\ \gamma(N-1) & \beta_0(N) & \dots & \beta_{d_2-1}(L) \\ 0 & \ddots & \ddots & \vdots \\ 0 & \dots & \gamma(L-1) & \beta_0(L) \end{pmatrix}$$

$$- \frac{\gamma(L)}{\alpha_{d_2}(L+1)} \begin{pmatrix} 0 & \dots & 0 & \gamma(N-1) \\ 0 & & & \alpha_0(N)-x \\ 0 & \ddots & & \vdots \\ 0 & \dots & 0 & \alpha_{d_2-1}(L) \end{pmatrix}$$

$$- \overset{N}{\mathbf{A}} \begin{pmatrix} \nabla_Q V_1'(x)_{N-d_2,N-1} & \dots & \nabla_Q V_1'(x)_{N-d_2,N+d_2-1} \\ \vdots & & \vdots \\ \nabla_Q V_1'(x)_{N,N-1} & \dots & \nabla_Q V_1'(x)_{N,N+d_2-1} \end{pmatrix}$$

$$\nabla_Q V_1'(x)_{mn} := \iint_\kappa dz dw \frac{V_1'(z) - V_1'(x)}{z-x} \psi_m(z) \phi_n(w) e^{zw}$$

**Theorem 3.** *(Riemann-Hilbert characterization)*

*3.1 (Jump discontinuities) The limits* $\underset{N}{\Psi}_\pm$, $\underset{N}{\widetilde{\Phi}}_\pm$ *when approaching the contours* $\Gamma_j$ *from the left (+) and right(−) are related by the following jump discontinuity conditions*

$$\underset{N}{\Psi}_+(x) = \underset{N}{\Psi}_-(x)\mathbf{H}^{(j)}$$
$$\underset{N}{\widetilde{\Phi}}_+(x) = \hat{\mathbf{H}}^{(j)}\underset{N}{\widetilde{\Phi}}_-(x)$$

*where*

$$\mathbf{H}^{(j)} := \mathbf{I} - 2\pi i \mathbf{e}_0 \kappa^T$$
$$\hat{\mathbf{H}}^{(j)} = (\mathbf{H}^{(j)})^{-1} = \mathbf{I} + 2\pi i \mathbf{e}_0 \kappa^T$$
$$\mathbf{e}_0 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \kappa := \begin{pmatrix} 0 \\ \kappa_{j1} \\ \vdots \\ \kappa_{jd_2} \end{pmatrix}$$

**3.2** *(Large argument asymptotics) The asymptotic form of* $\underset{N}{\widetilde{\Phi}}(x)$ *and* $\underset{N}{\Psi}(x)$ *as* $x \to \infty$ *in any given Stokes sector* $\mathcal{S}_j$, *is given by*

$$\underset{N}{\widetilde{\Phi}}(x) \sim K_j e^{T(q)} Y_N(q) \underset{N}{\Omega} q^G H_N$$
$$\underset{N}{\Psi}(x) \sim \overset{N}{\mathbf{A}}^{-1} H_N^{-1} q^{-G} \underset{N}{\Omega}^{-1} Y_N^{-1}(q) e^{-T(q)} K_j^{-1}$$

*where*

$$Y(q) = \mathbf{I} + \mathcal{O}\left(\frac{1}{q}\right).$$

*where*

$$T(q) := \mathrm{diag}(V_1(x) - n\ln x, S_0(q) + n\ln\mathrm{q}, \ldots S_{d_2-1}(q) + n\ln\mathrm{q})$$

$$H_N := \mathrm{diag}\left(\sqrt{h_{N-1}}, \frac{1}{\sqrt{h_N}}, \ldots \frac{1}{\sqrt{h_{N+d_2-1}}}\right)$$

$$S_\ell(q) := xy_\ell - V_2(y_l) - \frac{1}{2}\ln\left(\frac{V_2''(y_\ell)}{2\pi}\right)$$

*The* **saddle points** $\{y_\ell\}_{l=1\ldots d_2}$ *satisfy*

$$V_2'(y_\ell) = x,$$

$$y_\ell(q) = (v_{2+1})^{\frac{1}{d_2}}\omega^l q + \mathcal{O}(1),$$

$$q := x^{\frac{1}{d_r}}, \quad \omega := e^{\frac{2\pi i}{d_2}}$$

$$G := (0, 0, 1, \ldots d_2 - 1)$$

$$\underset{N}{\Omega} := \begin{pmatrix} 1 & 0 \\ 0 & \underset{N}{\Omega}^0 \end{pmatrix}$$

$$\underset{N}{\Omega} := \begin{pmatrix} 1 & 0 \\ 0 & \underset{N}{\Omega}^0 \end{pmatrix}$$

$$(\underset{N}{\Omega}^0)_{jk} := \omega^{(j-1)(k+N-1)}, \quad 1 \le j, k \le d_2.$$

**"Dual" Spectral curve**. *(Characteristic polynomial: $d_2+1$-fold branched covering of $x$-Riemann sphere or $d_1+1$-fold branched covering of $y$-Riemann sphere )*

$$E(x,y) := \det(y\mathbf{I} - \underset{N}{\mathbf{D_1}}(x)) = -(V_1'(x)-y)(V_2'(y)-x) - 1$$
$$+ \left\langle \mathrm{tr}\left( \frac{V_1'(x) - V_1'(M_1)}{x\mathbf{I} - M_1} \frac{V_1'(x) - V_1'(M_2)}{y\mathbf{I} - M_2} \right) \right\rangle = 0$$

**Theorem 4** *(Residue formulæ)*

$$\frac{\partial \ln(\mathbf{Z}_N)}{\partial u_a} = \underset{x=\infty}{\mathrm{res}}\, x^{-a} y dx$$
$$\frac{\partial \ln(\mathbf{Z}_N)}{\partial v_b} = \underset{y=\infty}{\mathrm{res}}\, y^{-b} x dy$$

**Large $N$ asymptotics**

*Apply the Riemann-Hilbert method, based upon the $g$-function*

$$\mathbf{g} = (g_0(x), g_1(x), \cdots g_{d_2}(x))$$

*where*

$$g_j(x) := \int_{b_0}^{p_j(x)} y dx$$

*is evaluated on the* **equilibrium curve**, *determined by the* **variational equations**

$$\frac{\partial \mathcal{F}_0}{\partial \epsilon_i} = 0, \quad i = 1, \ldots \text{genus}$$
$$\epsilon_j := \oint_{A_j} y dx, \quad \text{"filling fractions''}$$

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