WZW MODELS and GERBES

Krzysztof Gawędzki, Budapest, June. 2006

Main theme:

the strength of Lagrangian methods in CFT

One of the best illustrations:

Lochlainn's team work on Toda reductions of WZW models:

- L. Fehér, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui & A. Wipf
  "On Hamiltonian reductions of the Wess-Zumino-Novikov-Witten theories", Phys. Rep. **222** (1992), 1-64

A step back:

WZW models with non-simply connected targets

Problem: Wess-Zumino term with Kalb-Ramond $B = d^{-1}H$ field in a topologically not-trivial target

Needs proper math tools:

- for closed string amplitude: **gerbes**
- for open string amplitudes: **gerbe modules**

Crash course on line bundles

(with unitary connections):

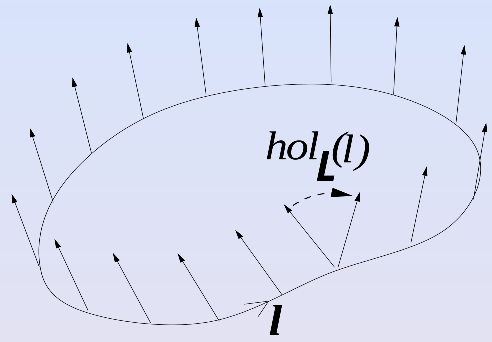
- For an exact 2-form $F = dA$ on space M

$$\int_S F = \int_{\partial S} A$$ **Stokes Theorem**

- If F is closed and has periods $\int_{c_2} F$ belonging to $2\pi \boldsymbol{Z}$ then

$$\int_S F = \frac{1}{i} \ln hol_{\mathcal{L}}(\partial S) \mod 2\pi$$

if $\mathcal{L}$ is a line bundle of curvature F where $hol_{\mathcal{L}}(\ell)$ stands for the **holonomy** of $\mathcal{L}$ along closed curve $\ell : S^1 \to M$



- $\mathcal{L}$ that are different (modulo isomorphism) are classified by

$$H^1(M,U(1)) \;=\; \pi_1(M)^*$$

- The **holonomy** of the closed loop $\ell : S^1 :\longrightarrow M$ is defined by:

$$hol_{\mathcal{L}}(\ell) \;=\; [\ell^*\mathcal{L}] \;\in\; H^1(S^1,U(1)) \;=\; U(1)$$

- **Local data** (A_i, g_{ij}) on a covering $(\mathcal{O}_i)$:

$$\begin{array}{lll} F = dA_i & \text{on} & \mathcal{O}_i \\ A_j - A_i \;=\; i\,d\ln\, g_{ij} & \text{on} & \mathcal{O}_{ij} \equiv \mathcal{O}_i \cap \mathcal{O}_j \\ g_{ij}g_{jk} \;=\; g_{ik} & \text{on} & \mathcal{O}_{ijk} \end{array}$$

- **Local formula** for the holonomy of the loop $\ell : S^1 \to M$:

$$hol_{\mathcal{L}}(\ell) = \exp\left[i \sum_b \int_{\ell(b)} A_{i_b}\right] \prod_{v \in b} g_{i_v i_b}(\ell(v))$$

where $\{b, v\}$ is a split of the circle S^1 into intervals b joined at vertices v s.t. $\ell(b) \subset \mathcal{O}_{i_b}$ and $\ell(v) \subset \mathcal{O}_{i_v}$

- **Feynman amplitude** of a particle in electromagnetic field F is given by

$$\mathcal{A}(\ell) = \exp\left[-\tfrac{1}{2}\|d\ell\|^2\right] hol_{\mathcal{L}}(\ell)$$

Crash course on (bundle) gerbes

(with unitary connections):

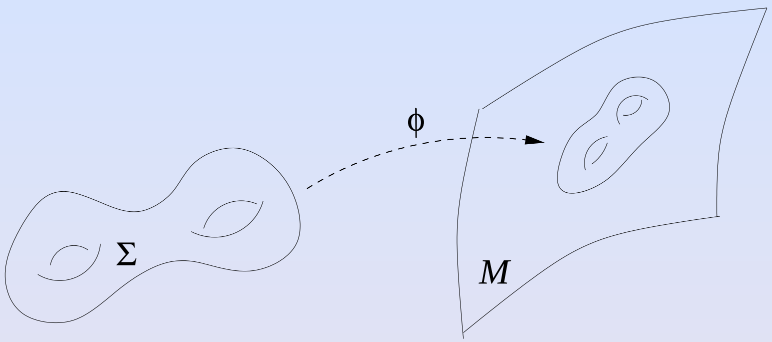
- For an exact 3-form $H = dB$ on space M

$$\int_V H = \int_{\partial V} B$$ **Stokes Theorem**

- If H is closed and has periods $\int_{c_3} H$ belonging to $2\pi\boldsymbol{Z}$ then

$$\int_V H = \frac{1}{i} \ln hol_{\mathcal{G}}(\partial V) \mod 2\pi$$

if $\mathcal{G}$ is a **gerbe** of **curvature** H where $hol_{\mathcal{G}}(\phi)$ stands for the **holonomy** of $\mathcal{G}$ along closed surface $\phi : \Sigma \to M$



- $\mathcal{G}$ that are different (modulo isomorphism) are classified by

$$H^2(M, U(1))$$

- The holonomy of the closed surface $\phi : \Sigma \to M$ is defined by:

$$hol_{\mathcal{G}}(\phi) \;=\; [\phi^*\mathcal{G}] \;\in\; H^2(\Sigma, U(1)) \;=\; U(1)$$

- Local data (B_i, A_{ij}, g_{ijk}) on a covering $(\mathcal{O}_i)$:

$$\begin{array}{ll} H = dB_i & \text{on} \quad \mathcal{O}_i \\ B_j - B_i \;=\; dA_{ij} & \text{on} \quad \mathcal{O}_{ij} \\ A_{ij} + A_{jk} - A_{ik} \;=\; i\, d\ln\, g_{ijk} & \text{on} \quad \mathcal{O}_{ijk} \\ g_{ijk}g_{ijl}^{-1}g_{ikl}g_{jkl}^{-1} \;=\; 1 & \text{on} \quad \mathcal{O}_{ijkl} \end{array}$$

- **Local formula** for the holonomy of the surface $\phi : \Sigma \to M$:

$$hol_{\mathcal{G}}(\phi) =$$
$$\exp\Big[i\sum_t \int_{\phi(t)} B_{i_t} + i\sum_{b\subset t}\int_{\phi(b)} A_{i_t i_b}\Big] \prod_{v\in b\subset t} g_{i_t i_b i_v}(\phi(v))$$

where $\{t, b, v\}$ is a triangulation of Σ into triangles t with edges b and vertices v s.t. $\ell(f) \subset \mathcal{O}_{i_f}$ for $f = t, b, v$

- **Feynman amplitude** of a closed string in Kalb-Ramond field H is given by

$$\mathcal{A}(\phi) = \exp\Big[-\tfrac{1}{2}\|d\phi\|^2\Big]\, hol_{\mathcal{G}}(\phi)$$

Application to the WZW models

$M = G$ is a simple, compact **Lie group**

$H_k = \frac{k}{12\pi} \operatorname{tr} (g^{-1} dg)^3$ is a closed 3-form on G

- For G **simply connected**, the periods of H_k are in $2\pi\boldsymbol{Z}$ iff the **level** k is an integer

  The corresponding **gerbe** is unique (up to isom.) and has been constructed:
  - for $SU(2)$ by K.G. (1986)
  - for $SU(N)$ by Chatterjee-Hitchin (1998)
  - for G general by Meinrenken (2002)

- For $G = \tilde{G}/Z$ **non-simply connected** (for Z a subgroup of the center of the covering group $\tilde{G}$), the periods of H_k are in $2\pi \boldsymbol{Z}$ iff the **level** k is an integer and satisfying **selection rules** found by Felder-K.G.-Kupiainen (1987)

  The corresponding **gerbe** $\mathcal{G}^k$ on G was constructed by K.G.-Reis (2003)

  It is unique but for $G = Spin(4n)/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2)$ where there are 2 non-isomorphic **gerbes** $\mathcal{G}^k_\pm$ since $H^2(G, U(1)) = \boldsymbol{Z}_2$

  In the latter case, **Witten**'s definition

$$hol_{\mathcal{G}}(\phi) = \exp\Big[i \int\limits_{\Phi(\mathcal{B})} H_k\Big]$$

  for Φ extending ϕ to $\mathcal{B}$ with $\partial\mathcal{B} = \Sigma$ does not always work

Quantization of the bulk WZW models

- By **transgression** (roughly integration along loops)

gerbe $\mathcal{G}$ on M $\quad - - - -> \quad$ line bundle $\mathcal{L}_{\mathcal{G}}$ on loop space LM

- Space of quantum states of the WZW model:

$$\mathcal{H} \;=\; \Gamma(\mathcal{L}_{\mathcal{G}^k}) \qquad \longleftarrow \quad \text{space of sections}$$

with a geometric action of the **current algebra** $\hat{\mathbf{g}} \times \hat{\mathbf{g}}$

- Decomposition into the highest weight (H.W.) level k irreps

$$\mathcal{H} \;\cong\; \bigoplus_{\lambda,\lambda'} M_{\lambda,\lambda'} \otimes \hat{V}_\lambda \otimes \hat{V}_{\lambda'}$$

obtained by finding the H.W. subspaces $M_{\lambda,\lambda'} \subset \Gamma(\mathcal{L}_{\mathcal{G}^k})$ $-->$ **spectrum** + **partition fcts** (Felder-K.G.-Kupiainen 1987)

Gerbe modules (Kapustin 1999)

- Let $\mathcal{G}$ be a **gerbe** with loc. data (B_i, A_{ij}, g_{ijk})

  A $\mathcal{G}$**-module** $\mathcal{E}$ (with unitary connection) is determined by local data $(\mathbf{A}_i, \mathbf{G}_{ij})$ with values in $N \times N$ matrices s.t.

  $$\mathbf{A}_j = \mathbf{G}_{ij}^{-1} \mathbf{A}_i \mathbf{G}_{ij} - i\, \mathbf{G}_{ij}^{-1} d\mathbf{G}_{ij} + A_{ij} \mathbf{1} = 0 \quad \text{on} \quad \mathcal{O}_{ij}$$

  $$\mathbf{G}_{ij}\, \mathbf{G}_{jk} = g_{ijk}\, \mathbf{G}_{ik} \quad \text{on} \quad \mathcal{O}_{ijk}$$

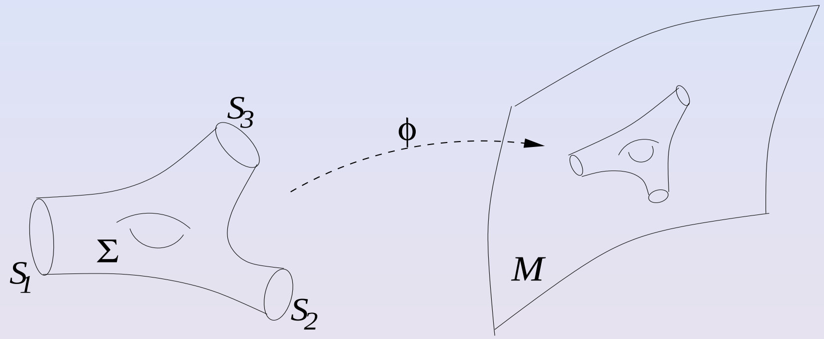
  $\equiv$ "**twisted gauge field**"

- For $\ell : S^1 \to M$, the **Wilson loop**

  $$hol_{\mathcal{E}}(\ell) = \operatorname{tr} \overrightarrow{\prod_{v \in b}} P \exp\Big[i \sum_b \int_{\ell(b)} \mathbf{A}_{i_b}\Big]\, \mathbf{G}_{i_v i_b}(\ell(v))$$

  is <u>not</u> unambiguously defined but

- If $\phi : \Sigma \to M$ for $\partial\Sigma = \sqcup S_\alpha$



and $\mathcal{E}_\alpha$ are $\mathcal{G}$-modules then for $\ell_\alpha = \phi|_{S_\alpha}$ the combination

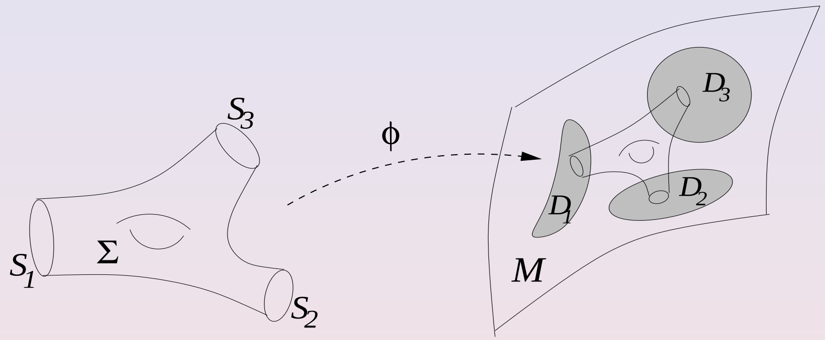
$$hol_{\mathcal{G}}(\phi) \prod_\alpha W_{\mathcal{E}_\alpha}(\ell_\alpha)$$

is <u>unambiguously</u> defined !!!

- **Problem !** $\mathcal{G}$-modules exist only for $\mathcal{G}$ with exact H

- **Solution:** One defines a $\mathcal{G}$**-brane** as a pair $(D,\mathcal{E}) \equiv \mathcal{D}$ where $D \subset M$ and $\mathcal{E}$ is a $\mathcal{G}\big|_D$-module

- If $\phi : \Sigma \to M$ for $\partial\Sigma = \sqcup S_\alpha$



and $\mathcal{D}_\alpha = (D_\alpha, \mathcal{E}_\alpha)$ are $\mathcal{G}$-branes s.t. $\phi(S_\alpha) \subset D_\alpha$ then

$$hol_{\mathcal{G}}(\phi) \prod_\alpha W_{\mathcal{E}_\alpha}(\ell_\alpha)$$

for $\ell_\alpha = \phi|_{S_\alpha}$ is still unambiguously defined

Example: **symmetric branes** in the WZW model

- These are $\mathcal{G}^k$-branes $\mathcal{D} = (D, \mathcal{E})$ s.t.

$$D = \{\, h\, \mathrm{e}^{2\pi i\lambda/k} h^{-1} \mid h \in G \,\} \equiv \mathcal{C}_\lambda$$

for λ a weight and the curvature of the $\mathcal{G}^k_D$-module $\mathcal{E}$ is

$$F = \tfrac{k}{4\pi}\, \mathrm{tr}\, (h^{-1}dh)\, \mathrm{e}^{2\pi i\lambda/k} (h^{-1}dh)\, \mathrm{e}^{-2\pi i\lambda/k}$$

- If $\phi : \Sigma \to G$, $\phi(S_\alpha) \subset D_\alpha$ and $\mathcal{D}_\alpha = (D_\alpha, \mathcal{E}_\alpha)$ are **symmetric $\mathcal{G}^k$-branes** then

$$\mathcal{A}(\phi) = \exp\Big[-\tfrac{k}{4\pi} \|d\phi\|^2 \Big]\ hol_{\mathcal{G}}(\phi)\ \prod_\alpha W_{\mathcal{E}_\alpha}(\ell_\alpha)$$

defines the **Feynman amplitudes** of the boundary WZW model preserving the diagonal current algebra symmetry

Classification of symmetric branes

(K.G.-Reis 2002, K.G. 2004)

- For **simply connected** G the symmetric $\mathcal{G}^k$-branes are of the form $\mathcal{D} = (\mathcal{C}_\lambda, \mathcal{E})$ with

$$\mathcal{E} = \boldsymbol{C}^N \otimes \mathcal{E}_\lambda^1$$

for the unique rank 1 $\mathcal{G}^k|_{\mathcal{C}_\lambda}$-module $\mathcal{E}_\lambda^1$

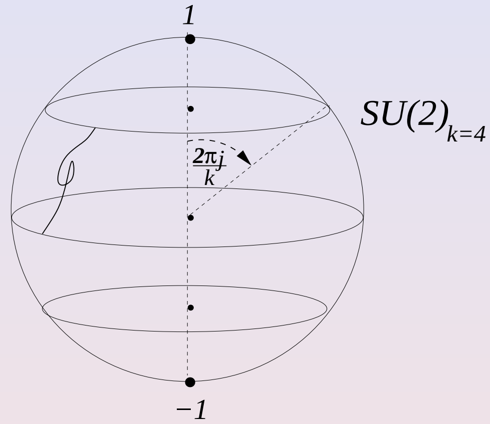
- For **non-simply connected** $G = \tilde{G}/Z$ the symmetric $\mathcal{G}^k$-branes supported by $\mathcal{C}_\lambda \cong \tilde{\mathcal{C}}_\lambda / Z_\lambda$ are $\mathcal{D} = (\mathcal{C}_\lambda, \mathcal{E})$ with

$$\mathcal{E} = \boldsymbol{C}^{n_1} \otimes \mathcal{E}_\lambda^1(i_1) \oplus \cdots \oplus \boldsymbol{C}^{n_I} \otimes \mathcal{E}_\lambda^1(i_I)$$

for $I = |Z_\lambda| = |\pi_1(\mathcal{C}_\lambda)|$ different rank 1 $\mathcal{G}^k|_{\mathcal{C}_\lambda}$-modules $\mathcal{E}_\lambda^1(i)$

Examples:

- For $G = SU(2) \cong S^3$ the conjugacy classes $\mathcal{C}_j$ are spheres $S^2 \subset S^3$ viewed under angles $\frac{2\pi j}{k}$ for $j = 0, \frac{1}{2}, \ldots, \frac{k}{2}$



Each carries a single $\mathcal{G}^k$-brane

- For $G = SO(3) \cong \boldsymbol{R}P^3$ the level k must be even

$\mathcal{C}_j \cong \tilde{\mathcal{C}}_j$ for $j = 0, \frac{1}{2}, \ldots, \frac{k-2}{4}$ carry one rank 1 sym. $\mathcal{G}^k$-brane

$\mathcal{C}_{\frac{k}{4}} \cong \tilde{\mathcal{C}}_{\frac{k}{4}} / \boldsymbol{Z}_2 \cong \boldsymbol{R}P^2$ carries two rank 1 symmetric $\mathcal{G}^k$-branes

Exceptional case:

- For $G = Spin(4n)/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2)$ and $\mathcal{C}_\lambda \cong \tilde{\mathcal{C}}_\lambda/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2)$ all symmetric $\mathcal{G}^k_-$-branes of the form $\mathcal{D} = (\mathcal{C}_\lambda, \mathcal{E})$ have

$$\mathcal{E} \;=\; \boldsymbol{C}^N \otimes \mathcal{E}^2_\lambda$$

where $\mathcal{E}^2_\lambda$ is the unique rank 2 $\mathcal{G}^k_-|_{\mathcal{C}_\lambda}$-module (there is an obstruction in $H^2(Z_\lambda, U(1)) = \boldsymbol{Z}_2$ to the existence of rank 1 branes)

$-->$ geometric generation of **non-abelian gauge symmetry**

Quantization of the boundary WZW models

- By **transgression**

gerbe $\mathcal{G}$ on M
a pair $(\mathcal{D}_0, \mathcal{D}_1)$
of $\mathcal{G}$−branes

$- - - - >$

vector bundle $\mathcal{E}_{\mathcal{D}_0}^{\mathcal{D}_1}$
on the space
of curves $I_{\mathcal{D}_0}^{\mathcal{D}_1}$

where $I_{D_0}^{D_1} = \{ \ell : [0, \pi] \to M \mid \ell(0) \in D_0 , \ \ell(\pi) \in D_1 \}$

- Space of the quantum states of the boundary WZW model with a geometric action of the **current algebra** $\hat{\mathbf{g}}$:

$$\mathcal{H}_{\mathcal{D}_0}^{\mathcal{D}_1} = \Gamma(\mathcal{E}_{\mathcal{D}_0}^{\mathcal{D}_1}) \qquad \leftarrow \quad \text{space of sections}$$

- By identifying the H.W. subspaces M_λ of sections, one gets

$$\mathcal{H}_{\mathcal{D}_0}^{\mathcal{D}_1} \cong \bigoplus_{\lambda} M_\lambda \otimes \hat{V}_\lambda$$

$-->$ **spectrum** + **partition fcts** + **bdary OPE** (K.G. 2004)

Orientifolds

- In order to define Feynman amplitudes on **non-orientable** surfaces in the topologically non-trivial Kalb-Ramond field H one needs additionally a **Jandl structure (JS)** on a gerbe $\mathcal{G}$ of curvature H (Schreiber-Schweigert-Waldorf 2005)

- **JS** is a triple (κ, ι, η) where
  - κ is an involution of M s.t. $\kappa^* H = -H$
  - ι is an isomorphism $\kappa^* \mathcal{G} \overset{\iota}{\cong} \mathcal{G}^*$
  - η is an equivalence of gerbe isomorphisms $\iota^2 \overset{\eta}{\cong} Id$

- On Lie groups G one takes $\kappa(g) = z g^{-1}$ for z in the center

- Given κ for simply connected G there are two different **JS**'s on $\mathcal{G}^k$ giving amplitudes that differ by $(-1)^{\#\text{ crosscaps}}$

- For non-simply connected $G = \tilde{G}/Z$ there may be obstructions to the existence of a **JS** with given κ

  If $Z = \boldsymbol{Z}_n$ and the obstruction vanishes then there are two **JS**'s on $\mathcal{G}^k$ for n odd and four for n even (K.G.-Suszek-Schweigert-Waldorf, work in progress)

Conclusions

- Gerbes (with JS) and gerbe modules encode the structure needed to define the **Feynman amplitudes** in the presence of topologically non-trivial **Kalb-Ramond** field H
- In the case of compact Lie groups, such structures may be completely classified
- In WZW models, due to the current algebra symmetry, the geometric analysis permits to extract directly information about the quantum theory
- Open problems include extension of the analysis to **SUSY** and **coset models** and, more importantly, the problem of dynamics of gerbes and gerbe modules and of its relation to **RG flows**, **brane condensation** and **twisted K-theory**