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Fundamental limitations on ‘warp drive’ spacetimes

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Abstract

‘Warp drive’ spacetimes are useful as ‘gedanken-experiments’ that force us to confront the foundations of general relativity, and among other things, to precisely formulate the notion of ‘superluminal’ communication. After carefully formulating the Alcubierre and Natário warp drive spacetimes, and verifying their non-perturbative violation of the classical energy conditions, we consider a more modest question and apply linearized gravity to the weak-field warp drive, testing the energy conditions to first and second orders of the warp-bubble velocity, v . Since we take the warp-bubble velocity to be non-relativistic, $v \ll c$, we are not primarily interested in the ‘superluminal’ features of the warp drive. Instead we focus on a secondary feature of the warp drive that has not previously been remarked upon—the warp drive (if it could be built) would be an example of a ‘reaction-less drive’. For both the Alcubierre and Natário warp drives we find that the occurrence of significant energy condition violations is not just a high-speed effect, but that the violations persist even at arbitrarily low speeds. A particularly interesting feature of this construction is that it is now meaningful to think of placing a finite mass spaceship at the centre of the warp bubble, and then see how the energy in the warp field compares with the mass–energy of the spaceship. There is no hope of doing this in Alcubierre’s original version of the warp field, since by definition the point at the centre of the warp bubble moves on a geodesic and is ‘massless’. That is, in Alcubierre’s original formalism and in the Natário formalism the spaceship is always treated as a test particle, while in the linearized theory we can treat the spaceship as a finite mass object. For both the Alcubierre and Natário warp drives we find that even at low speeds the net (negative) energy stored in the warp fields must be a significant fraction of the mass of the spaceship.

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1. Introduction

Alcubierre demonstrated that it is theoretically possible, within the framework of general relativity, to attain arbitrarily large velocities [1]. In fact, numerous solutions to the Einstein field equations are now known that allow ‘effective’ superluminal travel [2–11]. Despite the use of the term superluminal, it is not ‘really’ possible to travel faster than light, in any *local* sense. Providing a general *global* definition of superluminal travel is no trivial matter [12, 13], but it is clear that the spacetimes that allow ‘effective’ superluminal travel generically suffer from the severe drawback that they also involve significant negative energy densities. More precisely, superluminal effects are associated with the presence of *exotic* matter, that is, matter that violates the null energy condition (NEC). In fact, superluminal spacetimes violate all the known energy conditions, and Ken Olum demonstrated that negative energy densities and superluminal travel are intimately related [14]. Although most classical forms of matter are thought to obey the energy conditions, they are certainly violated by certain quantum fields [15]. Additionally, certain classical systems (such as non-minimally coupled scalar fields) have been found, which violate the null and the weak energy conditions [16, 17]. Finally, we mention that recent observations in cosmology strongly suggest that the cosmological fluid violates the strong energy condition (SEC), and provides tantalizing hints that the NEC *might* possibly be violated in a classical regime [18–20].

For warp drive spacetimes, by using the ‘quantum inequality’ deduced by Ford and Roman [21], it was soon verified that enormous amounts of energy are needed to sustain superluminal warp drive spacetimes [22, 23]. To reduce the enormous amounts of exotic matter needed in the superluminal warp drive, van den Broeck proposed a slight modification of the Alcubierre metric which considerably ameliorates the conditions of the solution [24]. It is also interesting to note that, by using the ‘quantum inequality’, enormous quantities of negative energy densities are needed to support the superluminal Krasnikov tube [8]. However, Gravel and Plante [25, 26] in a way similar in spirit to the van den Broeck analysis showed that it is theoretically possible to lower significantly the mass of the Krasnikov tube.

In counterpoint, in this paper we shall be interested in applying the linearized gravity to warp drive spacetimes, testing the energy conditions at first and second orders of the warp-bubble velocity. We will take the bubble velocity to be non-relativistic, $v \ll 1$. Thus we are not focussing our attention on the ‘superluminal’ aspects of the warp bubble, such as the appearance of horizons [27–29], and of closed timelike curves [30], but rather on a secondary unremarked effect: the warp drive (*if it can be realized in nature*) appears to be an example of a ‘reaction-less drive’ wherein the warp bubble moves by interacting with the geometry of spacetime instead of expending reaction mass.

A particularly interesting aspect of this construction is that one may place a finite mass spaceship at the origin and consequently analyse how the warp field compares with the mass–energy of the spaceship. This is not possible in the usual finite-strength warp field, since by definition the point at the centre of the warp bubble moves along a geodesic and is ‘massless’. That is, in the usual formalism the spaceship is always treated as a test particle, while in the linearized theory we can treat the spaceship as a finite mass object.

Because we do not make any *a priori* assumptions as to the ultimate source of the energy condition violations, we will not use (or need) the quantum inequalities. This means that the restrictions we derive on warp drive spacetimes are more generic than those derived using the quantum inequalities—the restrictions derived in this paper will hold regardless of whether the warp drive is assumed to be classical or quantum in its operation.

We do not mean to suggest that such a ‘reaction-less drive’ is achievable with current technology—indeed the analysis below will, even in the weak-field limit, place very stringent

conditions on the warp bubble. These conditions are so stringent that it appears unlikely that the ‘warp drive’ will ever prove technologically useful. The Alcubierre ‘warp drive’, and the related Natário ‘warp drive’, are likely to retain their status as useful ‘gedanken-experiments’—they are useful primarily as a theoretician’s probe of the foundations of general relativity, and we wish to sound a strong cautionary note against over-enthusiastic mis-interpretation of the technological situation.

2. Warp drive basics

Within the framework of general relativity, Alcubierre demonstrated that it is in principle possible to warp spacetime in a small *bubble-like* region, in such a way that the bubble may attain arbitrarily large velocities. Inspired by the inflationary phase of the early universe, the enormous speed of separation arises from the expansion of spacetime itself. The simplest model for hyper-fast travel is to create a local distortion of spacetime, producing an expansion behind the bubble, and an opposite contraction ahead of it. Natário’s version of the warp drive dispensed with the need for expansion at the cost of introducing a slightly more complicated metric.

The warp drive spacetime metric, in Cartesian coordinates, is given by (with $G = c = 1$)

$$ds^2 = -dt^2 + [d\vec{x} - \vec{\beta}(x, y, z - z_0(t)) dt] \cdot [d\vec{x} - \vec{\beta}(x, y, z - z_0(t)) dt]. \quad (1)$$

In terms of the well-known ADM formalism this corresponds to a spacetime wherein *space* is flat, while the ‘lapse function’ is identically unity, and the only non-trivial structure lies in the ‘shift vector’ $\beta(t, \vec{x})$. Thus warp drive spacetimes can also be viewed as specific examples of ‘shift-only’ spacetimes. The Alcubierre warp drive corresponds to taking the shift vector to always lie in the direction of motion

$$\vec{\beta}(x, y, z - z_0(t)) = v(t)\hat{z}f(x, y, z - z_0(t)), \quad (2)$$

in which $v(t) = dz_0(t)/dt$ is the velocity of the warp bubble, moving along the positive z -axis, whereas in the Natário warp drive the shift vector is constrained by being divergence-free

$$\nabla \cdot \vec{\beta}(x, y, z) = 0. \quad (3)$$

2.1. Alcubierre warp drive

In the Alcubierre warp drive the spacetime metric is

$$ds^2 = -dt^2 + dx^2 + dy^2 + [dz - v(t)f(x, y, z - z_0(t)) dt]^2. \quad (4)$$

The form function $f(x, y, z)$ possesses the general features of having the value $f = 0$ in the exterior and $f = 1$ in the interior of the bubble. The general class of form functions, $f(x, y, z)$, chosen by Alcubierre was spherically symmetric: $f(r)$ with $r = \sqrt{x^2 + y^2 + z^2}$. Then

$$f(x, y, z - z_0(t)) = f(r(t)) \quad \text{with} \quad r(t) = \{[(z - z_0(t))^2 + x^2 + y^2]^{1/2}\}. \quad (5)$$

Whenever a more specific example is required we adopt

$$f(r) = \frac{\tanh[\sigma(r + R)] - \tanh[\sigma(r - R)]}{2 \tanh(\sigma R)}, \quad (6)$$

in which $R > 0$ and $\sigma > 0$ are two arbitrary parameters. R is the ‘radius’ of the warp bubble, and σ can be interpreted as being inversely proportional to the bubble wall thickness. If σ is large, the form function rapidly approaches a *top hat* function, i.e.,

$$\lim_{\sigma \rightarrow \infty} f(r) = \begin{cases} 1, & \text{if } r \in [0, R], \\ 0, & \text{if } r \in (R, \infty). \end{cases} \quad (7)$$

It can be shown that observers with the 4-velocity

$$U^\mu = (1, 0, 0, vf), \quad U_\mu = (-1, 0, 0, 0). \quad (8)$$

move along geodesics, as their 4-acceleration is zero, i.e., $a^\mu = U^\nu U^\mu_{;\nu} = 0$. These observers are called Eulerian observers. Their 4-velocity is normal to the spatial hypersurface with $t = \text{const}$, and they are close analogues of the usual Eulerian observers of fluid mechanics, which move with the flow of the fluid. The spaceship, which in the original formulation is treated as a test particle which moves along the curve $z = z_0(t)$, can easily be seen to always move along a timelike curve, regardless of the value of $v(t)$. One can also verify that the proper time along this curve equals the coordinate time, by simply substituting $z = z_0(t)$ into equation (4). This reduces to $d\tau = dt$, taking into account $dx = dy = 0$ and $f(0) = 1$.

If we attempt to treat the spaceship as more than a test particle, we must confront the fact that by construction we have forced $f = 0$ outside the warp bubble. (Consider, for instance, the explicit form function of equation (6) in the limit $r \rightarrow \infty$.) This implies that the spacetime geometry is asymptotically Minkowski space, and in particular the ADM mass (defined by taking the limit as one moves to spacelike infinity i^0) is zero. That is, the ADM mass of the spaceship and the warp field generators must be exactly compensated by the ADM mass due to the stress–energy of the warp field itself. Viewed in this light it is now patently obvious that there must be significant violations of the classical energy conditions (at least in the original version of the warp-drive spacetime), and the interesting question becomes ‘where are these energy condition violations localized?’.

One of our tasks in the current paper will be to see if we can first avoid this exact cancellation of the ADM mass, and second, to see if we can make qualitative and quantitative statements concerning the localization and ‘total amount’ of energy condition violations. (A similar attempt at quantification of the ‘total amount’ of energy condition violation in traversable wormholes was recently presented in [31, 32].)

Consider a spaceship placed within the Alcubierre warp bubble. The expansion of the volume elements, $\theta = U^\mu_{;\mu}$, is given by $\theta = v(\partial f / \partial z)$. Taking into account equation (5), we have (for Alcubierre’s version of the warp bubble)

$$\theta = v \frac{z - z_0}{r} \frac{df(r)}{dr}. \quad (9)$$

The centre of the perturbation corresponds to the spaceship’s position $z_0(t)$. The volume elements are expanding behind the spaceship, and contracting in front of it. Appendix A contains a full calculation of all the orthonormal components of the Einstein tensor for the Alcubierre warp bubble. By using the Einstein field equation, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, we can make rather general statements regarding the nature of the stress energy required to support a warp bubble.

The WEC states $T_{\mu\nu} V^\mu V^\nu \geq 0$, in which V^μ is *any* timelike vector and $T_{\mu\nu}$ is the stress–energy tensor. Its physical interpretation is that the local energy density is positive. By continuity it implies the NEC. In particular, the WEC implies that $T_{\mu\nu} U^\mu U^\nu \geq 0$ where U^μ is the 4-velocity of the Eulerian observers discussed above. The calculations will be simplified using an orthonormal reference frame. Thus, from the results tabulated in appendix A we verify that for the warp drive metric, the WEC is violated, i.e.,

$$T_{\hat{\mu}\hat{\nu}} U^{\hat{\mu}} U^{\hat{\nu}} = -\frac{v^2}{32\pi} \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] < 0, \quad (10)$$

where $T_{\hat{\mu}\hat{\nu}}$ and $U^{\hat{\mu}}$ are, respectively, the stress–energy tensor and timelike Eulerian 4-velocity given in the orthonormal basis. Taking into account the Alcubierre form function (6), we have

$$T_{\hat{\mu}\hat{\nu}}U^{\hat{\mu}}U^{\hat{\nu}} = -\frac{1}{32\pi} \frac{v^2(x^2 + y^2)}{r^2} \left[\frac{df}{dr} \right]^2 < 0. \quad (11)$$

By considering the Einstein tensor component, $G_{\hat{t}\hat{t}}$, in an orthonormal basis (details given in appendix A), and taking into account the Einstein field equation, we verify that the energy density of the warp drive spacetime is given by

$$T^{\hat{t}\hat{t}} = T^{\mu\nu}U_{\mu}U_{\nu} = T^{\hat{t}\hat{t}} = T_{\hat{t}\hat{t}} = T_{\hat{\mu}\hat{\nu}}U^{\hat{\mu}}U^{\hat{\nu}}, \quad (12)$$

and so equals the energy density measured by the Eulerian observers, that is, equation (11). It is easy to verify that the energy density is distributed in a toroidal region around the z -axis, in the direction of travel of the warp bubble [23]. It is perhaps instructive to point out that the energy density for this class of spacetimes is nowhere positive. That the total ADM mass can nevertheless be zero is due to the intrinsic nonlinearity of the Einstein equations.

We can (in analogy with the definitions in [31, 32]) quantify the ‘total amount’ of energy condition violating matter in the warp bubble by defining

$$\begin{aligned} M_{\text{warp}} &= \int \rho_{\text{warp}} d^3x = \int T_{\mu\nu} U^{\mu} U^{\nu} d^3x \\ &= -\frac{v^2}{32\pi} \int \frac{x^2 + y^2}{r^2} \left[\frac{df}{dr} \right]^2 r^2 dr d^2\Omega = -\frac{v^2}{12} \int \left[\frac{df}{dr} \right]^2 r^2 dr. \end{aligned} \quad (13)$$

This is emphatically not the total mass of the spacetime, but it characterizes how much (negative) energy one needs to localize in the walls of the warp bubble. For the specific shape function (6) we can estimate

$$M_{\text{warp}} \approx -v^2 R^2 \sigma. \quad (14)$$

(The integral can be done exactly, but the exact result in terms of polylog functions is unhelpful.) Note that the energy requirements for the warp bubble scale quadratically with bubble velocity, quadratically with bubble size and inversely as the thickness of the bubble wall.

The NEC states that $T_{\mu\nu}k^{\mu}k^{\nu} \geq 0$, where k^{μ} is *any* arbitrary null vector and $T_{\mu\nu}$ is the stress–energy tensor. Taking into account the Einstein tensor components presented in appendix A, and the Einstein field equation, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, the NEC for a null vector oriented along the $\pm\hat{z}$ directions takes the following form:

$$T_{\mu\nu}k^{\mu}k^{\nu} = -\frac{v^2}{8\pi} \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] \pm \frac{v}{8\pi} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right). \quad (15)$$

In particular, if we average over the $\pm\hat{z}$ directions we have

$$\frac{1}{2} \{ T_{\mu\nu}k_{+\hat{z}}^{\mu}k_{+\hat{z}}^{\nu} + T_{\mu\nu}k_{-\hat{z}}^{\mu}k_{-\hat{z}}^{\nu} \} = -\frac{v^2}{8\pi} \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right], \quad (16)$$

which is manifestly negative, and so the NEC is violated for all v . Furthermore, note that even if we do not average, the coefficient of the term linear in v must be nonzero *somewhere* in the spacetime. Then at low velocities this term will dominate and at low velocities the unaveraged NEC will be violated in either the $+\hat{z}$ or $-\hat{z}$ directions.

To be a little more specific about how and where the NEC is violated, consider the Alcubierre form function. We have

$$T_{\mu\nu}k_{\pm\hat{z}}^{\mu}k_{\pm\hat{z}}^{\nu} = -\frac{1}{8\pi} \frac{v^2(x^2 + y^2)}{r^2} \left(\frac{df}{dr} \right)^2 \pm \frac{v}{8\pi} \left[\frac{x^2 + y^2 + 2(z - z_0(t))^2}{r^3} \frac{df}{dr} + \frac{x^2 + y^2}{r^2} \frac{d^2f}{dr^2} \right]. \quad (17)$$

The first term is manifestly negative everywhere throughout the space. As f decreases monotonically from the centre of the warp bubble, where it takes the value of $f = 1$, to the exterior of the bubble, with $f \approx 0$, we verify that df/dr is negative in this domain. The term d^2f/dr^2 is also negative in this region, as f attains its maximum in the interior of the bubble wall. Thus, the term in square brackets unavoidably assumes a negative value in this range, resulting in the violation of the NEC.

For a null vector oriented perpendicular to the direction of motion (for definiteness take $\hat{k} = \pm \hat{x}$) the NEC takes the following form:

$$T_{\mu\nu}k_{\pm\hat{x}}^\mu k_{\pm\hat{x}}^\nu = -\frac{v^2}{8\pi} \left[\frac{1}{2} \left(\frac{\partial f}{\partial y} \right)^2 + \left(\frac{\partial f}{\partial z} \right)^2 - (1-f) \frac{\partial^2 f}{\partial z^2} \right] \mp \frac{v}{8\pi} \left(\frac{\partial^2 f}{\partial x \partial z} \right). \quad (18)$$

Again note that the coefficient of the term linear in v must be nonzero *somewhere* in the spacetime. Then at low velocities this term will dominate, and at low velocities the NEC will be violated in one or other of the transverse directions. Upon considering the specific form of the spherically symmetric Alcubierre form function, we have

$$\begin{aligned} T_{\mu\nu}k_{\pm\hat{x}}^\mu k_{\pm\hat{x}}^\nu = & -\frac{v^2}{8\pi} \left[\frac{y^2 + 2(z - z_0(t))^2}{2r^2} \left(\frac{df}{dr} \right)^2 \right. \\ & \left. - (1-f) \left(\frac{x^2 + y^2}{r^3} \frac{df}{dr} + \frac{(z - z_0(t))^2}{r^2} \frac{d^2f}{dr^2} \right) \right] \\ & \mp \frac{v}{8\pi} \frac{x(z - z_0(t))}{r^2} \left(\frac{d^2f}{dr^2} - \frac{1}{r} \frac{df}{dr} \right). \end{aligned} \quad (19)$$

Again, the message to take from this is that localized NEC violations are ubiquitous and persist to arbitrarily low warp bubble velocities.

Using the ‘volume integral quantifier’ (as defined in [31, 32]), we may estimate the ‘total amount’ of averaged null energy condition violating matter in this spacetime, given by

$$\int T_{\mu\nu}k_{\pm\hat{z}}^\mu k_{\pm\hat{z}}^\nu d^3x \approx \int T_{\mu\nu}k_{\pm\hat{x}}^\mu k_{\pm\hat{x}}^\nu d^3x \approx -v^2 R^2 \sigma \approx M_{\text{warp}}. \quad (20)$$

The key things to note here are that the net volume integral of the $O(v)$ term is zero, and that the net volume average of the NEC violations is approximately the same as the net volume average of the WEC violations, which are $O(v^2)$.

2.2. Natário warp drive

The alternative version of the warp bubble due to José Natário demonstrates that the contraction/expansion referred to above is not always a feature of the warp drive [2]. In his construction a compression in the radial direction is exactly balanced by an expansion in the perpendicular direction, so that the shift vector is divergence-free $\nabla \cdot \beta = 0$. To verify this consider spherical coordinates (r, φ, ϕ) in the Euclidean 3-space (see [2] for details). The diagonal components of the extrinsic curvature are given by

$$K_{rr} = -2vf' \cos \varphi \quad \text{and} \quad K_{\varphi\varphi} = K_{\phi\phi} = vf' \cos \varphi, \quad (21)$$

respectively, where the prime denotes a derivative with respect to r . Taking into account the definition of the expansion of the volume element given by $\theta = K^i_i$, from equations (21), we simply have $\theta = K_{rr} + K_{\varphi\varphi} + K_{\phi\phi} = 0$. This analysis provides a certain insight into the geometry of the spacetime. For instance, consider the front of the warp bubble, with $\cos \varphi > 0$, such that $K_{rr} < 0$ (see [2] for details). This indicates a compression in the radial

direction, which is compensated by an expansion, $K_{\varphi\varphi} + K_{\phi\phi} = -K_{rr}$, in the perpendicular direction. Thus, this warp drive spacetime can be thought of as a warp bubble that pushes space aside and thereby ‘slides’ through space.

We now define β in the interior of the warp bubble to be v , and assume the radius of the warp bubble to be R . Furthermore we take β to fall to zero over a distance of order $1/\sigma$. Appendix B contains a full calculation of all the orthonormal components of the Einstein tensor for the Natário warp bubble, useful for analysis of energy condition violations in this class of spacetimes.

For the WEC we use the results of appendix B to write

$$T_{\mu\nu}U^\mu U^\nu = \frac{1}{8\pi}G_{\hat{t}\hat{t}} = -\frac{1}{16\pi}\text{Tr}(\mathbf{K}^2). \quad (22)$$

This is manifestly negative and so the WEC is violated everywhere throughout the spacetime.

We can again quantify the ‘total amount’ of energy condition violating matter in the warp bubble by defining

$$\begin{aligned} M_{\text{warp}} &= \int \rho_{\text{warp}} d^3x = \int T_{\mu\nu}U^\mu U^\nu d^3x \\ &= -\frac{1}{16\pi} \int \text{Tr}(\mathbf{K}^2) d^3x = -\frac{1}{16\pi} \int \beta_{(i,j)}\beta_{(i,j)} d^3x. \end{aligned} \quad (23)$$

Now β in the interior of the warp bubble is by definition v , while the size of the warp bubble is taken to be R . Furthermore gradients of β in the bubble walls are of order $v\sigma$, and the thickness of the bubble walls is of order $1/R$. So we can again estimate (as for the Alcubierre warp bubble)

$$M_{\text{warp}} \approx -v^2 R^2 \sigma. \quad (24)$$

With a bit more work we can verify that the NEC is violated throughout the spacetime. Consider a null vector pointing in the direction \hat{n} , so that $k^\mu = (1, \hat{n}^i)$. Then

$$T_{\mu\nu}k^\mu k^\nu = \frac{1}{8\pi}[G_{\hat{t}\hat{t}} + 2\hat{n}^i G_{\hat{t}\hat{i}} + \hat{n}^i \hat{n}^j G_{\hat{i}\hat{j}}]. \quad (25)$$

If the NEC is to hold, then this must be positive in both the \hat{n} and $-\hat{n}$ directions, so that we must have

$$\frac{1}{2}\{T_{\mu\nu}k_{+\hat{n}}^\mu k_{+\hat{n}}^\nu + T_{\mu\nu}k_{-\hat{n}}^\mu k_{-\hat{n}}^\nu\} = \frac{1}{8\pi}[G_{\hat{t}\hat{t}} + \hat{n}^i \hat{n}^j G_{\hat{i}\hat{j}}] \geq 0. \quad (26)$$

But now averaging over the x , y and z directions we see that the NEC requires

$$T_{\hat{t}\hat{t}} + \sum_i T_{\hat{i}\hat{i}} = \frac{1}{8\pi} \left[G_{\hat{t}\hat{t}} + \sum_i G_{\hat{i}\hat{i}} \right] \geq 0. \quad (27)$$

But from appendix B we have

$$G_{\hat{t}\hat{t}} + \sum_i G_{\hat{i}\hat{i}} = -2\text{Tr}(\mathbf{K}^2) \quad (28)$$

which is manifestly negative, and so the NEC is violated everywhere. Note that \mathbf{K} is $O(v)$ and so we again see that the NEC violations persist to arbitrarily low warp-bubble velocity.

3. Linearized gravity applied to the warp drive

Our goal now is to try to build a more realistic model of a warp drive spacetime where the warp bubble is interacting with a finite mass spaceship. To do so we first consider the linearized

theory [33–35] applied to warp drive spacetimes, for non-relativistic velocities, $v \ll 1$. A brief summary of linearized gravity is presented in appendix C. In linearized theory, the spacetime metric is given by $ds^2 = (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu$, with $h_{\mu\nu} \ll 1$ and $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. Taking into account equation (4), it is an exact statement that $h_{\mu\nu}$ has the following matrix elements:

$$(h_{\mu\nu}) = \begin{bmatrix} v^2 f^2 & 0 & 0 & -vf \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -vf & 0 & 0 & 0 \end{bmatrix}. \quad (29)$$

Now the results deduced from applying linearized theory are only accurate to first order in v . This is equivalent to neglecting the $h_{00} = v^2 f^2$ term in the matrix $(h_{\mu\nu})$, retaining only the first-order terms in v . Thus, we have the following approximation:

$$(h_{\mu\nu}) = \begin{bmatrix} 0 & 0 & 0 & -vf \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -vf & 0 & 0 & 0 \end{bmatrix}. \quad (30)$$

The trace of $h_{\mu\nu}$ is identically null, i.e., $h = h^\mu{}_\mu = 0$. Therefore, the trace reverse of $h_{\mu\nu}$, defined in equation (C.6), is given by $\bar{h}_{\mu\nu} = h_{\mu\nu}$, i.e., equation (30) itself.

The following relation will be useful in determining the linearized Einstein tensor,

$$\frac{\partial^2 \bar{h}_{03}}{\partial x^\alpha \partial x^\beta} = -v \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta}, \quad (31)$$

where we consider the bubble velocity parameter to be constant, $v = \text{constant}$. For simplicity we shall use the original form function given by Alcubierre, equation (5), so that equation (31) takes the form

$$\frac{\partial^2 \bar{h}_{03}}{\partial x^\alpha \partial x^\beta} = -v \left(\frac{\partial r}{\partial x^\alpha} \frac{\partial r}{\partial x^\beta} \frac{d^2 f}{dr^2} + \frac{\partial^2 r}{\partial x^\alpha \partial x^\beta} \frac{df}{dr} \right). \quad (32)$$

3.1. The weak energy condition (WEC)

The linearized theory applied to Alcubierre's warp drive, for non-relativistic velocities ($v \ll 1$), is an immediate application of the linearized Einstein tensor, equation (C.7), and there is no need to impose the Lorenz gauge³ [36]. The interest of this exercise lies in the application of the WEC. In linearized theory the 4-velocity can be approximated by $U^\mu = (1, 0, 0, 0)$, therefore the WEC reduces to

$$T_{\mu\nu} U^\mu U^\nu = T_{00} = \frac{1}{8\pi} G_{00}, \quad (33)$$

and from equation (C.7), we have

$$G_{00} = -\frac{1}{2} (\bar{h}_{00,\mu}{}^\mu - \bar{h}_{\mu\nu,}{}^{\mu\nu} - 2\bar{h}_{0\mu,0}{}^\mu). \quad (34)$$

The respective terms are given by

$$\bar{h}_{00,\mu}{}^\mu = 0, \quad (35)$$

$$\bar{h}_{\mu\nu,}{}^{\mu\nu} = -2\bar{h}_{03,03}, \quad (36)$$

³ The Lorenz gauge was apparently first used by the Danish physicist Ludwig Lorenz (1829–1891), though it is commonly misattributed to the Dutch physicist Hendrik Antoon Lorentz (1853–1928).

$$\bar{h}_{0\mu,0}{}^\mu = \bar{h}_{03,03}. \quad (37)$$

In general, if second time derivatives appear, they can be neglected because $\partial/\partial t$ is of the same order as $v\partial/\partial z$, so that $\partial_\mu \partial^\mu = \nabla^2 + O(v^2 \nabla^2)$. In fact, it is possible to prove that the Einstein tensor components G_{00} and G_{0i} do not contain second time derivatives of any generic $\bar{h}_{\mu\nu}$ [33]. That is, only the six equations $G_{ij} = 8\pi T_{ij}$ are true dynamical equations. In contrast, the equations $G_{0\mu} = 8\pi T_{0\mu}$ are called *constraint equations* because they are relations among the initial data for the other six equations; which prevent one from freely choosing the initial data.

Substituting equations (35)–(37) into equation (34), we have

$$G_{00} = O(v^2). \quad (38)$$

Thus, equation (33) is given by

$$T_{\mu\nu} U^\mu U^\nu = T_{00} = O(v^2), \quad (39)$$

and the WEC is identically ‘saturated’. Although in this approximation the WEC is not violated, it is on the verge of being so (to first order in v). This is compatible with the ‘exact’ non-perturbative calculation previously performed.

3.2. Negative energy density in boosted inertial frames

Despite the fact that the observers, with $U^\mu = (1, 0, 0, 0)$, measure zero energy density (more precisely $O(v^2)$), it can be shown that the observers which move with any other arbitrary velocity, $\tilde{\beta}$, along the positive z -axis measure a negative energy density (at $O(v)$). That is, $T_{\hat{0}\hat{0}} < 0$. The $\tilde{\beta}$ occurring here is completely independent of the shift vector $\beta(x, y, z - z_0(t))$, and is also completely independent of the warp bubble velocity v . Consider a Lorentz transformation, $x^{\hat{\mu}} = \Lambda^{\hat{\mu}}{}_{\nu} x^\nu$, with $\Lambda^{\hat{\mu}}{}_{\hat{\alpha}} \Lambda^{\hat{\alpha}}{}_{\nu} = \delta^{\hat{\mu}}{}_{\nu}$ and $\Lambda^{\hat{\mu}}{}_{\hat{\nu}}$ defined as

$$(\Lambda^{\hat{\mu}}{}_{\hat{\nu}}) = \begin{bmatrix} \gamma & 0 & 0 & \gamma \tilde{\beta} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma \tilde{\beta} & 0 & 0 & \gamma \end{bmatrix}, \quad (40)$$

with $\gamma = (1 - \tilde{\beta}^2)^{-1/2}$. The energy density measured by these observers is given by $T_{\hat{0}\hat{0}} = \Lambda^{\hat{\mu}}{}_{\hat{0}} \Lambda^{\hat{\nu}}{}_{\hat{0}} T_{\mu\nu}$. That is

$$T_{\hat{0}\hat{0}} = \gamma^2 T_{00} + 2\gamma^2 \tilde{\beta} T_{03} + \gamma^2 \tilde{\beta}^2 T_{33} \quad (41)$$

$$\begin{aligned} &= \gamma^2 \tilde{\beta} (2T_{03} + \tilde{\beta} T_{33}) \\ &= \frac{\gamma^2 \tilde{\beta}}{8\pi} (2G_{03} + \tilde{\beta} G_{33}) \end{aligned} \quad (42)$$

taking into account $T_{00} = 0$, due to equation (39). The respective Einstein tensor components, G_{03} and G_{33} , are given by

$$G_{03} = -\frac{1}{2}(\bar{h}_{03,11} + \bar{h}_{03,22}) + O(v^2), \quad (43)$$

$$G_{33} = O(v^2). \quad (44)$$

Finally, substituting these into equation (42), to a first-order approximation in terms of v , we have

$$T_{\hat{0}\hat{0}} = -\frac{\gamma^2 \tilde{\beta}}{8\pi} (\bar{h}_{03,11} + \bar{h}_{03,22}) + O(v^2). \quad (45)$$

Taking into account equation (31), we have

$$T_{\hat{0}\hat{0}} = \frac{\gamma^2 \tilde{\beta} v}{8\pi} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) + O(v^2). \quad (46)$$

Applying equation (32), we have the following relations,

$$\bar{h}_{03,11} = -v \left[\frac{x^2}{r^2} \frac{d^2 f}{dr^2} + \left(\frac{1}{r} - \frac{x^2}{r^3} \right) \frac{df}{dr} \right], \quad (47)$$

$$\bar{h}_{03,22} = -v \left[\frac{y^2}{r^2} \frac{d^2 f}{dr^2} + \left(\frac{1}{r} - \frac{y^2}{r^3} \right) \frac{df}{dr} \right], \quad (48)$$

and substituting into equation (45) we therefore have

$$T_{\hat{0}\hat{0}} = \frac{\gamma^2 \tilde{\beta} v}{8\pi} \left[\left(\frac{x^2 + y^2}{r^2} \right) \frac{d^2 f}{dr^2} + \left(\frac{x^2 + y^2 + 2(z - z_0(t))^2}{r^3} \right) \frac{df}{dr} \right] + O(v^2). \quad (49)$$

A number of general features can be extracted from the terms in square brackets, without specifying an explicit form of f . In particular, f decreases monotonically from its value at $r = 0$, $f = 1$, to $f \approx 0$ at $r \geq R$, so that df/dr is negative in this domain. The form function attains its maximum in the interior of the bubble wall, so that $d^2 f/dr^2$ is also negative in this region. Therefore there is a range of r in the immediate interior neighbourhood of the bubble wall that necessarily provides negative energy density, as seen by the observers considered above. Again we find that WEC violations persist to arbitrarily low warp bubble velocities.

3.3. The null energy condition (NEC)

The NEC states that $T_{\mu\nu} k^\mu k^\nu \geq 0$, where k^μ is a null vector. Considering $k^\mu = (1, 0, 0, \pm 1)$, we have

$$\begin{aligned} T_{\mu\nu} k^\mu k^\nu &= \frac{1}{8\pi} (G_{00} \pm 2G_{03} + G_{33}) \\ &= \mp \frac{1}{8\pi} (\bar{h}_{03,11} + \bar{h}_{03,22}) + O(v^2) \\ &= \pm \frac{v}{8\pi} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) + O(v^2). \end{aligned} \quad (50)$$

Whatever the value of the bracketed term, as long as it is nonzero the NEC will be violated (for small enough v) in either the $+z$ or $-z$ directions. This is compatible with the ‘exact’ non-perturbative calculation previously performed.

Using the general form function defined by Alcubierre, equation (50) reduces to

$$T_{\mu\nu} k^\mu k^\nu = \pm \frac{v}{8\pi} \left[\left(\frac{x^2 + y^2}{r^2} \right) \frac{d^2 f}{dr^2} + \left(\frac{x^2 + y^2 + 2(z - z_0(t))^2}{r^3} \right) \frac{df}{dr} \right] + O(v^2). \quad (51)$$

Equation (51) is proportional to the energy density, $T_{\hat{0}\hat{0}}$, of equation (49). We verify that the term in square brackets has a region for which it is negative, thus also violating the NEC in the immediate interior vicinity of the bubble wall.

3.4. Linearized gravity applied to the spaceship

The weak gravitational field of a static source, in particular of a spaceship, is given by the following metric:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 - 2\Phi(x, y, z)(dt^2 + dx^2 + dy^2 + dz^2). \quad (52)$$

Applying the linearized theory with $ds^2 = (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu$ and $h_{\mu\nu} \ll 1$, the matrix elements of $h_{\mu\nu}$ are given by

$$(h_{\mu\nu}) = \begin{bmatrix} -2\Phi & 0 & 0 & 0 \\ 0 & -2\Phi & 0 & 0 \\ 0 & 0 & -2\Phi & 0 \\ 0 & 0 & 0 & -2\Phi \end{bmatrix}. \quad (53)$$

The trace is given by $h = h^\mu{}_\mu = -4\Phi$. The elements of the trace reverse, $\bar{h}_{\mu\nu}$, are the following:

$$(\bar{h}_{\mu\nu}) = \begin{bmatrix} -4\Phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (54)$$

Applying the linearized Einstein tensor, equation (C.7), to the component, G_{00} , we have Poisson’s equation

$$\nabla^2 \Phi = 4\pi\rho, \quad (55)$$

in which ρ is the mass density of the spaceship. This case is comparable to the Newtonian limit, which is valid when the gravitational fields are too weak to produce velocities near the speed of light, that is, $|\Phi| \ll 1$ and $|v| \ll 1$. For such cases general relativity makes the same predictions as Newtonian gravity. For fields that change due to the movement of the source, for instance along the z -axis with a velocity of v , we verify that $\partial/\partial t$ is of the same order as $v\partial/\partial z$. It is easy to verify that the energy fluxes or the momentum densities, T_{0i} are proportional to first order in v , and the stresses, T_{ij} , to second order. Therefore the stress–energy tensor $T_{\mu\nu}$ obeys the inequalities $|T_{00}| \gg |T_{0i}| \gg |T_{ij}|$.

3.5. Spaceship immersed in the warp bubble

Consider now a spaceship in the interior of an Alcubierre warp bubble, which is moving along the positive z -axis with a non-relativistic constant velocity. That is, $v \ll 1$. The metric is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + [dz - vf(x, y, z - vt) dt]^2 - 2\Phi(x, y, z - vt)[dt^2 + dx^2 + dy^2 + (dz - vf(x, y, z - vt) dt)^2]. \quad (56)$$

If $\Phi = 0$, metric (56) reduces to the warp drive spacetime of equation (4). If $v = 0$, we have the metric representing the gravitational field of a static source, equation (52).

Applying the transformation, $z' = z - vt$, metric (56) takes the form

$$ds^2 = -dt^2 + dx^2 + dy^2 + [dz' - v\{f(x, y, z') - 1\} dt]^2 - 2\Phi(x, y, z')[dt^2 + dx^2 + dy^2 + (dz' - v\{f(x, y, z') - 1\} dt)^2]. \quad (57)$$

Note that the metric now looks ‘static’. Considering an observer in the interior of the bubble, co-moving with the spaceship, metric (57), with $f = 1$, reduces to

$$ds^2 \rightarrow -dt^2 + dx^2 + dy^2 + dz'^2 - 2\Phi(x, y, z')[dt^2 + dx^2 + dy^2 + dz'^2], \quad (58)$$

so that inside the bubble we simply have the weak gravitational field of a static source, that of the spaceship.

Outside the warp bubble, where $f = 0$, metric (57) reduces to

$$ds^2 \rightarrow -dt^2 + dx^2 + dy^2 + [dz' + v dt]^2 - 2\Phi(x, y, z')[dt^2 + dx^2 + dy^2 + (dz' + v dt)^2]. \quad (59)$$

3.5.1. First-order approximation. Applying the linearized theory, keeping terms linear in v and Φ but neglecting all superior order terms so that the approximation of equation (C.4) is valid, the matrix elements, $h_{\mu\nu}$, of the metric (56) are given by the following approximation:

$$(h_{\mu\nu}) = \begin{bmatrix} -2\Phi & 0 & 0 & -vf \\ 0 & -2\Phi & 0 & 0 \\ 0 & 0 & -2\Phi & 0 \\ -vf & 0 & 0 & -2\Phi \end{bmatrix}. \quad (60)$$

The trace of $h_{\mu\nu}$ is given by $h = h^\mu{}_\mu = -4\Phi$, and the trace-reversed elements, $\bar{h}_{\mu\nu}$, take the following form:

$$(\bar{h}_{\mu\nu}) = \begin{bmatrix} -4\Phi & 0 & 0 & -vf \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -vf & 0 & 0 & 0 \end{bmatrix}. \quad (61)$$

Therefore, from the WEC, $T_{\mu\nu}U^\mu U^\nu = T_{00}$, the trace-reversed elements that one needs to determine G_{00} (see equation (34)) are the following:

$$\bar{h}_{00,\mu}{}^\mu = -\bar{h}_{00,00} + \bar{h}_{00,11} + \bar{h}_{00,22} + \bar{h}_{00,33}, \quad (62)$$

$$\bar{h}_{\mu\nu,\mu}{}^{\mu\nu} = \bar{h}_{00,00} - 2\bar{h}_{03,03}, \quad (63)$$

$$\bar{h}_{0\mu,0}{}^\mu = -\bar{h}_{00,00} + \bar{h}_{03,03}. \quad (64)$$

Substituting equations (62)–(64) into equation (34), we have

$$G_{00} = 2\nabla^2\Phi + O(v^2, v\Phi, \Phi^2). \quad (65)$$

Thus, taking into account Poisson's equation, we verify that the WEC is given by

$$T_{\mu\nu}U^\mu U^\nu = \rho + O(v^2, v\Phi, \Phi^2), \quad (66)$$

where ρ is now the ordinary energy density of the spaceship which is manifestly positive. In linearized theory, the total ADM mass of the spacetime simply reduces to the mass of the spaceship, i.e.,

$$M_{\text{ADM}} = \int T_{00} d^3x = \int \rho d^3x + O(v^2, v\Phi, \Phi^2) = M_{\text{ship}} + O(v^2, v\Phi, \Phi^2). \quad (67)$$

The good news for the warp drive aficionados is that the dominant term is manifestly positive.

The NEC, with $k^\mu \equiv (1, 0, 0, \pm 1)$, takes the form

$$T_{\mu\nu}k^\mu k^\nu = \frac{1}{8\pi}(G_{00} \pm 2G_{03} + G_{33}). \quad (68)$$

The Einstein tensor components, G_{03} and G_{33} , are given by

$$G_{03} = -\frac{1}{2}(\bar{h}_{03,\mu}{}^\mu - \bar{h}_{0\mu,3}{}^\mu - \bar{h}_{3\mu,0}{}^\mu), \quad (69)$$

$$G_{33} = -\frac{1}{2}(\bar{h}_{\mu\nu,\mu}{}^{\mu\nu} - 2\bar{h}_{3\mu,3}{}^\mu), \quad (70)$$

with the respective trace-reversed terms

$$\bar{h}_{03,\mu}{}^\mu = -\bar{h}_{03,00} + \bar{h}_{03,11} + \bar{h}_{03,22} + \bar{h}_{03,33} \quad (71)$$

$$\bar{h}_{0\mu,3}{}^\mu = -\bar{h}_{00,30} + \bar{h}_{03,33} \quad (72)$$

$$\bar{h}_{3\mu,0}{}^\mu = -\bar{h}_{30,00} \quad (73)$$

$$\bar{h}_{3\mu,3}{}^\mu = -\bar{h}_{30,30} \quad (74)$$

and the term $\bar{h}_{\mu\nu}{}^{\mu\nu}$ is given by equation (63). Substituting these into equations (69), (70), we have

$$G_{03} = \frac{v}{2} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) + O(v^2, v\Phi, \Phi^2), \quad (75)$$

$$G_{33} = O(v^2, v\Phi, \Phi^2), \quad (76)$$

to first order in v and Φ , and neglecting the crossed terms $v\Phi$.

The null energy condition is thus given by

$$T_{\mu\nu}k^\mu k^\nu = \rho \pm \frac{v}{8\pi} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) + O(v^2, v\Phi, \Phi^2). \quad (77)$$

From this, one can deduce the existence of localized NEC violations even in the presence of a finite mass spaceship, and can also make deductions about the net volume-averaged NEC violations. First, note that for reasons of structural integrity one wants the spaceship itself to lie well inside the warp bubble, and not overlap with the walls of the warp bubble. But this means that the region where $\rho \neq 0$ does not overlap with the region where the $O(v)$ contribution due to the warp field is non-zero. So regardless of how massive the spaceship itself is, there will be regions in the wall of the warp bubble where localized violations of NEC certainly occur. If we now look at the volume integral of the NEC, the $O(v)$ contributions integrate to zero and we have

$$\int T_{\mu\nu}k_{\pm\hat{z}}^\mu k_{\pm\hat{z}}^\nu d^3x = \int \rho d^3x + O(v^2, v\Phi, \Phi^2) = M_{\text{ship}} + O(v^2, v\Phi, \Phi^2). \quad (78)$$

Consider a similar analysis for a null vector oriented perpendicular to the direction of motion, for instance, along the x -axis, so that $k^\mu = (1, \pm 1, 0, 0)$. The NEC then takes the form

$$T_{\mu\nu}k^\mu k^\nu = \frac{1}{8\pi} (G_{00} \pm 2G_{01} + G_{11}). \quad (79)$$

Taking into account equation (C.7), the respective Einstein tensor components are given by

$$G_{01} = -\frac{v}{2} \frac{\partial^2 f}{\partial x \partial z} + O(v^2, v\Phi, \Phi^2), \quad (80)$$

$$G_{11} = O(v^2, v\Phi, \Phi^2), \quad (81)$$

and G_{00} is given by equation (65). Thus, equation (79) takes the following form,

$$T_{\mu\nu}k^\mu k^\nu = \rho \mp \frac{v}{8\pi} \frac{\partial^2 f}{\partial x \partial z} + O(v^2, v\Phi, \Phi^2), \quad (82)$$

and considering the Alcubierre form function, we have

$$T_{\mu\nu}k^\mu k^\nu = \rho \mp \frac{v}{8\pi} \frac{x(z - z_0(t))}{r^2} \left(\frac{d^2 f}{dr^2} - \frac{1}{r} \frac{df}{dr} \right) + O(v^2, v\Phi, \Phi^2). \quad (83)$$

As for the situation when we considered null vectors aligned with the direction of motion, for these transverse null vectors we find localized NEC violations in the walls of the warp bubble. We also find that the volume integral of the $O(v)$ term is zero and that

$$\int T_{\mu\nu}k_{\pm\hat{x}}^\mu k_{\pm\hat{x}}^\nu d^3x = \int \rho d^3x + O(v^2, v\Phi, \Phi^2) = M_{\text{ship}} + O(v^2, v\Phi, \Phi^2). \quad (84)$$

The net result of this $O(v)$ calculation is that irrespective of the mass of the spaceship there will always be localized NEC violations in the wall of the warp bubble, and these localized NEC violations persist to arbitrarily low warp velocity. However at $O(v)$ the volume integral of the NEC violations is zero, and so we must look at higher order in v if we wish to deduce anything from the consideration of volume integrals to probe ‘net’ violations of the NEC.

3.5.2. Second-order approximation. Consider the approximation in which we keep the exact v dependence but linearize in the gravitational field of the spaceship Φ . The components of the Einstein tensor, relevant to determining the WEC and the NEC, are

$$G_{\hat{t}\hat{t}} = -\frac{v^2}{4} \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] + 2\nabla^2 \Phi + O(\Phi^2), \quad (85)$$

$$G_{\hat{z}\hat{z}} = -\frac{3}{4}v^2 \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] + O(\Phi^2), \quad (86)$$

$$G_{\hat{t}\hat{z}} = \frac{v}{2} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) + O(\Phi^2), \quad (87)$$

$$G_{\hat{t}\hat{x}} = -\frac{1}{2}v \frac{\partial^2 f}{\partial x \partial z} + O(\Phi^2), \quad (88)$$

$$G_{\hat{x}\hat{x}} = v^2 \left[\frac{1}{4} \left(\frac{\partial f}{\partial x} \right)^2 - \frac{1}{4} \left(\frac{\partial f}{\partial y} \right)^2 - \left(\frac{\partial f}{\partial z} \right)^2 + (1-f) \frac{\partial^2 f}{\partial z^2} \right] + O(\Phi^2). \quad (89)$$

The WEC is given by

$$T_{\hat{\mu}\hat{\nu}} U^{\hat{\mu}} U^{\hat{\nu}} = \rho - \frac{v^2}{32\pi} \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] + O(\Phi^2), \quad (90)$$

or by taking into account the Alcubierre form function, we have

$$T_{\mu\nu} U^\mu U^\nu = \rho - \frac{1}{32\pi} \frac{v^2(x^2 + y^2)}{r^2} \left(\frac{df}{dr} \right)^2 + O(\Phi^2). \quad (91)$$

Once again, using the ‘volume integral quantifier’, we find the following estimate,

$$\int T_{\hat{\mu}\hat{\nu}} U^{\hat{\mu}} U^{\hat{\nu}} d^3x = M_{\text{ship}} - v^2 R^2 \sigma + \int O(\Phi^2) d^3x, \quad (92)$$

which we can recast as

$$M_{\text{total}} = M_{\text{ship}} + M_{\text{warp}} + \int O(\Phi^2) d^3x, \quad (93)$$

where M_{total} is the net total integrated energy density, to the order of approximation considered, i.e., keeping the exact v dependence and linearizing in the gravitational field Φ .

Now suppose we demand that the volume integral of the WEC at least be positive, then

$$v^2 R^2 \sigma \leq M_{\text{ship}}. \quad (94)$$

This equation is effectively the quite reasonable condition that the net total energy stored in the warp field be less than the total mass–energy of the spaceship itself, which places a powerful constraint on the velocity of the warp bubble. Rewriting this in terms of the size of the spaceship R_{ship} and the thickness of the warp bubble walls $\Delta = 1/\sigma$, we have

$$v^2 \leq \frac{M_{\text{ship}}}{R_{\text{ship}}} \frac{R_{\text{ship}} \Delta}{R^2}. \quad (95)$$

For any reasonable spaceship this gives extremely low bounds on the warp bubble velocity.

In a similar manner, the NEC, with $k^\mu = (1, 0, 0, \pm 1)$, is given by

$$T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} = \rho \pm \frac{v}{8\pi} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) - \frac{v^2}{8\pi} \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] + O(\Phi^2). \quad (96)$$

Considering the ‘volume integral quantifier’, we verify that, as before, the exact solution in terms of polylogarithmic functions is unhelpful, although we may estimate that

$$\int T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} d^3x = M_{\text{ship}} - v^2 R^2 \sigma + \int O(\Phi^2) d^3x, \quad (97)$$

which is (to order $O(\Phi^2)$) the same integral we encountered when dealing with the WEC. This volume integrated NEC is now positive if

$$v^2 R^2 \sigma \leq M_{\text{ship}}. \quad (98)$$

Finally, considering a null vector oriented perpendicularly to the direction of motion (for definiteness take $\hat{k} = \pm \hat{x}$), the NEC takes the following form:

$$T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} = \rho - \frac{v^2}{8\pi} \left[\frac{1}{2} \left(\frac{\partial f}{\partial y} \right)^2 + \left(\frac{\partial f}{\partial z} \right)^2 - (1-f) \frac{\partial^2 f}{\partial z^2} \right] \mp \frac{v}{8\pi} \left(\frac{\partial^2 f}{\partial x \partial z} \right) + O(\Phi^2). \quad (99)$$

Once again, evaluating the ‘volume integral quantifier’, we have

$$\begin{aligned} \int T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} d^3x &= M_{\text{ship}} - \frac{v^2}{4} \int \left(\frac{df}{dr} \right)^2 r^2 dr \\ &+ \frac{v^2}{6} \int (1-f) \left(2r \frac{df}{dr} + r^2 \frac{d^2 f}{dr^2} \right) dr + \int O(\Phi^2) d^3x, \end{aligned} \quad (100)$$

which, as before, may be estimated as

$$\int T_{\hat{\mu}\hat{\nu}}k^{\hat{\mu}}k^{\hat{\nu}} d^3x \approx M_{\text{ship}} - v^2 R^2 \sigma + \int O(\Phi^2) d^3x. \quad (101)$$

If we do not want the total NEC violations in the warp field to exceed the mass of the spaceship itself we must again demand

$$v^2 R^2 \sigma \leq M_{\text{ship}}, \quad (102)$$

which places a strong constraint on the velocity of the warp bubble.

4. Summary and discussion

In this paper we have seen how the warp drive spacetimes (in particular, the Alcubierre and Natário warp drives) can be used as gedanken experiments to probe the foundations of general relativity. Though they are useful toy models for theoretical investigations, as potential technology they are greatly lacking. We have verified that the non-perturbative exact solutions of the warp drive spacetimes necessarily violate the classical energy conditions, and continue to do so for arbitrarily low warp bubble velocity—thus the energy condition violations in this class of spacetimes are generic to the form of the geometry under consideration and are not simply a side effect of the ‘superluminal’ properties.

Furthermore, by taking into account the notion of the ‘volume integral quantifier’, we have also verified that the ‘total amount’ of energy condition violating matter in the warp bubble is negative. Using linearized theory, we have built a more realistic model of the warp drive spacetime where the warp bubble interacts with a finite mass spaceship. We have tested and quantified the energy conditions to first and second order of the warp bubble

velocity. By doing so we have been able to safely ignore the causality problems associated with ‘superluminal’ motion, and so have focussed attention on a previously unremarked feature of the ‘warp drive’ spacetime. If it is possible to realize even a weak-field warp drive in nature, such a spacetime appears to be an example of a ‘reaction-less drive’. That is, the warp bubble moves by interacting with the geometry of spacetime instead of expending reaction mass, and the spaceship (which in linearized theory can be treated as a finite mass object placed within the warp bubble) is simply carried along with it. We have verified that in this case, the ‘total amount’ of energy condition violating matter (the ‘net’ negative energy of the warp field) must be an appreciable fraction of the positive mass of the spaceship carried along by the warp bubble. This places an extremely stringent condition on the warp drive spacetime, namely, that for all conceivably interesting situations the bubble velocity should be absurdly low, and it therefore appears unlikely that, by using this analysis, the warp drive will ever prove to be technologically useful. Finally, we point out that any attempt at building up a ‘strong-field’ warp drive starting from an approximately Minkowski spacetime will inevitably have to pass through a weak-field regime. Since the weak-field warp drives are already so tightly constrained, the analysis of this paper implies additional difficulties for developing a ‘strong field’ warp drive. In particular we wish to sound a strong cautionary note against over-enthusiastic mis-interpretation of the technological situation.

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Appendix A. The Einstein tensor for the Alcubierre warp drive

Appendix A.1. Generic form function

The Einstein tensor is given by $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. The Einstein tensor components of the warp drive spacetime, in Cartesian coordinates, in an orthonormal basis, $G_{\hat{\mu}\hat{\nu}}$, with a generic form function, $f(x, y, z - z_0(t))$, are given by

$$G_{\hat{t}\hat{t}} = -\frac{1}{4}v^2 \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right], \quad (\text{A.1})$$

$$G_{\hat{t}\hat{x}} = -\frac{1}{2}v \frac{\partial^2 f}{\partial x \partial z}, \quad (\text{A.2})$$

$$G_{\hat{t}\hat{y}} = -\frac{1}{2}v \frac{\partial^2 f}{\partial y \partial z}, \quad (\text{A.3})$$

$$G_{\hat{t}\hat{z}} = \frac{1}{2}v \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right), \quad (\text{A.4})$$

$$G_{\hat{x}\hat{x}} = v^2 \left[\frac{1}{4} \left(\frac{\partial f}{\partial x} \right)^2 - \frac{1}{4} \left(\frac{\partial f}{\partial y} \right)^2 - \left(\frac{\partial f}{\partial z} \right)^2 + (1 - f) \frac{\partial^2 f}{\partial z^2} \right], \quad (\text{A.5})$$

$$G_{\hat{x}\hat{y}} = \frac{1}{2}v^2 \left(\frac{\partial f}{\partial y} \right) \left(\frac{\partial f}{\partial x} \right), \quad (\text{A.6})$$

$$G_{\hat{x}\hat{z}} = v^2 \left[\left(\frac{\partial f}{\partial z} \right) \left(\frac{\partial f}{\partial x} \right) - \frac{1}{2}(1-f) \frac{\partial^2 f}{\partial x \partial z} \right], \quad (\text{A.7})$$

$$G_{\hat{y}\hat{z}} = v^2 \left[\left(\frac{\partial f}{\partial z} \right) \left(\frac{\partial f}{\partial y} \right) - \frac{1}{2}(1-f) \frac{\partial^2 f}{\partial y \partial z} \right], \quad (\text{A.8})$$

$$G_{\hat{y}\hat{y}} = v^2 \left[\frac{1}{4} \left(\frac{\partial f}{\partial y} \right)^2 - \frac{1}{4} \left(\frac{\partial f}{\partial x} \right)^2 - \left(\frac{\partial f}{\partial z} \right)^2 + (1-f) \frac{\partial^2 f}{\partial z^2} \right], \quad (\text{A.9})$$

$$G_{\hat{z}\hat{z}} = -\frac{3}{4} v^2 \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right]. \quad (\text{A.10})$$

Appendix A.2. Alcubierre form function

Taking into account Alcubierre’s choice of the form function, i.e., $f(r)$ with $r = \sqrt{x^2 + y^2 + [z - z_0(t)]^2}$, the Einstein tensor components of the warp drive spacetime, in Cartesian coordinates, in an orthonormal basis, $G_{\hat{\mu}\hat{\nu}}$, take the following form:

$$G_{\hat{t}\hat{t}} = -\frac{v^2}{4} \frac{(x^2 + y^2)}{r^2} \left(\frac{df}{dr} \right)^2, \quad (\text{A.11})$$

$$G_{\hat{t}\hat{x}} = -\frac{v}{2} \frac{x(z - z_0(t))}{r^2} \left(\frac{d^2 f}{dr^2} - \frac{1}{r} \frac{df}{dr} \right), \quad (\text{A.12})$$

$$G_{\hat{t}\hat{y}} = -\frac{v}{2} \frac{y(z - z_0(t))}{r^2} \left(\frac{d^2 f}{dr^2} - \frac{1}{r} \frac{df}{dr} \right), \quad (\text{A.13})$$

$$G_{\hat{t}\hat{z}} = \frac{v}{2} \left[\frac{x^2 + y^2}{r^2} \frac{d^2 f}{dr^2} + \frac{x^2 + y^2 + 2(z - z_0(t))^2}{r^3} \frac{df}{dr} \right], \quad (\text{A.14})$$

$$G_{\hat{x}\hat{x}} = \frac{v^2}{4} \frac{x^2 - y^2 - 4(z - z_0(t))^2}{r^2} \left(\frac{df}{dr} \right)^2 + v^2(1-f) \left[\frac{(z - z_0(t))^2}{r^2} \frac{d^2 f}{dr^2} + \frac{x^2 + y^2}{r^3} \frac{df}{dr} \right], \quad (\text{A.15})$$

$$G_{\hat{x}\hat{y}} = \frac{v^2}{2} \frac{xy}{r^2} \left(\frac{df}{dr} \right)^2, \quad (\text{A.16})$$

$$G_{\hat{x}\hat{z}} = \frac{v^2}{2} \frac{x(z - z_0(t))}{r^2} \left[2 \left(\frac{df}{dr} \right)^2 - (1-f) \left(\frac{d^2 f}{dr^2} - \frac{1}{r} \frac{df}{dr} \right) \right], \quad (\text{A.17})$$

$$G_{\hat{y}\hat{z}} = \frac{v^2}{2} \frac{y(z - z_0(t))}{r^2} \left[2 \left(\frac{df}{dr} \right)^2 - (1-f) \left(\frac{d^2 f}{dr^2} - \frac{1}{r} \frac{df}{dr} \right) \right], \quad (\text{A.18})$$

$$G_{\hat{y}\hat{y}} = \frac{v^2}{4} \frac{y^2 - x^2 - 4(z - z_0(t))^2}{r^2} \left(\frac{df}{dr} \right)^2 + v^2(1-f) \left[\frac{(z - z_0(t))^2}{r^2} \frac{d^2 f}{dr^2} + \frac{x^2 + y^2}{r^3} \frac{df}{dr} \right], \quad (\text{A.19})$$

$$G_{\hat{z}\hat{z}} = -\frac{3}{4} v^2 \frac{(x^2 + y^2)}{r^2} \left(\frac{df}{dr} \right)^2. \quad (\text{A.20})$$

Appendix A.3. Linearized Einstein tensor for the Alcubierre warp drive

Linearizing in the warp velocity is very easy. The only nonzero components are

$$G_{\hat{t}\hat{x}} = -\frac{1}{2}v \frac{\partial^2 f}{\partial x \partial z}, \quad (\text{A.21})$$

$$G_{\hat{t}\hat{y}} = -\frac{1}{2}v \frac{\partial^2 f}{\partial y \partial z}, \quad (\text{A.22})$$

$$G_{\hat{t}\hat{z}} = \frac{1}{2}v \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right). \quad (\text{A.23})$$

All other components are $O(v^2)$.

For the Alcubierre form function $f(r)$ we find that at linear order in warp bubble velocity

$$G_{\hat{t}\hat{x}} = -\frac{v}{2} \frac{x(z - z_0(t))}{r^2} \left(\frac{d^2 f}{dr^2} - \frac{1}{r} \frac{df}{dr} \right), \quad (\text{A.24})$$

$$G_{\hat{t}\hat{y}} = -\frac{v}{2} \frac{y(z - z_0(t))}{r^2} \left(\frac{d^2 f}{dr^2} - \frac{1}{r} \frac{df}{dr} \right), \quad (\text{A.25})$$

$$G_{\hat{t}\hat{z}} = \frac{v}{2} \left[\frac{x^2 + y^2}{r^2} \frac{d^2 f}{dr^2} + \frac{x^2 + y^2 + 2(z - z_0(t))^2}{r^3} \frac{df}{dr} \right]. \quad (\text{A.26})$$

All other components are $O(v^2)$.

Appendix B. The Einstein tensor for the Natário warp drive

The Einstein tensor for the Natário warp drive is relatively easy to calculate once one realizes that it can be dealt with within the framework of ‘shift-only’ spacetimes. The extrinsic curvature of the constant- t time slices is

$$K_{ij} = \beta_{(i;j)} = \frac{1}{2}[\partial_i \beta_j + \partial_j \beta_i] \quad (\text{B.1})$$

and the no expansion condition implies

$$K = \text{Tr}(\mathbf{K}) = 0 \quad (\text{B.2})$$

so the trace of extrinsic curvature is zero. The intrinsic curvature of the constant- t time slices is by construction zero (since they are flat). It is useful to introduce a vorticity tensor

$$\Omega_{ij} = -\beta_{[i;j]} = \frac{1}{2}[\partial_i \beta_j - \partial_j \beta_i] \quad (\text{B.3})$$

and then a tedious but straightforward computation (which follows a variant of the Gauss–Codazzi decomposition) yields

$$R_{ijkl} = K_{ik} K_{jl} - K_{il} K_{jk}, \quad (\text{B.4})$$

$$R_{tijk} = -\partial_i \Omega_{jk} + \beta_l (K_{kl} K_{ij} - K_{jl} K_{ik}), \quad (\text{B.5})$$

$$R_{titt} = -\partial_t K_{ij} + (\mathbf{K}\mathbf{\Omega} + \mathbf{\Omega}\mathbf{K})_{ij} - (\mathbf{K}^2)_{ij} - \beta_k \beta_{k,ij} + \beta_k \beta_l (K_{kl} K_{ij} - K_{jk} K_{il}). \quad (\text{B.6})$$

Here we have defined

$$(\mathbf{K}\mathbf{\Omega} + \mathbf{\Omega}\mathbf{K})_{ij} \equiv K_{ik} \Omega_{kj} + \Omega_{ik} K_{kj}, \quad (\text{B.7})$$

and similarly

$$(\mathbf{K}^2)_{ij} \equiv K_{ik}K_{kj}. \quad (\text{B.8})$$

The computation reported here for the Natário warp drive is identical in spirit (and differs in only notation and minor technical details) to the computation of the ‘acoustic’ Riemann tensor for the ‘acoustic geometry’ reported in [37, 38]. Specifically, when sound waves propagate in a constant-density constant speed-of-sound background fluid flow the acoustic metric is identical in form to Natário’s metric.

The Einstein tensor is now (in an orthonormal basis)

$$G_{\hat{t}\hat{t}} = -\frac{1}{2} \text{Tr}(\mathbf{K}^2), \quad (\text{B.9})$$

$$G_{\hat{t}\hat{i}} = \frac{1}{2} \Delta \beta_i, \quad (\text{B.10})$$

$$G_{\hat{i}\hat{j}} = \frac{d}{dt} K_{ij} - \frac{1}{2} \delta_{ij} \text{Tr}(\mathbf{K}^2) - (\mathbf{K}\boldsymbol{\Omega} + \boldsymbol{\Omega}\mathbf{K})_{ij}, \quad (\text{B.11})$$

where d/dt denotes the usual advective derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \beta^i \frac{\partial}{\partial x^i}. \quad (\text{B.12})$$

In view of the fact that for the Natário spacetime all the time dependence comes from motion of the warp bubble, so that all fields have a spacetime dependence of the form $\beta(x, y, z - z_0(t))$, we can deduce

$$\frac{d}{dt} \rightarrow -v(t) \frac{\partial}{\partial z} + \beta^i \frac{\partial}{\partial x^i} = -[v(t)\hat{z} - \vec{\beta}] \cdot \nabla \quad (\text{B.13})$$

so that

$$G_{\hat{i}\hat{j}} \rightarrow -[v(t)\hat{z} - \vec{\beta}] \cdot \nabla K_{ij} - \frac{1}{2} \delta_{ij} \text{Tr}(\mathbf{K}^2) - (\mathbf{K}\boldsymbol{\Omega} + \boldsymbol{\Omega}\mathbf{K})_{ij}. \quad (\text{B.14})$$

If we now linearize, then the only surviving term in the Einstein tensor is $G_{\hat{t}\hat{i}} = \frac{1}{2} \Delta \beta_i$ as all other terms are $O(v^2)$.

A particularly simple special case of the Natário spacetime is obtained if the shift vector β is taken to be irrotational as well as being divergenceless [38]. In this special case $\boldsymbol{\Omega} \rightarrow 0$ and also $\Delta \beta_i \rightarrow 0$, so that (in an orthonormal basis)

$$G_{\hat{t}\hat{t}} = -\frac{1}{2} \text{Tr}(\mathbf{K}^2), \quad (\text{B.15})$$

$$G_{\hat{t}\hat{i}} = 0, \quad (\text{B.16})$$

$$G_{\hat{i}\hat{j}} = -[v(t)\hat{z} - \vec{\beta}] \cdot \nabla K_{ij} - \frac{1}{2} \delta_{ij} \text{Tr}(\mathbf{K}^2). \quad (\text{B.17})$$

When linearizing this special case we find that *all* components of the Einstein tensor are $O(v^2)$.

Appendix C. Linearized gravity

For a weak gravitational field, linearized around flat Minkowski spacetime, we can obtain the spacetime metric as [33–35]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (\text{C.1})$$

with $h_{\mu\nu} \ll 1$. It is convenient to raise and lower tensor indices with $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, respectively, rather than with $g^{\mu\nu}$ and $g_{\mu\nu}$. We shall adopt this notational convention, keeping in mind that the tensor $g^{\mu\nu}$ still denotes the inverse metric, and not $\eta^{\mu\alpha}\eta^{\nu\beta}g_{\alpha\beta}$. It should also be noted

that in linear approximation we have $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, since the composition of $g_{\mu\nu}$ and $g^{\mu\nu}$ differs from the identity operator only by terms quadratic in $h_{\mu\nu}$.

The linearized Einstein equation can be obtained in a straightforward manner as follows. Adopting the form of equation (C.1) for the metric components, the resulting connection coefficients, when linearized in the metric perturbation $h_{\mu\nu}$, yield

$$\begin{aligned}\Gamma^\mu_{\alpha\beta} &= \frac{1}{2}\eta^{\mu\nu}(h_{\alpha\nu,\beta} + h_{\beta\nu,\alpha} - h_{\alpha\beta,\nu}) \\ &= \frac{1}{2}(h^\mu_{\alpha,\beta} + h^\mu_{\beta,\alpha} - h_{\alpha\beta}{}^\mu{}_\nu).\end{aligned}\quad (\text{C.2})$$

A similar linearization of the Ricci tensor, $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$, given as a function of the Christoffel symbols, i.e.,

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}, \quad (\text{C.3})$$

results in the following approximation,

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} = \frac{1}{2}(h^\alpha_{\mu\nu,\alpha} + h^\alpha_{\nu,\mu\alpha} - h_{\mu\nu,\alpha}{}^\alpha), \quad (\text{C.4})$$

where $h = h^\alpha_\alpha = \eta^{\alpha\beta}h_{\beta\alpha}$ is the trace of $h_{\alpha\beta}$. Applying a further contraction to discover the Ricci scalar, $R = g^{\mu\nu}R_{\mu\nu} \simeq \eta^{\mu\nu}R_{\mu\nu}$, the Einstein tensor to linear order is given by

$$G_{\mu\nu} = \frac{1}{2}[h_{\mu\alpha,\nu}{}^\alpha + h_{\nu\alpha,\mu}{}^\alpha - h_{\mu\nu,\alpha}{}^\alpha - h_{,\mu\nu} - \eta_{\mu\nu}(h_{\alpha\beta}{}^{\alpha\beta} - h_{,\beta}{}^\beta)]. \quad (\text{C.5})$$

Equation (C.5) can be simplified by defining the *trace reverse* of $h_{\alpha\beta}$, given by

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h, \quad (\text{C.6})$$

with $\bar{h} = \bar{h}^\alpha_\alpha = -h$. Therefore, in terms of $\bar{h}_{\alpha\beta}$, the linearized Einstein tensor reads

$$G_{\alpha\beta} = -\frac{1}{2}[\bar{h}_{\alpha\beta,\mu}{}^\mu + \eta_{\alpha\beta}\bar{h}_{\mu\nu}{}^{\mu\nu} - \bar{h}_{\alpha\mu,\beta}{}^\mu - \bar{h}_{\beta\mu,\alpha}{}^\mu + O(\bar{h}^2_{\alpha\beta})], \quad (\text{C.7})$$

and the linearized Einstein equation is found to be

$$-\bar{h}_{\alpha\beta,\mu}{}^\mu - \eta_{\alpha\beta}\bar{h}_{\mu\nu}{}^{\mu\nu} + \bar{h}_{\alpha\mu,\beta}{}^\mu + \bar{h}_{\beta\mu,\alpha}{}^\mu = 16\pi T_{\alpha\beta}. \quad (\text{C.8})$$

Without loss of generality, one can impose the gauge condition, i.e.,

$$\bar{h}^{\mu\nu}{}_{,\nu} = 0, \quad (\text{C.9})$$

which is similar to the tensor analogue of the Lorenz gauge $A^\mu{}_{,\mu} = 0$ of electromagnetism [36]. The Einstein equation then simplifies to

$$-\bar{h}_{\alpha\beta,\mu}{}^\mu = 16\pi T_{\alpha\beta}. \quad (\text{C.10})$$

The gauge condition, equation (C.9), the field equation, equation (C.10), and the definition of the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \quad (\text{C.11})$$

are the fundamental equations of the linearized theory of gravity written in the Lorenz gauge [36].

References

- [1] Alcubierre M 1994 The warp drive: hyper-fast travel within general relativity *Class. Quantum Grav.* **11** L73–7 (*Preprint gr-qc/0009013*)
- [2] Natário J 2002 Warp drive with zero expansion *Class. Quantum Grav.* **19** 1157 (*Preprint gr-qc/0110086*)
- [3] Morris M and Thorne K S 1988 Wormholes in spacetime and their use for interstellar travel: a tool for teaching general relativity *Am. J. Phys.* **56** 395–412
- [4] Visser M 1989 Traversable wormholes: some simple examples *Phys. Rev. D* **39** 3182

- [5] Visser M 1989 Traversable wormholes from surgically modified Schwarzschild space-times *Nucl. Phys. B* **328** 203
- [6] Visser M 1995 *Lorentzian Wormholes: From Einstein to Hawking* (New York: AIP)
- [7] Krasnikov S V 1998 Hyper-fast interstellar travel in general relativity *Phys. Rev. D* **57** 4760 (*Preprint gr-qc/9511068*)
- [8] Everett A E and Roman T A 1997 A superluminal subway: the Krasnikov tube *Phys. Rev. D* **56** 2100 (*Preprint gr-qc/9702049*)
- [9] Lemos J P S, Lobo F S N and de Oliveira S Q 2003 Morris–Thorne wormholes with a cosmological constant *Phys. Rev. D* **68** 064004 (*Preprint gr-qc/0302049*)
- [10] Lemos J P S and Lobo F S N 2004 Plane symmetric traversable wormholes in an anti-de Sitter background *Phys. Rev. D* **69** 104007 (*Preprint gr-qc/0402099*)
- [11] For a review article, see for example: Lobo F and Crawford P 2003 Weak energy condition violation and superluminal travel *Current Trends in Relativistic Astrophysics, Theoretical, Numerical, Observational (Lecture Notes in Physics vol 617)* ed L Fernández *et al* (Berlin: Springer) pp 277–91 (*Preprint gr-qc/0204038*)
- [12] Visser M, Bassett B and Liberati S 1999 Perturbative superluminal censorship and the null energy condition *Proc. 8th Canadian Conf. on General Relativity and Relativistic Astrophysics* (New York: AIP) (*Preprint gr-qc/9908023*)
- [13] Visser M, Bassett B and Liberati S 2000 Superluminal censorship *Nucl. Phys. Proc. Suppl.* **88** 267–70 (*Preprint gr-qc/9810026*)
- [14] Olum K 1998 Superluminal travel requires negative energy density *Phys. Rev. Lett.* **81** 3567–70 (*Preprint gr-qc/9805003*)
- [15] Barcelo C and Visser M 2002 Twilight for the energy conditions? *Int. J. Mod. Phys. D* **11** 1553 (*Preprint gr-qc/0205066*)
- [16] Barcelo C and Visser M 2000 Scalar fields, energy conditions, and traversable wormholes *Class. Quantum Grav.* **17** 3843 (*Preprint gr-qc/0003025*)
- [17] Barcelo C and Visser M 1999 Traversable wormholes from massless conformally coupled scalar fields *Phys. Lett. B* **466** 127 (*Preprint gr-qc/9908029*)
- [18] Riess A G *et al* 2004 Type Ia supernova discoveries at $z > 1$ from the Hubble Space Telescope: evidence for past deceleration and constraints on dark energy evolution *Astrophys. J.* **607** 665–87 (*Preprint astro-ph/0402512*)
- [19] Visser M 2004 Jerk, snap, and the cosmological equation of state *Class. Quantum Grav.* **21** 2603 (*Preprint gr-qc/0309109*)
- [20] Caldwell R R, Kamionkowski M and Weinberg N N 2003 Phantom energy and cosmic doomsday *Phys. Rev. Lett.* **91** 071301 (*Preprint astro-ph/0302506*)
- [21] Ford L H and Roman T A 1995 Averaged energy conditions and quantum inequalities *Phys. Rev. D* **51** 4277 (*Preprint gr-qc/9410043*)
- [22] Ford L H and Roman T A 1996 Quantum field theory constrains traversable wormhole geometries *Phys. Rev. D* **53** 5496 (*Preprint gr-qc/9510071*)
- [23] Pfenning M J and Ford L H 1997 The unphysical nature of warp drive *Class. Quantum Grav.* **14** 1743 (*Preprint gr-qc/9702026*)
- [24] van den Broeck C 1999 A warp drive with more reasonable total energy requirements *Class. Quantum Grav.* **16** 3973 (*Preprint gr-qc/9905084*)
- [25] Gravel P and Plante J 2004 Simple and double walled Krasnikov tubes: I. Tubes with low masses *Class. Quantum Grav.* **21** L7
- [26] Gravel P 2004 Simple and double walled Krasnikov tubes: II. Primordial microtubes and homogenization *Class. Quantum Grav.* **21** 767
- [27] Hiscock W A 1997 Quantum effects in the Alcubierre warp drive spacetime *Class. Quantum Grav.* **14** L183 (*Preprint gr-qc/9707024*)
- [28] Clark C, Hiscock W A and Larson S L 1999 Null geodesics in the Alcubierre warp drive spacetime: the view from the bridge *Class. Quantum Grav.* **16** 3965 (*Preprint gr-qc/9907019*)
- [29] González-Díaz P F 2000 On the warp drive space-time *Phys. Rev. D* **62** 044005 (*Preprint gr-qc/9907026*)
- [30] Everett A E 1996 Warp drive and causality *Phys. Rev. D* **53** 7365
- [31] Visser M, Kar S and Dadhich N 2003 Traversable wormholes with arbitrarily small energy condition violations *Phys. Rev. Lett.* **90** 201102 (*Preprint gr-qc/0301003*)
- [32] Kar S, Dadhich N and Visser M 2004 Quantifying energy condition violations in traversable wormholes *Pramana* **63** 859–64 (*Preprint gr-qc/0405103*)
- [33] Schutz B F S 1990 *A First Course in General Relativity* (Cambridge: Cambridge University Press)
- [34] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (New York: Freeman)

- [35] Wald R M 1984 *General Relativity* (Chicago, IL: University of Chicago Press)
- [36] Lorenz L 1867 On the identity of the vibrations of light with electrical currents *Phil. Mag.* **34** 287–301
van Bladel J 1991 Lorenz or Lorentz? *IEEE Antennas Propag. Mag.* **33** 69
- [37] Fischer U R and Visser M 2003 On the space-time curvature experienced by quasiparticle excitations in the Painleve–Gullstrand effective geometry *Ann. Phys., NY* **304** 22 (*Preprint* cond-mat/0205139)
- [38] Fischer U R and Visser M 2003 Warped space-time for phonons moving in a perfect nonrelativistic fluid *Europhys. Lett.* **62** 1 (*Preprint* gr-qc/0211029)