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THE USE OF CONFORMAL TWO-STRUCTURES IN INITIAL VALUE PROBLEMS IN GENERAL RELATIVITY

by

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Abstract

A unified formalism for considering several types of initial value problems in general relativity is presented based on the use of conformal two-geometries to carry the two degrees of freedom of the gravitational field. A four-dimensional region of space-time is foliated and fibrated by a two-parameter family of rigged spacelike two-surfaces gotten by dragging a single two-surface along the integral curves of two commuting vector fields. The full Riemannian four-geometry at each point of the region can be reconstructed from the intrinsic and extrinsic geometries of the two-surfaces and of the rigging planes orthogonal to them. The Einstein field equations, projected into the surface and rigging plane, are derived from a Palatini variational principle. The rigging projections of the field equations determine the conformal factor everywhere from data given on a single two-surface. The traceless part of the surface projection becomes a pair of wave equations for the components of the conformal two-metric. The remaining equations, together with certain kinematical conditions on the foliation and fibration, determine the geometry of the rigging space. A generic initial value problem is considered

using an initial hypersurface that results from dragging a single two-surface along the trajectories of one of the vector fields. For a spacelike vector field, the field equations that determine the evolution of the three-geometry of the resulting space-like initial hypersurface are the well-known constraint equations. For a null vector field, they are the subsidiary equations of the characteristic initial value problem. The solution of the linearized version of the field equations is considered. A decoupling gauge is found in which the wave equation for the linearized conformal two-structure decouples from the other components. It is shown that sufficient data can be given on a single two-surface to formally prolong a solution to the field equations into a four-dimensional neighborhood of the surface. The most important part of the two-surface data consists of a denumerably infinite set of totally symmetric traceless tensors which determine the conformal two-structure.

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To my father - SR

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CHAPTER I

INITIAL VALUE PROBLEMS IN GENERAL RELATIVITY

1.1 Introduction

A theory admits an initial value formulation if it is possible to uniquely predict its dynamical evolution, within an equivalence class of physically indistinguishable solutions, from specified initial data given on some hypersurface or pair of intersecting hypersurfaces in the space of independent variables of the theory. The Initial Value Problem for a theory consists of finding such a set of initial data on a given hypersurface or hypersurfaces. As originally conceived, General Relativity is a classical theory and it seems reasonable to expect that, like other classical theories, it has such a formulation. Initial value problems for the Einstein equations have been discussed on space-like, null and time-like hypersurfaces or combinations of such surfaces[1].

There are important physical reasons for investigating the initial value problem for the gravitational field. Foremost among these is the problem of identifying the true degrees of freedom of the gravitational field. The true degrees of freedom represent the radiation modes of the field and their excitation signifies the presence of gravitational radiation.

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1) Theories, like general relativity, involving hyperbolic partial differential equations possess real characteristic (or null) surfaces. No amount of data, given on a <u>single</u> null hypersurface can predict the evolution of the fields off of such a hypersurface. Data must be specified on the null hypersurface plus some other intersecting hypersurface, which may be null, space-like or time-like. However, Penrose [1963] has shown how to specify data on a null cone, including the vertex.

The true degrees of freedom are important for defining the energy of the field and in attempting to construct observables[1]. Quantization of general relativity should involve commutation relations between quantities representing the true degrees of freedom.

A more recent motivation for studying the initial value problem is to obtain numerical solutions, using computers, that might, for example, model the gravitational radiation emitted by binary stellar systems (see, for example, Smarr [1979] and d'Inverno [1980]).

A problem related to the initial value problem is this: Suppose we perturb the initial data slightly. Then, we expect the dynamical evolution of the theory to be only slightly perturbed from its original evolution. This requirement, called the stability property, is an intuitively reasonable physical restriction and it can be stated precisely mathematically. An initial value problem is said to be well-posed if a unique and stable solution can be shown to exist corresponding to the set of initial data.

The well-known difficulty in solving the general relativistic initial value problem stems from two related aspects of the theory: First, the non-linearity of the field equations; physically interpreted, this means that the gravitational field is self-interacting. Second, the theory's invariance under a gauge group consisting of the set of diffeomorphisms of space-time. This aspect expresses the requirement that the theory be generally covariant. These properties of the field equations follow from the desire to formulate a covariant theory solely from the metric tensor and its first and second derivatives. Einstein's theory is

1) In the Hamiltonian sense, the observables are constants of the motion; they commute with the Hamiltonian and, in general relativity, are invariant under coordinate transformations. See, for example, Bergmann [1961].

the simplest generally covariant field theory meeting these conditions. General covariance leads to two important consequences. The first is that the evolution of some of the field variables of the theory are not determined by the field equations. The second is that the initial data for the remaining determinate elements are not freely specifiable, but are subject to four differential constraint equations. A similar situation holds for Maxwell's electromagnetic theory; but here, the non-linearity of the field equations makes an explicit identification of the gravitational degrees of freedom a difficult problem.

The similarities between Einstein's and Maxwell's equations arise from the fact that the gauge groups of both theories are function groups: i.e., groups whose elements are arbitrary functions of a set of $\underline{s}$ parameters. One of Noether's theorems [Trautman, 1964] shows that the field equations obey a set of $\underline{s}$ differential identities, called the Generalized Bianchi identities. These identities imply that the field equations are not independent, and this poses some difficulty in formulating a Cauchy problem. One cannot simply solve for the highest-order derivatives appearing in all of the equations. A Cauchy problem can be formulated only when it is possible to calculate all partial derivatives of the field at all points of the initial hypersurface given all partial derivatives up to some finite order. If $\underline{m}$ is the order of the highest derivative appearing in the field equations, then, on the initial surface, one has to prescribe all partial derivatives up to order m-1, as Cauchy data. By repeated differentiation of the field equations, all other partial derivatives are calculable. (When the field equations have analytic coefficients and the Cauchy data is analytic, then we can construct an analytic solutions from the knowledge of all partial derivatives). If one cannot solve for some of the derivatives, then the Cauchy data is insufficient. A system of equations in which it is possible to solve for the highest order derivatives of all the dependent variables is said to be in Cauchy-Kovalevsky form. The Einstein equations, like Maxwell's equations, are not in Cauchy-Kovalevsky form. This is basically because of the four differential constraints from the Bianchi identities, discussed in the previous paragraph[1].

Since there are components of the gravitational field whose solution is not determined by the field equations in a given coordinate system, they must be specified off of the initial hypersurface before the field equations can be solved. Their choice is exactly analogous to the gauge freedom in Maxwell's theory.

There is no unique way to determine which quantities are truly dynamical and which ones are pure gauge quantities. In general relativity, gauge conditions are called coordinate conditions. One such gauge choice imposes harmonic coordinate conditions in which each of the coordinates satisfies the covariant wave equation. Most of the work on existence proofs for the space-like gravitational initial value problem (the Cauchy Problem) has been carried out in harmonic coordinates.

On the other hand, most of the analyses of the geometric content of the Einstein fields equations and its initial value problem are done in the (3+1)-formulation of the field equations. The (3+1)-formalism decomposes space-time into a family of space-like three-surfaces [Stachel, 1962,1969]. Such a family, called a foliation in the language of differential geometry, is threaded or fibrated by the trajectories of a time-like vector field. The evolution of the initial data, given on a single member of the foliation, is governed by equations written in terms of the Lie derivatives with respect to the time-like vector field. The (3+1)-formulation has been generalized by O'Murchadha [1973]

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1) An excellent discussion of these points is found in Trautman [1964].

to anholonomic hypersurface elements orthogonal to a unit time-like vector field and by Stachel [1980] to arbitrary orthogonal vector fields. While the (3+1)-formalism does not imply any coordinate conditions, it facilitates a particular class of gauge conditions imposed on the so-called 'lapse' function and 'shift' vector which in adapted coordinates are closely related to the g_{00} and g_{0i} components, respectively, of the metric tensor.

The question of determining the number of degrees of freedom of the gravitational field, that is the number of independent components of the field, can be stated in terms of an initial value problem. For the Cauchy problem, the number of degrees of freedom is one-half the number of freely specifiable initial data functions set on the hypersurface. The number of degrees of freedom in the gravitational case turns out to be two since, as we shall show in Section 1.2, one has to give four arbitrary functions per point on the hypersurface. This number agrees with the number suggested, for instance, by Fourier mode counting in the linearized approximation of general relativity.

The cases in which data is set on a portion of a null hypersurface (plus a portion of another intersecting hypersurface) are called characteristic initial value problems. More important (for us) than the well-posedness of this type of problem[1] is the geometric interpretation of the initial data in these cases. In the double null formulation, discussed in Section 1.3, the initial null hypersurfaces are foliated by one-parameter families of space-like two-surfaces (a 2+1-decomposition). The full four-geometry induces on each member of the foliation a positive-definite two-metric so that each initial surface carries a one-parameter family of Riemannian two-surfaces with positive definite metric. The

1) See Müller zum Hagen and Siefert [1977] for a discussion of the well-posedness of characteristic initial value problems.

two-metric of each two-surface may be decomposed into a conformal two-metric, with determinant unity, and a conformal factor. The set of conformal two-metrics given on the family of two-surfaces on each null hypersurface is the freely specifiable initial data for the characteristic initial value problem. (Note that one has to give two functions per point on the two initial hypersurfaces, rather than the four functions per point on one hypersurface as in the Cauchy problem.) These results motivated D'Inverno and Stachel [1978] to see if a similar analysis were possible for the Cauchy problem. After foliating the initial Cauchy surface by a one-parameter family of two-surfaces, they showed that the constraint equations imposed no restrictions on the conformally-invariant part of the two-metric on these two-surfaces, or on its Lie derivative with respect to the normal direction. This suggests that a conformally-invariant family of two-metrics together with its corresponding Lie derivatives in the direction normal to the Cauchy surface might serve as the freely-specifiable initial data for the Cauchy problem as well. They designated any one-parameter family of conformally-invariant two-metrics a "conformal two-structure".

The aim of this dissertation is to extend the work of D'Inverno and Stachel by presenting a formalism in which, for all types of initial value problems, the gravitational degrees of freedom are carried by the conformal two-structure. This is done by using the {2+2}-formalism first introduced by D'Inverno and Stachel [1978] and developed extensively by Smallwood [1980] and d'Inverno and Smallwood [1980]. By (2+2)-formalism we mean the breakup of various quantities with respect to a foliation of space-time by a two-parameter family of space-like two-surfaces and its fibration by a pair of commuting vector fields which span a family of transvecting time-like two-planes. (Stachel [1984a] has considered the general n+m breakup).

We proceed as follows. In Chapter One, we discuss the concept of degrees of freedom, using various physical fields as examples, and relate it to the initial value problem. Then we will look at the initial value problem of general relativity for analytic initial data and solutions. The simplest (and earliest) existence and uniqueness theorems were proved for this case (Darmois [1927]). After this, we turn to the Cauchy problem in harmonic coordinates and state the standard existence and uniqueness theorems. Next, the (3+1)-decomposition of the field equations is discussed. The geometrical interpretation of the dynamical variables and the constraint equations becomes clear. We use the (3+1)-formulation to count the number of degrees of freedom of the gravitational field. Last, we contrast two methods of dealing with the constraint equations and isolating the true degrees of freedom of the gravitational field. The first method, due to D'Inverno and Stachel [1978], develops the formalism of conformal two-structures on Cauchy surfaces; its generalization to the full four-dimensional geometry forms the body of this thesis. The second method, due to York and others, uses conformal techniques on space-like 3-geometries.

Chapter Two provides a general discussion of the geometrical underpinning of our formalism, the theory of imbedding and rigging of submanifolds. The essentially new features of this thesis are presented in Chapter Three. After deriving the field equations using a Palatini variational principle, we write them as equations governing the evolution of the geometry of two-surfaces along a pair of commuting vector fields, and analyze what initial data must be given on a single initial two-surface in order that the gravitational field in a four-dimensional region be uniquely defined. Several examples from the linearized version of the Two+two formalism are considered in Chapter 4. We show, using examples, that to completely determine the gravitational field in a four-dimensional region, one has to specify the following quantities on an initial space-like two-surface: a scalar field (and two independent derivatives off the two-surface) representing the conformal scale factor of the two-surface; a two-dimensional vector field, and a denumerably-infinite set of symmetric traceless two-tensors which is formally equivalent to the linearized conformal two-structure. Chapter Five considers the double-null initial value problem in the two+two formalism. We show that essentially the same initial data is needed for the exact case on a single two-surface as was required in its linearized counterpart. While the two+two formalism has not been investigated for the existence of solutions other than analytic ones, it helps us understand why the initial data in the space-like initial value problem differs from that in the characteristic case and how the conformal two-structure can represent the gravitational degrees of freedom in both cases.

Regarding conventions and notation, we choose the signature of out space-time metric to be (-+++). Furthermore, we adopt Schouten's [1954] convention for defining the Riemann and Ricci tensors. Lower case Greek letters (e.g., $\mu, \nu = 0,3$) will serve as indices for four-dimensional space-time quantities. Lower case Roman letters ($a, i = 1,2,3$) will be used for quantities defined on a three-surface. while upper-case Roman letters ($A=2,3$) will be used for quantities defined on a two-dimensional surfaces. Lower-case Roman bold letters will serve as rigging indices and run from e.g. $\mathbf{x}, \mathbf{y} = 0,1$.

1.2 The Cauchy Problem and the Degrees of Freedom of the Gravitational Field

The purpose of this section is to outline the treatment of the Einstein equations of general relativity as a system of partial differential equations that has a well-posed Cauchy problem, if the initial data is correctly set. Such a system of equations, describing the time evolution of a physical system, is usually either of hyperbolic or parabolic type. It is not obvious that the Cauchy problem is an appropriate initial value problem for the Einstein equations because, at first sight, they do not fall into either class. However, when the field equations are expressed in a harmonic coordinate system[1], they take the form of hyperbolic equations for the components of the metric tensor; and for such equations, the Cauchy problem is appropriate. As we shall see, any solution of the Einstein equations is equivalent, under some member of the pseudo-group of coordinate transformations, to a solution satisfying the field equations in harmonic coordinates. In this sense, the Einstein equations have a well-posed Cauchy problem.

We mentioned in Section 1.1 that the number of degrees of freedom per point of the gravitational field is one-half the number of functions that are freely specifiable as initial data in the Cauchy problem. The concept of degrees of freedom of a field directly generalizes that for systems with a finite number of degrees of freedom. Consider a non-relativistic system of $\underline{N}$ (uncharged) point particles which interact with each other and are also acted upon by external forces which depend only on position and velocity. Such structureless particles are completely characterized by their mass $\underline{m}$, position $\mathbf{x}$ and velocity $\dot{\mathbf{x}}$. The degrees of free-

1) Harmonic coordinates satisfy $\nabla^{\mu}\nabla_{\mu}x^{\nu} = 0$.

dom of this system are a set of quantities which can serve to distinguish different physical trajectories of the system compatible with the equations of motion. One such representation of the degrees of freedom is the set of positions and velocities of all the particles at one moment of time. Another representation is given by certain collective motions of the system, termed normal modes -- the degrees of freedom being the amplitude and phase of each mode. For simplicity, let us allow only interactions between the particles which depend upon nothing but the relative distances. The most general equations of motion are then

$$m_i \ddot{\mathbf{x}}_i = \sum_j \mathbf{f}_{ij}(\mathbf{x}_i - \mathbf{x}_j) + \mathbf{f}_{ext}(\mathbf{x}_i) \qquad [1.2\text{-}1]$$

Furthermore, if all the forces are derivable from a potential, we have

$$m_i \ddot{\mathbf{x}}_i = -\nabla_i \sum_j \Phi_{ij}(\mathbf{x}_i - \mathbf{x}_j) - \nabla\Phi_{ext}(\mathbf{x}_i) \qquad [1.2\text{-}2]$$

If some assumptions of smoothness are put on the potential functions Φ_{ij} and Φ_{ext}, then by the fundamental existence theorems for systems of ordinary differential equations [see, for example, Lefschetz, 1927], a unique solution $\{\mathbf{x}_i(t)\}$ exists and depends continuously on 6N numbers: the initial positions $\{\mathbf{x}_i(t_0)\}$ and velocities $\{\dot{\mathbf{x}}_i(t_0)\}$ of the particles of the system. Because there is a one-to-one correspondence between the set of initial data and the set of solutions, each solution is distinguished by its initial data set. We say the initial data embodies the degrees of freedom of the system. The number of degrees of freedom is defined to be $3N = 6N/2$, half the number of freely specifiable initial conditions.

Many classical mechanics textbooks introduce continuous fields by deriving the equations of motion of a string from a co-linear array of point particles connected by massless springs by passing to a limit in which the number of particles become infinite while the distance between them vanishes. The positions and velocities of the particles go over to the displacement from equilibrium and the instantaneous velocity of each point of the string. The masses go over to the density of the string, and the spring forces to the tension of the string. The equations of motion of the point particles go over to the one-dimensional wave equation for the string. Existence and uniqueness proofs for the one-dimensional wave equation are discussed in, for example, Bers, John and Schecter [1964]. We generalize the concept of the number of degrees of freedom to this continuous system by calling it the number of degrees of freedom per point of the string. Because two numbers need to be given for each point of the string, there is one degree of freedom per point. For the three-dimensional scalar wave equation in Minkowski space, the existence and uniqueness of solutions is most easily proved using the Leray theory of solutions of hyperbolic equations (see Wald [1985]). The set of initial data needed to completely specify a solution is the field and its first normal (timelike) derivative on a single (space-like) hypersurface. This means, similarly, that there is one degree of freedom per point of the hypersurface.

The next physical system that we shall analyze in terms of its initial value problem is Maxwell's equations for the electromagnetic field potentials in Minkowski space. We shall show that when written in a Minkowski coordinate system[1] and

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1) The Minkowski metric in Minkowski coordinates is the diagonal metric $\eta_{\mu\nu}$ = [-1 1 1 1]

an appropriate gauge, they possess a well-posed Cauchy problem. The data needed to uniquely determine a solution of the equations consists of four arbitrary numbers per point of an initial space-like hypersurface which corresponds to an initial time. Thus Maxwell's equations have two degrees of freedom.

The electromagnetic field is described covariantly by a second-rank anti-symmetric tensor field $F_{\mu\nu}$. The components of $F_{\mu\nu}$ in the inertial frame corresponding to the adopted Minkowski coordinate system are the electric field and the magnetic induction:

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix}$$

The field equations with sources break up into two sets (see, for example, Jackson [1975]):

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad [1.2\text{-}3]$$

$$\partial_{[\mu} F_{\nu\lambda]} = 0 \qquad [1.2\text{-}4]$$

j^ν is the source current 4-vector for the Maxwell field. It is a conserved quantity

$$\partial_\mu j^\mu = 0. \qquad [1.2\text{-}5]$$

The conservation of the current 4-vector is an integrability condition of the first set of field equations [1.2-3], due to the antisymmetry of the field:

$$\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\mu j^\mu = 0 \qquad [1.2\text{-}6]$$

We will analyze the Cauchy initial value problem in a (3+1)-formalism with respect to an inertial frame, i.e., a family of flat space-like hypersurfaces which foliate Minkowski space. The hypersurfaces $\mathbf{S}_t$ given by $t = x^0 =$ constant in a Minkowski coordinate system form such a family. We take $\mathbf{S}_0$ as our initial hypersurface. The covariant normal to any hypersurface $\mathbf{S}$ is the unit vector

$$n_\nu = \partial x^0/\partial x^\nu = \delta^\nu{}_0 \qquad [1.2\text{-}7]$$

Its contravariant form is

$$n^\mu = n_\nu \eta^{\mu\nu} = \eta^{0\nu} = -\delta^\nu{}_0 \qquad [1.2\text{-}8]$$

The second set of Maxwell's equations [1.2-4] implies the existence (at least locally) of a four-vector potential A_μ related to the electromagnetic field by

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu \qquad [1.2\text{-}9]$$

In terms of these potentials, equation 1.2-3 becomes:

$$\eta^{\mu\nu}\partial_\mu \partial_\nu A_\sigma - \partial_\sigma \partial_\nu A^\nu = j_\sigma \qquad [1.2\text{-}10]$$

which is set of four coupled second-order partial differential equations for the potentials A_σ.

In order to write the field equations in (3+1)-form, all four-dimensional quantities are decomposed into a part normal and a part tangent to each member of the family of hypersurfaces $\{\mathbf{S}_t\}$. The projection operator into the normal direction is $n_\mu n^\nu$. Quantities which arise by projecting into the hypersurface are termed <u>spatial</u> quantities. The normal part of the 4-vector potential is the scalar field

$$\Phi = A_\mu n^\mu = -A_0 \qquad [1.2\text{-}11]$$

The tangential component of A_μ is the three-vector given by

$$\mathbf{A} = (A_1, A_2, A_3) \qquad [1.2\text{-}12]$$

The current 4-vector j^μ can similarly be decomposed:

$$\rho = j^\mu n_\mu = j^0$$
$$\mathbf{j} = (j^1, j^2, j^3) \qquad [1.2\text{-}13]$$

so that the four-vector conservation law for j^μ becomes

$$\partial_t \rho + \nabla\cdot\mathbf{j} = 0 \qquad [1.2\text{-}14]$$

In terms of the potentials, the electric and magnetic field vectors are

$$\mathbf{E} = -\nabla\Phi - \partial_t \mathbf{A} \qquad [1.2\text{-}15]$$

$$\mathbf{B} = \nabla\times\mathbf{A} \qquad [1.2\text{-}16]$$

We shall project the field equation 1.2-10 into the normal and tangential directions to the hypersurfaces. The normal projection of equation 1.2-10 is:

$$\nabla^2\Phi + \partial_t(\nabla\cdot\mathbf{A}) = \rho \qquad [1.2\text{-}17]$$

The spatial projection is:

$$\nabla^2 \mathbf{A} - \partial^2_{tt}\mathbf{A} - \nabla(\partial_t \Phi + \nabla \cdot \mathbf{A}) = \mathbf{j} \qquad [1.2\text{-}18]$$

The set of equations 1.2-17 and 1.2-18 is not in Cauchy-Kovalevsky form because the highest-order time derivative of Φ (the first time derivative) cannot be solved for explicitly in terms of all the other derivatives.

The definition of $F_{\mu\nu}$ in terms of the four-vector potential shows that the gradient of a scalar function can be added to A_σ without changing the physically meaningful fields. This is the well-known gauge transformation

$$A_\sigma \rightarrow A_\sigma + \partial_\sigma \chi \qquad [1.2\text{-}19]$$

All four-vector potentials which are connected by a gauge transformation form an equivalence class which defines the same physical state.

In (3+1)-form, the gauge transformation is:

$$\Phi \rightarrow \tilde{\Phi} = \Phi - \partial_t \chi \qquad [1.2\text{-}20]$$

$$\mathbf{A} \rightarrow \tilde{\mathbf{A}} = \mathbf{A} + \nabla \chi \qquad [1.2\text{-}21]$$

We can choose χ so that it vanishes together with its time derivatives up to any finite order on the initial hypersurface, but has non-zero higher derivatives. Two different sets of potentials belonging to the same physical equivalence class will then agree on the initial hypersurface up to that order but will differ in higher-order derivatives.

This shows that, unless one singles out a particular member of an equivalence class, there can be no well-posed Cauchy problem for Maxwell's equations written in terms of the potentials. We single out a member of the equivalence class by imposing some gauge condition. One example is the transverse or Coulomb gauge:

$$\nabla \cdot \mathbf{A} = 0 \qquad [1.2\text{-}22]$$

This gauge condition transforms equation [1.2-17] into an equation for the scalar potential alone:

$$\nabla^2 \Phi = \rho \qquad [1.2\text{-}23]$$

while equation [1.2-18] becomes

$$\nabla^2 \mathbf{A} - \partial^2{}_{tt}\mathbf{A} - \nabla(\partial_t \Phi) = \mathbf{j} \qquad [1.2\text{-}24]$$

We now apply the Helmholtz decomposition[1] to the vector potential

$$\mathbf{A} = \mathbf{A}^{tr} + \mathbf{A}^{long} \qquad [1.2\text{-}25]$$

Since $\mathbf{A}$ satisfies the gauge condition $\nabla \cdot \mathbf{A} = 0$, we have $\nabla \cdot \mathbf{A}^{long} = 0$. A longitudinal vector satisfying this condition

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1) Any three-vector $\mathbf{G}$ obeying certain boundary conditions at infinity can be decomposed into transverse and longitudinal parts:

$$\mathbf{G} = \mathbf{G}^{tr} + \mathbf{G}^{long}$$

where

$$\nabla \cdot \mathbf{G}^{tr} = 0$$

$$\nabla \times \mathbf{G}^{long} = 0$$

The Helmholtz decomposition of a vector field is discussed, for example, in Jackson [1975].

is necessarily constant. We may without loss of generality set this constant to zero, hence $\mathbf{A}^{long} = 0$ and $\mathbf{A} = \mathbf{A}^{tr}$.

One can show that the third term on the left-hand side of equation 1.2-24 is $-\mathbf{j}^{long}$ so that equation 1.2-24 may be written entirely in terms of transverse fields and currents[1]:

$$\nabla^2 \mathbf{A}^{tr} - \partial^2_{tt} \mathbf{A}^{tr} = \mathbf{j}^{tr} \qquad [1.2\text{-}26]$$

Equation 1.2-26 has a unique solution provided that we specify $\{\mathbf{A}^{tr}, \partial_t \mathbf{A}^{tr}\}$ on $x^0 = 0$. However, this alone would not guarantee that the vector potential obeys the constraint everywhere. We prove that this is so. On the initial hypersurface, we choose initial data that satisfies

$$\nabla \cdot \mathbf{A} = 0 \text{ and } \nabla \cdot \partial_t \mathbf{A} = 0 \qquad [1.2\text{-}27]$$

Taking the divergence of equation 1.2-26 yields

$$\nabla^2 \Lambda - \partial^2_{tt} \Lambda = 0 \qquad [1.2\text{-}28]$$

with $\Lambda = \nabla \cdot \mathbf{A}$. With the set of initial data, equation 1.2-28 has the unique solution $\Lambda = 0$ everywhere; hence, the gauge condition is maintained during the evolution of the initial data.

Using spherical coordinates, one can show that, in the

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1) We decompose $\mathbf{j} = \mathbf{j}^{tr} + \mathbf{j}^{long}$. Since the conservation equation [1.2-14] involves only the longitudinal part of the current vector

$$\partial_t \rho + \nabla \cdot \mathbf{j}^{long} = 0$$

and since $\nabla^2 \Phi = \rho$, we have

$$\nabla \cdot (\partial_t \nabla \Phi + \mathbf{j}^{long}) = 0$$

By definition, the quantity $\mathbf{f} \equiv \partial_t \nabla \Phi + \mathbf{j}^{long}$ also satisfies

$$\nabla \times \mathbf{f} = 0$$

so we may set $\mathbf{f} = 0$ and we hence

$$\partial_t \nabla \Phi = -\mathbf{j}^{long}$$

Coulomb gauge, A_θ and A_φ serve as the freely specifiable initial data for the radiative part of the Maxwell field. They and their time derivatives are the data that can be given at an initial time in order to determine a unique solution to the field equations. In addition, some lower-dimensional data, A_r and $\partial_t A_r$, need to be given on a single two-surface imbedded in the initial time surface[1] which encloses the source of the field. Boundary conditions for Φ also need to be given; for instance, Φ prescribed on a timelike tube r = constant. The fact that four numbers per point need to be given on the initial hypersurface means that the Maxwell field has two degrees of freedom. The additional degree of freedom is the remnant of the gauge invariance of the theory and of the longitudinal coupling to the charge.

We now turn to the Cauchy problem for the vacuum Einstein equations:

$$G_{\mu\nu} = R_{\mu\nu} - 1/2 g_{\mu\nu} R = 0 \qquad [1.2\text{-}29]$$

where $G_{\mu\nu}$ and $R_{\mu\nu}$ are the Einstein and Ricci tensors of the space-time and where

$$R = g^{\mu\nu} R_{\mu\nu} \qquad [1.2\text{-}30]$$

These constitute a set of 10 coupled second-order, quasilinear partial differential equations for the ten com-

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1) The initial data constraints in spherical coordinates are

$$r^{-2}\partial_r(r^2 A_r) + (\sin\theta)^{-1}\partial_\theta(\sin\theta A_\theta) + \partial_\varphi(\sin\theta\ A_\varphi) = 0$$

$$r^{-2}\partial_r(r^2 A_{r,t}) + (\sin\theta)^{-1}\partial_\theta(\sin\theta\ A_{\theta,t}) + \partial_\varphi(\sin\theta\ A_{\varphi,t}) = 0$$

They determine A_r and $A_{r,t}$ on the initial-time hypersurface S_0 from their values on a single two-surface contained within it and the freely specifiable initial data on S_0.

ponents of the metric tensor $g_{\mu\nu}$. We shall consider only solutions having a Lorentzian metric of signature -+++. We also assume that the manifold on which the equations are set is orientable and time-orientable[1]. The smoothness of the manifold must be at least piecewise C^2, so that the Riemann tensor may be defined.

In Section 1.1 it was stated that the Einstein field equations are not in Cauchy-Kovalevsky form, so that the Cauchy data alone is insufficient to extend the components of the metric field uniquely off the initial surface. To understand why, suppose we have a solution to the field equations in some region $\mathcal{R}$ of a space-time. On a space-like hypersurface in $\mathcal{R}$, the metric and its normal derivative can be computed. Any coordinate transformation which, together with a finite number of its derivatives, reduces to the identity on the hypersurface will preserve the metric and this number of its normal derivatives on the hypersurface, but change the higher-order normal derivatives of the metric. Hence, no finite set of Cauchy data, on the hypersurface, can determine a solution of the field equations if general covariance is maintained. However, all such formally different solutions, connected to each other via coordinate transformation, are physically equivalent to one another. Two metrics $\mathbf{g}_1$ and $\mathbf{g}_2$ are physically equivalent if there exists a diffeomorphism $\varphi:\mathbf{M} \rightarrow \mathbf{M}$ which takes $\mathbf{g}_1$ into $\mathbf{g}_2$:

$$\varphi_* \mathbf{g}_1 = \mathbf{g}_2.$$

The solutions of the field equations are unique only up to such a diffeomorphism[2] (see Hawking and Ellis [1973] or Sachs

1) See Appendix A.

2) This diffeomorphism is sometimes referred to as an <u>isometry</u>, a concept distinct from isometries defined by Killing vectors.

and Wu [1977]). Following Sachs and Wu, we define a gravitational field as an equivalence class of space-times, where the equivalence of the space-times is defined by the existence of an orientation- and time-orientation preserving diffeomorphism. As noted previously, we may recover formal determinism by working in some gauge such as the harmonic gauge.

The simplest analysis of the Einstein field equations is performed by adapting a particular coordinate system to the initial hypersurface. Suppose **S** is a spacelike hypersurface and we use a coordinate system in which **S** is defined by $x^0 = 0$ (at least locally). The remaining three coordinates (x^k) then form a coordinate system for **S**. Our goal is to determine the metric off of **S** as a function of the metric on **S**. In the adapted coordinate system, the metrical relations between points on **S** is given by the g_{ij} components of the metric tensor. We shall require that g_{ij} be positive definite. Of the ten field equations, the four given by $G^0{}_\mu=0$, do not contain any second derivatives with respect to time. The remaining field equations, $G_{ij} = 0$, contain second time derivatives only of the g_{ij}. With the coordinate freedom at our disposal, we may contruct a coordinate system in such a way as to specify $g_{\mu 0}$ both on and off the initial hypersurface, consistent with the coordinate conditions we have already adapted to **S**. It follows that the $g_{0\mu}$ are non-dynamical variables, and merely represent coordinate information.

For instance, we may set $g_{00} = -1$ and $g_{0i} = 0$ on **S**. A coordinate system in which these conditions hold can be extended into a finite neighborhood of the initial surface by dragging the three spatial coodinates along the geodesic curves which pass normally through each point of **S** with the geodesic length parameter serves as the remaining (time) coordinate (being chosen equal to x^0 on **S**). These coordinates are called <u>geodesic normal coordinates</u>. The transformation to geodesic normal coordinates preserves the initial values of g_{ij} and $g_{ij,0}$.

With this choice of coordinates, the six equations $G_{ij}=0$ may be put into Cauchy-Kovalevsky form:

$$g_{ij,00} = F(g_{ij}, g_{ij,0}, g_{ij,k}, g_{ij,km}) \qquad [1.2\text{-}31]$$

In this form, it is clear that the Cauchy data for this system are g_{ij} and $g_{ij,0}$.

The four equations $G^0{}_\mu=0$ now involve only g_{ij}, $g_{ij,0}$ and their spatial derivatives since all other metric components are fixed by the coordinate conditions. The equations $G^0{}_\mu=0$ are functions of the Cauchy data only and hence are constraints upon this data.

Properly prescribed Cauchy data satisfying the constraint equations must determine the subsequent evolution of the field. It follows that the evolved field must automatically satisfy the constraint equations. The proof follows from four differential identities satisfied by the field equations, called the contracted Bianchi identities:

$$\nabla_\mu G^{\mu\nu} = 0 \qquad [1.2\text{-}32]$$

In our adapted coordinate system, the Bianchi identities (assuming the evolution equations $G_{ij} = 0$ hold everywhere) become

$$\partial_0 G^0{}_\mu + X^{\lambda i}{}_\mu G^0{}_{\lambda,i} + Y^\sigma{}_\mu G^0{}_\sigma = 0 \qquad [1.2\text{-}33]$$

where the functions X and Y are built up from the metric tensor and its first derivatives. Equations 1.2-33 are a system of first-order partial differential equations which possesses a unique solution provided that $G^0{}_\sigma$ are prescribed on $x^0 = 0$ and that the coefficients are continuous (which is true by hypothesis). In particular, $G^0{}_\sigma = 0$ is the only solution of these equations for $G^0{}_\sigma = 0$ on $x^0 = 0$. Thus, the constraint equations continue to be satisfied off **S** if they are satisfied on **S** and $G_{ij} = 0$ are satisfied off **S**.

The earliest solutions to the Cauchy problem considered were analytic. In this case, we can apply the standard Cauchy-Kovalevsky theorem for analytic solutions evolving from analytic data which proves local existence and uniqueness (see Courant and Hilbert [1962]). For analytic data, we note that successive time derivatives of equation 1.2-31 can be taken. For each iteration, the term on the left-hand-side can be set equal to a term on the right-hand-side which contains time derivatives of one lower order. An analytic function of n-variables is uniquely defined in a neighborhood of a point if all of its partial derivatives with respect to the n-variables are given at that point. In our case n=4 and the g_{ij} and $g_{ij,0}$ are specified in a neighborhood within **S** of $p \in$ **S**. But since the g_{ij} are analytic, all the partial derivatives of g_{ij} and $g_{ij,0}$ with respect to the spatial coordinates are also known at **p**. As is easily seen, the field equations allow us to compute all possible time derivatives of g_{ij} to all orders. They also allow us compute all mixed derivatives of time order >2. Thus all possible partial derivatives are known at p. The partial derivatives may be used to construct a Taylor series expansion of g_{ij} about **p** which is a solution of the field equations. We are ensured that the solution exists within some radius of convergence around p. The uniqueness of the solution follows from the uniqueness theorems for analytic functions. Analyticity also insures that, if this construction were performed at a neighboring point, the solutions would be identical in the region of overlap of their respective domains of convergence.

There are, however, fundamental limitations to the analytic case. Solutions with discontinuities are necessarily globally non-analytic and hyperbolic equations always possess such solutions. This observation is somewhat mitigated by the recognition that a sequence of analytic functions can arbitrarily closely approximate such solutions (see Choquet-Bruhat, de-Witt-Morette, and Dillard-Bleick, [1977] or Courant and Hilbert [1962]). A more important problem with analytic functions concerns their domain of dependence. Consider a region $\mathcal{V}$ of the initial hypersurface. The set of points of **M** which can be reached via time-like or null curves only from points within $\mathcal{V}$ is called the domain of dependence of $\mathcal{V}$ and written $D(\mathcal{V})$. Physical intuition suggests that initial data set on $\mathcal{V}$ should completely determine the fields only within $D(\mathcal{V})$. The condition of analyticity is so strong that the fields within $\mathcal{V}$ determine the fields outside $D(\mathcal{V})$ as well.

For the non-analytic case, existence and uniqueness proofs for the Cauchy problem for the Einstein Equations have been obtained using harmonic coordinates (cf. footnote, p.9) (See, for example, Fischer and Marsden [1979] or Choquet and York [1980]). Harmonic coordinate systems also satisfy the conditions

$$\Gamma^{\kappa}(x) \equiv g^{\mu\nu}\Gamma^{\kappa}{}_{\mu\nu}(x) = 0, \text{ or } (\sqrt{-g}\, g^{\mu\nu})_{,\nu} = 0 \qquad [1.2\text{-}34]$$

Any initial value problem can be reduced locally to one in a harmonic coodinate system. To see this, let $(\mathbf{U},\varphi)$ be a coordinate system (in the usual notation, $\mathbf{U}$ is an open neighborhood and $\varphi\colon \mathbf{U} \to R^4$ is a mapping from $p \in \mathbf{U}$ into the set of four numbers $\{y^\nu\}$) such that a space-like hypersurface **S** is given by $y^0 = 0$ on **S**. The values of y^ν on **S** are $(0, y^i)$ and those of $y^\nu{}_{,0}$ are $\delta^\nu{}_0$. Using these as initial conditions, the harmonic coordinate condition may be solved in some region $\mathcal{R}$ for four scalar fields $\{x^\nu\}$ which will satisfy:

$$x^\nu = y^\nu \quad \text{and } x^\nu{}_{,0} = y^\nu{}_{,0} \text{ on } \mathbf{S} \text{ and}$$
$$\text{such that } |\partial x^\nu / \partial y^\mu| \neq 0 \qquad [1.2\text{-}35]$$

The $\{x^\nu\}$ may be used as coordinates and $x^\nu(y^\mu)$ can be considered a coordinate transformation. In the new coordinate system, the values of $g_{\mu\nu}$ and $g_{\mu\nu,0}$ are the same as they were in the old coordinate system. (The imposition of the harmonic coordinate condition still leaves available some gauge freedom, namely the transformation to other harmonic coordinate systems).

For vacuum fields the vanishing of the Einstein tensor is equivalent to the vanishing of the Ricci tensor, which can be written in the form:

$$R_{\mu\nu} = -1/2\ g^{\alpha\beta} g_{\mu\nu,\alpha\beta} + 1/2\ g_{\mu\alpha}\Gamma^\kappa{}_{,\nu} + 1/2\ g_{\nu\alpha}\Gamma^\kappa{}_{,\mu} + H_{\mu\nu} \qquad [1.2\text{-}36]$$

where $H_{\mu\nu}$ is a polynomial function of $g_{\mu\nu}$ and $g_{\mu\nu,\kappa}$.

Hence, in harmonic coordinate systems, the Einstein equations are equivalent to the so-called reduced Einstein equations:

$$R^{(h)}{}_{\mu\nu} = -1/2\ g^{\alpha\beta} g_{\mu\nu,\alpha\beta} + H_{\mu\nu}\ , \qquad [1.2\text{-}37]$$

which are a strictly hyperbolic set of quasilinear partial differential equations with identical characteristics for each component of the metric. Proofs of the existence and uniqueness of solutions of this set of equations are discussed in, for example, Wald [1985], using the Leray theory, and in Fischer and Marsden [1979], using the theory of linear semi-groups. Both of these approaches require that the metric tensor belong to the Sobolev space[1] H^s ($s \geq 3$) of functions.

The advantage of using harmonic coordinates is that it is sufficient to solve the Cauchy problem for ${R^{(h)}}_{\mu\nu} = 0$. A solution $g_{\mu\nu}$ to the reduced Einsteins equation is a solution of Einstein's equations only if $\Gamma^\kappa = 0$ in the region in which the solution exists. This will be true if: (i) $\Gamma^\kappa = 0$ on **S** and (ii) the constraint equations hold on **S** (see Choquet-Bruhat [1952]). Condition (ii) is assumed to hold by virtue of the field equations. Condition (i) can always be satisfied, as we have seen above. The constraint equations of the original theory thus become initial conditions to be satisfied in order that the harmonic gauge condition propagate of **S**. This also proves "physical uniqueness" of the solution, since any two metrics with the same initial data can be transformed to harmonic coordinates in which formal uniqueness holds. We refer the reader to Fischer and Marsden [1979] for the proof of the local theorem of existence and uniqueness for Einstein's equations for non-analytic solutions. Discussing the initial value problem in a particular coordinate system obscures much of its geometrical content, as well as the geometrical significance of the dynamical quantities. We seek a formulation which is independent of particular coordinate systems. To do this, we

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1) Sobolev spaces are vector spaces of functions which satisfy certain differentiability conditions. We have no need for a technical discussion except to say that functions of class C^∞ form a dense subset of the Sobolev space H^3.

view the initial value problem in terms of the evolution of the metric of an abstract three-manifold, invoking the language of foliations and fibrations. In this formulation, an arbitrary three-manifold with positive definite metric generalizes the notion of the "state of the system at an initial time". "Evolution in time" is replaced by a one-parameter family of imbeddings that map the three-manifold into the four-dimensional space-time. We replace time derivative (a coordinate-dependent concept) with the Lie derivative with respect to a time-like vector field, which not only is a covariant concept but is independent of any metrical structure on the manifold. One gains more geometrical insight into the constraint equations, since they can be written as geometrical relations between the first and second fundamental forms of the initial hypersurface.

The method of analysis we are about to describe is not suited to proving existence or uniqueness theorems, which are best handled in particular coordinate systems, making them more amenable to the standard tools of analysis. Our limited task is to specify and geometrically interpret a set of initial data which serves to physically characterize distinct solutions. We can do this by assuming that we are given a solution, and determining which data are needed to define that solution uniquely.

The Cauchy problem may be presented abstractly and more geometrically in the following way. An initial data set is a triplet $\mathcal{J}=(\mathcal{S},{}^{(3)}\mathbf{g},\mathbf{h})$ where $\mathcal{S}$ is a three-dimensional manifold, ${}^{(3)}\mathbf{g}$ is a three-dimensional positive-definite Riemannian metric on $\mathcal{S}$, and $\mathbf{h}$ a covariant symmetric two-tensor on $\mathcal{S}$. A development of $\mathcal{J}$ is a space-time $(\mathbf{M},\mathbf{g})$ such that there exists an imbedding map $B:\mathcal{S} \to \mathbf{M}$ which satisfies the condition that the pullback of $\mathbf{g}$ from $B(\mathcal{S})$ to $\mathcal{S}$ and the pullback of the second fundamental form (these terms are defined below) on

$B(\mathcal{S})$ to $\mathcal{S}$ agree with ${}^{(3)}\mathbf{g}$ and with $\mathbf{h}$ respectively. We shall denote by $\mathbf{S}$ the set $B(\mathcal{S})$ of image points of $\mathcal{S}$ under $\mathbf{B}$. Since the metric determines the space-time structure, one does not know in advance what the domain of dependence of the initial surface will be. It is for this reason that the solution consists of the pair $(\mathbf{M},\mathbf{g})$ rather than $\mathbf{g}$ alone. The Cauchy problem is that of finding a development of $\mathcal{J}$ such that $\mathbf{g}$ satisfies the Einstein equations in M. Provided the initial data satisfy certain constraints, there will exist a maximal development of $\mathcal{S}$, such that it is an extension of any other development of $\mathcal{S}$. This theorem is due to Choquet-Bruhat [1952] and is also presented in Fischer and Marsden [1979].

We will now consider globally hyperbolic space-times $(M,\mathbf{g})$. Only here does a global Cauchy problem make sense. In this case, there exists a space-like hypersurface having a domain of dependence that is the whole manifold (the topology of the space-time is $R\times S$, S being the topology of the Cauchy surface; see Choquet-Bruhat and York, 1979). Many space-times, such as the Reissner-Nordstrom solutions, do not admit a Cauchy surface; but even then, we can always work in small enough regions so that the Cauchy problem is meaningful. The topological assumptions imply that the initial Cauchy surface belongs to a 1-parameter family of Cauchy Surfaces ($\mathbf{S}_t$ with $\mathbf{S}_0 \equiv \mathbf{S}$) which foliates the space-time. A foliation is a parametrized family of surfaces such that every point of the manifold lies on one and only one member of the family. The parameter t can be considered as a global time function. For an extensive treatise on the theory of foliations, see Reinhart [1983].

The global topology also implies that there exists a globally-defined (but non-unique) vector field $\mathbf{V}^\nu$ satisfying

$V^{\nu}\nabla_{\nu}t = 1$. This vector field defines a <u>fibration</u>, a one-parameter family of diffeomorphisms which carries the initial hypersurface into any member of the family

$$\Phi_t\colon \mathbf{S}_0 \to \mathbf{S}_t$$

We can think of **V** as deforming the initial surface by pulling or dragging each of its points along the integral curves of **V**, creating the foliation. The four-dimensional metric induces a three-dimensional positive definite metric on each member of the foliation. The induced metric and extrinsic curvature evolve in time as we go through the members of the foliation.

Before analyzing the field equations, we discuss in detail the geometry of imbedded three-surfaces. As we mentioned, the initial hypersurface $\mathbf{S} = \mathbf{S}_0$ can be regarded as the image, under an imbedding map **B**, of an abstract manifold $\mathcal{S}$:

$$\mathrm{B}\colon \mathcal{S} \to \mathbf{M} \qquad \text{(a sufficiently smooth map)},$$

so that $\mathrm{B}(\mathcal{S}) = \mathbf{S}_0$.

In a coordinate system, the imbedding map is expressed as

$$x^{\mu} = B^{\mu}(y^{a}) \qquad [1.2\text{-}38]$$

where $\{y^a\}$ and $\{x^{\mu}\}$ form local coordinate systems on $\mathcal{S}$ and **M**, respectively.

B defines another map $\mathbf{B}^*$, the so-called <u>pull-back</u> map, which maps covectors belonging to the cotangent space of any point on $\mathrm{B}(\mathcal{S})$ into the cotangent space of its image on $\mathcal{S}$:

$$\mathbf{B}^*\colon T^*_{B(p)}\mathbf{M} \to T^*_{p}\mathcal{S} \qquad [1.2\text{-}39]$$

where $T^*_p\mathcal{S}$ and $T^*_{B(p)}\mathbf{M}$ are the cotangent spaces of $\mathcal{S}$ and **M** at p and **B**(p), respectively.

In coordinates, the pull-back map is written as

$$B^\mu{}_a = \partial_a B^\mu(y^b) \qquad [1.2\text{-}40]$$

and for a pulled-back covector, we have

$$m_a = B^\mu{}_a m_\mu \qquad [1.2\text{-}41]$$

where $\mathbf{m}$ is a covector in $\mathbf{M}$ on $\mathbf{S}$. Considered as a matrix, the pull-back map is of rank three.

$\mathbf{B}$ also defines the <u>push-forward</u> or <u>differential</u> map, $B_\star$, which carries vectors from the tangent space of a point of $\mathcal{S}$ into the tangent space of its image in $\mathbf{M}$.

$$B_\star : T_p\mathcal{S} \rightarrow T_{B(p)}\mathbf{M} \qquad [1.2\text{-}42]$$

where $T_p\mathcal{S}$ and $T_{B(p)}\mathbf{M}$ are the tangent spaces of $\mathcal{S}$ and $\mathbf{M}$ at p and $\mathbf{B}(p)$, respectively.

The coordinate expression for $\mathbf{B}^\star$ is also $B^\mu{}_a$, so for any vector v^a in $\mathcal{S}$, we have

$$v^\mu = B^\mu{}_a v^a \qquad [1.2\text{-}43]$$

A positive-definite three-metric is induced on $\mathcal{S}$ by the pull-back of $\mathbf{g}$:

$$^{(3)}\mathbf{g} = B^\star\mathbf{g} \text{ or in coordinates } {}^{(3)}g_{ab} = B^\mu{}_a{}^\nu{}_b g_{\mu\nu} \qquad [1.2\text{-}44]$$

using the notation $B^\mu{}_a{}^\nu{}_b \equiv B^\mu{}_a B^\nu{}_b$. The induced metric has inverse defined by

$$^{(3)}g^{ab}\,{}^{(3)}g_{bc} = \delta^a{}_c \qquad [1.2\text{-}45]$$

and an induced metrical connection given by the Christoffel symbols constructed from ${}^{(3)}\mathbf{g}$.

Because it is of rank three, the imbedding map defines a timelike covariant vector field $\mathbf{w}$ on $\mathbf{S}$, satisfying $B^{\mu}{}_{a} w_{\mu} = 0$. w_{μ} is only defined up to a scalar factor at each point on $\mathbf{S}$ and can be completely defined by fixing its magnitude on $\mathbf{S}$:

$$\rho^{-2} = -\,\mathbf{g}(\mathbf{w},\mathbf{w}) \qquad [1.2\text{-}46]$$

Locally, $\mathbf{w}$ is proportional to the gradient of a scalar field. (We refer the reader to Appendix A for the complete statement of this result)

$$\mathbf{w} = s\mathbf{d}\psi \quad \text{or} \quad w_{\mu} = s\partial_{\mu}\psi. \qquad [1.2\text{-}47]$$

With out loss of phyiscal significance, we can choose $\mathbf{w}$ to be the gradient of a scalar field which we take to be the time parameter t, simplifying the later analysis.

Another tensor field on $\mathcal{S}$ can be defined by pulling back the covariant derivative of $\mathbf{w}$

$$\mathbf{h} = -B^{\star}(\nabla\mathbf{w}) \quad \text{or} \quad h_{ab} = -B^{\mu}{}_{a}{}^{\nu}{}_{b}\, \nabla_{\mu} w_{\nu} \qquad [1.2\text{-}48]$$

This tensor describes the bending of $\mathcal{S}$ within M and is called the <u>extrinsic curvature</u> or <u>second fundamental form</u> of $\mathcal{S}$. While the covariant derivative appears to involve derivatives in arbitrary directions, $\mathbf{h}$ is in fact independent of the prolongation of $\mathbf{w}$ off the hypersurface. The extrinsic curvature is usually defined using the unit normal $\mathbf{n}$, where $n_{\mu} = \rho w_{\mu}$. Our choice is just ρ^{-1} times the usual second fundamental form (the extrinsic curvature, as we have defined it, exists in affine spaces for which orthogonality is not defined). Equation 1.2-47 implies that $\mathbf{h}$ is symmetric with respect to its indices.

The maps B_* and B^* cannot be inverted since they are of rank three. To carry covariant quantities from $\mathcal{S}$ to **S** and contravariant quantities from **S** to $\mathcal{S}$, we must rig the hypersurface. A <u>rigged</u> hypersurface is defined as one that has a contravariant vector defined at each point of the hypersurface, which is nowhere tangent to it. The rigging vector **r** must hence satisfy $r^\mu w_\mu \neq 0$.

Because of the way **w**, the surface one-form, and **V**, the deformation vector, are defined they satisfy:

$$V^\mu w_\mu = 1 \qquad [1.2\text{-}49]$$

We could use **V** to rig the initial hypersurface. but, in a metrical space, a preferred rigging direction exists which is orthogonal to the hypersurface. Let **N** be any vector orthogonal to **S**:

$$N^\mu B^\nu{}_a g_{\mu\nu} = 0 \qquad [1.2\text{-}50]$$

which we can also normalize via

$$N^\mu w_\mu = 1 \qquad [1.2\text{-}51]$$

Then the vector field **V** satisfies

$$\mathbf{V} = \mathbf{N} + \boldsymbol{\sigma} \qquad [1.2\text{-}52]$$

where σ is a vector field tangent to **S** i.e. $N^\mu \sigma_\mu = 0$. The interpretation of equation 1.2-52 is that the motion from one hypersurface to the next can be decomposed into a motion perpendicular to the surface and a shift parallel to the surface, given by the so-called <u>shift</u> vector σ.

The magnitudes of **N** and **w** can be related since their covariant components are proportional, as follows from the fact that $B^{\mu}{}_{a}$ annhilates both. One can easily show that

$$N_{\mu} = -\rho^2 w_{\mu} \text{ and hence } \mathbf{g(N,N)} = -\rho^2 \qquad [1.2\text{-}53]$$

Equation 1.2-53 justifies calling ρ the <u>lapse</u> function because it gives the metrical distance between infinitesimally close members of $\{\mathbf{S}_t\}$ along an orthogonal vector field connecting them.

The projection operator $C^{\mu}{}_{\nu} \equiv N^{\mu}w_{\nu}$ projects any vector into the rigging (normal) direction, while $B^{\mu}{}_{\nu} \equiv \delta^{\mu}{}_{\nu} - C^{\mu}{}_{\nu}$ projects into the hypersurface. Any arbitrary vector **A** can be decomposed into

$$A^{\mu} = B^{\mu}{}_{\nu}A^{\nu} + C^{\mu}{}_{\nu}A^{\nu} = {}'A^{\mu} + {}''A^{\mu} \qquad [1.2\text{-}54]$$

(In the future, the prime (') will indicate a quantity lying in the hypersurface while the double prime (") will indicate a quantity parallel to the rigging.)

Likewise, any arbitrary covector **z** can be decomposed into:

$$z_{\mu} = B^{\nu}{}_{\mu}z_{\nu} + C^{\nu}{}_{\mu}z_{\nu} = {}'z_{\mu} + {}''z_{\mu} \qquad [1.2\text{-}55]$$

These projection operators can operate on quantities of higher rank. For example, an arbitrary covariant tensor of rank two will have four projections

$$A_{\mu\lambda} = B^{\nu}{}_{\mu}{}^{\kappa}{}_{\lambda}A_{\nu\kappa} + B^{\nu}{}_{\mu}C^{\kappa}{}_{\lambda}A_{\nu\kappa} + C^{\nu}{}_{\mu}B^{\kappa}{}_{\lambda}A_{\nu\kappa} + C^{\nu}{}_{\mu}{}^{\kappa}{}_{\lambda}A_{\nu\kappa} \qquad [1.2\text{-}56]$$

The metric tensor has only two projections because of the orthogonality of the rigging to the three-space **S**:

$$g_{\mu\nu} = {}'g_{\mu\nu} + {}''g_{\mu\nu} \qquad [1.2\text{-}57]$$

The contravariant metric tensor can be built up from quantities that we have previously defined

$$g^{\mu\nu} = {}^{(3)}g^{ab}B^{\mu}{}_{a}{}^{\nu}{}_{b} - \rho^{-2}N^{\mu}N^{\nu} \qquad [1.2\text{-}58]$$

$$= {}^{(3)}g^{ab}B^{\mu}{}_{a}{}^{\nu}{}_{b} - \rho^{-2}(V^{\mu} - \sigma^{\mu})(V^{\nu} - \sigma^{\nu}) \qquad [1.2\text{-}59]$$

Thus the metric tensor is equivalent to the ten quantities g^{ab}, ρ and σ^{a}. In a coordinate system adapted to the surface and deformation vector field, one can show

$$\rho^{-2} = g^{00} \qquad [1.2\text{-}60]$$

$$\sigma^{a} = \rho^{2}g^{a0} \qquad [1.2\text{-}61]$$

relating the lapse function and shift vector to some components of the metric tensor.

Since **w** is a gradient, and using equation 1.2-51 andthe fact that $\sigma^{\mu}w_{\mu} = 0$, we have

$$\pounds_{\mathbf{N}}\mathbf{w} = 0 \text{ and } \pounds_{\sigma}\mathbf{w} = 0 \text{ and } \pounds_{\mathbf{V}}\mathbf{w} = 0 \qquad [1.2\text{-}62]$$

One can also show that

$$N^{\mu}\nabla_{\nu}w_{\mu} = -\rho^{-2}\nabla_{\nu}\rho \qquad [1.2\text{-}63]$$

We can now compute all the projections of the covariant derivative of **w**

$$\nabla_\kappa w_\mu = B^\alpha{}_\kappa{}^\beta{}_\mu \nabla_\alpha w_\beta + B^\alpha{}_\kappa C^\beta{}_\mu \nabla_\alpha w_\beta + C^\alpha{}_\kappa B^\beta{}_\mu \nabla_\alpha w_\beta + C^\alpha{}_\kappa{}^\beta{}_\mu \nabla_\alpha w_\beta \quad [1.2\text{-}64]$$

The first term is the extrinsic curvature modulo sign. The other terms are:

$$B^\alpha{}_\kappa C^\beta{}_\mu \nabla_\alpha w_\beta = -B^\alpha{}_\kappa w_\mu \, \rho^{-1} \nabla_\alpha \rho = -w_\mu \rho^{-1} {}' \nabla_\kappa \rho = C^\alpha{}_\kappa B^\beta{}_\mu \nabla_\alpha w_\beta \quad [1.2\text{-}65]$$

$$C^\alpha{}_\kappa{}^\beta{}_\mu \nabla_\alpha w_\beta = - w_\kappa w_\mu \rho^{-1} N^\nu \nabla_\nu \rho = - w_\kappa w_\mu \rho^{-1} \pounds_{\mathbf{N}} \rho \quad [1.2\text{-}66]$$

Thus:

$$\nabla_\kappa w_\mu = -h_{\kappa\mu} - 2\rho^{-1} w_{(\kappa} {}' \nabla_{\mu)} \rho - w_\kappa w_\mu \rho^{-1} \pounds_{\mathbf{N}} \rho \quad [1.2\text{-}67]$$

The Lie derivatives of objects lying in $\mathcal{S}$ with respect to vector fields pointing out of $\mathcal{S}$ are defined, as in Schouten [1954], by carrying them over to **M** using the connecting quantities, acting on them with the Lie derivative, and projecting back into $\mathcal{S}$. For the induced three-metric tensor in particular we have:

$$\begin{aligned}
\pounds_{\mathbf{N}} {}' g_{ab} &= B^\mu{}_a{}^\nu{}_b \pounds_{\mathbf{N}} B^\rho{}_a{}^\sigma{}_b {}' g_{\rho\sigma} \\
&= B^\mu{}_a{}^\nu{}_b \pounds_{\mathbf{N}} (g_{\mu\nu} - {}'' g_{\mu\nu}) = B^\mu{}_a{}^\nu{}_b \pounds_{\mathbf{N}} g_{\mu\nu} \\
&= 2B^\mu{}_a{}^\nu{}_b \nabla_{(\mu} N_{\nu)} \\
&= -2\rho^2 B^\mu{}_a{}^\nu{}_b \nabla_{(\mu} w_{\nu)} \\
&= 2\rho^2 h_{ab}
\end{aligned} \quad [1.2\text{-}68]$$

In a similar way, a covariant derivative on $\mathcal{S}$ can be induced from the covariant derivative on **M**: let $T^{ab}{}_c$ be an arbitrary geometrical object on $\mathcal{S}$. Using the appropriate connecting quantities, carry this object over to **S**. Then take covariant derivatives of this 4-geometrical object in directions lying within **S**, using the projector $B^{\nu}{}_{\mu}$. Finally, carry the derivative back to $\mathcal{S}$ using the connecting quantities. As examples, the covariant derivatives of a vector and covector **v** and **z** of $\mathcal{S}$ are defined by

$$D_a v^b = B^{\nu}{}_a{}^b{}_\mu \nabla_\nu {}'v^\mu \text{ where } {}'v^\mu = B^\mu{}_a v^a \qquad [1.2\text{-}69]$$

$$D_a z_b = B^{\nu}{}_a{}^\mu{}_b \nabla_\nu {}'z_\mu \text{ where } {}'z_\mu = B^a{}_\mu z_a \qquad [1.2\text{-}70]$$

It is not hard to show that the covariant derivative induced in this way is identical to the covariant derivative defined from the induced metric via its Christoffel symbols if and only if the rigging is orthogonal to **S**. With the metric and covariant derivative defined, the full intrinsic geometry of $\mathcal{S}$ is known. The Riemann tensor on $\mathcal{S}$ and all of its contractions can now be calculated.

The Riemann tensor on $\mathcal{S}$ is related to the Riemann tensor on **M** by Gauss's equation (see Schouten, [1954], p. 237):

$$^{(3)}R_{abc}{}^d = B^{\kappa}{}_a{}^\mu{}_b{}^\nu{}_c{}^d{}_\sigma R_{\kappa\mu\nu}{}^\sigma - 2\rho^2\, h_{[a}{}^d h_{b]c} \qquad [1.2\text{-}71]$$

Another relation between the geometry of $\mathcal{S}$ and the geometry of **M** is given by Codazzi's equation (see Schouten [1954])

$$2D_{[a}h_{b]c} = B^{\kappa}{}_a{}^\mu{}_b{}^\nu{}_c w_\sigma R_{\kappa\mu\nu}{}^\sigma + \rho^{-1}h_{bc}D_a\rho \; - \rho^{-1}h_{ba}D_c\rho \qquad [1.2\text{-}72]$$

Together, the Gauss and Codazzi equations are the integrability conditions for the local isometric imbedding of a Riemannian manifold into a Riemannian manifold of one higher dimension.

After these preliminaries, we turn to the field equations.

The Einstein equations break up into two sets. The first set, termed the evolution equations, are given by the projection of the Ricci tensor totally into the hypersurface

$$B^{\mu}{}_{b}{}^{\nu}{}_{c}R_{\mu\nu} = B^{\mu}{}_{b}{}^{\nu}{}_{c}w_{\sigma}N^{\kappa}R_{\kappa\mu\nu}{}^{\sigma} + B^{\mu}{}_{b}{}^{\nu}{}_{c}{}^{\kappa}{}_{d}{}^{d}{}_{\sigma}R_{\kappa\mu\nu}{}^{\sigma} = 0 \qquad [1.2\text{-}73]$$

The first term on the right hand side can be gotten from the calculation of the Lie derivative of **h**

$$\begin{aligned}
\pounds_{\mathbf{N}}h_{ab} &= -\ B^{\mu}{}_{a}{}^{\nu}{}_{b}\pounds_{\mathbf{N}}(B^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}\nabla_{\rho}N_{\sigma}) \\
&= -\ B^{\mu}{}_{a}{}^{\nu}{}_{b}\pounds_{\mathbf{N}}(B^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}\rho^{-2}\nabla_{\rho}w_{\sigma}) \\
&= h_{ab}\pounds_{\mathbf{N}}\rho^{-2} + \rho^{-2}B^{\mu}{}_{a}{}^{\nu}{}_{b}\pounds_{\mathbf{N}}\nabla_{\mu}w_{\nu} + B^{\mu}{}_{a}{}^{\sigma}{}_{b}\ \rho^{-2}(\nabla_{\rho}w_{\sigma})\,L_{\mathbf{N}}B^{\rho}{}_{\mu} \\
&\quad + B^{\rho}{}_{a}{}^{\nu}{}_{b}\ \rho^{-2}\ \nabla_{\rho}w_{\sigma}\ \pounds_{\mathbf{N}}B^{\sigma}{}_{\nu}
\end{aligned} \qquad [1.2\text{-}74]$$

The last two terms on the right vanish because:

$$\pounds_{\mathbf{N}}B^{\sigma}{}_{\nu} = -\ \pounds_{\mathbf{N}}C^{\sigma}{}_{\nu} = -\pounds_{\mathbf{N}}N^{\sigma}w_{\nu} = -N^{\sigma}\pounds_{\mathbf{N}}w_{\nu} = 0 \qquad [1.2\text{-}75]$$

To evaluate the second term on the right, we use an identity involving the commutator of the Lie and covariant derivative

$$\pounds_{\mathbf{N}}\nabla_{\mu}w_{\nu} = \nabla_{\mu}\pounds_{\mathbf{N}}w_{\nu} - (\nabla_{\mu}\nabla_{\nu}N^{\kappa} + N^{\sigma}R_{\sigma\mu\nu}{}^{\kappa})\,w_{\kappa}\ ; \qquad [1.2\text{-}76]$$

The first term vanishes, giving finally

$$\pounds_{\mathbf{N}}h_{ab} = h_{ab}\pounds_{\mathbf{N}}\rho^{-2} - \rho^{-2}\ B^{\mu}{}_{a}{}^{\nu}{}_{b}w_{\kappa}\nabla_{\mu}\nabla_{\nu}N^{\kappa} - \rho^{-2}B^{\mu}{}_{a}{}^{\nu}{}_{b}N^{\sigma}R_{\sigma\mu\nu}{}^{\kappa}w_{\kappa} \qquad [1.2\text{-}77]$$

So the twice-normal-twice-surface projection of the Riemann tensor is given by

$$B^{\mu}{}_{a}{}^{\nu}{}_{b}N^{\sigma}R_{\sigma\mu\nu}{}^{\kappa}w_{\kappa} = -\rho^{2}\pounds_{\mathbf{N}}h_{ab} - 2\rho^{-1}h_{ab}\pounds_{\mathbf{N}}\rho - B^{\mu}{}_{a}{}^{\nu}{}_{b}w_{\kappa}\nabla_{\mu}\nabla_{\nu}N^{\kappa} \qquad [1.2\text{-}78]$$

The last term on the right hand side of equation 1.2-78 can be rewritten, after some algebra,

$$B^{\mu}{}_{a}{}^{\nu}{}_{b}w_{\kappa}\nabla_{\mu}\nabla_{\nu}N^{\kappa} = \rho^{-1}D_{a}D_{b}\rho - h_{ab}\rho^{-1}\pounds_{\mathbf{N}}\rho + \rho^{2}h_{ac}h_{b}{}^{c} \qquad [1.2\text{-}79]$$

Thus the first term on the right-hand side of equation 1.2-73 becomes

$$B^{\mu}{}_{a}{}^{\nu}{}_{b}N^{\sigma}R_{\sigma\mu\nu}{}^{\kappa}w_{\kappa} = -\rho^{2}\pounds_{\mathbf{N}}h_{ab} - h_{ab}\rho^{-1}\pounds_{\mathbf{N}}\rho - \rho^{-1}D_{a}D_{b}\rho - \rho^{2}h_{ac}h_{b}{}^{c} \qquad [1.2\text{-}80]$$

The second term of equation 1.2-73 can be evaluated from the contraction of Gauss's equation [1.2-71]

$$^{(3)}R_{ab} \equiv {}^{(3)}R_{dbc}{}^{d} = B^{\kappa}{}_{d}{}^{\mu}{}_{a}{}^{\nu}{}_{b}{}^{d}{}_{\sigma}R_{\kappa\mu\nu}{}^{\sigma} - 2\rho^{2}h_{[d}{}^{d}h_{a]b} \qquad [1.2\text{-}81]$$

yielding

$$B^{\kappa}{}_{d}{}^{\mu}{}_{a}{}^{\nu}{}_{b}{}^{d}{}_{\sigma}R_{\kappa\mu\nu}{}^{\sigma} = {}^{(3)}R_{ab} + \rho^{2}h_{d}{}^{d}h_{ab} - \rho^{2}h_{a}{}^{d}h_{db} \qquad [1.2\text{-}82]$$

When 1.2-80 and 1.2-82 are substituted into equation 1.2-73 we finally get

$$-\rho^{2}\pounds_{\mathbf{N}}h_{ab} - h_{ab}\rho^{-1}\pounds_{\mathbf{N}}\rho - \rho^{-1}D_{a}D_{b}\rho - \rho^{2}h_{ac}h_{b}{}^{c} + {}^{(3)}R_{ab}$$
$$+ \rho^{2}h_{d}{}^{d}h_{ab} - \rho^{2}h_{a}{}^{d}h_{db} = 0 \qquad [1.2\text{-}83]$$

Equations 1.2-68 and 1.2-83 form a first order system of partial differential equations for ${}^{(3)}\mathbf{g}$ and $\mathbf{h}$, which we may express in terms of the Lie derivative with respect to $\mathbf{V}$ using

$$\pounds_{\mathbf{N}} = \pounds_{\mathbf{V}} - \pounds_{\sigma} \qquad [1.2\text{-}84]$$

They propagate the three-metric and extrinsic curvatures along the vector field $\mathbf{V}$.

The second set of field equations, the remaining four equations $G_{\mu\nu}N^{\nu} = 0$, contain no second- (or higher-)order Lie derivatives of any quantities. They are constraint equation on the initial data. We shall now proceed to write them in covariant form in the 3+1 breakup.

Calculating the Ricci scalar in the 3+1 breakup gives:

$$2\rho^{-2}N^{\mu}N^{\lambda}R_{\mu\lambda} + R = {}'g^{\mu\lambda}{}'g^{\kappa\nu}R_{\kappa\mu\lambda\nu}\,; \qquad [1.2\text{-}85]$$

which can be rewritten as

$$2\rho^{-2}N^{\mu}N^{\lambda}G_{\mu\lambda} = {}'g^{\mu\lambda}{}'g^{\kappa\nu}R_{\kappa\mu\lambda\nu} \qquad [1.2\text{-}86]$$

Making use of the contracted Gauss's equation [1.2-82] and the field equations, we have the first constraint equation:

$$2\rho^{-2}N^{\mu}N^{\lambda}G_{\mu\lambda} = {}^{(3)}R + \rho^{2}h^{2} - \rho^{2}h^{ab}h_{ab} = 0 \qquad [1.2\text{-}87]$$

The second set of constraint equations is gotten from the surface-normal projection of the field equations

$$N^{\nu}B^{\mu}{}_{a}G_{\mu\nu} = 0$$

$$= N^{\nu}B^{\mu}{}_{a}G_{\mu\nu} = N^{\nu}B^{\mu}{}_{a}\kappa_{b}{}^{\alpha}{}_{c}g^{bc}R_{\kappa\mu\nu\alpha}$$

$$= \rho^2 w_\nu B^\mu{}_a \kappa_b{}^\alpha{}_c g^{bc} R_{\kappa\mu\alpha}{}^\nu \quad [1.2\text{-}88]$$

Using Codazzi's equation 1.2-72 we get

$$N^\nu B^\mu{}_a G_{\mu\nu} = \rho^2 g^{bc}[2D_{[a}h_{b]c} - 2\rho^{-1}h_{b[c}D_{a]}\rho] = 0 \quad [1.2\text{-}89]$$

or

$$g^{bc}D_{[a}\rho^{-1}h_{b]c} = 0 \quad [1.2\text{-}90]$$

The constraint equations are usually interpreted as conditions which the initial data ${}^{(3)}\mathbf{g}$ and $\mathbf{h}$ must satisfy. Strictly speaking, the initial data for the set of partial differential equations which form the Einstein equations is $\mathbf{g}$ and $£_\mathbf{v}\mathbf{g}$. It is when we prescribe the relation between $£_\mathbf{v}\mathbf{g}$ and $\mathbf{h}$ by choosing values for ρ and σ that constraints are placed on the initial data. One can alternatively choose $\mathbf{g}$ and $£_\mathbf{v}\mathbf{g}$ arbitrarily and use the constraint equations to determine ρ and σ. The geometrical interpretation of this viewpoint is that the 3-geometry on two neigboring surfaces is given arbitrarily; the lapse and shift functions complete the determination of the 4-geometry between the two surfaces by determining the unit normal vector $\mathbf{n}$ to the initial surface. An example of this approach is the maximal slicing condition, discussed by York [1979] and others. Here, all the 3-geometries of the foliation are chosen to satisfy the condition $g^{ba}h_{ba} = 0$. When this condition is substituted into the constraint equations, a set of elliptical differential equations results for ρ and σ.

Having identified the constraint equations above [1.2-87,1.2-89], we now indicate one way to solve them, if only in a formal sense, following d'Inverno and Stachel

[1978]. Let **S** again denote the initial Cauchy surface. Just as we foliated the four-dimensional space-time with a one-parameter family of three-surfaces, we foliate **S** with a one-parameter family of space-like two-surfaces $\varphi(x^i) = p$, where p is a parameter distinguishing members of the foliation (see Figure 1-1) (for the remainder of this section, the components of the three-metric on **S** will be written as g_{ij} rather than ${}^{(3)}g_{ij}$). Let us adapt coordinates to the foliation so that its members are given by $x^1 = p$ and the remaining two coordinates on **S**, A=2,3, are coordinates of the two-surfaces. In this coordinate system, the covariant normal $\mathbf{d}\varphi$ is given by:

$$s_i = \partial_i\varphi = \delta^1{}_i \qquad [1.2\text{-}91]$$

In this coordinate system, the connecting quantity is given by $B^i{}_A = \delta^i{}_A$. It induces a positive definite two-metric g_{AB} on each two-surface, whose components in adapted coordinates, are the (A,B)-components of the three-metric.

We can fibrate the foliation using a vector field **u** tangent to **S** satisfying

$$u^k s_k = 1 \qquad [1.2\text{-}92]$$

This enables us to identify points on different members of the foliation. We can extend the coordinates of any single two-surface to the entire foliation by holding them fixed on the trajectories of the fibrating vector field. In these coordinates, **u** takes the form $u^k = \delta^k{}_1$.

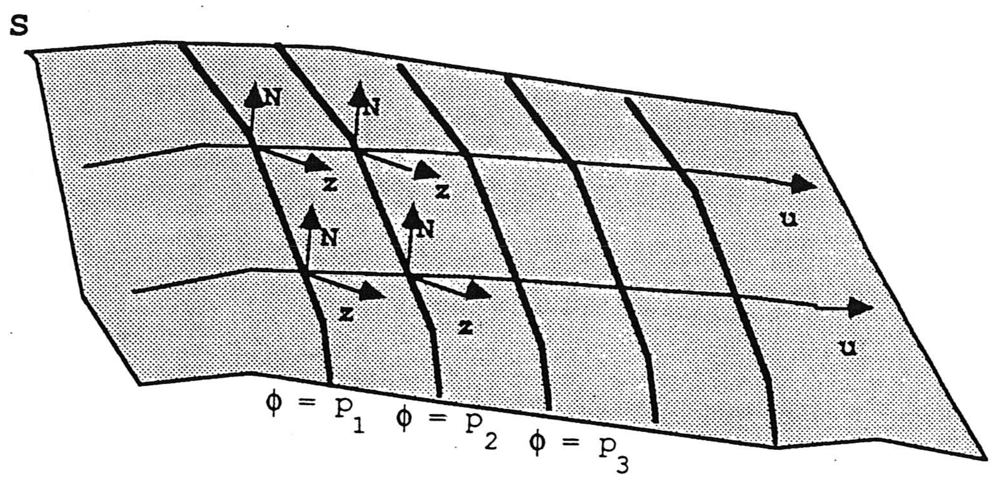
Letting e^λ denote the determinant of g_{AB}, define a conformally related quantity

$$\tilde{g}_{AB} = e^{-\lambda/2} g_{AB} \qquad [1.2\text{-}93]$$

Since the determinant of $\tilde{g}_{AB}$ is unity, there are only two independent components of $\tilde{g}_{AB}$. There now exists a one-parameter family of conformal two-metrics on the foliation $\tilde{g}_{AB}(x^A, x^1)$. The trace of the derivative of $\tilde{g}_{AB}$ with respect to x^1 vanishes.

$$\tilde{g}^{AB}\,\tilde{g}_{AB,1} = |\tilde{g}_{AB}|_{,1} = 0 \qquad [1.2\text{-}94]$$

Such a family is called a <u>conformal two-structure</u> by D'Inverno and Stachel, and may be used to represent the true degrees of freedom of the gravitational field, as will be shown.



1
Figure 1-1 Foliation of the Cauchy Surface By a One-Parameter Family of Two-Surfaces

We can rig the foliation by using a vector field **z**, orthogonal to the two-surfaces of the foliation, and tangent to **S**. **z** is uniquely defined if we adopt the normality condition:

$$z^i s_i = 1 \qquad [1.2\text{-}95]$$

Using the rigging vector, we can now define the projection operators

$$C^i{}_j = Z^i s_j \qquad [1.2\text{-}96]$$

$$B^i{}_j = \delta^i{}_j - C^i{}_j \qquad [1.2\text{-}97]$$

into the rigging direction and two-surfaces, respectively.

In analogy with the 3+1 breakup, the fibrating vector field can be uniquely decomposed

$$u^i = Z^i + \beta^i, \qquad [1.2\text{-}98]$$

where β^i is tangent to the two-surfaces $\varphi(x) = p$, that is $\beta^i s_i = 0$, which in adapted coordinates implies $\beta^1 = 0$. Using this decomposition, and the coordinate conditions, we can write the projection operators in the adapted coordinate system

$$C^i{}_j = (\delta^i{}_1 - \beta^i)\delta^1{}_j \qquad [1.2\text{-}99]$$

$$B^i{}_j = \delta^i{}_j - (\delta^i{}_1 - \beta^i)\delta^1{}_j \qquad [1.2\text{-}100]$$

Then, in adapted coordinates, the three-metric is equivalent to the four sets of functions:

$$g_{ij} \Leftrightarrow \tilde{g}_{AB}(x^A, x^1),\ \lambda(x^A, x^1),\ \alpha^2(x^A, x^1),\ \beta^A(x^A, x^1)$$

since we can use the projection operators to decompose the three-metric into

$$g_{ij}dx^i dx^j = \alpha^2(dx^1)^2 + e^{\lambda/2}\tilde{g}_{AB}(dx^A + \beta^A dx^1)(dx^B + \beta^B dx^1)$$

$$[1.2\text{-}101]$$

where $\alpha^2 \equiv g_{ij}u^i u^j = g_{11} \qquad [1.2\text{-}102]$

α and β^A are the lapse function and shift vectors of the fibrating vector field **u**.

The extrinsic curvature h_{ab}, a tensor field on **S** can also be decomposed into the parts:

$$h_{AB} \equiv B^i{}_A{}^j{}_B h_{ij} = 1/2\rho^{-2}\pounds_{\mathbf{N}}\, g_{AB} \qquad [1.2\text{-}103]$$

$$h^C{}_C \equiv g^{AB}h_{AB} = 1/2\ \rho^{-2}\pounds_{\mathbf{N}}\lambda \qquad [1.2\text{-}104]$$

$$\tilde{h}_{AB} = \exp(-1/2\lambda)(h_{AB} - 1/2g_{AB}h^C{}_C) = 1/2\rho^{-2}\pounds_{\mathbf{N}}\tilde{g}_{AB} \qquad [1.2\text{-}105]$$

$$\Theta \equiv h_{ij}z^iz^j = 1/2\rho^{-2}\pounds_{\mathbf{N}}\alpha^2 \qquad [1.2\text{-}106]$$

$$h_A \equiv B^i{}_A z^j h_{ij} \Rightarrow h^A = 1/2\rho^{-2}\pounds_{\mathbf{N}}\beta^A \qquad [1.2\text{-}107]$$

so that, assuming $\rho = 1$ and $\sigma^i = 0$, we have:

$$h_{ij} = B^A{}_i B_j \exp(1/2\lambda)\,\tilde{h}_{AB} + 1/2B^A{}_i B_j g_{AB}h_C{}^C + 2B^A{}_{(i}s_{j)}h_A + \Theta s_i s_j \qquad [1.2\text{-}108]$$

Putting this decomposition, and the corresponding one for the three-dimensional metric [1.2-101] into the constraint equations [1.2-87 and 1.2-89], d'Inverno and Stachel arrive at the following equations:

- the constraint $G_{\mu\nu}N^\mu N^\nu = 0$ becomes:

$$\pounds^2{}_{\mathbf{Z}}\lambda + 3/8(\pounds_{\mathbf{Z}}\lambda) - \alpha^{-1}(\pounds_{\mathbf{Z}}\alpha)(\pounds_{\mathbf{Z}}\lambda) - \alpha^2\exp(-1/2\lambda)\,{}^{(2)}\tilde{R}$$
$$+ \exp(-1/2\lambda)\left[1/2\ \alpha^2\tilde{\nabla}^2\lambda + 2\alpha\nabla^2\alpha + \alpha^2\exp(1/2\lambda)\,\tilde{\chi}^{AB}\tilde{\chi}_{AB} + 2h^Ah_A\right.$$
$$\left. - \alpha^2\exp(-1/2\lambda)\ \tilde{h}^{AB}\tilde{h}_{AB}\right] + 2\Theta h^A{}_A + 1/2\alpha^2h^A{}_Ah^B{}_B = 0 \qquad [1.2\text{-}109]$$

- the constraint equations $G_{\mu\nu}N^\mu B^\nu{}_b = 0$ become

$$\pounds_{\mathbf{Z}}h^A{}_A + (1/4h^A{}_A - \alpha^{-2}\Theta)\pounds_{\mathbf{Z}}\lambda - \alpha\tilde{\chi}^{AB}\tilde{h}_{AB}$$
$$- \alpha^{-1}\exp(-1/2\lambda)\,\tilde{\nabla}(\alpha h^A) = 0 \qquad [1.2\text{-}110]$$

$$\pounds_{\mathbf{Z}}(\alpha^{-1}h_A) + 1/2\alpha^{-1}h_A\pounds_{\mathbf{Z}}\lambda + \exp(-1/2\lambda)\nabla_B[\alpha\exp(1/2\lambda)\,\tilde{h}^B{}_A]$$
$$- (\alpha^{-1}\Theta)_{,A} - 1/2\alpha^{-2}(\alpha^{-1}h^B{}_B)_{,A} = 0 \qquad [1.2\text{-}111]$$

where $\tilde{\chi}_{AB} = -1/2\alpha^{-2}\pounds_{\mathbf{Z}}\tilde{g}_{AB}$; ∇_A denotes covariant differentiation with respect to the two-surface metric; $\tilde{\nabla}^2 = \tilde{g}^{AB}\tilde{\nabla}_A\tilde{\nabla}_B$; and all tilded quantities are built out of $\tilde{g}_{AB}$ and its inverse $\tilde{g}^{AB}$.

Several possible integration schemes for this set of equations are discussed by D'Inverno and Stachel [1978]. If we take the conformal two-structure as the gravitational degrees of freedom, then the quantities $\tilde{g}_{AB}$ and $\tilde{h}_{AB}$ are the freely-specifiable initial data. The lapse function α and shift vector β^A merely define the family of two-surfaces and are not truly dynamical; they may be conveniently set to $\alpha = 1$ and $\beta^A = 0$. This leaves five functions yet to be determined on **S**: λ, Θ, h^A and $h^A{}_A$. We have the freedom to choose one function that will specify the initial surface, leaving four unknowns and four constraint equations. Formally, they then solve the constraint equations by specifying sufficient initial data for these four equations on one initial two-surface. They conclude that the conformal two-structure can represent the true degrees of freedom of the gravitational field, which are specified by four independent quantities per space-time point of the hypersurface, and that all other components of the initial three-metric and its velocity are either determined by the constraint equations or represent the choice of an initial hypersurface itself.

In a series of papers, O'Murchada and York [1974] present an alternative method for solving the constraint equations. It is based upon the recognition that the conformal geometry of a space-like hypersurface can be taken as the dynamical degrees of freedom of the gravitational field. In the Hamiltonian formulation of general relativity (see Dirac [1958] and Arnowitt, Deser and Misner [1962]), the canonically conjugate dynamical variables are the three-metric g_{ab} and three-tensor density π^{ab} constructed from the extrinsic curvature:

$$\pi^{ab} = \sqrt{g}\,(Kg^{ab} - K^{ab}) \qquad [1.2\text{-}112]$$

where we use the symbol K^{ab} instead of h^{ab} to denote the extrinsic curvature constructed using the unit covariant normal instead of an arbitrary covariant normal as we did above. Note that g is the determinant of the surface three-metric.

We may write the vacuum constraint equations 1.2-87 and 1.2-89 in terms of g_{ab} and, for convenience, the momentum tensor p^{ab} related to the momentum tensor density π^{ab} by

$$\pi^{ab} = \sqrt{g}\; p^{ab} \qquad [1.2\text{-}113]$$

The constraint equations are now the Momentum constraints

$$\nabla_a p^{ab} = 0 \qquad [1.2\text{-}114]$$

and Hamiltonian Constraint

$$p^{ab}p_{ab} - 1/2\; p^2 - {}^{(3)}R = 0, \qquad [1.2\text{-}115]$$

so named because it determines the evolution of g_{ab} and p^{ab} via the canonical equations of motion.

The momentum tensor can be orthogonally and covariantly decomposed into transverse-traceless and longitudinal parts analogous to the Helmholtz decomposition discussed above for the electromagnetic potential. Since the transverse-traceless part, by definition, satisfies the equation 1.2-114, the momentum constraint is only on the longitudinal part of the momentum.

In order to solve the Hamiltonian constraint, they conformally rescale the three-metric g_{ab} on **S** to a conformally-related metric and show that the Hamiltonian constraint determines the conformal scaling factor.

Together, the momentum and Hamiltonian constraints form a coupled set of quasilinear elliptic equations for determining the conformal scale factor and longitudinal parts of the momentum. O'Murchada [1980] has examined the conditions under which the system decouples and has solved the Hamiltonian constraint for some cases. The freely-specifiable initial data is the conformally-invariant metric $\tilde{g}_{ab} = g^{-1/3}g_{ab}$ and the transverse-traceless part of the momentum. Now $\tilde{g}_{ab}$ has five independent components and since we are free to do coordinate transformations on **S**, we are free to choose three of the remaining components of the three-metric as coordinate conditions. This leaves two components of the three-metric comprising part of the dynamical information. There are also only two independent components of the transverse-traceless momentum. Hence, four functions can be prescribed on **S** to determine the dynamical evolution. This again shows that there are two degrees of freedom for the gravitational field.

It is important to note that conformal three-geometry techniques are not applicable to null surfaces since the null three-geometry is degenerate. On the other hand, the conformal two-structure technique, discussed previously, is applicable to null surfaces, as we shall see in the next section. For this reason, it is a useful formalism for unifying all possible initial value problems. The reader will recall that the constraint equations for the electromagnetic field, similar to the momentum constaints for gravity, could be solved by using spherical coordinates on the Cauchy surface. Geometrically, this amounts to foliating the Cauchy surface using a one-parameter family of two-spheres. The freely-specifiable Cauchy data is a two-vector field, and its normal derivatives, on the members of the foliation; this is the electromagnetic analog of the conformal two-structure. An analogous treatment may possibly be devised to link the conformal three-geometry approach to the conformal two-structure method for the Cauchy problem.

1.3 <u>The Characteristic Initial Value Problem in General Relativity</u>

In order to bypass the problem of solving the constraint equations which arise in the Cauchy problem, one may consider the initial value problem on null hypersurfaces[1]. As we remarked in Section 1.1, it is not possible to set initial data on a single null hypersurface leading to a unique solution of the initial value problem; data must also be given on an additional hypersurface - null, timelike or spacelike. An exception to this statement is that the problem may be posed on a complete null cone with vertex (see Penrose [1963] and Penrose and Rindler [1984]). In this section we shall consider the double null initial value problem, in which initial data is given on two intersecting null hypersurfaces. It was originally consider by Sachs [1962] and is the only characteristic initial value problem for which existence and uniqueness theorems have been proved for non-analytic solutions (see Müller zum Hagen and Seifert [1977]). We review Sachs' approach here.

The initial data is set on two null hypersurfaces, $\mathcal{U}$ and $\mathcal{V}$, which intersect on a two-dimensional space-like surface Σ. We construct two distinct families of null hypersurfaces, one containing $\mathcal{U}$ as a member and the other containing $\mathcal{V}$. The evolution of the initial data is to be given with respect to two vector fields which are tangent to the null hypersurfaces (see Figure 1-2), as we shall show below.

We start by considering one of the initial null hyper-

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1) See, for example, Sachs [1962], Bondi, van der Burg and Metzner [1962] and Gambini and Restuccia [1978], Tamburino and Winocur [1966] and Aragone and Chela-Flores [1975]. Recent work has been done on defining a Hamiltonian on null hypersurfaces (Nagarajan and Goldberg [1985] and Goldberg [1984]).

surfaces $\mathcal{U}$, which can be described by some function $u(x^\mu) = 0$. The vector field k^μ defined on $\mathcal{U}$ by

$$k^\mu = g^{\mu\nu}u_{,\nu} \qquad [1.3\text{-}1]$$

is a null vector field since

$$k^\mu k_\mu = g^{\mu\nu}u_{,\mu}u_{,\nu} = 0 \qquad [1.3\text{-}2]$$

and is tangent to $\mathcal{U}$ itself. On $\mathcal{U}$, the vector field **k** is tangent to a family of integral curves parametrized by $*v$. Through each point of $\mathcal{U}$ passes one and only one such curve. Given $u(x^\mu)$, the parametrization is unique up to an additive constant, so we choose $*v = 0$ on Σ.

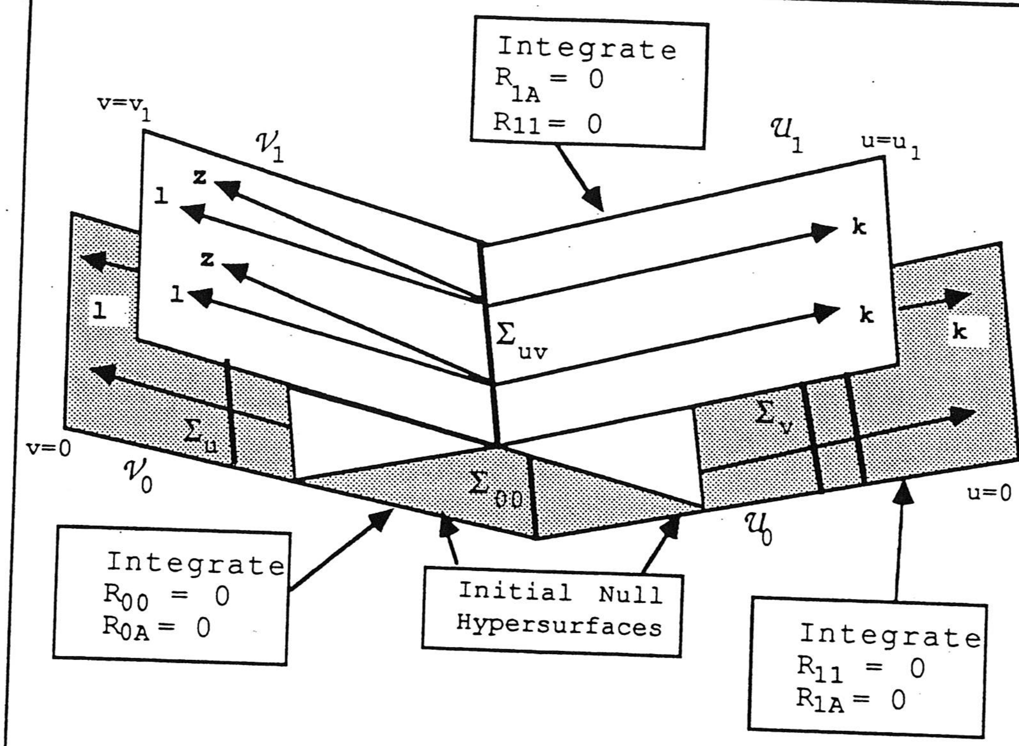

Figure 1-2 The Double Null Initial Value Problem

Similarly, the hypersurface $\mathcal{V}$ is defined by a function

$v(x^\mu) = 0$ and gives rise to a null vector field l^μ tangent to $\mathcal{V}$. Again, there is a family of integral curves parametrized by $*u$ which is tangent to $\mathbf{l}$ on $\mathcal{V}$ and we choose $*u = 0$ on Σ. Since we will use $*v$ and $*u$ to define the families of null hypersurfaces mentioned above, we will identify $*v = v$ and $*u = u$.

The functions $u(x^\mu)$ and $v(x^\mu)$ are not unique, and we can use this freedom to fix $\mathbf{k}$ and $\mathbf{l}$ so that

$$k^\mu l_\mu = -1 \text{ on } \Sigma \qquad [1.3\text{-}3]$$

We shall foliate $\mathcal{U}$ by a one-parameter family of two-surfaces $\{\Sigma_v\}$. A member of this family is defined by dragging all the points of Σ a fixed parameter distance $\underline{v}$ along the integral curves of $\mathbf{k}$. The hypersurface $\mathcal{V}$ can be foliated in the same way using $\mathbf{l}$ and Σ, giving rise to another one-parameter family of two-surfaces $\{\Sigma_u\}$.

At any two-surface Σ_v on $\mathcal{U}$, there is a unique null hypersurface, other than $\mathcal{U}$, which intersect $\mathcal{U}$. We denote this hypersurface by $\mathcal{V}_v$. At any two-surface Σ_u on $\mathcal{V}$, there is also an intersecting null hypersurface $\mathcal{U}_u$. The initial hypersurface $\mathcal{V}$ corresponds to $\mathcal{V}_0$ while $\mathcal{U}$ corresponds to $\mathcal{U}_0$. By this construction, we have extended the definitions of $u(x^\mu)$ and $v(x^\mu)$ into a four-dimensional region and have defined two families of null hypersurfaces $\{\mathcal{U}_u\}$ and $\{\mathcal{V}_v\}$ with $\mathcal{U} = \mathcal{U}_0$ and $\mathcal{V} = \mathcal{V}_0$. This also gives rise to a two-parameter family of space-like two-surfaces $\{\Sigma_{uv}\}$ defined by the intersection of the $\mathcal{U}_u$ and $\mathcal{V}_v$ null hypersurfaces; Σ

corresponds to Σ_{00}. The two-parameter family of two-surfaces $\{\Sigma_{uv}\}$ is called a <u>foliation of co-dimension-two</u>. The vector fields **k** and **l** are now defined in a four-dimensional region and are tangent to congruences of null geodesics. They are also orthogonal to each two-surface Σ_{uv}, since u and v are constant on Σ_{uv} and we can rig Σ_{uv} using **k** and **l**. A projection operator into the rigging directions is given by

$$C^{\mu}{}_{\nu} = [k^{\rho}l_{\rho}]^{-1}(l^{\mu}k_{\nu} + l^{\mu}k_{\nu}) \qquad [1.3\text{-}4]$$

while a projection operator into the two-surface is

$$B^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - C^{\mu}{}_{\nu} \qquad [1.3\text{-}5]$$

The induced metric on the members of the family of two-surfaces is defined using the projection operator $B^{\mu}{}_{\nu}$ acting on $g_{\mu\nu}$.

The vector fields **k** and **l**, defined as the tangent vector fields to the $\{\mathcal{U}_u\}$ and $\{\mathcal{V}_v\}$ families of null hypersurfaces, are unique, which is also true of the two-parameter family of two-surfaces $\{\Sigma_{uv}\}$. However, the vector fields **k** and **l** do not, in general, commute. The geometrical interpretation of this is the following: we drag all the points of Σ a parameter distance v along **k** to define Σ_v on $\mathcal{U}$ and then drag points of Σ_v a parameter distance u along **l** to define Σ_{vu} on $\mathcal{V}_v$. Alternatively, we can drag all the points of Σ a parameter distance u along **l** to define Σ_u on $\mathcal{V}$ and then drag points of Σ_u along **k** to define the same two-surface Σ_{vu} on $\mathcal{U}_u$. While the two two-surfaces defined this way are the same, we end up at different points on the surface. To arrive at the same point, we must use a pair of commuting vector fields which are tangent to the null hypersurfaces. We can, without loss of generality, choose one of them to be **k** itself. Then, the most general vector field **z**, commuting with **k** and tangent to the members of the family $\{\mathcal{V}_v\}$, has the form

$$z^\mu = \alpha l^\mu + C^\mu \qquad [1.3\text{-}6]$$

where C^μ is tangent to each two-surface Σ_{uv}. α and C^μ are the lapse function and shift vector of the foliation of $\mathcal{V}_v$. If we choose the parametrization of **z** to be the u-value of the member of $\{\Sigma_{uv}\}$ it intersects, then **z** satisfies the normalization condition

$$z^\mu u_{,\mu} = -1 \qquad [1.3\text{-}7]$$

Without loss of generality, we can take $\alpha = 1$ and $\mathbf{C} = 0$ on $\mathcal{V}$, i.e. $\mathbf{z} = \mathbf{l}$ on $\mathcal{V}$.

Following Sachs, we work in a a coordinate system adapted to the families of null hypersurfaces by choosing $x^0 = u$ and $x^1 = v$. Then, since the surfaces x^0 = constant and x^1 = constant are null hypersurfaces, the metric components g^{00} and g^{11} vanish.

The remaining two coordinates are defined in a four-dimensional region by taking the coordinates on Σ to be constant along the trajectories of the vector fields **k** and **z**. The condition that the coordinates of Σ be constant along the trajectories of **k** is

$$k^\mu x^A{}_{,\mu} = 0 \qquad [1.3\text{-}8]$$

In the adapted coordinates, this becomes

$$g^{0A} = 0 \qquad [1.3\text{-}9]$$

Similarly, $z^\mu x^A{}_{,\mu} = 0$ implies

$$\alpha g^{1A} + C^A = 0 \qquad [1.3\text{-}10]$$

From 1.3-7, we have

$$\alpha g^{01} = -1 \qquad [1.3\text{-}11]$$

Equations 1.3-10 and 1.3-11 relate the lapse function α and the shift vector C^A of the foliation of $\{\mathcal{V}_v\}$ to components of the metric tensor in the adapted coordinate system.

On $\mathcal{V}$, where **z = 1**, we thus have

$$g^{01} = -1 \text{ and } g^{1A} = 0 \qquad [1.3\text{-}12]$$

In the adapted coordinate system, the projection operator $C^{\mu}{}_{\nu}$ takes the form

$$C^{\mu}{}_{\nu} = \delta^{\mu}{}_{0}\delta^{0}{}_{\nu} + \delta^{\mu}{}_{1}\delta^{1}{}_{\nu} \qquad [1.3\text{-}13]$$

One can easily show that the induced metric on a two-surface is given by the g_{AB}-components of the four-dimensional metric tensor.

The metrical line element takes the form,

$$ds^2 = -e^{-2q}dudv + e^{2h}\tilde{g}_{AB}(dx^A + C^A du)(dx^B + C^B du); \qquad [1.3\text{-}14]$$

where h is defined by $e^{4h} = \det[g_{AB}]$, $\det[\tilde{g}_{AB}] = 1$ and the function q is defined by $e^{-2q} = -g_{01} = \alpha^{-1}$. We follow Sachs' notation for the conformal scale factor rather than use λ as in Section 1.2.

The quantities q, $\tilde{g}_{AB}$, h and C^A constitute six independent functions of four coordinates. From the way the coordinate system is constructed, q = 0 on $\mathcal{U}$ and $\mathcal{V}$ and $C^A = 0$ on $\mathcal{V}$.

Sachs divides the source-free field equations into four groups

$R_{AB} = 0$	Propagation Equations	[1.3-15]
$R_{11} = 0$	Hypersurface Equations	[1.3-16]
$R_{1A} = 0$		[1.3-17]
$R_{00} = 0$	Subsidiary Equations	[1.3-18]
$R_{0A} = 0$		[1.3-19]
$R_{01} = 0$	Trivial Equation	[1.3-20]

Using the Bianchi identities, he proves that, if the propagation and hypersurface equations hold in a neighborhood $\mathcal{W}$ of Σ bounded from below by $\mathcal{U}$ and $\mathcal{V}$ and if the subsidiary equations hold on $\mathcal{V}$, then the subsidiary equations and the trivial equation hold automatically in $\mathcal{W}$.

The subsidiary equation $R_{00} = 0$ [1.3-18] on $\mathcal{V}$ takes the form

$$h_{,00} - (h_{,0})^2 - 1/2\, \tilde{g}^{AB}{}_{,0}\tilde{g}_{AB,0} = 0 \text{ where } X_{,0} = \partial X/\partial u \qquad [1.3\text{-}21]$$

If $\tilde{g}_{AB}$ is given on $\mathcal{V}$ and h and $h_{,0}$ are given on Σ, then equation 1.3-21 can be used to solve for h on all of $\mathcal{V}$.

The remaining subsidiary equations $R_{0A} = 0$ determine $C_{A,1}$ everywhere on $\mathcal{V}$ in terms of $C_{A,1}$ on Σ, where $X_{,1} \equiv \partial X/\partial v$.

$$C_{A,01} + F_1C_{A,1} + F_2 = 0. \qquad [1.3\text{-}22]$$

F_1 and F_2 are functions of $\tilde{g}_{AB}$ and h on $\mathcal{V}$. Summarizing, equations 1.3-21 and 1.3-22 determine h and $C_{A,1}$ on $\mathcal{V}$ in terms of h, $h_{,0}$, $C_{A,1}$ on Σ and $\tilde{g}_{AB}$ on $\mathcal{V}$.

The hypersurface equation $R_{11} = 0$ [1.3-16] on $\mathcal{U}$ takes the form

$$h_{,11} + 2h_{,1}q_{,1} - (h_{,1})^2 - 1/2\ \tilde{g}^{AB}{}_{,1}\tilde{g}_{AB,1} = 0 \qquad [1.3\text{-}23]$$

Since $q = 0$ on $\mathcal{U}$, equation 1.3-23 can be used to solve for h on $\mathcal{U}$ in terms of h and $h_{,1}$ on Σ and $\tilde{g}_{AB}$ on $\mathcal{U}$.

The two remaining hypersurface equations $R_{1A} = 0$ take the form on $\mathcal{U}$

$$C_{A,11} + F_3 C_{A,1} + F_4 = 0 \qquad [1.3\text{-}24]$$

where F_3 and F_4 are functions of h, $\tilde{g}_{AB}$ and q on $\mathcal{U}$. Thus, one can solve for C_A on $\mathcal{U}$ using equation 1.3-24 in terms of h, $\tilde{g}_{AB}$ and q on $\mathcal{U}$ and $C_{A,1}$ on Σ (On any other hypersurface, $\mathcal{U}_u$, equation 1.3-23 will be used to solve for q instead of h (provided $h_{,1} \neq 0$). To do this, we must know h and $\tilde{g}_{AB}$ on $\mathcal{U}_u$. Equation 1.3-24 can still be used for C_A if h, $\tilde{g}_{AB}$ and q are known on $\mathcal{U}_u$).

To summarize, we have the means to solve for h, q and C_A on $\mathcal{U}$ and $\mathcal{V}$ in terms of h, $h_{,0}$, $h_{,1}$, q and $C_{A,1}$ on Σ and $\tilde{g}_{AB}$ on $\mathcal{U}$ and $\mathcal{V}$. Note that $\tilde{g}_{AB}$ is defined independently on $\mathcal{U}$ and $\mathcal{V}$ but must match on Σ. We may use the local conformal flatness of any two-geometry to let us choose the coordinate system (x^A) on Σ such that

$$\tilde{g}_{AB} = \delta_{AB} \qquad [1.3\text{-}25]$$

The components of g_{AB} (that is, $e^{2h}\tilde{g}_{AB}$) are propagated off $\mathcal{U}$ and $\mathcal{V}$ using equation 1.3-15 which takes the form

$$g_{AB',01} + F_5 g_{AB',0} + F_6 = 0 \qquad [1.3\text{-}26]$$

F_5 and F_6 are known on any hypersurface $\mathcal{U}_u$ if q, h, $\tilde{g}_{AB}$, C_A are known there.

Thus $g_{AB',0}$ is known on $\mathcal{U}$ if $g_{AB',0}$ is given on $\mathcal{V}$ which is the case. Knowledge of $g_{AB',0}$ is equivalent to knowing g_{AB} on the first neighboring hypersurface $\mathcal{U}_1$. We can then use $R_{11} = 0$ to solve for q on $\mathcal{U}_1$ and then use $R_{1A} = 0$ to solve for C_A on $\mathcal{U}_1$. This procedure can be formally repeated to give g_{AB} on all hypersurfaces within a neighborhood of $\mathcal{U}$ in terms of the initial data

$C_{A',1}$, h, h_0 and h_1 on Σ

$\tilde{g}_{AB}$ on $\mathcal{U}$ and $\mathcal{V}$

and the coordinate conditions, $q = 0$ on $\mathcal{U}$ and $\mathcal{V}$, and $C_A = 0$ on $\mathcal{V}$. The coordinate conditions merely describe how the coordinate systems on the two initial hypersurfaces are to be layed out, and have no dynamical content. What is important is that the initial data to be supplied on each of the pair of null hypersurfaces is the <u>conformal two-structure</u>. All other data is to be given on a <u>single two-surface</u> as in the Cauchy problem.

As we mentioned above, an existence and uniqueness theorem for the double-null initial value problem was proved by Müller zum Hagen and Seifert [1977}. Leaving aside the details, if $\tilde{g}_{AB}$ are of Sobolev class H^{2s+2} on $\mathcal{V}$ and $\mathcal{U}$; if h and $C_{A',1}$ are of class H^{2s+1} on Σ and if h_0 and h_1 are of class H^{2s} on Σ, then there exists a region $\mathcal{W}$ in which we have a metric of class H^s satisfying the vacuum Einstein equations and which is maximal and unique up to isometries. The differentiability properties of the solution is an important difference between the Cauchy and double null initial value problems. We stated in Section 1.2 that if the initial data for the Cauchy problem belonged to Sobolev class H^s, then so did the solution. In the double-null initial value problem, the Sobolev class of the initial data needs to be H^{2s+2} for the solution to be H^s.

The double-null initial value problem also differs from the Cauchy problem in that two functions of three parameters need to be given on <u>two</u> hypersurfaces, rather than four functions of three-parameters on a single hypersurface. Unlike the Cauchy problem, the functions do not involve derivatives of the field off the surface but only values of the field on the surface. This halving of initial data is typical of the characteristic initial value problem and is discussed, for example, in Bers, John and Schecter [1964]. For characteristic initial value problems, the number of degrees of freedom is equal to the number of independent functions which must be given on each of the initial hypersurfaces; in this case it is two.

CHAPTER II

THE TWO + TWO FORMALISM

2.1 The Geometry of Rigged Two-Surfaces

We have seen how formal unification of the Cauchy and characteristic initial value problems for the gravitational field can be achieved by utilizing one-parameter families of space-like two-surfaces to foliate the hypersurface or hypersurfaces on which the initial data is set. For the Cauchy Problem, the freely-specifiable initial data consists of the conformally-invariant two-metric and its Lie-derivative with respect to a hypersurface-orthogonal vector field given on each member of the foliation of the initial space-like hypersurface. While we did not consider the evolution of the initial data in this formalism, it is not hard to see that each subsequent space-like hypersurface can also be foliated by a one-parameter family of space-like two surfaces. Thus, one can consider the problem to be one of determining the evolution of the conformal two-structures as we travel from hypersurface to hypersurface. A similar result holds for the double null characteristic initial value problem, where we did, in fact, consider the problem of evolution.

The hypersurfaces considered in Sections 1.2 and 1.3 are special cases of a more general formalism. The foliation of the initial hypersurface arises from dragging a single two-surface along a transvecting vector field. That is, by moving all the points of the initial two-surface the same parameter distance along the integral curves of the vector field, we create a one-parameter family of two-surfaces. We refer to the dragging of a surface as a deformation of that surface. Each two-surface of the first foliated surface is now dragged by a second vector field, filling up a four-dimensional region of space-time. As we shall show, the two vector fields must commute in order that the two-surface passing through a point of space-time be unique. A foliation that depending on two parameters is called a foliation of co-dimension two. Since families of space-like two-surfaces play a fundamental role in the two+two analysis, we shall devote this chapter to studing their geometric properties.

A space-like two-surface can be represented geometrically by the imbedding of an abstract two-dimensional manifold $\mathcal{N}$ with positive-definite metric into a space-time manifold **M**. $\mathcal{N}$ will serve as the initial two-surface which is to be deformed. We can carry over directly the definitions that were used in Section 1.2 for the imbedding of a three-dimensional space-like manifold. In the two+two case, an imbedding of a two-dimensional manifold $\mathcal{N}$ in **M** is a set $(\mathbf{B},\mathcal{N},\mathbf{M})$ where **B** is a mapping:

$$\mathbf{B}: \mathcal{N}\rightarrow\mathbf{M} \quad q\in\mathcal{N}\rightarrow p\in\mathbf{M} = \mathbf{B}(q) \qquad [2.1\text{-}1]$$

In local coordinates the imbedding map is represented by the set of four functions of two-parameters

$$x^{\mu} = B^{\mu}(y^{A}) \qquad [2.1\text{-}2]$$

where $\{x^{\mu};\mu=0,1,2,3\}$ is a local coordinate system on **M** that contains p and $\{y^{A};A=2,3\}$ is a local coordinate system on $\mathcal{N}$ that contains q. The set of points $\mathbf{B}[\mathcal{N}]$ which constitutes the image of $\mathcal{N}$ under **B** forms a two-surface in **M** and is denoted by **N**. We want **N** to be a smooth surface in **M**, which will be the case if **B** is a smooth bijection of $\mathcal{N}$ onto **N** (see Choquet et al, [1977]). This way of representing a surface is sometimes called the <u>parametrized representation</u>. The definitions of the pull-back and differential maps in this case are straighforward generalizations of the 3+1-case.

Because $\mathcal{N}$ is a two-dimensional manifold, the Jacobian matrix of the imbedding map **B** [2.1-2] has rank two, implying that there exists a two-dimensional subspace of the cotangent space at each point **N** which is annhilated by B^* (see Appendix A). We denote this space by $\mathcal{T}^*$. Any covector m_μ of $\mathcal{T}^*$ satisfies

$$B^{\mu}{}_{B} m_{\mu} = 0 \qquad [2.1\text{-}3]$$

A basis for $\mathcal{T}^*$ is given by any two linearly-independent covectors $w^{\mathbf{x}}{}_{\mu}$ ($\mathbf{x}$ = 0,1) which satisfy equation 2.1-3. The basis covectors $w^{\mathbf{x}}{}_{\mu}$ are defined only up to a nonsingular linear transformation

$$\overline{w}^{\mathbf{x}}{}_{\mu} = A^{\mathbf{x}}{}_{\mathbf{y}} w^{\mathbf{y}}{}_{\mu} \; ; \; \det[A^{\mathbf{x}}{}_{\mathbf{y}}] \neq 0 \qquad [2.1\text{-}4]$$

The members of the set $\{w^{\mathbf{x}}{}_{\mu}\}$ are called <u>pseudo-normal</u> covectors to the surface. The word pseudo-normal is used to emphasize that this is not really a metrical concept. A necessary and sufficient condition that a set of quantities $w^{\mathbf{y}}{}_{\mu}$ satisfying 2.1-3 be pseudo-normals to a surface is

$$B^{\mu}{}_{A}{}^{\nu}{}_{B} w^{\mathbf{y}}{}_{[\mu,\nu]} = 0 \qquad [2.1\text{-}5]$$

It is not hard to show the necessity of 2.1-5; it follows directly from 2.1-2 and 2.1-3. Its sufficiency is the content of Frobenius' Theorem, which is stated in Appendix A and discussed below.

<u>Rigged Surfaces</u>

At each point of the imbedded surface, there is a subset of the tangent space consisting of vectors tangent to the surface. Given an arbitrary vector of **M**, it is possible to tell whether it is tangent to the surface: find a vector in $\mathcal{N}$ whose push-forward agrees with the original vector. A simpler test for a vector r^{μ} tangent to **N** is if $r^{\mu}w^{\mathbf{x}}{}_{\mu} = 0$. It is not possible, however, to take an arbitrary vector at a point of **N** and decompose it uniquely into a part which is tangent to **N** and a part which is not. In order to do this, one must <u>rig</u> the surface, a technique applicable to non-metrical spaces as well as metrical spaces. We used this method in the 3+1 breakup in Section 1.2. There, a contravariant vector, orthogonal to **S** (but not normalized to unity), was used to rig the three-surface. We follow a similar procedure in the 2+2 analysis. A complete discussion of rigged spaces is given by Schouten [1954].

A two-surface **N** is rigged by giving a two-dimensional subspace $\mathcal{T}$ of the tangent space of M at each of its points, which contains no vector tangent to **N**. Operationally, one gives a pair of linearly independent basis vectors $t_{\mathbf{x}}{}^{\mu}$ which span $\mathcal{T}$ (i.e., which are not tangent to **N**). They may be normalized by requiring

$$t_{\mathbf{x}}{}^{\mu}w^{\mathbf{y}}{}_{\mu} = \delta_{\mathbf{x}}{}^{\mathbf{y}} \qquad [2.1\text{-}6]$$

Then one can define the operator

$$C^{\mu}{}_{\nu} = t_{\mathbf{x}}{}^{\mu}w^{\mathbf{x}}{}_{\nu} \qquad [2.1\text{-}7]$$

which is independent of the choice of basis vectors, provided the normality condition 2.1-6 holds. The operator $C^{\mu}{}_{\nu}$, acting on an arbitrary vector v^{μ}, defines two vectors:

$$'v^{\mu} = (\delta^{\mu}{}_{\nu} - C^{\mu}{}_{\nu})\,v^{\mu} \qquad [2.1\text{-}8]$$

$$''v^{\mu} = C^{\mu}{}_{\nu}v^{\mu} \qquad [2.1\text{-}9]$$

such that[1]

$$v^{\mu} = {}'v^{\mu} + {}''v^{\mu} \qquad [2.1\text{-}10]$$

Given a rigging, this decomposition of v^{μ} is unique.

It is not hard to show, from equations 2.1-7 and 2.1-8 that $'v^{\mu}$ is tangent to **N** since $'v^{\mu}w^{\mathbf{x}}{}_{\mu} = 0$. This lead us to define

$$B^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - C^{\mu}{}_{\nu} \qquad [2.1\text{-}11]$$

as the projection operator into the surface. The vector $''v^{\mu}$ belongs to $\mathcal{T}$ because it is a linear combination of the rigging vectors:

$$''v^{\mu} = C^{\mu}{}_{\nu}v^{\mu} = t_{\mathbf{x}}{}^{\mu}(w^{\mathbf{x}}{}_{\nu}v^{\nu}) \qquad [2.1\text{-}12]$$

One can readily see that $B^{\mu}{}_{\nu}$ and $C^{\mu}{}_{\nu}$ are projection operators satisfying:

$$B^{\nu}{}_{\mu}B^{\mu}{}_{\rho} = B^{\nu}{}_{\rho} \qquad [2.1\text{-}13]$$

$$C^{\nu}{}_{\mu}C^{\mu}{}_{\rho} = C^{\nu}{}_{\rho} \qquad [2.1\text{-}14]$$

$$B^{\nu}{}_{\mu}C^{\mu}{}_{\rho} = 0 \qquad [2.1\text{-}15]$$

$$C^{\nu}{}_{\mu}B^{\mu}{}_{\rho} = 0 \qquad [2.1\text{-}16]$$

These projection operators can be applied in a straightforward manner to geometrical objects which may involve terms with one index in one space and another index in the other. For example, the decomposition of the metric tensor is given by:

$$g_{\mu\nu} = (B^{\rho}{}_{\mu} + C^{\rho}{}_{\mu})(B^{\sigma}{}_{\nu} + C^{\sigma}{}_{\nu})\ g_{\rho\sigma}$$

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1) As in Section 1.2 we use the prime (') prefix to denote an object tangent to the surface and a double prime (") prefix to denote an object tangent to the rigging space.

$$= B^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu} g_{\rho\sigma} + B^{\rho}{}_{\mu} C^{\sigma}{}_{\nu} g_{\rho\sigma} + C^{\rho}{}_{\mu} B^{\sigma}{}_{\nu} g_{\rho\sigma} + C^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu} g_{\rho\sigma}$$

$$= {}' g_{\rho\sigma} + B^{\rho}{}_{\mu} C^{\sigma}{}_{\nu} g_{\rho\sigma} + C^{\rho}{}_{\mu} B^{\sigma}{}_{\nu} g_{\rho\sigma} + {}'' g_{\rho\sigma} \qquad [2.1\text{-}17]$$

(using the notation $B^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu} = B^{\rho}{}_{\mu} B^{\sigma}{}_{\nu}$ introduced in Section 1.2).

Under coordinate transformations of **M**, the connecting quantities $B^{\mu}{}_{A}$ transform as contravariant vectors, while under coordinate transformations on $\mathcal{N}$, they transform as covariant vectors. Therefore, $t_{\mathbf{x}}{}^{\mu}$ and $B^{\mu}{}_{A}$ comprise a set of basis vectors $\{e_m{}^{\mu}\}$ spanning the full tangent space of M (for this section, roman indices run from 0 to 4). There exists a dual basis $\{\omega^{m}{}_{\mu}\}$ defined by

$$\mathbf{e}_m{}^{\mu} \omega^{n}{}_{\mu} = \delta^{n}{}_{m} \qquad [2.1\text{-}18]$$

The $w^{\mathbf{x}}{}_{\mu}$ are already members of this dual basis due to equation 2.1-6. The other two dual covectors, which we denote $B^{A}{}_{\mu}$, are defined by

$$B^{A}{}_{\mu} t_{\mathbf{x}}{}^{\mu} = 0 \text{ and } B^{A}{}_{\mu} B^{\mu}{}_{B} = \delta^{A}{}_{B} \qquad [2.1\text{-}19]$$

Under $\mathcal{N}$-coordinate transformations, $B^{A}{}_{\mu}$ transforms as a contravariant vector.

Any contravariant vector v^{μ} defined on **N**, when acted upon by $B^{A}{}_{\mu}$, is mapped into a contravariant vector v^{A} in $\mathcal{N}$. The $B^{A}{}_{\mu}$ are connecting quantities whose abstract representation is

$$B^{-1}{}_{*}: T_{*}\mathbf{M} \rightarrow T_{*}\mathcal{N} \qquad [2.1\text{-}20]$$

For example, $v^{A} = B^{A}{}_{\mu} v^{\mu}$ is a vector field on $\mathcal{N}$.

The same connecting quantity carries covariant vectors from $\mathcal{N}$ to **M**, hence we have

$$B^{-1*}: T^{*}\mathcal{N} \rightarrow T^{*}\mathbf{M} \qquad [2.1\text{-}21]$$

$s_{\mu} = B^{A}{}_{\mu} s_{A}$ is a covariant vector field on **N**.

The action of both of these maps on arbitrary tensors is defined in the usual way.

In general, the $\mathcal{T}$ spaces do not fit together to form two-surfaces so we must treat them as anholonomic surface elements with the rigging vectors serving as an anholonomic basis for $\mathcal{T}$. From this viewpoint, connecting quantities are defined between $\mathcal{T}$ and **M** just as they were between $\mathcal{N}$ and **M**. Only here, since there is no surface, the connecting quantities are just the rigging vectors and their duals. We define a set of connecting quantities $C_{\mathbf{x}}{}^{\mu}$ and $C^{\mathbf{x}}{}_{\mu}$ by

$$C_{\mathbf{x}}{}^{\mu} = t_{\mathbf{x}}{}^{\mu} \text{ and } C^{\mathbf{x}}{}_{\mu} = w^{\mathbf{x}}{}_{\mu} \qquad [2.1\text{-}22]$$

They are analogous to the $B_{A}{}^{\mu}$ and $B^{A}{}_{\mu}$ and transform tensorially under rigging transformations of the form 2.1-4.

It follows from the above results that we can expand $\delta_{\nu}{}^{\mu}$ in terms of the projections operators

$$\begin{aligned} \delta^{\mu}{}_{\nu} &= B^{\mu}{}_{\nu} + C^{\mu}{}_{\nu} \\ &= B_{A}{}^{\mu}{}_{\nu}{}^{A} + C_{\mathbf{x}}{}^{\mu}{}_{\nu}{}^{\mathbf{x}} \end{aligned} \qquad [2.1\text{-}23]$$

We shall use this expression frequently.

One of the requirements for a truly covariant 2+2 formalism is that all equations transform tensorially with respect to $\mathcal{T}$-indices as well as $\mathcal{N}$-indices. This means that all relations should depend only upon the rigging space and respect to $\mathcal{T}$-indices as well as $\mathcal{N}$-indices. This means that all relations should depend only upon the rigging space and not upon the basis which spans the space. This is the reason we have introduced the connecting quantities rather than analyzing components of a quantity with respect to a given rigging basis.

To proceed further, we assume that **N** belongs to a two-parameter family of two-surfaces that fills up a four-dimensional region of space-time. The exact specification of this family will be determined below, but for now we only need to know that all connecting quantities are smooth fields defined in a four-dimensional region. The tetrad $\{B_A{}^\mu, C_{\mathbf{x}}{}^\mu\}$ and its dual $\{B^A{}_\mu, C^{\mathbf{x}}{}_\mu\}$ form an anholonomic coordinate system for the space-time although the subset $\{B_A{}^\mu\}$ is holonomic by itself.

A very important set of quantities, the so-called <u>objects of anholonomicity</u>, are defined by:

$$\Omega^k{}_{ij} = e^\mu{}_i{}^\lambda{}_j \partial_{[\mu}\omega^k{}_{\lambda]} \qquad [2.1\text{-}24]$$

where the $e^\mu{}_i$ and $\omega^k{}_\lambda$ are generic connecting quantities standing for $B_A{}^\mu, C_{\mathbf{x}}{}^\mu$ and $B^A{}_\mu, C^{\mathbf{x}}{}_\mu$. The objects of anholonomicity can also be expressed in terms of the Lie Bracket of the basis vectors

$$\Omega^k{}_{ij} = -1/2\ \omega^k{}_\mu[e_i, e_j]^\mu \qquad [2.1\text{-}25]$$

The objects of anholonomicity divide up into six sets:

(A) $$\Omega^C{}_{AB} = B^\mu{}_A{}^\lambda{}_B \partial_{[\mu} B^C{}_{\lambda]} \qquad [2.1\text{-}26a]$$

(B) $$\Omega^{\mathbf{x}}{}_{AB} = B^\mu{}_A{}^\lambda{}_B \partial_{[\mu} C^{\mathbf{x}}{}_{\lambda]} = B^\mu{}_A{}^\lambda{}_B \partial_{[\mu} w^{\mathbf{x}}{}_{\lambda]} \qquad [2.1\text{-}26b]$$

(C) $$\Omega^C{}_{\mathbf{x}B} = C_{\mathbf{x}}{}^\mu B^\lambda{}_B \partial_{[\mu} B^C{}_{\lambda]} \qquad [2.1\text{-}26c]$$

(D) $$\Omega^{\mathbf{y}}{}_{\mathbf{x}B} = C_{\mathbf{x}}{}^\mu B^\lambda{}_B \partial_{[\mu} C^{\mathbf{x}}{}_{\lambda]} = C_{\mathbf{x}}{}^\mu B^\lambda{}_B \partial_{[\mu} w^{\mathbf{x}}{}_{\lambda]} \qquad [2.1\text{-}26d]$$

(E) $\Omega^{A}{}_{xy} = C^{\mu}{}_{x}{}^{\lambda}{}_{y}\partial_{[\mu}B^{A}{}_{\lambda]} = C_{x}{}^{\mu}C_{y}{}^{\lambda}\partial_{[\mu}B^{A}{}_{\lambda]}$ [2.1-26e]

(F) $\Omega^{z}{}_{xy} = C^{\mu}{}_{x}{}^{\lambda}{}_{y}\partial_{[\mu}C^{z}{}_{\lambda]} = C_{x}{}^{\mu}C_{y}{}^{\lambda}\partial_{[\mu}w^{z}{}_{\lambda]}$ [2.1-26f]

The objects in set (A) vanish because we are using a coordinate basis on $\mathcal{N}$. The objects in set (B) vanish by virtue of equation 2.1-5: i.e., the pseudo-normals are surface-forming. It is a consequence of Frobenius' theorem, presented in Appendix A, that the vanishing of $\Omega^{x}{}_{AB}$ is the necessary and sufficient condition that the connecting quantities are surface-forming. Another consequence of this theorem is that the basis covectors $C^{x}{}_{\mu}$ are linear combinations of gradients of functions. In fact, according to the theorem there exists a set of basis vectors defined in a four-dimensional region which are equal to the gradients of a pair of scalar fields[1] (as we saw in the double-null case)

$$C^{x}{}_{\mu} \ {*}{=}\ \partial_{\mu}\varphi^{x}(x^{\nu}) \qquad (x = 0,1) \qquad [2.1\text{-}26]$$

All points which lie on a single two-surface satisfy the relations

$$\varphi^{x}(x^{\nu}) = \text{constant} = \alpha^{x} \qquad [2.1\text{-}27]$$

These relations provide an alternative means of specifying a family of imbedded surfaces. This represention of a surface is sometimes referred to as the <u>null formalism</u> (which has nothing to do with the metrical concept of 'null'). It is equivalent to the parametrized representation, as we explain in Appendix A. The objects in set (B) transform tensorially with respect to surface and rigging transformation.

The objects in case (C) and (D) vanish only when one of the rigging vectors together with a surface vector are surface-forming. In general this is not the case. The $\Omega^{C}{}_{xB}$

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1) The notation '*=' denotes equality in a particular coordinate system or basis.

(set (C)) do not transform tensorially with respect to surface surface indices but do so with respect to rigging indices. Similarly, the $\Omega^{\mathbf{y}}{}_{\mathbf{x}B}$ (set (D)) do not transform tensorially with respect to rigging indices but do so with respect to surface indices.

As we mentioned before, the spaces $\mathcal{T}$ spanned by the rigging vectors $\{C_{\mathbf{x}}{}^{\mu}\}$ at each point do not mesh to form a smooth two-surface. Hence the objects in sets (E) $\Omega^{A}{}_{\mathbf{xy}}$ do not in general vanish. This is equivalent to the failure to close of the Lie bracket of the basis vectors of the rigging space:

$$[C_0, C_1]^{\mu}\ B^{A}{}_{\mu} \neq 0 \quad \Leftrightarrow \Omega^{A}{}_{\mathbf{xy}} \neq 0 \qquad [2.1\text{-}28]$$

Set (E) objects, $\Omega^{C}{}_{\mathbf{xy}}$, transform tensorially with respect to surface and rigging transformations. The objects in set (F) $\Omega^{\mathbf{z}}{}_{\mathbf{xy}}$ do not transform tensorially at all, and do not vanish for arbitrary rigging bases.

The anholonomic objects play a role in defining the Lie derivative for the spaces $\mathcal{N}$ and $\mathcal{T}$. We adopt a general procedure for defining Lie derivatives (and covariant derivatives, as we shall see in Section 2.2). We carry quantities from $\mathcal{N}$ or $\mathcal{T}$ over to **M**, perform the required differentiation, when possible, in **M**, and then carry the result back to $\mathcal{N}$ or $\mathcal{T}$, again using the connecting quantities. For example, for two vectors m^A and z^A of $\mathcal{N}$, the Lie derivative of m^A with respect to z^A is defined by:

$$\begin{aligned} \pounds_{\mathbf{z}} m^{A} &= B^{A}{}_{\mu} \pounds_{\mathbf{z}} {}'m^{\mu} = B^{A}{}_{\mu} [{}'z^{\nu}\partial_{\nu}{}'m^{\mu} - {}'m^{\nu}\partial_{\nu}{}'z^{\mu}] \\ &= [{}'z^{\nu}\partial_{\nu} m^{A} - {}'m^{\nu}\partial_{\nu} z^{A}] - {}'m^{\mu}{}'z^{\nu}\partial_{\nu} B^{A}{}_{\mu} - {}'m^{\nu}{}'z^{\mu}\partial_{\nu} B^{A}{}_{\mu} \end{aligned}$$

$$= [z^B\partial_B m^A - m^B\partial_B z^A] - {}'m^{\mu}{}' z^{\nu}[\partial_\nu B^A{}_\mu - \partial_\nu B^A{}_\mu]$$

$$= [z^B\partial_B m^A - m^B\partial_B z^A] - 2m^C z^B \Omega^A{}_{BC}$$

$$= [z^B\partial_B m^A - m^B\partial_B z^A] \qquad [2.1\text{-}29]$$

Since $\Omega^A{}_{BC} = 0$, this is the usual definition of the Lie derivative defined intrinsically on $\mathcal{N}$. Likewise, for a covariant vector we have

$$\pounds_{\mathbf{z}} p_A = [z^B\partial_B p_A + p_B\partial_A z^B] \qquad [2.1\text{-}30]$$

For rigging space vectors $p^{\mathbf{x}}$ and $s^{\mathbf{x}}$, the Lie derivative is defined by:

$$\pounds_{\mathbf{s}} p^{\mathbf{x}} = C^{\mathbf{x}}{}_\mu \pounds_{\mathbf{s}} "p^\mu = C^{\mathbf{x}}{}_\mu ["s^\sigma\partial_\sigma "p^\mu - "p^\sigma\partial_\sigma "s^\mu]$$

$$= ["s^\sigma\partial_\sigma p^{\mathbf{x}} - "p^\sigma\partial_\sigma s^{\mathbf{x}}] - ["p^\mu" s^\sigma\partial_\sigma C^{\mathbf{x}}{}_\mu - "p^\mu" s^\sigma\partial_\mu C^{\mathbf{x}}{}_\sigma]$$

$$= [s^{\mathbf{z}}\partial_{\mathbf{z}} p^{\mathbf{x}} - p^{\mathbf{z}}\partial_{\mathbf{z}} s^{\mathbf{x}}] -$$

$$[p^{\mathbf{y}} s^{\mathbf{z}} C_{\mathbf{y}}{}^\mu C_{\mathbf{z}}{}^\sigma \partial_\sigma C^{\mathbf{x}}{}_\mu - p^{\mathbf{y}} s^{\mathbf{z}} C_{\mathbf{z}}{}^\mu C_{\mathbf{y}}{}^\sigma \partial_\sigma C^{\mathbf{x}}{}_\mu]$$

$$= [s^{\mathbf{z}}\partial_{\mathbf{z}} p^{\mathbf{x}} - p^{\mathbf{z}}\partial_{\mathbf{z}} s^{\mathbf{x}}] - 2p^{\mathbf{y}} s^{\mathbf{z}} C_{\mathbf{z}}{}^\mu C_{\mathbf{y}}{}^\sigma \partial_{[\sigma} C^{\mathbf{x}}{}_{\mu]}$$

$$= [s^{\mathbf{z}}\partial_{\mathbf{z}} p^{\mathbf{x}} - p^{\mathbf{z}}\partial_{\mathbf{z}} s^{\mathbf{x}}] - 2s^{\mathbf{z}}\Omega^{\mathbf{x}}{}_{\mathbf{zy}} p^{\mathbf{y}} \qquad [2.1\text{-}31]$$

where we define $\partial_{\mathbf{x}} = C_{\mathbf{x}}{}^\mu \partial_\mu$. This expression is formally like the usual expression for the Lie derivative, except for the presence of an extra term on the right-hand side. This term is due to the anholonomic property of the rigging vectors and serves to make the expression on the right-hand side transform tensorially. Similarly we can define the Lie derivative of $q_{\mathbf{x}}$

$$\pounds_{\mathbf{s}} q_{\mathbf{x}} = [s^{\mathbf{z}}\partial_{\mathbf{z}} q_{\mathbf{x}} + q_{\mathbf{z}}\partial_{\mathbf{x}} s^{\mathbf{z}}] + 2s^{\mathbf{z}}\Omega^{\mathbf{y}}{}_{\mathbf{zx}} q_{\mathbf{y}} \qquad [2.1\text{-}32]$$

Mixed Lie derivatives can be defined as well. Suppose that s^A is a surface vector while $r^{\mathbf{x}}$ is a rigging vector. Define the Lie derivative by

$$\pounds_{\mathbf{r}} s^A = B^A{}_\mu \pounds_{\mathbf{r}} {}' s^\mu = B^A{}_\mu ({}'' r^\nu \partial_\nu {}' s^\mu - {}' s^\nu \partial_\nu {}'' r^\mu)$$

$$= ({}'' r^\nu \partial_\nu s^A - {}'' r^\nu {}' s^\mu \partial_\nu B^A{}_\mu + {}'' r^\mu {}' s^\nu \partial_\nu B^A{}_\mu)$$

$$= r^{\mathbf{x}} \partial_{\mathbf{x}} s^A - r^{\mathbf{x}} s^C (C_{\mathbf{x}}{}^\nu B_C{}^\mu \partial_\nu B^A{}_\mu - C_{\mathbf{x}}{}^\mu B_C{}^\nu \partial_\nu B^A{}_\mu)$$

$$= r^{\mathbf{x}} \partial_{\mathbf{x}} s^A - 2 r^{\mathbf{x}} s^C \Omega^A{}_{\mathbf{x}C}$$

$$= r^{\mathbf{x}} \partial_{\mathbf{x}} s^A + 2 r^{\mathbf{x}} \Omega^A{}_{C\mathbf{x}} s^C \qquad [2.1\text{-}33]$$

For a covariant two-surface vector the Lie derivative is

$$\pounds_{\mathbf{r}} p_A = r^{\mathbf{x}} \partial_{\mathbf{x}} p_A - 2 r^{\mathbf{x}} \Omega^C{}_{A\mathbf{x}} p_C \qquad [2.1\text{-}34]$$

The Lie derivatives of rigging vectors and covectors with respect to surface vectors are given by

$$\pounds_{\mathbf{v}} r^{\mathbf{x}} = v^A \partial_A r^{\mathbf{x}} + 2 v^A \Omega^{\mathbf{x}}{}_{\mathbf{z}A} r^{\mathbf{z}} \qquad [2.1\text{-}35]$$

$$\pounds_{\mathbf{v}} u_{\mathbf{x}} = v^A \partial_A u_{\mathbf{x}} - 2 v^A \Omega^{\mathbf{z}}{}_{\mathbf{x}A} u_{\mathbf{z}} \qquad [2.1\text{-}36]$$

What is interesting about these Lie derivatives is that they are covariant, but do not depend on either an affine connection, as does the ordinary covariant derivative or on the derivative of the vector field, as would the regular Lie derivative. Using these relations as a starting point, Schouten [1954] introduces a new set of derivatives which are covariant with respect to rigging and surface transformations. They are

$$\mathfrak{D}_{\mathbf{x}} s^A = \partial_{\mathbf{x}} s^A + 2 \Omega^A{}_{C\mathbf{x}} s^C \qquad [2.1\text{-}37]$$

$$\mathfrak{D}_{\mathbf{x}} p_A = \partial_{\mathbf{x}} p_A - 2 \Omega^C{}_{A\mathbf{x}} p_C \qquad [2.1\text{-}38]$$

$$\mathcal{D}_A r^{\mathbf{x}} = \partial_A r^{\mathbf{x}} + 2\Omega^{\mathbf{x}}{}_{\mathbf{z}A} r^{\mathbf{z}} \qquad [2.1\text{-}39]$$

$$\mathcal{D}_A u_{\mathbf{x}} = \partial_A u_{\mathbf{x}} - 2\Omega^{\mathbf{z}}{}_{\mathbf{x}A} r_{\mathbf{z}} \qquad [2.1\text{-}40]$$

Then the Lie derivatives 2.1-33 - 2.1-36 can be written as

$$£_{\mathbf{r}} s^A = r^{\mathbf{x}} \mathcal{D}_{\mathbf{x}} s^A \qquad [2.1\text{-}41]$$

$$£_{\mathbf{r}} p_A = r^{\mathbf{x}} \mathcal{D}_{\mathbf{x}} p_A \qquad [2.1\text{-}42]$$

$$£_{\mathbf{v}} r^{\mathbf{x}} = v^A \mathcal{D}_A r^{\mathbf{x}} \qquad [2.1\text{-}43]$$

$$£_{\mathbf{v}} u_{\mathbf{x}} = v^A \mathcal{D}_A u_{\mathbf{x}} \qquad [2.1\text{-}44]$$

The $\mathcal{D}$-derivatives can be extended to arbitrary objects since the latter can be written as sums of exterior products of covariant and contravariant vectors. If q is a scalar density on $\mathcal{N}$ of weight w, the $\mathcal{D}_{\mathbf{x}}$-derivative of q is

$$\mathcal{D}_{\mathbf{x}} q = \partial_{\mathbf{x}} q - 2w\Omega^C{}_{C\mathbf{x}} q \qquad [2.1\text{-}45]$$

Finally, we define the Lie derivative of surface quantities of $\mathcal{N}$ with respect to an arbitrary vector field **z** of **M**. For a vector p^B and covector q_B we have

$$£_{\mathbf{z}} p^B = B^B{}_\mu £_{\mathbf{z}}' p^\mu \qquad [2.1\text{-}46]$$

$$£_{\mathbf{z}} q_B = B^\mu{}_B £_{\mathbf{z}}' q_\mu \qquad [2.1\text{-}47]$$

Note that the quantities $£_{\mathbf{z}}' p^\mu$ and $£_{\mathbf{z}}' q_\mu$ are not tangent to **N**; their projections $C^{\mathbf{x}}{}_\mu £_{\mathbf{z}}' p^\mu$ and $C^\mu{}_{\mathbf{x}} £_{\mathbf{z}}' q_\mu$ do not in general vanish.

<u>Deformation of a Surface</u>

We shall now construct two-parameter families of two-surfaces by the process of <u>deformation</u> of a single two-surface. A deformation of a subspace results from the displacement of its points. We show how to make this notion more precise. Note that displacement is defined for general geometrical objects on manifolds, with displacements of subspaces as a special case. Our discussion follows Schouten [1954]. Displacements arise from transporting, in some manner to be determined, a geometrical object along a vector field on the manifold, whose integral curves generate a one-parameter family of point transformations. The parametrization of the integral curves is uniquely defined up to an additive constant. We shall call this vector field the deformation vector field V^{μ}.

Let us start off with a single two-dimensional submanifold **N** which arises, as we discussed above, by imbedding an abstract two-manifold $\mathcal{N}$ in **M**. Define a vector field **V** in some four-dimensional region of **M**. This vector field is assumed to transvect **N**, i.e. it is nowhere tangent to **N**. A displacement or deformation of a subspace results from the dragging of the points of the subspace along the streamlines of the deformation vector field. Each point on the surface is carried an equal parameter distance along the integral curves of the vector field (see Figure 2-1). To put this more mathematically, a vector field defines a one-parameter family of diffeomorphisms of the manifold into itself, given by mapping each point of **M** into a point a fixed parameter distance along the integral curves

$$\sigma_t : \mathbf{M} \to \mathbf{M} \qquad [2.1\text{-}48]$$

A deformation of a surface is equivalent to a one-parameter family of imbedding maps $\mathbf{B}_t$ constructed by composing σ_t and **B**:

$$\mathbf{B}_t = (\sigma_t \circ \mathbf{B}) : \quad \mathcal{N} \to \mathbf{M} \qquad [2.1\text{-}49]$$

The set of points belonging to some member of the family of two-surfaces ($\mathbf{S} = \{\cup_t \mathbf{B}_t[\mathcal{N}] : \forall t\}$) forms a three-dimensional submanifold of **M**.

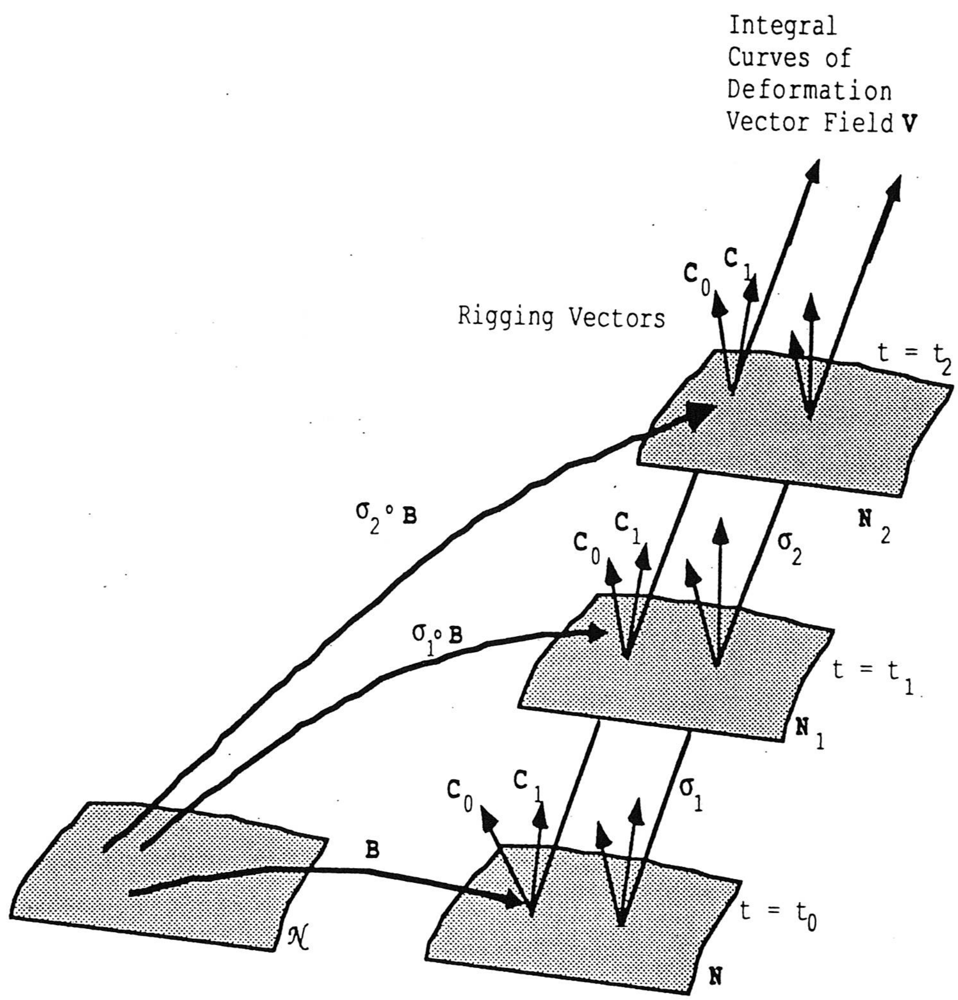

Figure 2-1 Deformation of a Two-surface by a Vector Field

Now take another vector field **U** which is linearly independent of **V** and such that **U** transvects **S**. It also defines a family of integral curves and a one-parameter family of diffeomorphisms

$$\mu_s : \mathbf{M} \to \mathbf{M} \qquad [2.1\text{-}50]$$

Each two-surface belonging to the family which makes up **S** can be dragged along by **U**. In this way we fill up a four-dimensional region of **M** with two-surfaces. In general, if we drag a two-surface by a vector field **V** and then by another vector field **U**, we will end up with a different two-surface than if we did the dragging in reverse order. In order to insure that the final two-surface be independent of the order of dragging, we require that the composite maps commute

$$\sigma_t \circ \mu_s = \mu_s \circ \sigma_t \qquad [2.1\text{-}51]$$

The necessary and sufficient condition for this to hold is that the vector fields commute (the commuting vector fields are themselves tangent to a family of integral manifolds).

$$[\mathbf{U},\mathbf{V}] = 0 \qquad [2.1\text{-}52]$$

(see, for example, Abraham, Marsden and Ratiu [1983]).
The set of map

$$\mathbf{B}_{t,s} = (\mu_s \circ \sigma_t \circ \mathbf{B}) : \mathcal{N} \to \mathbf{M} \qquad [2.1\text{-}53]$$

defines a two-parameter family of a two-surfaces. Each member of the family has a unique pair of parameters (s,t) which are necessarily constant on the set of points belonging to that member. Hence each two-surface belonging to the family can be represented by the pair of functions $t(x^\mu)$ = constant and $s(x^\mu)$ = constant, which correspond to the surfaces $\varphi^{\mathbf{x}}(x^\mu)$ defined above

$$\varphi^0 = s(x^\mu) = \text{constant}$$

$$\varphi^1 = t(x^\mu) = \text{constant} \qquad [2.1\text{-}54]$$

The gradients of the functions $\varphi^{\mathbf{x}}(x^{\mu})$ span $\mathcal{T}^{*}$ and bring us back to the null representation of a surface, defining a foliation of codimension-two of **M**. The deformation vector fields are then said to <u>fibrate the foliation</u>. The <u>fibration</u> establishes a one-to-one correspondance between points on different members of a foliation induced by the integral curves of the vector fields defining the foliation.

Earlier, we defined the Lie derivative of an arbitrary surface quantity of $\mathcal{N}$ with respect to an arbitrary vector field **z** of **M**. For a vector p^B and covector q_B we had

$$£_{\mathbf{z}} p^B = B^B{}_{\mu} £_{\mathbf{z}}{}' p^{\mu} \qquad [2.1\text{-}55]$$

$$£_{\mathbf{z}} q_B = B^{\mu}{}_B £_{\mathbf{z}}{}' q_{\mu} \qquad [2.1\text{-}56]$$

The quantities $£_{\mathbf{z}}{}'p^{\mu}$ and $£_{\mathbf{z}}{}'q_{\mu}$ are not tangent to **N**: their projections $C^{\mathbf{x}}{}_{\mu}£_{\mathbf{z}}{}'p^{\mu}$ and $C^{\mu}{}_{\mathbf{x}}£_{\mathbf{z}}{}'q_{\mu}$ do not in general vanish. However, when **z** is one of the two deformation vectors, these quantities do vanish. To show this, consider what happens when we drag the points of **N** along the integral curves of the deformation vector fields. Any vector γ^{μ} tangent to the surface ($\gamma^{\mu} = {}'\gamma^{\mu}$) will remain tangent to the deformed surface. A sufficient condition for this is that the tangent vector is Lie dragged along the curves:

$$£_{\mathbf{V}}{}'\gamma^{\mu} = 0 \qquad [2.1\text{-}57]$$

But we could have a transformation of ${}'\gamma^{\mu}$ within the surface itself, so a necessary and sufficient condition is

$$C^{\mathbf{x}}{}_{\mu}£_{\mathbf{V}}{}'\gamma^{\mu} = 0 \qquad [2.1\text{-}58]$$

Since ${}'\gamma^{\mu}$ is arbitrary and ${}'\gamma^{\mu} = \gamma^{A}B^{\mu}{}_{A}$, we have

$$C^{\mathbf{x}}{}_{\mu}£_{\mathbf{V}}\gamma^{A}B^{\mu}{}_{A} = \gamma^{A}C^{\mathbf{x}}{}_{\mu}£_{\mathbf{V}}B^{\mu}{}_{A} = 0, \qquad [2.1\text{-}59]$$

which implies that

$$C^{\mathbf{x}}{}_{\mu}\pounds_{\mathbf{V}}B^{\mu}{}_{A} = 0 \qquad [2.1\text{-}60]$$

Writing this equation out gives

$$\begin{aligned}
C^{\mathbf{x}}{}_{\mu}\pounds_{\mathbf{V}}B^{\mu}{}_{A} &= C^{\mathbf{x}}{}_{\mu}(V^{\rho}\partial_{\rho}B^{\mu}{}_{A} - B^{\rho}{}_{A}\partial_{\rho}V^{\mu}) \\
&= -B^{\rho}{}_{A}\partial_{\rho}V^{\mathbf{x}} + V^{\mu}B^{\rho}{}_{A}\partial_{\rho}C^{\mathbf{x}}{}_{\mu} + C^{\mathbf{x}}{}_{\mu}V^{\rho}\partial_{\rho}B^{\mu}{}_{A} \\
&= -\partial_{A}V^{\mathbf{x}} + 2V^{\mu}B^{\rho}{}_{A}\partial_{[\rho}C^{\mathbf{x}}{}_{\mu]} \\
&= -\partial_{A}V^{\mathbf{x}} + 2V^{B}B^{\rho}{}_{A}{}^{\mu}{}_{B}\partial_{[\rho}C^{\mathbf{x}}{}_{\mu]} + 2V^{\mathbf{z}}B^{\rho}{}_{A}C_{\mathbf{z}}{}^{\mu}\partial_{[\rho}C^{\mathbf{x}}{}_{\mu]} \\
&= -\partial_{A}V^{\mathbf{x}} - 2V^{\mathbf{z}}\Omega^{\mathbf{x}}{}_{\mathbf{z}A} + 2V^{B}\Omega^{\mathbf{x}}{}_{AB} \\
&= -\mathcal{D}_{A}V^{\mathbf{x}} + 2V^{B}\Omega^{\mathbf{x}}{}_{AB} \\
&= -\mathcal{D}_{A}V^{\mathbf{x}} = 0 \qquad [2.1\text{-}61]
\end{aligned}$$

since $\Omega^{\mathbf{x}}{}_{AB} = 0$ for a surface. The condition that **V** is a deformation vector field is thus

$$\mathcal{D}_{A}V^{\mathbf{x}} = 0 \qquad [2.1\text{-}62]$$

This is the basic deformation equation and is actually the covariant generalization of equation 1.2-49.

On the other hand, for a general vector field, and for **V** in particular, one has

$$B^{B}{}_{\mu}\pounds_{\mathbf{V}}B^{\mu}{}_{A} = B^{B}{}_{\mu}{}^{\nu}{}_{A}\,\pounds_{\mathbf{V}}B^{\mu}{}_{\nu} = -B^{B}{}_{\mu}{}^{\nu}{}_{A}\pounds_{\mathbf{V}}C^{\mu}{}_{\nu} = 0 \qquad [2.1\text{-}63]$$

Combining 2.1-60 and 2.1-63, we obtain

$$\pounds_{\mathbf{V}}B^{\mu}{}_{A} = 0 \qquad [2.1\text{-}64]$$

which is completely equivalent to 2.1-62. Another way of seeing this is that $B_A{}^\mu$ is defined everywhere from its values on **N** by propagating it, using the push-forward map associated with $(\sigma_t \circ \mathbf{B})$. From the definition of the Lie derivative of contravariant vector given in Appendix A, this is the same as saying that its Lie derivative vanishes.

A simple consequence of equation 2.1-60 is that

$$B^\mu{}_A \pounds_{\mathbf{V}} C^{\mathbf{x}}{}_\mu = 0 \qquad [2.1\text{-}65]$$

There is no corresponding equation for $C^\mu{}_{\mathbf{y}} \pounds_{\mathbf{V}} C^{\mathbf{x}}{}_\mu$ because under rigging transformations of the form 2.1-4, it does not behave like a tensor. For some choice of rigging basis it can be made to vanish, but not for every choice.

Another important result that we shall now prove is that the Lie derivatives of the rigging projection of the deformation vectors vanish. If **U** and **V** are the two deformation vectors, then

$$\begin{aligned} \pounds_{\mathbf{U}} V^{\mathbf{x}} &= C^{\mathbf{x}}{}_\mu \pounds_{\mathbf{U}}{}'' V^\mu = -C^{\mathbf{x}}{}_\mu \pounds_{\mathbf{U}}{}' V^\mu = -C^{\mathbf{x}}{}_\mu \pounds_{\mathbf{U}} (B^\mu{}_\nu V^\nu) \\ &= -V^\nu C^{\mathbf{x}}{}_\mu \pounds_{\mathbf{U}} B^\mu{}_\nu = -V^\nu C^{\mathbf{x}}{}_\mu \pounds_{\mathbf{U}} (B^\mu{}_A{}^A{}_\nu) \\ &= -V^\nu C^{\mathbf{x}}{}_\mu B^\mu{}_A \pounds_{\mathbf{U}} B^A{}_\nu = 0 \end{aligned} \qquad [2.1\text{-}66]$$

since $\pounds_{\mathbf{U}} B^\mu{}_A = 0$ and $C^{\mathbf{x}}{}_\mu B^\mu{}_A = 0$.

Similarly, one has

$$\pounds_{\mathbf{U}} U^{\mathbf{x}} = \pounds_{\mathbf{V}} V^{\mathbf{x}} = \pounds_{\mathbf{V}} U^{\mathbf{x}} = 0 \qquad [2.1\text{-}67]$$

(No similar relation $\pounds_{\mathbf{U}} V^A = 0$ holds.) From equation 2.1-62 and these relations, it follows that

$$\pounds_{''\mathbf{U}} V^{\mathbf{x}} = \pounds_{''\mathbf{U}} U^{\mathbf{x}} = \pounds_{''\mathbf{V}} V^{\mathbf{x}} = \pounds_{''\mathbf{V}} U^{\mathbf{x}} = 0 \qquad [2.1\text{-}68]$$

as well. This completes the discussion of the deformation of surfaces.

For later work, it is convenient to obtain expressions for ~~the~~ some of the anholonomic objects in terms of components of the deformation vector fields. This can be done quite easily by expanding the commutator of the deformation vectors in terms of the rigging basis vectors.

From the vanishing of the commutator of the deformation vectors, $[\mathbf{U},\mathbf{V}]^{\mu} = 0$, we get

$$
\begin{aligned}
[\mathbf{U},\mathbf{V}]^{\mu} = 0 &= [U^{\mathbf{x}}C_{\mathbf{x}} + U^{A}B_{A}, V^{\mathbf{y}}C_{\mathbf{y}} + V^{C}B_{C}]^{\mu} \\
&= [U^{\mathbf{x}}C_{\mathbf{x}}, V^{\mathbf{y}}C_{\mathbf{y}}]^{\mu} + [U^{\mathbf{x}}C_{\mathbf{x}}, V^{C}B_{C}]^{\mu} \\
&\qquad + [U^{A}B_{A}, V^{\mathbf{y}}C_{\mathbf{y}}]^{\mu} + [U^{A}B_{A}, V^{C}B_{C}]^{\mu} \\
&= U^{\mathbf{x}}V^{\mathbf{y}}[C_{\mathbf{x}}, C_{\mathbf{y}}]^{\mu} + U^{\mathbf{x}}C_{\mathbf{y}}{}^{\mu}\partial_{\mathbf{x}}V^{\mathbf{y}} \\
&\quad - V^{\mathbf{y}}C_{\mathbf{x}}{}^{\mu}\partial_{\mathbf{y}}U^{\mathbf{x}} + U^{\mathbf{x}}V^{C}[C_{\mathbf{x}}, B_{C}]^{\mu} \\
&\quad + U^{\mathbf{x}}B_{C}{}^{\mu}\partial_{\mathbf{x}}V^{C} - V^{C}C_{\mathbf{x}}{}^{\mu}\partial_{C}U^{\mathbf{x}} - V^{\mathbf{x}}U^{C}[C_{\mathbf{x}}, B_{C}]^{\mu} \\
&\quad - V^{\mathbf{x}}B_{C}{}^{\mu}\partial_{\mathbf{x}}U^{C} + V^{C}C_{\mathbf{x}}{}^{\mu}\partial_{C}U^{\mathbf{x}} \\
&\quad + U^{A}B_{C}{}^{\mu}\partial_{A}V^{C} - V^{A}B_{C}{}^{\mu}\partial_{A}U^{C} = 0
\end{aligned}
\qquad [2.1\text{-}69]
$$

Projecting this equation into the surface gives,

$$
\begin{aligned}
&-2U^{\mathbf{x}}V^{\mathbf{y}}\Omega^{B}{}_{\mathbf{xy}} - 2U^{\mathbf{x}}V^{C}\Omega^{B}{}_{\mathbf{x}C} \\
&\qquad + U^{\mathbf{x}}\partial_{\mathbf{x}}V^{B} + 2V^{\mathbf{x}}U^{C}\Omega^{B}{}_{\mathbf{x}C} \\
&\qquad - V^{\mathbf{x}}\partial_{\mathbf{x}}U^{B} + U^{A}\partial_{A}V^{B} - V^{A}\partial_{A}U^{B} = 0
\end{aligned}
$$

or

$$
-2U^{\mathbf{x}}V^{\mathbf{y}}\Omega^{B}{}_{\mathbf{xy}} + U^{\mathbf{x}}\mathfrak{D}_{\mathbf{x}}V^{B} - V^{\mathbf{x}}\mathfrak{D}_{\mathbf{x}}U^{B} + U^{A}\partial_{A}V^{B} - V^{A}\partial_{A}U^{B} = 0 \qquad [2.1\text{-}70]
$$

Because $\Omega^{B}{}_{\mathbf{xy}}$ is antisymmetric, it has the form

$$
\Omega^{B}{}_{\mathbf{xy}} = \varepsilon_{\mathbf{xy}}\Omega^{B}, \qquad [2.1\text{-}71]
$$

where $\Omega^B = 1/2\varepsilon^{xy}\Omega^B{}_{xy}$ is a two-vector with respect to surface coordinate transformations, but behaves like a scalar density of weight +1 with respect to rigging transformations. We can explicitly solve 2.1-70 for $\Omega^B{}_{xy}$. First, write

$$-2U^xV^y\Omega^B{}_{xy} = -2U^xV^y\varepsilon_{xy}\Omega^B = -2\zeta\Omega^B \qquad [2.1\text{-}72]$$

where we define a two-dimensional cross-product of **U** and **V**

$$\zeta \equiv U^xV^y\varepsilon_{xy} \qquad [2.1\text{-}73]$$

Then equation 2.1-70 has the following equivalent forms:

$$\Omega^B = (2\zeta)^{-1}(U^x\mathcal{D}_xV^B - V^x\mathcal{D}_xU^B + U^A\partial_AV^B - V^A\partial_AU^B), \qquad [2.1\text{-}74]$$

$$= (2\zeta)^{-1}(U^x\mathcal{D}_xV^B - V^x\mathcal{D}_xU^B + \pounds'_{U}V^B) \qquad [2.1\text{-}75]$$

$$= -(2\zeta)^{-1}(\pounds_{V}U^B - U^x\mathcal{D}_xV^B) \qquad [2.1\text{-}76]$$

$$= (2\zeta)^{-1}(\pounds_{U}V^B - V^x\mathcal{D}_xU^B) \qquad [2.1\text{-}77]$$

The projection of the commutator of the deformation vectors into the rigging space gives

$$-2U^xV^y\Omega^z{}_{xy} + U^x\partial_xV^z - V^y\partial_yU^z - 2U^xV^C\Omega^z{}_{xC}$$
$$- V^C\partial_CU^z + 2V^xU^C\Omega^z{}_{xC} + U^C\partial_CV^z = 0, \qquad [2.1\text{-}78]$$

or equivalently

$$-2U^xV^y\Omega^z{}_{xy} + U^x\partial_xV^z - V^y\partial_yU^z + U^C\mathcal{D}_CV^z - V^C\mathcal{D}_CU^z = 0, \qquad [2.1\text{-}79]$$

which we can explicitly solve for $\Omega^z{}_{xy}$. Since, $\mathcal{D}_CU^z = \mathcal{D}_CV^z = 0$, equation 2.1-79 becomes

$$-2U^xV^y\Omega^z{}_{xy} + U^x\partial_xV^z - V^y\partial_yU^z = 0 \qquad [2.1\text{-}80]$$

which is just the vanishing of the commutator of the rigging projections of the deformation vectors: something we have already proved. Again, one can readily show that $\Omega^{\mathbf{z}}{}_{\mathbf{xy}}$ has the form

$$\Omega^{\mathbf{z}}{}_{\mathbf{xy}} = \varepsilon_{\mathbf{xy}}(2\zeta)^{-1}(U^{\mathbf{w}}\partial_{\mathbf{w}}V^{\mathbf{z}} - V^{\mathbf{w}}\partial_{\mathbf{w}}U^{\mathbf{z}}) \qquad [2.1\text{-}81]$$

In a metrical space, a preferred rigging exists given by vectors which are orthogonal to **N**. We will use only this type of rigging. The requirement that $C_{\mathbf{x}}{}^{\mu}$ be orthogonal to **N** is given by

$$C_{\mathbf{x}}{}^{\mu}g_{\mu\nu}B_A{}^{\nu} = 0 \qquad [2.1\text{-}82]$$

which is equivalent to

$$B^{\rho}{}_{\mu}C^{\sigma}{}_{\nu}g_{\rho\sigma} = 0 \qquad [2.1\text{-}83]$$

Since the manifold $\mathcal{N}$ is spacelike, the rigging basis vectors span a timelike two-plane. For orthogonal riggings, the decomposition of the metric (equation 2.1-17) simplifies to

$$g_{\rho\sigma} = {}'g_{\rho\sigma} + {}''g_{\rho\sigma} \qquad [2.1\text{-}84]$$

We can define the following quantities:

$$g_{AB} = B^{\mu}{}_{A}{}^{\nu}{}_{B}\, g_{\mu\nu} \qquad [2.1\text{-}85]$$

$$g_{\mathbf{xy}} = C_{\mathbf{x}}{}^{\mu}{}_{\mathbf{y}}{}^{\nu}g_{\mu\nu} \qquad [2.1\text{-}86]$$

Then we can write

$$g_{\mu\nu} = g_{AB}B^{A}{}_{\mu}{}^{B}{}_{\nu} + g_{\mathbf{xy}}C^{\mathbf{x}}{}_{\mu}{}^{\mathbf{y}}{}_{\nu} \qquad [2.1\text{-}87]$$

The quantity g_{AB} is the pull-back of the metric tensor of **M**, via the map **B**, to $\mathcal{N}$. We shall interpret this in Section 2.2 as the metric tensor of $\mathcal{N}$. The quantity $g_{\mathbf{xy}}$ is the dyad components of the metric tensor with respect to the rigging vectors. We shall also interpret the quantities $g_{\mathbf{xy}}$ as components of the metric tensor of $\mathcal{T}$ with respect to the anholonomic basis.

2.2 The Intrinsic Geometry of $\mathcal{N}$ and $\mathcal{T}$

An imbedded two-manifold $\mathcal{N}$ has an intrinsic geometry that is inherited from the geometry of the space-time. This intrinsic geometry, which defines distances between points of $\mathcal{N}$, is completely determined by pulling-back the metric tensor of **M** to $\mathcal{N}$ using the imbedding map. Let two infinitesimally close points on $\mathcal{N}$ be labelled y^A and $y^A + dy^A$, respectively. Under the imbedding map, these points are mapped into $x^\mu = B^\mu(y^A)$ and $x^\mu + dx^\mu = B^\mu(y^A) + B^\mu{}_A dy^A$. The distance between the image points, as points of **M**, is given by:

$$
\begin{aligned}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} B^\mu{}_A dy^A B^\nu{}_B dy^B \\
&= (B^\mu{}_A dy^A)(B^\nu{}_B dy^B) g_{\mu\nu} = (B^\mu{}_A{}^\nu{}_B g_{\mu\nu}) dy^A dy^B \\
&= g_{AB} dy^A dy^B,
\end{aligned}
\qquad [2.2\text{-}1]
$$

where

$$g_{AB} \equiv B^\mu{}_A{}^\nu{}_B g_{\mu\nu} \qquad [2.2\text{-}2]$$

is the pull-back of $g_{\mu\nu}$ (written $B^*\mathbf{g}$ in coordinate-free notation). The quantity g_{AB} is the metric tensor of $\mathcal{N}$. Its tensorial character follows from the transformation properties of the connecting quantities $B^\mu{}_A$ as described in Section 2.1: under $\mathcal{N}$-coordinate transformations, g_{AB} transforms like a covariant tensor of order two. Since the conformal two-structure is built up from families of space-like two-surfaces, we assume that $\mathcal{N}$ is everywhere space-like and g_{AB} is a rank-two positive-definite tensor field on $\mathcal{N}$ with signature (++).

The metric tensor g_{BC} has a unique inverse ${}^{(2)}g^{AB}$, defined by

$$^{(2)}g^{AB}g_{BC} = \delta^{A}{}_{C} \qquad [2.2\text{-}3]$$

and also defines a covariant derivative operation $^{(2)}\nabla_{A}$:

$$^{(2)}\nabla_{A}{}^{(2)}g^{BC} = {}^{(2)}\nabla_{A}g_{BC} = 0 \qquad [2.2\text{-}4]$$

The unique symmetric affine connection which satisfies this equation is given by the Christoffel symbols

$$^{(2)}\Gamma^{A}{}_{BC} = 1/2\,{}^{(2)}g^{AD}\{-g_{BC,D} + g_{DB,C} + g_{CD,B}\} \qquad [2.2\text{-}5]$$

The Riemann curvature tensor of the affine connection is defined by the commutator of the covariant derivatives. We have

$$\nabla_{[A}\nabla_{B]}p^{D} = 1/2\,{}^{(2)}R_{ABC}{}^{D}\, p^{C} \qquad [2.2\text{-}6]$$

for any arbitrary vector p^{C} belonging to $\mathcal{N}$. Explicitly, the Riemann curvature tensor is given in terms of the Christoffel connection as

$$'R_{ABC}{}^{D} = 2\partial_{[A}{}^{(2)}\Gamma^{D}{}_{B]C} + 2\,{}^{(2)}\Gamma^{D}{}_{[A|E|}{}^{(2)}\Gamma^{E}{}_{B]C} \qquad [2.2\text{-}7]$$

The rigging played no part in the above analysis; the entire intrinsic geometry of $\mathcal{N}$ can be defined solely in terms of the imbedding pull-back map and the metrical geometry of **M**.

When **N** is rigged, the contravariant metric tensor $g^{\mu\nu}$ can also be pulled-back to $\mathcal{N}$ defining a symmetric second-rank contravariant tensor on $\mathcal{N}$:

$$g^{AB} \equiv B^A{}_\mu B^B{}_\nu g^{\mu\nu} \qquad [2.2\text{-}8]$$

The contravariant $\mathcal{N}$-tensor g^{AB} and the contravariant $\mathcal{N}$-tensor ${}^{(2)}g^{AB}$ defined above are identical precisely because the rigging we have chosen is orthogonal to **N**. To prove this, consider

$$g^{AB}g_{BC} = B^A{}_\mu B^B{}_\nu g^{\mu\nu} B^\sigma{}_B B^\rho{}_C\, g_{\sigma\rho} = B^A{}_\mu B^\sigma{}_\nu B^\rho{}_C g^{\mu\nu} g_{\sigma\rho}$$

$$= B^A{}_\mu B^\rho{}_C (\delta^\sigma{}_\nu - C^\sigma{}_\nu) g^{\mu\nu} g_{\sigma\rho}$$

$$= B^A{}_\mu B^\mu{}_C - B^A{}_\mu C^x{}_\nu g^{\mu\nu} (g_{\sigma\rho} B^\rho{}_C C_x{}^\sigma) \qquad [2.2\text{-}9]$$

For orthogonal rigging spaces, the last term on the right vanishes. Hence

$$g^{AB}g_{BC} = B^A{}_\mu B^\mu{}_C = \delta^A{}_C \qquad [2.2\text{-}10]$$

proving that, in our case, the induced contravariant metric is the same as the inverse of the induced covariant metric

$$g^{AB} = {}^{(2)}g^{AB} \qquad [2.2\text{-}11]$$

This gives an unambiguous means of raising and lowering surface indices. Henceforth, we shall drop the super-prefix (${}^{(2)}$) on g^{AB}.

With an orthogonal rigging, the connecting quantities $B^\mu{}_A$ and $B^B{}_\mu$ are metrically related

$$B^B{}_\mu = g^{AB} g_{\mu\nu} B^\nu{}_A \qquad [2.2\text{-}12]$$

The proof follows easily by showing that the left-hand side of equation 2.2-12 satisfies the same duality conditions that defined $B^B{}_\mu$ in Section 2.1.

Another way to define an affine connection on a rigged surface is to induce one from an existing connection on **M**. This procedure is applicable to non-metric affine spaces. We do this by showing how to define the covariant derivative of a covariant vector field p_B on $\mathcal{N}$ similar to the way we defined Lie derivatives in Section 2.1. Using $B^B{}_\nu$, carry p_B over to **N** creating a vector field $'p_\nu = B^B{}_\nu p_B$. Then, arbitrarily prolonging $'p_\nu$ off **N**, the covariant derivative on **M** of this vector field is defined. Pulling back the covariant derivative to $\mathcal{N}$ using $B^\mu{}_A$ defines the covariant derivative $D_A p_B$ on $\mathcal{N}$. Thus

$$D_A p_B = B^{\mu}{}_A{}^{\nu}{}_B \nabla_\mu {}'p_\nu \qquad [2.2\text{-}13]$$

Since the right-hand side of equation 2.2-13 involves derivatives only in directions tangent to **N**, the left-hand side is independent of the prolongation. Exapanding the right-hand side of equation 2.2-13, we get

$$\begin{aligned} D_A p_B &= B^{\mu}{}_A{}^{\nu}{}_B (\partial_\mu {}'p_\nu + \Gamma^\rho{}_{\mu\nu} {}'p_\rho) \\ &= \partial_A p_B - {}'p_\nu B^\mu{}_A \partial_\mu B^\nu{}_B + B^{\mu}{}_A{}^{\nu}{}_B \Gamma^\rho{}_{\mu\nu} {}'p_\rho \\ &= \partial_A p_B - p_C B^C{}_\nu{}^\mu{}_A \partial_\mu B^\nu{}_B + B^{\mu}{}_A{}^{\nu}{}_B \Gamma^\rho{}_{\mu\nu} p_C B^C{}_\rho \\ &= \partial_A p_B + {}'\Gamma^C{}_{AB}\, p_C \end{aligned} \qquad [2.2\text{-}14]$$

The last term on the right-hand side of equation 2.2-14 is the induced affine connection on $\mathcal{N}$

$$'\Gamma^{C}{}_{AB} = B^{\mu}{}_{A}{}^{\nu}{}_{B}\Gamma^{\rho}{}_{\mu\nu}B^{C}{}_{\rho} - B^{C}{}_{\nu}{}^{\mu}{}_{A}\partial_{\mu}B^{\nu}{}_{B}$$

$$= -B^{\nu}{}_{A}{}^{\mu}{}_{B}\nabla_{\nu}B^{C}{}_{\mu} \qquad [2.2\text{-}15]$$

The symmetry of $'\Gamma^{C}{}_{AB}$ with respect to (A,B) is evident if we write 2.2-15 as

$$'\Gamma^{C}{}_{AB} = B^{\mu}{}_{A}{}^{\nu}{}_{B}\Gamma^{\rho}{}_{\mu\nu}B^{C}{}_{\rho} - B^{C}{}_{\nu}\partial_{A}B^{\nu}{}_{B}$$

$$= B^{\mu}{}_{A}{}^{\nu}{}_{B}\Gamma^{\rho}{}_{\mu\nu}B^{C}{}_{\rho} - B^{C}{}_{\nu}\partial_{A}\partial_{B}x^{\nu} \qquad [2.2\text{-}16]$$

where both terms on the right-hand side of equation 2.2-16 are symmetric with respect to thise indices.

We shall prove below that, in a metric space with orthogonal rigging, the induced affine connection and the Christoffel connection on $\mathcal{N}$ are the same. However, we will maintain the conceptual distinction between the two types of connections (and use different symbols) because many of the results we shall prove on extrinsic curvatures (defined in the next section) can be expressed in terms of the affine geometry alone.

The covariant derivative of a contravariant vector r^B of $\mathcal{N}$ is defined in the same manner:

$$D_A r^B = B^{\mu}{}_{A}{}^{B}{}_{\nu}\nabla_{\mu}{}'r^{\nu}, \qquad [2.2\text{-}17]$$

with

$$'r^{\nu} = B^{\nu}{}_{B}r^{B}$$

The extension of the covariant derivative to tensors with a general number of indices is straightforward since any tensor can be written as the sum of exterior products of contravariant and covariant vectors.

Because we have rigged **N** orthogonally, the covariant derivative with respect to the induced affine connection of the two-metric tensor g_{BC} vanishes:

$$D_A g_{BC} = B^{\mu}{}_{A} {}^{\nu}{}_{B} {}^{\kappa}{}_{C} \nabla_{\mu}{}' g_{\nu\kappa} = -B^{\mu}{}_{A} {}^{\nu}{}_{B} {}^{\kappa}{}_{C} \nabla_{\mu}{}'' g_{\nu\kappa} = 0 \qquad [2.2\text{-}18]$$

Because there is only one symmetric connection satisfying equation 2.2-18, the induced covariant derivative is the same as that defined by the Christoffel connection above. We have

$$D_A = {}^{(2)}\nabla_A \qquad [2.2\text{-}19]$$

The contravariant Levi-Civita symbol ε^{AB} on $\mathcal{N}$ is defined by

$$\varepsilon^{23} = -\varepsilon^{32} = 1$$
$$\varepsilon^{22} = \varepsilon^{33} = 0 \qquad [2.2\text{-}20]$$

It transforms as a totally antisymmetric tensor density of weight +1 under $\mathcal{N}$-coordinate transformations. The covariant Levi-Civita symbol ε_{AB} is an antisymmetric tensor density of weight -1. We can use the Levi-Civita tensor densities to construct the determinant of the two-metric g_{BD} of $\mathcal{N}$

$$\gamma^2 = \det|g_{AB}| = g_{22}g_{33} - g_{23}g_{32} = 1/2\varepsilon^{AB}\varepsilon^{CD}g_{AC}g_{BD} \qquad [2.2\text{-}21]$$

making γ^2 a scalar density of weight +2. Its square root γ is a scalar density of weight +1. The contravariant and covariant metric tensors are related by

$$g^{AC} = \gamma^{-2}\varepsilon^{AB}\varepsilon^{CD}g_{BD} \qquad [2.2\text{-}22]$$

and

$$g_{AC} = \gamma^{2}\varepsilon_{AB}\varepsilon_{CD}g^{BD} \qquad [2.2\text{-}23]$$

Some relations involving the Levi-Civita symbols are

$$\varepsilon^{AB}\varepsilon_{CD} = \delta^{A}{}_{C}\delta^{B}{}_{D} - \delta^{A}{}_{D}\delta^{B}{}_{C} = \delta^{A}{}_{C}{}^{B}{}_{D} \qquad [2.2\text{-}24]$$

$$\varepsilon^{AB}\varepsilon_{CB} = \delta^{A}{}_{C} \qquad [2.2\text{-}25]$$

$$\varepsilon^{AB}\varepsilon_{AB} = 2 \qquad [2.2\text{-}26]$$

One can readily show that

$$D_{A}\varepsilon_{BC} = D_{A}\varepsilon^{BC} = \mathfrak{D}_{x}\varepsilon_{BC} = \mathfrak{D}_{x}\varepsilon^{BC} = 0 \qquad [1.2\text{-}27]$$

We introduce the quantity $\tilde{g}_{AB}$ defined by

$$\tilde{g}_{AB} = \gamma^{-1}g_{AB} \qquad [2.2\text{-}28]$$

which transforms like a tensor density of weight -1 under $\mathcal{N}$-coordinate transformations. Its determinant being unity, $\tilde{g}_{AB}$ is invariant under conformal rescaling of the metric tensor; i.e., when g_{AB} undergoes the transformation

$$g_{AB} \rightarrow \Omega^{2}g_{AB}, \qquad [2.2\text{-}29]$$

$\tilde{g}_{AB}$ remains unchanged

$$\tilde{g}_{AB} \rightarrow \tilde{g}_{AB} \qquad [2.2\text{-}30]$$

The inverse quantity

$$\tilde{g}^{AB} = \gamma g^{AB} \qquad [2.2\text{-}31]$$

is a tensor density of weight +1 and is also conformally-invariant. We denote by $\tilde{g}^{AB}$ the conformal metric tensor of $\mathcal{N}$.

The rigging space has its own metric $g_{\mathbf{xy}}$ defined by

$$g_{\mathbf{xy}} = C^{\nu}{}_{\mathbf{x}}{}^{\mu}{}_{\mathbf{y}} g_{\nu\mu}, \qquad [2.2\text{-}32]$$

as well as a contravariant metric

$$g^{\mathbf{xy}} = C^{\mathbf{x}}{}_{\nu}{}^{\mathbf{y}}{}_{\mu} g^{\nu\mu} \qquad [2.2\text{-}33]$$

$g_{\mathbf{xy}}$ and $g^{\mathbf{xy}}$ transform as covariant and contravariant tensors, respectively, under rigging transformations given by 2.1-4. They define the magnitudes of objects belonging to the rigging space $\mathcal{T}$. They are inverses of each other because the rigging is orthogonal to $\mathcal{N}$ and can be used to raise and lower rigging indices.

One can easily show that

$$C_{\mathbf{x}}{}^{\nu} = g^{\nu\mu} g_{\mathbf{xy}} C^{\mathbf{y}}{}_{\mu} \qquad [2.2\text{-}34]$$

A Levi-Civita or alternating symbol can be defined on $\mathcal{T}$ just as it was on $\mathcal{N}$. Define $\varepsilon^{\mathbf{xy}}$ and $\varepsilon_{\mathbf{xy}}$ by:

$$\varepsilon^{01} = -\varepsilon^{10} = \varepsilon_{01} = -\varepsilon_{10} = 1$$

$$\varepsilon^{00} = \varepsilon^{11} = \varepsilon_{00} = \varepsilon_{11} = 0 \qquad [2.2\text{-}35]$$

$\varepsilon^{\mathbf{xy}}$ is a tensor density of weight +1 while $\varepsilon_{\mathbf{xy}}$ is a tensor density of weight -1 under rigging transformations. The determinant of $g_{\mathbf{xy}}$ is a tensor density of weight +2 under rigging transformations. If we introduce the scalar density of weight +2 by

$$\rho^2 = -\det[g_{\mathbf{xy}}] = -1/2\ \varepsilon^{\mathbf{xz}}\varepsilon^{\mathbf{yw}} g_{\mathbf{xy}} g_{\mathbf{zw}} \quad [2.2\text{-}36]$$

then ρ^2 is positive because the rigging two-plane is timelike. Relations analogous to 2.2-24, 25 and 26 hold for these Levi-Civita symbols.

We also have

$$g_{\mathbf{xy}} = -\rho^2 \varepsilon_{\mathbf{xz}}\ \varepsilon_{\mathbf{yw}} g^{\mathbf{zw}} \quad [2.2\text{-}37]$$

$$g^{\mathbf{zw}} = -\rho^{-2} \varepsilon^{\mathbf{zx}} \varepsilon^{\mathbf{wy}} g_{\mathbf{xy}} \quad [2.2\text{-}38]$$

and

$$\varepsilon^{\mathbf{xz}} g_{\mathbf{xy}} = -\rho^2 \varepsilon_{\mathbf{yx}} g^{\mathbf{zx}} \quad [2.2\text{-}39]$$

In the rigging space, an induced anholonomic covariant derivative exists. Let q^μ be a vector field tangent to the rigging space, so that

$$q^\mu = q^{\mathbf{x}} C_{\mathbf{x}}{}^{\mu}$$

Define the covariant derivative of $q^{\mathbf{y}}$ by:

$$D_{\mathbf{x}} q^{\mathbf{y}} = C^{\nu}{}_{\mathbf{x}} C^{\mathbf{y}}{}_{\mu} \nabla_\nu q^\mu \quad [2.2\text{-}40]$$

Then

$$D_{\mathbf{x}} q^{\mathbf{y}} = C_{\mathbf{x}}{}^{\nu} C^{\mathbf{y}}{}_{\mu} \nabla_\nu q^\mu = \partial_{\mathbf{x}} q^{\mathbf{y}} - (C_{\mathbf{x}}{}^{\nu} C_{\mathbf{z}}{}^{\mu} \nabla_\nu C^{\mathbf{y}}{}_{\mu}) q^{\mathbf{z}} \quad [2.2\text{-}41]$$

Introducing the induced connection on $\mathcal{T}$

$$''\Gamma^{\mathbf{y}}{}_{\mathbf{xz}} = -\ C_{\mathbf{x}}{}^{\nu} C_{\mathbf{z}}{}^{\mu} \nabla_\nu C^{\mathbf{y}}{}_{\mu} \quad [2.2\text{-}42]$$

we get

$$D_x q^y = \partial_x q^y + "\Gamma^y{}_{xz} q^z \qquad [2.2\text{-}43]$$

This covariant derivative is compatible with the induced metric as can be seen by the fact that:

$$D_z g^{xy} = C_z{}^\nu{}_\mu{}^x{}_\sigma{}^y \nabla_\nu "g^{\mu\nu} = - C_z{}^\nu{}_\mu{}^x{}_\sigma{}^y \nabla_\nu 'g^{\mu\nu} = 0 \qquad [2.2\text{-}44]$$

Similarly

$$D_z g_{xy} = 0 \qquad [2.2\text{-}45]$$

The affine connection $"\Gamma^x{}_{yz}$ on $\mathcal{T}$ is not torsion-free, but contains terms attributible to the anholonomicity of the rigging space

$$"\Gamma^x{}_{yz} = 1/2\, g^{xw}\{-g_{yz,w} + g_{wy,z} + g_{zw,y}\} + g^{xw}\Omega_{\{ywz\}} \qquad [2.2\text{-}46]$$

where we define

$$\Omega_{\{ywz\}} \equiv \Omega_{ywz} - \Omega_{wzy} + \Omega_{zyw} \qquad [2.2\text{-}47]$$

with

$$\Omega_{yzw} \equiv \Omega^r{}_{yz} g_{rw} \qquad [2.2\text{-}48]$$

where $\Omega^r{}_{yz}$ is the anholonomic object defined in Section 2.1.

The antisymmetric part of $"\Gamma^y{}_{xz}$ is minus the object of anholonomicity

$$"\Gamma^y{}_{[xz]} = -\Omega^y{}_{xz} \qquad [2.2\text{-}49]$$

but the anholonomic object also enters into the symmetric part.

It is not as straightforward to define a Riemann tensor on

$\mathcal{T}$ as it is on $\mathcal{N}$. The commutator of two covariant derivatives has a more complicated expression due to the anholonomicity of the space. We have

$$D_{\mathbf{x}}D_{\mathbf{y}}p^{\mathbf{z}} = \partial_{\mathbf{x}}(\partial_{\mathbf{y}}p^{\mathbf{z}} + {}''\Gamma^{\mathbf{z}}{}_{\mathbf{yw}}p^{\mathbf{w}}) + {}''\Gamma^{\mathbf{z}}{}_{\mathbf{xw}}(\partial_{\mathbf{y}}p^{\mathbf{w}} + {}''\Gamma^{\mathbf{w}}{}_{\mathbf{ys}}p^{\mathbf{s}})$$

$$- {}''\Gamma^{\mathbf{w}}{}_{\mathbf{xy}}(\partial_{\mathbf{w}}p^{\mathbf{z}} + {}''\Gamma^{\mathbf{z}}{}_{\mathbf{ws}}p^{\mathbf{s}})$$

$$= \partial_{\mathbf{x}}\partial_{\mathbf{y}}p^{\mathbf{z}} + {}''\Gamma^{\mathbf{z}}{}_{\mathbf{yw}}\partial_{\mathbf{x}}p^{\mathbf{w}} + {}''\Gamma^{\mathbf{z}}{}_{\mathbf{xw}}\partial_{\mathbf{y}}p^{\mathbf{w}} - {}''\Gamma^{\mathbf{w}}{}_{\mathbf{xy}}\partial_{\mathbf{w}}p^{\mathbf{z}}$$

$$+ (\partial_{\mathbf{x}}{}''\Gamma^{\mathbf{z}}{}_{\mathbf{yw}})p^{\mathbf{w}} + {}''\Gamma^{\mathbf{z}}{}_{\mathbf{xw}}{}''\Gamma^{\mathbf{w}}{}_{\mathbf{ys}}p^{\mathbf{s}} - {}''\Gamma^{\mathbf{w}}{}_{\mathbf{xy}}{}''\Gamma^{\mathbf{z}}{}_{\mathbf{ws}}p^{\mathbf{s}}$$

[2.2-50]

When this equation is antisymmetrized over **x** and **y**, the second and third terms on the right-hand side cancel, giving

$$D_{[\mathbf{x}}D_{\mathbf{y}]}p^{\mathbf{z}} = \partial_{[\mathbf{x}}\partial_{\mathbf{y}]}p^{\mathbf{z}} - {}''\Gamma^{\mathbf{w}}{}_{[\mathbf{xy}]}\partial_{\mathbf{w}}p^{\mathbf{z}} + \{\partial_{[\mathbf{x}}{}''\Gamma^{\mathbf{z}}{}_{\mathbf{y}]\mathbf{w}}$$

$$+ {}''\Gamma^{\mathbf{z}}{}_{[\mathbf{x}|\mathbf{s}|}{}''\Gamma^{\mathbf{s}}{}_{\mathbf{y}]\mathbf{w}} - {}''\Gamma^{\mathbf{s}}{}_{[\mathbf{xy}]}{}''\Gamma^{\mathbf{z}}{}_{\mathbf{sw}}\}p^{\mathbf{w}} \quad [2.2\text{-}51]$$

Unlike the holonomic case, the term on the right-hand side cannot be defined as the Riemann tensor since it depends on derivatives of $p^{\mathbf{z}}$. We follow Schouten in modifying the equation so as to define the proper Riemann tensor. First note that one can show

$$\partial_{[\mathbf{x}}\partial_{\mathbf{y}]}p^{\mathbf{z}} = - \Omega^{A}{}_{\mathbf{xy}}\partial_{A}p^{\mathbf{z}} - \Omega^{\mathbf{w}}{}_{\mathbf{xy}}\partial_{\mathbf{w}}p^{\mathbf{z}} \quad [2.2\text{-}52]$$

and, since

$${}''\Gamma^{\mathbf{w}}{}_{[\mathbf{xy}]} = - \Omega^{\mathbf{w}}{}_{\mathbf{xy}}, \quad [2.2\text{-}53]$$

equation 2.5-51 can be rewritten to

$$D_{[\mathbf{x}}D_{\mathbf{y}]}p^{\mathbf{z}} = - \Omega^{A}{}_{\mathbf{xy}}\partial_{A}p^{\mathbf{z}}$$

$$+ \left\{\partial_{[\mathbf{x}}{}''\Gamma^{\mathbf{z}}{}_{\mathbf{y}]\mathbf{w}} + {}''\Gamma^{\mathbf{z}}{}_{[\mathbf{x}|\mathbf{s}|}{}''\Gamma^{\mathbf{s}}{}_{\mathbf{y}]\mathbf{w}} - {}''\Gamma^{\mathbf{s}}{}_{[\mathbf{xy}]}{}''\Gamma^{\mathbf{z}}{}_{\mathbf{sw}}\right\}p^{\mathbf{w}}$$

[2.2-54]

Using the "covariant" $\mathfrak{D}$-derivative, we can write equation 2.2-54 in the form

$$D_{[\mathbf{x}}D_{\mathbf{y}]}p^{\mathbf{z}} + \Omega^{A}{}_{\mathbf{xy}}\,\mathfrak{D}_{A}p^{\mathbf{z}} = \{\partial_{[\mathbf{x}}{}''\Gamma^{\mathbf{z}}{}_{\mathbf{y}]\mathbf{w}} + {}''\Gamma^{\mathbf{z}}{}_{[\mathbf{x}|\mathbf{s}|}{}''\Gamma^{\mathbf{s}}{}_{\mathbf{y}]\mathbf{w}} - {}''\Gamma^{\mathbf{s}}{}_{[\mathbf{xy}]}{}''\Gamma^{\mathbf{z}}{}_{\mathbf{sw}}\}p^{\mathbf{w}} + 2\Omega^{A}{}_{\mathbf{xy}}\Omega^{\mathbf{z}}{}_{\mathbf{w}A}p^{\mathbf{w}} \qquad [2.2\text{-}55]$$

The right-hand side of equation 2.2-55 provides the correct definition of the Riemann tensor of the anholonomic rigging space.

$$''R_{\mathbf{xyw}}{}^{\mathbf{z}} = 2\partial_{[\mathbf{x}}{}''\Gamma^{\mathbf{z}}{}_{\mathbf{y}]\mathbf{w}} + 2''\Gamma^{\mathbf{z}}{}_{[\mathbf{x}|\mathbf{s}|}{}''\Gamma^{\mathbf{s}}{}_{\mathbf{y}]\mathbf{w}} - 2''\Gamma^{\mathbf{s}}{}_{[\mathbf{xy}]}{}''\Gamma^{\mathbf{z}}{}_{\mathbf{sw}} + 4\Omega^{A}{}_{\mathbf{xy}}\Omega^{\mathbf{z}}{}_{\mathbf{w}A} \qquad [2.2\text{-}56]$$

2.3 The Extrinsic Geometry of $\mathcal{N}$ and $\mathcal{T}$

Whereas the intrinsic geometry, given by the metric tensor and induced connection, defines 'distance' and 'parallelism' within the surface, the extrinsic geometry describes the bending of the surface in the imbedding manifold. Bending is a concept that is easy to visualize in a metric space: the normal vector to the surface changes direction as one moves from point to point on the surface. However, the concept generalizes to affine spaces as well, and the first part of this section is devoted to describing extrinsic geometry in affine spaces. Later on, we shall specialize the results to metric spaces.

Consider a contravariant $\mathcal{N}$-vector z^A at a point $q \in \mathcal{N}$. Parallel transport z^A to a neighboring point $q_1 \in \mathcal{N}$ along a path in $\mathcal{N}$. Denote by s^B the tangent vector to the path which is parametrized by λ. z^A may also be considered a vector in **M**, $z^\mu = z^A B_A{}^\mu$, defined at $\mathbf{B}(q)$. The path is also a path in **M** and its tangent vector is

$$s^\nu = s^B B^\nu{}_B \qquad [2.3\text{-}1]$$

For an infinitesimal displacement $s^\nu d\lambda$, the components of the parallel transported vector z^μ at q_1 are given by

$$^*z^\nu(q_1) = z^\nu(q) - \Gamma^\nu{}_{\kappa\rho} z^\kappa s^\rho \, d\lambda \qquad [2.3\text{-}2]$$

Since $C^{x}{}_\mu$ is a field on **N**, its components at q_1 are

$$C^{\mathbf{x}}{}_{\mu}(q_1) = C^{\mathbf{x}}{}_{\mu}(q) + C^{\mathbf{x}}{}_{\mu,\sigma}(q)\, s^{\sigma} d\lambda \qquad [2.3\text{-}3]$$

Then we have at q_1, the rigging projection of the parallel transported vector

$$^{*}z^{\nu}(q_1)\, C^{\mathbf{x}}{}_{\nu}(q_1) = z^{\nu}(q)\, C^{\mathbf{x}}{}_{\nu}(q) + z^{\nu}(q)\, C^{\mathbf{x}}{}_{\nu,\sigma}(q)\, s^{\sigma} d\lambda - C^{\mathbf{x}}{}_{\mu}(q)\, \Gamma^{\mu}{}_{\kappa\rho} z^{\kappa} s^{\rho} d\lambda \qquad [2.3\text{-}4]$$

The first term on the right-hand side vanishes at q because $z^{\nu}(q)$ is tangent to the surface there. We have

$$\begin{aligned} ^{*}z^{\nu}(q_1)\, C^{\mathbf{x}}{}_{\nu}(q_1) &= z^{\nu}(q)\, C^{\mathbf{x}}{}_{\nu,\sigma}(q)\, s^{\sigma} d\lambda - C^{\mathbf{x}}{}_{\mu}(q)\, \Gamma^{\mu}{}_{\kappa\rho} z^{\kappa} s^{\rho} d\lambda \\ &= \nabla_{\sigma} C^{\mathbf{x}}{}_{\nu} z^{\nu} s^{\sigma} d\lambda = z^{A} s^{B} (B^{\nu}{}_{A} \sigma_{B} \nabla_{\sigma} C^{\mathbf{x}}{}_{\nu})\, d\lambda \\ &= -z^{A} s^{B} (C^{\mathbf{x}}{}_{\nu} B^{\sigma}{}_{B} \nabla_{\sigma} B^{\nu}{}_{A})\, d\lambda \end{aligned} \qquad [2.3\text{-}5]$$

The term in parenthesis on the last line of equation 2.3-5 has a simple interpretation. If it vanishes, then the vector z^{A} remains tangent to $\mathcal{N}$ as it is parallel transported. If it doesn't vanish, then z^{A} begins to point out of the surface as it is parallel transported. The term in parenthesis, related to the <u>extrinsic curvature</u> of the surface, is denoted $H_{AB}{}^{\mathbf{x}}$. It is the projection into the rigging of a surface basis vector parallel-transported along another surface basis vector.

In a similar manner, a vector that belongs to the rigging space at q can be parallel-transported along the same curve. In general, it will start to bend over into the surface. How much it does is determined by projections of covariant derivatives of the rigging basis vectors. A systematic study of the extrinsic geometry of a surface in an affinely connected manifold can be done by looking at such projections of the covariant derivatives of both the surface and rigging basis vectors. Weyl [1927] performed just such an analysis (the projections of the covariant derivatives of the basis vectors define the connection coefficients or Ricci rotation coefficients in the adapted basis). For the moment, we do not assume a metric tensor exists, but do assume the existence of an affine connection. Without a metric tensor, the rigging vectors cannot be assumed orthogonal to the surface since orthogonality is not defined.

We have the follow projections; with the symbols we shall use for them

I) Surface basis vector $B^{\nu}{}_{A}$ transported along a surface basis vector $B^{\nu}{}_{B}$

$B^{\nu}{}_{B}\nabla_{\nu}B^{\mu}{}_{A}$ projected onto surface: $B^{\nu}{}_{B}C_{\mu}\nabla_{\nu}B^{\mu}{}_{A} \equiv {}'\Gamma^{C}{}_{AB}$ [2.3-6]

projected onto rigging: $C^{\mathbf{x}}{}_{\mu}B^{\nu}{}_{B}\nabla_{\nu}B^{\mu}{}_{A} \equiv H_{BA}{}^{\mathbf{x}}$ [2.3-7]

II) Surface basis vector $B^{\nu}{}_{A}$ transported along a rigging basis vector $C^{\nu}{}_{\mathbf{x}}$

$C^{\nu}{}_{\mathbf{x}}\nabla_{\nu}B^{\mu}{}_{A}$ projected onto surface: $B^{C}{}_{\mu}C^{\nu}{}_{\mathbf{x}}\nabla_{\nu}B^{\mu}{}_{A} \equiv \gamma^{C}{}_{A\mathbf{x}}$ [2.3-8]

projected onto rigging: $C^{\mathbf{x}}{}_{\mu}C^{\nu}{}_{\mathbf{y}}\nabla_{\nu}B^{\mu}{}_{A} \equiv -{}^{*}L_{\mathbf{y}}{}^{\mathbf{x}}{}_{A}$

[2.3-9]

III) Rigging basis vector $C^{\nu}{}_{\mathbf{x}}$ transported along a rigging basis vector $C^{\nu}{}_{\mathbf{y}}$

$C^{\nu}{}_{\mathbf{y}}\nabla_{\nu}C^{\mu}{}_{\mathbf{x}}$ projected onto surface: $B^{C}{}_{\mu}C^{\nu}{}_{\mathbf{y}}\nabla_{\nu}C^{\mu}{}_{\mathbf{x}} \equiv {}^{*}H_{\mathbf{yx}}{}^{C}$

[2.3-10]

projected onto rigging: $C^{\mathbf{z}}{}_{\mu} C^{\nu}{}_{\mathbf{y}} \nabla_{\nu} C^{\mu}{}_{\mathbf{x}} \equiv {}''\Gamma_{\mathbf{xy}}{}^{\mathbf{z}}$

[2.3-11]

IV) Rigging basis vector $C^{\nu}{}_{\mathbf{x}}$ transported along a surface basis vector $B^{\nu}{}_{B}$

$B^{\nu}{}_{B} \nabla_{\nu} C^{\mu}{}_{\mathbf{x}}$ projected onto surface: $B^{C}{}_{\mu} B^{\nu}{}_{B} \nabla_{\nu} C^{\mu}{}_{\mathbf{x}} \equiv -L_{B}{}^{C}{}_{\mathbf{x}}$

[2.3-12]

projected onto rigging: $C^{\mathbf{z}}{}_{\mu} B^{\nu}{}_{B} \nabla_{\nu} C^{\mu}{}_{\mathbf{x}} \equiv \gamma_{B}{}^{\mathbf{z}}{}_{\mathbf{x}}$

[2.3-13]

Using the definition of the anholonomic object in Section 2.1, one can relate the L's and the γ's.

$$\gamma_{A}{}^{\mathbf{x}}{}_{\mathbf{y}} = -{}^{*}L_{\mathbf{y}}{}^{\mathbf{x}}{}_{A} + 2\Omega^{\mathbf{x}}{}_{\mathbf{y}A} \qquad [2.3\text{-}14]$$

$$\gamma_{\mathbf{x}}{}^{A}{}_{B} = -L_{B}{}^{A}{}_{\mathbf{x}} + 2\Omega^{A}{}_{B\mathbf{x}} \qquad [2.3\text{-}15]$$

The projections $'\Gamma^{C}{}_{AB}$ and $''\Gamma_{\mathbf{xy}}{}^{\mathbf{z}}$, discussed in Section 2.2, are just the induced affine connections on $\mathcal{N}$ and $\mathcal{T}$ respectively.

The quantities **H** and **L** have two indices in $\mathcal{N}$ and one index in $\mathcal{T}$ while the quantities ***H** and ***L** have one index in $\mathcal{N}$ and two indices in $\mathcal{T}$. $H_{BA}{}^{\mathbf{x}}$, $L_{B}{}^{A}{}_{\mathbf{x}}$, ${}^{*}H_{\mathbf{xy}}{}^{A}$ and ${}^{*}L_{\mathbf{x}}{}^{\mathbf{y}}{}_{A}$, transform as tensors under coordinate or basis transformations of $\mathcal{N}$ and $\mathcal{T}$. Just as we interpreted $H_{BA}{}^{\mathbf{x}}$ earlier, we can see that $L_{B}{}^{A}{}_{\mathbf{x}}$ determines how much a rigging vector, parallel-transported along a curve in $\mathcal{N}$ begins to bend over into $\mathcal{N}$. Similar interpretations hold for ${}^{*}H_{\mathbf{xy}}{}^{A}$ and ${}^{*}L_{\mathbf{x}}{}^{\mathbf{y}}{}_{A}$.

The symmetric part of $H_{BA}{}^{\mathbf{x}}$ is called the <u>extrinsic curvature tensor</u> of the surface, while the symmetric part of $^{*}H_{\mathbf{xy}}{}^{A}$ is the extrinsic curvature of the rigging space.

The antisymmetric parts of $H_{[AB]}{}^{\mathbf{x}}$ and $^{*}H_{[\mathbf{xy}]}{}^{A}$ are essentially the anholonomic objects

$$H_{[AB]}{}^{\mathbf{x}} = -\Omega_{AB}{}^{\mathbf{x}} \qquad [2.3\text{-}16]$$

$$^{*}H_{[\mathbf{xy}]}{}^{A} = -\Omega_{\mathbf{xy}}{}^{A} \qquad [2.3\text{-}17]$$

(In our work, we are assuming that $H_{[AB]}{}^{\mathbf{x}} = 0$.)

Schouten [1954], on the other hand, defines the extrinsic curvatures of the surface and rigging spaces directly in terms of the covariant derivatives of the projectors $B^{\mu}{}_{\nu}$ and $C^{\mu}{}_{\nu}$.

Using the fact that

$$\nabla_{\mu}B^{\mu}{}_{\nu} = -\nabla_{\mu}C^{\mu}{}_{\nu} \qquad [2.3\text{-}18]$$

he introduces (following Schouten [1954]) the three-index curvature tensors

$$H_{\mu\sigma}{}^{\nu} = B^{\alpha}{}_{\mu}{}^{\beta}{}_{\sigma}\nabla_{\alpha}B^{\nu}{}_{\beta} = -B^{\alpha}{}_{\mu}{}^{\beta}{}_{\sigma}\nabla_{\alpha}C^{\nu}{}_{\beta} \qquad [2.3\text{-}19]$$

$$L_{\mu}{}^{\nu}{}_{\sigma} = B^{\alpha}{}_{\mu}{}^{\nu}{}_{\beta}\nabla_{\alpha}B^{\beta}{}_{\sigma} = -B^{\alpha}{}_{\mu}{}^{\nu}{}_{\beta}\nabla_{\alpha}C^{\beta}{}_{\sigma} \qquad [2.3\text{-}20]$$

$$^{*}H_{\mu\sigma}{}^{\nu} = C^{\alpha}{}_{\mu}{}^{\beta}{}_{\sigma}\nabla_{\alpha}C^{\nu}{}_{\beta} = -C^{\alpha}{}_{\mu}{}^{\beta}{}_{\sigma}\nabla_{\alpha}B^{\nu}{}_{\beta} \qquad [2.3\text{-}21]$$

$$^{*}L_{\mu}{}^{\nu}{}_{\sigma} = C^{\alpha}{}_{\mu}{}^{\nu}{}_{\beta}\nabla_{\alpha}C^{\beta}{}_{\sigma} = -C^{\alpha}{}_{\mu}{}^{\nu}{}_{\beta}\nabla_{\alpha}B^{\beta}{}_{\sigma} \qquad [2.3\text{-}22]$$

The covariant derivatives of $B^{\beta}{}_{\sigma}$ and $C^{\beta}{}_{\sigma}$ can be written in terms of these quantities

$$\nabla_{\alpha}B^{\beta}{}_{\sigma} = -\nabla_{\alpha}C^{\beta}{}_{\sigma} = H_{\alpha\sigma}{}^{\beta} + L_{\alpha}{}^{\beta}{}_{\sigma} - {}^{*}H_{\alpha\sigma}{}^{\beta} - {}^{*}L_{\alpha}{}^{\beta}{}_{\sigma} \qquad [2.3\text{-}23]$$

The two sets of quantities are clearly related. In terms of projections onto $\mathcal{N}$ and $\mathcal{T}$, $H_{\alpha\sigma}{}^{\beta}$ and $L_{\alpha}{}^{\beta}{}_{\sigma}$ become

$$H_{AB}{}^{\mathbf{x}} = B^{\mu}{}_{A}{}^{\sigma}{}_{B}C^{\mathbf{x}}{}_{\nu}H_{\mu\sigma}{}^{\nu} = -B^{\mu}{}_{A}{}^{\sigma}{}_{B}C^{\mathbf{x}}{}_{\nu}\nabla_{\mu}C^{\nu}{}_{\sigma} = -B^{\mu}{}_{A}{}^{\sigma}{}_{B}\nabla_{\mu}C^{\mathbf{x}}{}_{\sigma}$$
$$= -C^{\mathbf{x}}{}_{\sigma}B^{\mu}{}_{A}\nabla_{\mu}B^{\sigma}{}_{B} \qquad [2.3\text{-}24]$$

$$L_{A}{}^{B}{}_{\mathbf{x}} = B^{\mu}{}_{A}{}^{B}{}_{\nu}C_{\mathbf{x}}{}^{\sigma}L_{\mu}{}^{\nu}{}_{\sigma} = -B^{\mu}{}_{A}{}^{B}{}_{\nu}C_{\mathbf{x}}{}^{\sigma}\nabla_{\mu}C^{\nu}{}_{\sigma} = -B^{\mu}{}_{A}{}^{B}{}_{\nu}\nabla_{\mu}C_{\mathbf{x}}{}^{\nu}$$
$$[2.3\text{-}25]$$

while for ${}^{*}H_{\alpha\sigma}{}^{\beta}$ and ${}^{*}L_{\alpha}{}^{\beta}{}_{\sigma}$, we have

$${}^{*}H_{\mathbf{xy}}{}^{A} = C^{\mu}{}_{\mathbf{x}}{}^{\sigma}{}_{\mathbf{y}}B^{A}{}_{\nu}{}^{*}H_{\mu\sigma}{}^{\nu} = -C^{\mu}{}_{\mathbf{x}}{}^{\sigma}{}_{\mathbf{y}}B^{A}{}_{\nu}C^{\alpha}{}_{\mu}{}^{\beta}{}_{\sigma}\nabla_{\alpha}B^{\nu}{}_{\beta}$$
$$= -C^{\alpha}{}_{\mathbf{x}}{}^{\beta}{}_{\mathbf{y}}\nabla_{\alpha}B^{A}{}_{\beta} = C^{\alpha}{}_{\mathbf{x}}B^{A}{}_{\beta}\nabla_{\alpha}C^{\beta}{}_{\mathbf{y}} \qquad [2.3\text{-}26]$$

$${}^{*}L_{\mathbf{x}}{}^{\mathbf{y}}{}_{A} = C^{\mu}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\nu}B^{\sigma}{}_{A}{}^{*}L_{\mu}{}^{\nu}{}_{\sigma} = -C^{\mu}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\nu}B^{\sigma}{}_{A}C^{\alpha}{}_{\mu}{}^{\nu}{}_{\beta}\nabla_{\alpha}B^{\beta}{}_{\sigma}$$
$$= -C^{\mu}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\nu}B^{\sigma}{}_{A}\nabla_{\mu}B^{\nu}{}_{\sigma} = -C^{\mu}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\nu}\nabla_{\mu}B^{\nu}{}_{A} = C^{\mu}{}_{\mathbf{x}}B^{\nu}{}_{A}\nabla_{\mu}C^{\mathbf{y}}{}_{\nu} \qquad [2.3\text{-}27]$$

which are the same quantities defined by equations 2.3-7, 2.3-9, 2.3-10 and 2.3-12.

We can interpret the γ's as follows. The projections $\gamma^{C}{}_{A\mathbf{x}}$ define $\mathcal{N}$-covariant derivative for objects with indices in the rigging space $\mathcal{T}$. For a contravariant rigging-space vector $s^{\mathbf{x}}$ it is defined by

$$D_A s^{\mathbf{x}} = B^{\mu}{}_A C^{\mathbf{x}}{}_{\nu} \nabla_{\mu}'' s^{\nu}$$

$$= \partial_A s^{\mathbf{x}} - s^{\mathbf{y}} B^{\mu}{}_A C_{\mathbf{y}}{}^{\nu} \nabla_{\mu} C^{\mathbf{x}}{}_{\nu}$$

$$= \partial_A s^{\mathbf{x}} + s^{\mathbf{y}} C^{\mathbf{x}}{}_{\nu} B^{\mu}{}_A \nabla_{\mu} C_{\mathbf{y}}{}^{\nu}$$

$$= \partial_A s^{\mathbf{x}} + \gamma_A{}^{\mathbf{x}}{}_{\mathbf{y}} s^{\mathbf{y}} \qquad [2.3\text{-}28]$$

Using the relation between $\gamma_A{}^{\mathbf{x}}{}_{\mathbf{y}}$ and ${}^{*}L_{\mathbf{y}}{}^{\mathbf{x}}{}_A$, equation 2.3-14, we get

$$D_A s^{\mathbf{x}} = \partial_A s^{\mathbf{x}} - s^{\mathbf{y}\,*} L_{\mathbf{y}}{}^{\mathbf{x}}{}_A + 2 s^{\mathbf{y}} \Omega^{\mathbf{x}}{}_{\mathbf{y}A}$$

$$= \mathfrak{D}_A s^{\mathbf{x}} - s^{\mathbf{y}\,*} L_{\mathbf{y}}{}^{\mathbf{x}}{}_A \qquad [2.3\text{-}29]$$

Similarly, a covariant derivative of rigging-space covectors is defined by

$$D_A q_{\mathbf{x}} = B^{\mu}{}_A C^{\nu}{}_{\mathbf{x}} \nabla_{\mu}'' q_{\nu}$$

$$= B^{\mu}{}_A C_{\mathbf{x}}{}^{\nu} \nabla_{\mu}'' q_{\nu} = \partial_A q_{\mathbf{x}} - q_{\mathbf{y}} B^{\mu}{}_A C^{\mathbf{y}}{}_{\nu} \nabla_{\mu} C_{\mathbf{x}}{}^{\nu}$$

$$= \partial_A q_{\mathbf{x}} - q_{\mathbf{y}} \gamma_A{}^{\mathbf{y}}{}_{\mathbf{x}} \qquad [2.3\text{-}30]$$

Again we have

$$D_A q_{\mathbf{x}} = \mathfrak{D}_A q_{\mathbf{x}} + {}^{*}L_{\mathbf{x}}{}^{\mathbf{y}}{}_A q \qquad [2.3\text{-}31]$$

In a completely parallel manner, a covariant derivative of surface quantities with respect to the rigging index can be defined. For a vector z^A belonging to $\mathcal{N}$, we have:

$$D_{\mathbf{x}} z^A = B^A{}_{\mu} C^{\nu}{}_{\mathbf{x}} \nabla_{\nu}' z^{\mu} \qquad [2.3\text{-}32]$$

which can be written in the form:

$$D_{\mathbf{x}}z^A = \partial_{\mathbf{x}}z^A - z^B B^\mu{}_B C_{\mathbf{x}}{}^\rho \nabla_\rho B^A{}_\mu$$

$$= \partial_{\mathbf{x}}z^A + \gamma_{\mathbf{x}}{}^A{}_B z^B$$

$$= \mathfrak{D}_{\mathbf{x}}z^A - L_B{}^A{}_{\mathbf{x}} z^B \qquad [2.3\text{-}33]$$

For covariant vectors we have

$$D_{\mathbf{x}}s_A = \partial_{\mathbf{x}}s_A - \gamma_{\mathbf{x}}{}^B{}_A s_B$$

$$= \mathfrak{D}_{\mathbf{x}}s_A + L_A{}^B{}_{\mathbf{x}} s_B \qquad [2.3\text{-}34]$$

A covariant derivative of space-time objects can also be defined. For a vector in the four-dimensional manifold define

$$D_A s^\nu = B^\mu{}_A \nabla_\mu s^\nu = B^\mu{}_A \nabla_\mu ({}'s^\nu + {}''s^\nu)$$

$$= B^\mu{}_A{}^\nu{}_\sigma \nabla_\mu {}'s^\sigma + B^\mu{}_A C^\nu{}_\sigma \nabla_\mu {}'s^\sigma + B^\mu{}_A{}^\nu{}_\sigma \nabla_\mu {}''s^\sigma + B^\mu{}_A C^\nu{}_\sigma \nabla_\mu {}''s^\sigma$$

$$= D_A s^B B^\nu{}_B + s^B H_{AB}{}^{\mathbf{x}} C^\nu{}_{\mathbf{x}} - s^{\mathbf{x}} L_A{}^B{}_{\mathbf{x}} B^\nu{}_B + D_A s^{\mathbf{x}} C^\nu{}_{\mathbf{x}}$$

[2.3-35]

The action of the $\mathfrak{D}$-derivatives can be extended to objects belonging to the same space as the index by requiring that it be equal to the regular covariant derivative, i.e.,

$$\mathfrak{D}_{\mathbf{x}}z^{\mathbf{y}} \equiv D_{\mathbf{x}}z^{\mathbf{y}}$$

$$\mathfrak{D}_{\mathbf{x}}r_{\mathbf{y}} \equiv D_{\mathbf{x}}r_{\mathbf{y}} \qquad [2.3\text{-}36]$$

$$\mathfrak{D}_A m^C \equiv D_A m^C$$

$$\mathfrak{D}_A q_C \equiv D_A q_C \qquad [2.3\text{-}37]$$

The action of $\mathfrak{D}$ on mixed objects follows from the fact that both $\mathfrak{D}_{\mathbf{x}}$ and $\mathfrak{D}_A$ obey Leibniz' rule. We can determine the effect of $\mathfrak{D}_{\mathbf{x}}$ on a mixed tensor by considering the outer product $m_A s^{\mathbf{x}}$. We then have

$$\mathfrak{D}_{\mathbf{x}}(m_A s^{\mathbf{y}}) = s^{\mathbf{y}}\mathfrak{D}_{\mathbf{x}} m_A + m_A \mathfrak{D}_{\mathbf{x}} s^{\mathbf{y}}$$

$$\mathfrak{D}_{\mathbf{x}}(m_A s^{\mathbf{y}}) = s^{\mathbf{y}}(\partial_{\mathbf{x}} m_A - 2\Omega^B{}_{\mathbf{x}A} m_B) + m_A(\partial_{\mathbf{x}} s^{\mathbf{y}} + {}''\Gamma^{\mathbf{y}}{}_{\mathbf{xz}} s^{\mathbf{z}})$$

$$\mathfrak{D}_{\mathbf{x}}(m_A s^{\mathbf{y}}) = \partial_{\mathbf{x}}(m_A s^{\mathbf{y}}) - 2\Omega^B{}_{\mathbf{x}A}(m_B s^{\mathbf{y}}) + {}''\Gamma^{\mathbf{y}}{}_{\mathbf{xz}}(s^{\mathbf{z}} m_A)$$

[2.3-38]

For an arbitrary mixed tensor, such as $T^{\mathbf{y}}{}_{AB}$, this gives the expression

$$\mathfrak{D}_{\mathbf{x}} T^{\mathbf{y}}{}_{AB} = \partial_{\mathbf{x}} T^{\mathbf{y}}{}_{AB} - 2\Omega^C{}_{\mathbf{x}B} T^{\mathbf{y}}{}_{AC} - 2\Omega_{\mathbf{x}A} T^{\mathbf{y}}{}_{CB} + {}''\Gamma^{\mathbf{y}}{}_{\mathbf{xz}} T^{\mathbf{z}}{}_{AB}$$

[2.3-39]

and

$$\mathfrak{D}_C T^{\mathbf{y}}{}_{AB} = \partial_C T^{\mathbf{y}}{}_{AB} - \Gamma_C{}^D{}_A T^{\mathbf{y}}{}_{DB} - \Gamma_C{}^D{}_B T^{\mathbf{y}}{}_{AD} + 2\Omega^{\mathbf{y}}{}_{\mathbf{x}C} T^{\mathbf{z}}{}_{AB}$$

[2.3-40]

The Lie derivatives can be expanded in terms of covariant $\mathfrak{D}$-derivatives. For example,

$$£'_{\mathbf{z}} T^{\mathbf{y}}{}_A{}^B = z^C \mathfrak{D}_C T^{\mathbf{y}}{}_A{}^B - T^{\mathbf{y}}{}_A{}^C \mathfrak{D}_C z^B + T^{\mathbf{y}}{}_{CB} \mathfrak{D}_A z^C \qquad [2.3\text{-}41]$$

or

$$£_{"\mathbf{z}}T^{\mathbf{y}}{}_{A\mathbf{x}} = z^{\mathbf{w}}\mathcal{D}_{\mathbf{w}}T^{\mathbf{y}}{}_{A\mathbf{x}} + T^{\mathbf{y}}{}_{A\mathbf{w}}\mathcal{D}_{\mathbf{x}}z^{\mathbf{w}} - T^{\mathbf{w}}{}_{A\mathbf{x}}\mathcal{D}_{\mathbf{w}}z^{\mathbf{y}} \qquad [2.3\text{-}42]$$

The Ricci tensor is defined by

$$R_{\mu\nu} = R_{\alpha\mu\nu}{}^{\alpha} \qquad [2.3\text{-}43]$$

and its independent projections are:

$$R_{AB} = B^{\mu}{}_{A}{}^{\nu}{}_{B}R_{\alpha\mu\nu}{}^{\alpha} = R_{\mathbf{x}AB}{}^{\mathbf{x}} + R_{CAB}{}^{C} \qquad [2.3\text{-}44]$$

$$R_{\mathbf{x}A} = C_{\mathbf{x}}{}^{\mu}B^{\nu}{}_{A}R_{\alpha\mu\nu}{}^{\alpha} = R_{\mathbf{zx}A}{}^{\mathbf{z}} + R_{C\mathbf{x}A}{}^{C} \qquad [2.3\text{-}45]$$

$$R_{\mathbf{xy}} = C_{\mathbf{x}}{}^{\mu}{}_{\mathbf{y}}{}^{\nu}R_{\alpha\mu\nu}{}^{\alpha} = R_{\mathbf{zxy}}{}^{\mathbf{z}} + R_{C\mathbf{xy}}{}^{C} \qquad [2.3\text{-}46]$$

We shall consider the Ricci tensor later in Chapter 3; we are now only interested in the projections of the Riemann tensor which appear in equations 2.3-44, 2.3-45 and 2.3-46. We shall relate these to the intrinsic and extrinsic geometries of the surface and/or rigging space by the Gauss and Codazzi equations, and another relation which involves $\mathcal{D}$-derivatives of the extrinsic curvature. We shall now investigate these equations.

The Riemann tensor of $\mathcal{N}$ was defined in equation [2.2-7] in terms of the commutator of surface covariant derivatives. However, because the covariant derivatives are pulled-back from **M**, the Riemann tensor can also be written in terms of projections of the commutator of the covariant derivatives of **M**. This brings in the Riemann tensor of **M** and leads to a relation between the Riemann tensors of $\mathcal{N}$ and **M** known as <u>Gauss' equation</u>

$$'R_{DCB}{}^{A} = L_{D}{}^{A}{}_{\mathbf{x}}H^{\mathbf{x}}{}_{CB} - L_{C}{}^{A}{}_{\mathbf{x}}H^{\mathbf{x}}{}_{DB} + R_{DCB}{}^{A} \qquad [2.3\text{-}47]$$

where $'R_{DCB}{}^{A}$ is the Riemann tensor of $\mathcal{N}$. Gauss' equation is proved in Appendix B. The last term on the right-hand side of equation 2.3-47 is the term need in equation 2.3-44.

Another set of equations which relate $\mathcal{N}$-quantities to projections of the four-dimensional Riemann tensor is <u>Codazzi's equations</u>, also derived in Appendix B,

$$2D_{[C}H_{A]B}{}^{\mathbf{x}} = R_{CAB}{}^{\mathbf{x}} \qquad [2.3\text{-}48]$$

There are corresponding Gauss and Codazzi equations for the rigging spaces. These are somewhat more complicated as the rigging spaces are not holonomic. (We refer the reader to Schouten for a complete discussion). They are <u>Gauss' equation</u> for $\mathcal{T}$

$$"R_{\mathbf{xyz}}{}^{\mathbf{w}} = -2{}^{*}H_{[\mathbf{x}|\mathbf{z}|}{}^{A\,*}L_{\mathbf{y}]}{}^{\mathbf{w}}{}_{A} + 2\Omega_{\mathbf{xy}}{}^{A\,*}L_{\mathbf{z}}{}^{\mathbf{w}}{}_{A} + R_{\mathbf{xyz}}{}^{\mathbf{w}} \qquad [2.3\text{-}49]$$

and <u>Codazzi's equations</u> for $\mathcal{T}$

$$2D_{[\mathbf{x}}{}^{*}H_{\mathbf{y}]\mathbf{z}}{}^{A} = -2{}^{*}H_{[\mathbf{xy}]}{}^{B}L_{B}{}^{A}{}_{\mathbf{z}} + R_{\mathbf{xyz}}{}^{A} \qquad [2.3\text{-}50]$$

where $"R_{\mathbf{xyz}}{}^{\mathbf{w}}$ is the Riemann tensor of $\mathcal{T}$.

The remaining projections of the Riemann tensor which play an important role in the analysis of the field equations and need to be evaluated are $R_{\mathbf{x}AB}{}^{\mathbf{y}}$ and $R_{A\mathbf{xy}}{}^{A}$. In Appendix B, we show that

$$D_{\mathbf{x}}H_{AB}{}^{\mathbf{y}} = {}^{*}L_{\mathbf{x}}{}^{\mathbf{z}}{}_{A}{}^{*}L_{\mathbf{z}}{}^{\mathbf{y}}{}_{B} - D_{A}{}^{*}L_{\mathbf{x}}{}^{\mathbf{y}}{}_{B} + L_{A}{}^{C}{}_{\mathbf{x}}H_{CB}{}^{\mathbf{y}} + R_{\mathbf{x}AB}{}^{\mathbf{y}} \qquad [2.3\text{-}51]$$

We can express this in terms of the $\mathfrak{D}$-derivatives if we use the relations:

$$\mathfrak{D}_{A}{}^{*}L_{\mathbf{x}}{}^{\mathbf{y}}{}_{B} = D_{A}{}^{*}L_{\mathbf{x}}{}^{\mathbf{y}}{}_{B} - {}^{*}L_{\mathbf{z}}{}^{\mathbf{y}}{}_{B}{}^{*}L_{\mathbf{x}}{}^{\mathbf{z}}{}_{A} + {}^{*}L_{\mathbf{x}}{}^{\mathbf{z}}{}_{B}{}^{*}L_{\mathbf{z}}{}^{\mathbf{y}}{}_{A} \qquad [2.3\text{-}52]$$

and

$$\mathcal{D}_{\mathbf{x}} H_{AB}{}^{\mathbf{y}} = D_{\mathbf{x}} H_{AB}{}^{\mathbf{y}} - H_{CB}{}^{\mathbf{y}} L_A{}^C{}_{\mathbf{x}} - H_{AC}{}^{\mathbf{y}} L_B{}^C{}_{\mathbf{x}}; \qquad [2.3\text{-}53]$$

yielding

$$\mathcal{D}_{\mathbf{x}} H_{AB}{}^{\mathbf{y}} = -\mathcal{D}_A {}^{*}L_{\mathbf{x}}{}^{\mathbf{y}}{}_B + {}^{*}L_{\mathbf{x}}{}^{\mathbf{z}}{}_B {}^{*}L_{\mathbf{z}}{}^{\mathbf{y}}{}_A - H_{AC}{}^{\mathbf{y}} L_B{}^C{}_{\mathbf{x}} + R_{\mathbf{x}AB}{}^{\mathbf{y}} \qquad [2.3\text{-}54]$$

For the rigging extrinsic curvature

$$\mathcal{D}_C {}^{*}H_{\mathbf{xy}}{}^D = -\mathcal{D}_{\mathbf{x}} L_C{}^D{}_{\mathbf{y}} + L_C{}^E{}_{\mathbf{y}} L_E{}^D{}_{\mathbf{x}} - {}^{*}H_{\mathbf{xz}}{}^D {}^{*}L_{\mathbf{y}}{}^{\mathbf{z}}{}_C + R_{C\mathbf{xy}}{}^D \qquad [2.3\text{-}55]$$

These equations are also derived in Appendix B.

Metric Spaces

In a metric space, $L_A{}^B{}_{\mathbf{x}}$ and $H_{AB}{}^{\mathbf{x}}$ are related:

$$H^{\mathbf{x}}{}_{AB} = -B^{\mu}{}_A{}^{\nu}{}_B \nabla_{\mu} C^{\mathbf{x}}{}_{\nu} = -B^{\mu}{}_A{}^{\nu}{}_B g_{\nu\rho} \nabla_{\mu} (g^{\mathbf{xy}} C_{\mathbf{y}}{}^{\rho})$$

$$= -B^{\mu}{}_A{}^{\nu}{}_B g_{\nu\rho}\, g^{\mathbf{xy}} \nabla_{\mu} C_{\mathbf{y}}{}^{\rho} = -g_{BC} g^{\mathbf{xy}} B^{\mu}{}_A{}^C{}_{\rho} \nabla_{\mu} C_{\mathbf{y}}{}^{\rho}$$

$$= g_{BC}\, g^{\mathbf{xy}} L_A{}^C{}_{\mathbf{y}} \qquad [2.3\text{-}56]$$

Likewise, for the rigging extrinsic curvatures we have

$${}^{*}H_{\mathbf{xy}}{}^A = g^{AB} g_{\mathbf{yz}} {}^{*}L_{\mathbf{x}}{}^{\mathbf{z}}{}_B \qquad [2.3\text{-}57]$$

Since, in a metric space, $H_{AB}{}^{\mathbf{x}}$ and $L_A{}^B{}_{\mathbf{x}}$ (${}^{*}H_{\mathbf{xy}}{}^A$ and ${}^{*}L_{\mathbf{x}}{}^{\mathbf{y}}{}_A$) are related to each other by simply raising and lowering indices with the induced metric tensors, we drop any distinction between them and use the symbol $\mathbf{H}$ exclusively.

One can also define the <u>mean extrinsic curvature</u> $H^{\mathbf{x}}$ of $\mathcal{N}$ as the trace of $H^{\mathbf{x}}{}_{AB}$:

$$H^{\mathbf{x}} \equiv g^{AB}H^{\mathbf{x}}{}_{AB} \qquad [2.3\text{-}58]$$

The <u>mean extrinsic curvature</u> ${}^{*}H^{A}$ of $\mathcal{T}$ is the trace of $H^{\mathbf{x}}{}_{AB}$:

$${}^{*}H^{A} \equiv g^{\mathbf{xy}}\,{}^{*}H_{\mathbf{xy}}{}^{A} \qquad [2.3\text{-}59]$$

Just as the ordinary covariant derivatives of the induced metric tensors vanish, the mixed covariant derivatives of the metric tensors vanish. Since $''g_{\nu\kappa} = g_{\nu\kappa} - {}'g_{\nu\kappa}$ we have, from the definition of the mixed covariant derivative

$$D_A g_{\mathbf{xy}} \equiv B^{\mu}{}_{A} C^{\nu}{}_{\mathbf{x}}{}^{\kappa}{}_{\mathbf{y}} \nabla_{\mu}\, ''g_{\nu\kappa} = -B^{\mu}{}_{A} C^{\nu}{}_{\mathbf{x}}{}^{\kappa}{}_{\mathbf{y}} \nabla_{\mu}\, 'g_{\nu\kappa}$$

$$= B^{\mu}{}_{A} C^{\kappa}{}_{\mathbf{y}}\, 'g_{\nu\kappa} \nabla_{\mu} C^{\nu}{}_{\mathbf{x}} = 0 \qquad [2.3\text{-}60]$$

Simlarly, one can show that

$$D_A g^{\mathbf{xy}} = D_{\mathbf{x}} g_{AB} = D_{\mathbf{x}} g^{AB} = 0 \qquad [2.3\text{-}61]$$

Applying equation 2.3-34 to the metric tensor itself, one gets a relation the between $\mathfrak{D}$-derivative of the metric tensor and the extrinsic curvature tensor

$$D_{\mathbf{x}} g_{AB} = \mathfrak{D}_{\mathbf{x}} g_{AB} + H_{A}{}^{C}{}_{\mathbf{x}}\, g_{CB} + H_{B}{}^{C}{}_{\mathbf{x}}\, g_{AC} = 0 \qquad [2.3\text{-}62]$$

and thus

$$\mathfrak{D}_{\mathbf{x}} g_{AB} = -\, H_{A}{}^{C}{}_{\mathbf{x}}\, g_{CB} - H_{B}{}^{C}{}_{\mathbf{x}}\, g_{AC}$$

$$= -\, H_{AB\mathbf{x}} - H_{BA\mathbf{x}} = -\, 2H_{(AB)\mathbf{x}} = -\, 2H_{AB\mathbf{x}} \qquad [2.3\text{-}63]$$

The $\mathfrak{D}$-derivative of the contravariant metric tensor is related to the raised extrinsic curvature by

$$\mathcal{D}_{\mathbf{x}} g^{AB} = 2 g_{\mathbf{xy}} H^{AB\mathbf{y}} \qquad [2.3\text{-}64]$$

For the rigging space, we have

$$\mathcal{D}_{A} g_{\mathbf{xy}} = -2\,{}^{*}H_{(\mathbf{xy})A}$$

and

$$\mathcal{D}_{A} g^{\mathbf{xy}} = 2\,{}^{*}H^{(\mathbf{xy})}{}_{A} \qquad [2.3\text{-}65]$$

The action of $\mathcal{D}_{\mathbf{x}}$ on the conformal two-geometry will now be considered. Since the $\mathcal{D}$-derivatives of the Levi-Civita tensor densities vanish, we can calculate the action of $\mathcal{D}_{\mathbf{x}}$ on the conformal scale factor:

$$\mathcal{D}_{\mathbf{x}} \gamma^2 = \varepsilon^{AB}\varepsilon^{CD} g_{AC} \mathcal{D}_{\mathbf{x}}\, g_{BD} = \gamma^2 g^{BD}\, \mathcal{D}_{\mathbf{x}} g_{BD} = -2\gamma^2 g^{BD}\; g_{\mathbf{xy}} H_{BD}{}^{\mathbf{y}}$$

$$= -2\gamma^2 g_{\mathbf{xy}} H^{\mathbf{y}} \qquad [2.3\text{-}66]$$

$$\mathcal{D}_{\mathbf{x}} \gamma = -\gamma g_{\mathbf{xy}} H^{\mathbf{y}} \qquad [2.3\text{-}67]$$

$$D_{\mathbf{x}} \gamma^{-1} = \gamma^{-1} g_{\mathbf{xy}} H^{\mathbf{y}} \qquad [2.3\text{-}68]$$

This relates the $\mathcal{D}$-derivative of the conformal scale factor to the mean extrinsic curvature of the two-surface.

The action of $\mathcal{D}_{\mathbf{x}}$ on $\tilde{g}_{AB}$ is

$$\mathcal{D}_{\mathbf{x}} \tilde{g}_{AB} = \mathcal{D}_{\mathbf{x}}(\gamma^{-1} g_{AB}) = \gamma^{-1}\mathcal{D}_{\mathbf{x}} g_{AB} + g_{AB}\mathcal{D}_{\mathbf{x}}\gamma^{-1}$$

$$= -2\gamma^{-1} g_{\mathbf{xy}} H_{AB}{}^{\mathbf{y}} + g_{AB}\gamma^{-1} g_{\mathbf{xy}} H^{\mathbf{y}}$$

$$= -2\gamma^{-1} g_{\mathbf{xy}} (H_{AB}{}^{\mathbf{y}} - 1/2\; g_{AB} H^{\mathbf{y}})$$

$$= -2\gamma^{-1} g_{\mathbf{xy}} \Lambda^{C}{}_{A}{}^{D}{}_{B} H_{CD}{}^{\mathbf{y}}$$

$$= -2\gamma^{-1} g_{\mathbf{xy}} \overline{H}_{CD}{}^{\mathbf{y}}.$$

$$= \gamma^{-1}\Lambda^{C}{}_{A}{}^{D}{}_{B}\mathbf{D}_{\mathbf{x}}g_{AB} \qquad [2.3\text{-}69]$$

where $\Lambda^{C}{}_{A}{}^{D}{}_{B} \equiv \delta^{C}{}_{A}\delta^{D}{}_{B} - 1/2g_{AB}g^{CD}$ is the trace-removing operator on two-surface quantities and $\overline{H}_{CD}{}^{\mathbf{y}} \equiv \Lambda^{C}{}_{A}{}^{D}{}_{B}H_{CD}{}^{\mathbf{y}}$ is the traceless part of $H_{CD}{}^{\mathbf{y}}$.

Sometimes it is more convenient to use the tensor density

$$\tilde{H}^{\mathbf{y}}{}_{AB} = \gamma^{-1}(H_{AB}{}^{\mathbf{y}} - 1/2\ g_{AB}H^{\mathbf{y}}) = \gamma^{-1}\Lambda^{C}{}_{A}{}^{D}{}_{B}\ H_{CD}{}^{\mathbf{y}}$$
$$= \gamma^{-1}\overline{H}_{CD}{}^{\mathbf{y}} \qquad [2.3\text{-}70]$$

which yields

$$\mathbf{D}_{\mathbf{x}}\tilde{g}_{AB} = -2g_{\mathbf{xy}}\tilde{H}_{AB}{}^{\mathbf{y}} \qquad [2.3\text{-}71]$$

The traceless tensor density $\tilde{H}^{\mathbf{y}}{}_{AB}$ is related to the conformal two-metric in the same way that the extrinsic curvature is related to the regular two-metric. We call $\tilde{H}^{\mathbf{y}}{}_{AB}$ the <u>conformal extrinsic curvature</u> tensor.

Defining the raised conformal metric

$$\tilde{g}^{AB} = \gamma g^{AB} \qquad [2.3\text{-}72]$$

we have

$$\mathbf{D}_{\mathbf{x}}\tilde{g}^{AB} = 2\gamma g_{\mathbf{xy}}(H^{AB\mathbf{y}} - 1/2g^{AB}H^{\mathbf{y}}) \qquad [2.3\text{-}73]$$
$$= 2g_{\mathbf{xy}}\tilde{H}^{AB\mathbf{y}},$$

where $\tilde{H}^{\mathbf{y}AB} = \gamma\Lambda^{A}{}_{C}{}^{B}{}_{D}\ H^{CD\mathbf{y}}$ [2.3-74]

The mean extrinsic curvature of the rigging space is related to the determinant of the rigging metric by

$$^{*}H_{A} \equiv -1/(2\rho^{2})\mathbf{D}_{A}\rho^{2}$$
$$\equiv -(1/\rho)\mathbf{D}_{A}\rho \qquad [2.3\text{-}75]$$

We turn to equation 2.3-54, to consider how it looks in a metric space. Operate on both sides with $g_{\mathbf{xy}}$ to get (recalling $\mathfrak{D}_{\mathbf{w}} g_{\mathbf{xy}} = 0$)

$$\mathfrak{D}_{\mathbf{x}} H_{AB\mathbf{y}} = - g_{\mathbf{yz}} \mathfrak{D}_A {}^{\star}H_{\mathbf{x}}{}^{\mathbf{z}}{}_B + {}^{\star}H_{\mathbf{x}}{}^{\mathbf{z}}{}_B {}^{\star}H_{\mathbf{zy}A} - H_{AC\mathbf{y}} H_B{}^C{}_{\mathbf{x}} + R_{\mathbf{x}AB\mathbf{y}}$$

[2.3-76]

Substituting $H_{AB\mathbf{y}} = -1/2 \mathfrak{D}_{\mathbf{y}} g_{AB}$ we get

$$-1/2 \mathfrak{D}_{\mathbf{x}} \mathfrak{D}_{\mathbf{y}} g_{AB} = R_{\mathbf{x}AB\mathbf{y}} - g_{\mathbf{yz}} \mathfrak{D}_A {}^{\star}H_{\mathbf{x}}{}^{\mathbf{z}}{}_B + {}^{\star}H_{\mathbf{x}}{}^{\mathbf{z}}{}_B {}^{\star}H_{\mathbf{zy}A} - H_{AC\mathbf{y}} H_B{}^C{}_{\mathbf{x}}$$

[2.3-77]

or

$$R_{\mathbf{x}AB\mathbf{y}} = -1/2 \mathfrak{D}_{\mathbf{x}} \mathfrak{D}_{\mathbf{y}} g_{AB} + g_{\mathbf{yz}} \mathfrak{D}_A {}^{\star}H_{\mathbf{x}}{}^{\mathbf{z}}{}_B - {}^{\star}H_{\mathbf{x}}{}^{\mathbf{z}}{}_B {}^{\star}H_{\mathbf{zy}A} + H_{AC\mathbf{y}} H_B{}^C{}_{\mathbf{x}}$$

[2.3-78]

which is the desired result.

In the next chapter, we shall use the expressions for the projections derived in this section to write out the Two+Two formulation of the Einstein Field equations.

CHAPTER III

THE EINSTEIN FIELD EQUATIONS

3.1 Measurability of the Conformal Two-Metric

The conformal two-metric is a directly measurable quantity. Consider a small object of known size and shape and an observer at a distance from the object and consider a bundle of light rays from the object that are view by the observer. The measured shape is the projection on a two-dimensional screen in the instantaneous rest-frame of the observer and orthogonal to the trajectories of the bundle of rays. We now relate the measured shape to the conformal two-metric at the object's location.

We proceed by constructing an appropriate coordinate system based upon the trajectory of the observer $x^\mu(\tau)$. For simpplicity, we take the tangent vector **u** to the observer's trajectory

$$u^\mu \equiv dx^\mu/d\tau \qquad [3.1-1]$$

to be a future-pointing unit time-like geodesic. Thus

$$u^\mu u_\mu = -1 \qquad [3.1-2]$$

$$u^\mu \nabla_\mu u^\nu = 0 \qquad [3.1-3]$$

Then we can take τ to be the proper time along the observer's trajectory. At any instant of the observer's time, say τ_0, the image of the object consists of the endpoints of a bundle of null geodesics which intercept the trajectory of the observer at the event $x^\mu(\tau_0)$ and therefore lie on the past null cone of that event. We can then define a function w which is constant on the null cone and whose value is given by τ, i.e. $w = \tau$. A set of null cones, each emanating from the trajectory of the observer, can be used to define a local coordinate w.

Two more coordinates are defined in the following manner. Let $\{\mathbf{u}, {}^{\perp}\mathbf{e}_i;\ i=1,3\}$ represent an orthogonal tetrad which is parallel propagated along the world line of the observer. The spacelike triad $\{{}^{\perp}\mathbf{e}_i\}$ is orthogonal to $\mathbf{u}$ and maintains this relation due to the parallel propagation, defining a non-rotating reference frame. (The assumption of a geodesic trajectory makes the propagation of the non-rotating frame simple; otherwise we would have to resort to Fermi-Walker transport). At each instant of proper time, we can use $\{{}^{\perp}\mathbf{e}_i\}$ to define a spherical coordinate system (θ, ϕ). Each null geodesic can be labelled by the measured value of (θ, ϕ) as it strikes the observer. The (θ, ϕ)-coordinate system can be extended along the null geodesics by defining them to be constant along the null geodesics. This can be done at every instant of proper time.

A fourth and final coordinate y is needed and there are many ways in which it can be defined. A simple choice is the preferred affine parameter v of the null geodesics normalized in such a way as to be zero at the observer's position and also satisfying $u^{\mu} w_{\mu} = -1$. A second way is to use the so-called area or luminosity distance r. Another way is to use either one of these prescriptions on the initial null cone but define y off the null surface in some other way. The final form of the coordinate system (x^{μ}) is

$$x^0 = w,\ x^1 = y,\ x^2 = \theta,\ x^3 = \phi$$

In the coordinate system so described, the metric tensor has the general form

$$g_{\mu\nu} = \begin{bmatrix} \alpha & \beta & v_2 & v_3 \\ \beta & 0 & 0 & 0 \\ v_2 & 0 & g_{22} & g_{23} \\ v_3 & 0 & g_{23} & g_{33} \end{bmatrix} \qquad [3.1\text{-}4]$$

The submatrix

$$g_{AB} = \begin{bmatrix} g_{22} & g_{23} \\ g_{23} & g_{33} \end{bmatrix} \qquad [3.1\text{-}5]$$

is just the induced two-metric on the family of two-surfaces given by w = constant and y = constant and its determinant is

$$\gamma^2 = \det[g_{AB}] = g_{22}g_{33} - (g_{23})^2 \qquad [3.1\text{-}6]$$

The <u>conformal two-metric</u>, $\tilde{g}_{AB}$, is defined as in Section 2.1

$$\tilde{g}_{AB} = g_{AB}/\gamma \qquad (A,B = 2,3) \qquad [3.1\text{-}7]$$

such that

$$\det[\tilde{g}_{AB}] = 1 \qquad [3.1\text{-}8]$$

In this coordinate system,

$$dl^2 = g_{AB}(w_0, y, \theta_0, \phi_0)\, dx^A dx^B \qquad [3.1\text{-}9]$$

is the distance between two infinitesimally close points on an object of known size and shape, called a "standard figure", observed at proper time τ_0, and at a distance <u>y</u> down the null geodesic in the direction (θ_0, ϕ_0). dl^2 represents a known quantity; But dx^A represents the observed angular displacements which are measurable at the observer's location. g_{AB} can thus be calculated from a sufficient number of measurements of dx^A, using the known values of dl^2.

The area distance <u>r</u> is defined as the ratio of the cross-sectional area of the object dS_0 at the source to the observed solid angle $d\Omega$ at the observer

$$dS_0 = r^2 d\Omega_0 \qquad [3.1\text{-}10]$$

and is readily measurable. From this definition, one can show that r satisfies

$$r^4 \sin^2\theta = \gamma^2 \qquad [3.1\text{-}11]$$

and, therefore, γ is measurable as well.

The image suffers some distortion as it propagates along the light path. The measure of distortion is given by the difference between g_{AB}/r^2 (= $\sin\theta\, \tilde{g}_{AB}/\gamma$) and the metric of the unit sphere. Hence, by calculating this distortion, $\tilde{g}_{AB}$ is directly measurable. Such an interpretation of the conformal two-metric has been given by Ellis et al [1985] in considering the maximum amount of information about the structure of the universe that can be obtained from astronomical observations.

The arguments presented above can be modified to measure the conformal two-structure on a null hypersurface, such as the one used in Section 1.3. Here, we consider a null hypersurface filled up with standard figures. At any point on a two-dimensional cross-section of the null hypersurface (a large screen!), there will appear a one-parameter family of distorted images (the parameter being r, the area distance to the source). The collection of images over the whole two-surface is equivalent to the conformal two-structure.

3.2 The Derivation of the Einstein Field Equations Using a Palatini Variational Principle

It is well-known that Einstein's field equations can be derived from a variational principle. The Hilbert action for the gravitational field is

$$I = \int \sqrt{-g}\,(R + L_F)\,d^4x \qquad [3.2\text{-}1]$$

$$= \int (\sqrt{-g}\,g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}\,L_F)\,d^4x \qquad [3.2\text{-}2]$$

where $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar of space-time and $\sqrt{-g}\,L_F$ is the source Lagrangian density. We shall use a Palatini-type variational principle, where the Ricci tensor is constructed from an affine connection which is varied independently of the metric tensor (see, for example, Misner, Thorne and Wheeler [1973], p.492). By varying $g_{\mu\nu}$ we derive the field equations. The variation of the affine connection gives not only the relations between the induced metric tensors and induced affine connections on both $\mathcal{N}$ and $\mathcal{T}$ (proved in Section 2.2) but also the relationships between the extrinsic curvature tensors and the induced metrics (proved in Section 2.3). We assume minimal coupling between the sources and the gravitational field, i.e. $\sqrt{-g}\,L_F$ depends only on $g_{\mu\nu}$ and not on its derivatives nor on the affine connection.

We consider an arbitrary four-dimensional region of space-time. The fields are varied in the interior of the region but not on its boundary. The total variation, due to the variations of the metric and the affine connection, is

$$\delta I = \int(\delta\sqrt{-g}\ g^{\mu\nu} + \sqrt{-g}\delta g^{\mu\nu})\,R_{\mu\nu}\,d^4x\ +$$

$$\int(\delta\sqrt{-g}\ L_F + \sqrt{-g}\,(\delta L_F/\delta g^{\mu\nu})\ \ \delta g^{\mu\nu}\ d^4x$$

$$+ \int\sqrt{-g}\ g^{\mu\nu}\delta R_{\mu\nu}(\Gamma)\,d^4x \qquad [3.2\text{-}3]$$

It is not hard to show that

$$\delta\sqrt{-g} = -1/2\ \sqrt{-g}\ \ g_{\alpha\beta}\delta g^{\alpha\beta} \qquad [3.2\text{-}4]$$

so that equation 3.2-3 becomes

$$\delta I = \int(\sqrt{-g}\,(R_{\mu\nu} - 1/2g_{\mu\nu}R + \delta L_F/\delta g^{\mu\nu} - 1/2g_{\mu\nu}L_F\,)\,\delta g^{\mu\nu}d^4x$$

$$+ \int\sqrt{-g}\ \ g^{\mu\nu}\delta R_{\mu\nu}(\Gamma)\,d^4x \qquad [3.2\text{-}5]$$

If we define the canonical stress-energy tensor by

$$T_{\mu\nu} = \delta L_F/\delta g^{\mu\nu} - 1/2g_{\mu\nu}L_F$$

then the variation in the action can be written

$$\delta I = \int(\sqrt{-g}\,(G_{\mu\nu} - T_{\mu\nu})\,\delta g^{\mu\nu}d^4x + \int\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}(\Gamma)\,d^4x \qquad [3.2\text{-}6]$$

Using the decomposition of the metric tensor and the definition of the conformal two-metric developed in Section 2.2, we can write the metric tensor in the form

$$g^{\mu\nu} = \tilde{g}^{AB}\,\gamma^{-1}B^{\mu}{}_{A}{}^{\nu}{}_{B} + g^{\mathbf{xy}}C_{\mathbf{x}}{}^{\mu}{}_{\mathbf{y}}{}^{\nu} \qquad [3.2\text{-}7]$$

The variation $\delta g^{\mu\nu}$ can be expressed in terms of variations of $\tilde{g}^{AB}$, γ, $g^{\mathbf{xy}}$ and $C_{\mathbf{x}}^{\mu}$. $B^{\mu}{}_{A}$ is not varied since we leave the families of two-surfaces unchanged during the variations.

The variation of the connecting quantities has the general form

$$\delta B^{\mu}{}_{A} = 0 \qquad [3.2\text{-}8]$$

$$C_{\mathbf{x}}{}^{\mu} \rightarrow C_{\mathbf{x}}{}^{\mu} + \delta C_{\mathbf{x}}{}^{\mu} \qquad [3.2\text{-}9]$$

$$C^{\mathbf{x}}{}_{\mu} \rightarrow C^{\mathbf{x}}{}_{\mu} + \delta C^{\mathbf{x}}{}_{\mu} \qquad [3.2\text{-}10]$$

$$B^{A}{}_{\mu} \rightarrow B^{A}{}_{\mu} + \delta B^{A}{}_{\mu} \qquad [3.2\text{-}11]$$

The fact that the varied connecting quantities must maintain the same non-metrical duality relations (see Section 2.1) as the original quantities gives us 16 conditions that need to be satisfied by the 24 variations $\delta B^{A}{}_{\mu}$, $\delta C^{\mathbf{x}}{}_{\mu}$, and $\delta C_{\mathbf{x}}{}^{\mu}$.

The first duality condition is

$$B^{\mu}{}_{A}(C^{\mathbf{x}}{}_{\mu} + \delta C^{\mathbf{x}}{}_{\mu}) = 0 \qquad [3.2\text{-}12]$$

which implies that $\delta C^{\mathbf{x}}{}_{\mu}$ belongs to the rigging plane. We therefore define a set of four infinitesimal quantities $c^{\mathbf{x}}{}_{\mathbf{y}}$ by

$$c^{\mathbf{x}}{}_{\mathbf{y}} = \delta C^{\mathbf{x}}{}_{\mu} C^{\mu}{}_{\mathbf{y}} \qquad [3.2\text{-}13]$$

so that we can express

$$\delta C^{\mathbf{x}}{}_{\mu} = c^{\mathbf{x}}{}_{\mathbf{y}} C^{\mathbf{y}}{}_{\mu} \qquad [3.2\text{-}14]$$

The second duality condition is

$$B^{\mu}{}_{A}(B^{B}{}_{\mu} + \delta B^{B}{}_{\mu}) = \delta_{A}{}^{B} \qquad [3.2\text{-}15]$$

which implies that $\delta B^B{}_\mu$ also belongs to the rigging space; so we define four more infinitesimal quantities

$$b^A{}_{\mathbf{y}} = C_{\mathbf{y}}{}^\mu \delta B^B{}_\mu \qquad [3.2\text{-}16]$$

so that

$$\delta B^A{}_\mu = b^A{}_{\mathbf{y}} C^{\mathbf{y}}{}_\mu \qquad [3.2\text{-}17]$$

Third duality condition yields nothing new, while the last one gives

$$\delta C_{\mathbf{y}}{}^\mu = - B^\mu{}_A b^A{}_{\mathbf{y}} - c^{\mathbf{x}}{}_{\mathbf{y}} C_{\mathbf{x}}{}^\mu \qquad [3.2\text{-}18]$$

Summarizing, the variation of the connecting quantities is determined by the eight quantities, $c^{\mathbf{x}}{}_{\mathbf{y}}$ and $b^A{}_{\mathbf{y}}$ whereby

$$\delta C^{\mathbf{y}}{}_\mu = c^{\mathbf{y}}{}_{\mathbf{x}} C^{\mathbf{x}}{}_\mu \qquad [3.2\text{-}19]$$

$$\delta B^A{}_\mu = b^A{}_{\mathbf{x}} C^{\mathbf{x}}{}_\mu \qquad [3.2\text{-}20]$$

$$\delta C_{\mathbf{y}}{}^\mu = - B^\mu{}_A b^A{}_{\mathbf{y}} - c^{\mathbf{x}}{}_{\mathbf{y}} C_{\mathbf{x}}{}^\mu \qquad [3.2\text{-}21]$$

We also assume that under the variation, the rigging vectors maintain their orthogonality to the two-surfaces.

We have thus shown that the variation $\delta g^{\mu\nu}$ takes the form

$$\delta g^{\mu\nu} = \delta \tilde{g}^{AB} \gamma^{-1} B^\mu{}_A{}^\nu{}_B - \delta\gamma g^{AB}\gamma^{-2} B^\mu{}_A{}^\nu{}_B + \delta g^{\mathbf{xy}} C_{\mathbf{x}}{}^\mu{}_{\mathbf{y}}{}^\nu$$
$$+ g^{\mathbf{xy}} \delta C_{\mathbf{x}}{}^\mu C_{\mathbf{y}}{}^\nu + g^{\mathbf{xy}} C_{\mathbf{x}}{}^\mu \delta C_{\mathbf{y}}{}^\nu \qquad [3.2\text{-}22]$$

Inserting this form into the first variational term in the integrand of equation 3.2-5, we get

$$(G_{\mu\nu} - T_{\mu\nu})\delta g^{\mu\nu} = \delta\tilde{g}^{AB}\gamma^{-1}(G_{AB} - T_{AB}) - \delta\gamma(G_{AB} - T_{AB})g^{AB}$$
$$+ \delta g^{\mathbf{xy}}(G_{\mathbf{xy}} - T_{\mathbf{xy}}) - 2g^{\mathbf{xy}}(B^{\mu}{}_{A}b^{A}{}_{\mathbf{x}} + c^{\mathbf{z}}{}_{\mathbf{x}}C_{\mathbf{z}}{}^{\mu})C_{\mathbf{y}}{}^{\nu}(G_{\mu\nu} - T_{\mu\nu})$$

$$= \delta\tilde{g}^{AB}\gamma^{-1}(G_{AB} - T_{AB}) - \delta\gamma(G_{AB} - T_{AB})g^{AB}$$
$$+ \delta g^{\mathbf{xy}}(G_{\mathbf{xy}} - T_{\mathbf{xy}}) - 2b^{A}{}_{\mathbf{x}}g^{\mathbf{xy}}(G_{A\mathbf{y}} - T_{A\mathbf{y}})$$
$$- 2c^{\mathbf{z}}{}_{\mathbf{x}}g^{\mathbf{xy}}(G_{\mathbf{xy}} - T_{\mathbf{xy}}) \qquad [3.2\text{-}23]$$

To the first term on the right hand side of equation 3.2-23 we can add the term

$$-1/2\delta\tilde{g}^{AB}g_{AB}g^{CD}(G_{CD} - T_{CD})\gamma^{-1},$$

since this term is identically zero because $\delta\tilde{g}^{AB}g_{AB}$ is proportional to $\delta(\det[\tilde{g}_{AB}])$, which vanishes. Equation 3.2-23 then becomes:

$$(G_{\mu\nu} - T_{\mu\nu})\delta g^{\mu\nu} = \delta\tilde{g}^{AB}[\gamma^{-1}(G_{AB} - 1/2g_{AB}g^{CD}G_{CD})$$
$$- \gamma^{-1}(T_{AB} - 1/2g_{AB}g^{CD}T_{CD})]$$
$$- \delta\gamma(G_{AB} - T_{AB})g^{AB}$$
$$- 2b^{A}{}_{\mathbf{x}}(G_{A}{}^{\mathbf{x}} - T_{A}{}^{\mathbf{x}}) + [\delta g^{\mathbf{xy}} - 2c^{\mathbf{xy}}](G_{\mathbf{xy}} - T_{\mathbf{xy}}) \qquad [3.2\text{-}24]$$

The variation of the following quantities gives rise to the corresponding field equations:

$$\delta\tilde{g}^{AB} \quad \Rightarrow \quad \tilde{G}_{AB} = \tilde{T}_{AB} \qquad [3.2\text{-}25]$$

$$\delta\gamma \quad \Rightarrow \quad G_{AB}g^{AB} = T_{AB}g^{AB} \qquad [3.2\text{-}26]$$

$$[\delta g^{xy} - 2c^{(xy)}] \Rightarrow G_{xy} = T_{xy} \qquad [3.2\text{-}27]$$

$$b^{A}{}_{x} \Rightarrow G_{A}{}^{x} = T_{A}{}^{x} \qquad [3.2\text{-}28]$$

(Note that the variation of the rigging metric is not independent of the variation of the rigging vectors).

We now investigate the variation in the action due to the variation of Γ. The contribution of this to the action is

$$\delta I_{\Gamma} = \int \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu}(\Gamma)\, d^4x \qquad [3.2\text{-}29]$$

The Riemann tensor in a holonomic coordinate system takes the form

$$R_{\lambda\mu\nu}{}^{\kappa} = 2\partial_{[\lambda}\Gamma^{\kappa}{}_{\mu]\nu} + 2\Gamma^{\kappa}{}_{[\lambda|\rho|}\Gamma^{\rho}{}_{\mu]\nu} \qquad [3.2\text{-}30]$$

and the Ricci tensor is

$$R_{\mu\nu} = 2\partial_{[\lambda}\Gamma^{\lambda}{}_{\mu]\nu} + 2\Gamma^{\lambda}{}_{[\lambda|\rho|}\Gamma^{\rho}{}_{\mu]\nu} \qquad [3.2\text{-}31]$$

The variation in the Ricci tensor due to the variation in Γ is

$$\delta R_{\mu\nu} = 2\partial_{[\lambda}\delta\Gamma^{\lambda}{}_{\mu]\nu} + 2\delta\Gamma^{\lambda}{}_{[\lambda|\rho|}\Gamma^{\rho}{}_{\mu]\nu} + 2\Gamma^{\lambda}{}_{[\lambda|\rho|}\delta\Gamma^{\rho}{}_{\mu]\nu} \qquad [3.2\text{-}32]$$

The variation $\delta\Gamma^{\lambda}{}_{\mu\nu}$ is a tensor since the difference between two affine connections is a tensor. It is immediately seen that

$$\delta R_{\mu\nu} = \nabla_{\lambda}\delta\Gamma^{\lambda}{}_{\mu\nu} - \nabla_{\mu}\delta\Gamma^{\lambda}{}_{\lambda\nu} \qquad [3.2\text{-}33]$$

The variational integral thus becomes

$$\delta I_{\Gamma} = \int \sqrt{-g}\, g^{\mu\nu} [\nabla_{\lambda}\delta\Gamma^{\lambda}{}_{\mu\nu} - \nabla_{\mu}\delta\Gamma^{\lambda}{}_{\lambda\nu}]\, d^4x \qquad [3.2\text{-}34]$$

Integrating by parts to bring the metric tensor inside the covariant derivative,

$$\delta I_\Gamma = \int \{ [\nabla_\lambda (\delta\Gamma^\lambda{}_{\mu\nu} \sqrt{-g}\; g^{\mu\nu}) - \nabla_\mu (\delta\Gamma^\lambda{}_{\lambda\nu} \sqrt{-g}\; g^{\mu\nu})]$$

$$- [\delta\Gamma^\lambda{}_{\mu\nu} \nabla_\lambda (\sqrt{-g}\; g^{\mu\nu}) - \delta\Gamma^\lambda{}_{\lambda\nu} \nabla_\mu (\sqrt{-g}\; g^{\mu\nu})] \} \, d^4x \, .$$

[3.2-35]

The first and second terms of this equation vanish, because they are divergences; and, by Gauss's theorem, can be written as boundary variations, which vanish by assumption.

Equation 3.2-35 can thus be written as

$$\delta I_\Gamma = -\int [\delta\Gamma^\lambda{}_{\mu\nu} \nabla_\lambda (\sqrt{-g}\; g^{\mu\nu}) - \delta\Gamma^\lambda{}_{\lambda\nu} \nabla_\mu (\sqrt{-g}\; g^{\mu\nu})] \} \, d^4x$$

[3.2-36]

After some index manipulation, this becomes

$$\delta I_\Gamma = -\int \delta\Gamma^\lambda{}_{\mu\nu} [\nabla_\lambda (\sqrt{-g}\; g^{\mu\nu}) - \delta^{\mu}_{\lambda} \nabla_\rho (\sqrt{-g}\; g^{\rho\nu})] \} \, d^4x$$

[3.2-37]

The vanishing of the action leads to the vanishing of the coefficient of the variation $\delta\Gamma^\lambda{}_{\mu\nu}$, i.e.,

$$\nabla_\lambda (\sqrt{-g}\; g^{\mu\nu}) - \delta^{(\mu}{}_\lambda \nabla_\rho (\sqrt{-g}\; g^{|\rho|\nu)}) = 0 \qquad [3.2\text{-}38]$$

The unique solution of this equation is the vanishing of

$$\nabla_\lambda g^{\mu\nu} = 0 \qquad [3.2\text{-}39]$$

This is the usual result of the Palatini variational principle for Einstein's equations: the four-dimensional affine connection is the Christoffel symbol of the four-dimensional metric tensor. We shall not use this result directly, but we shall use the fact that the Riemann tensor now has all the algebraic symmetries it has in a metric space.

Our approach is to consider all projections of 3.2-39 into the surface and rigging directions. The complete set of projections of 3.2-39 is

$$1.\ B^{\mu}{}_{A}{}^{\nu}{}_{B}{}^{\kappa}{}_{C}\nabla_{\mu}g_{\nu\kappa} = B^{\mu}{}_{A}{}^{\nu}{}_{B}{}^{\kappa}{}_{C}\nabla_{\mu}({}'g_{\nu\kappa} + {}''g_{\nu\kappa})$$

$$= B^{\mu}{}_{A}{}^{\nu}{}_{B}{}^{\kappa}{}_{C}\nabla_{\mu}{}'g_{\nu\kappa} = D_{A}g_{BC} = 0 \qquad [3.2\text{-}40]$$

This proves that the induced connection of $\mathcal{N}$ is the metric connection, as we showed directly in Section 2.2.

$$2.\ B^{\mu}{}_{A}{}^{\nu}{}_{B}C_{\mathbf{x}}{}^{\kappa}\nabla_{\mu}g_{\nu\kappa} = B^{\mu}{}_{A}{}^{\nu}{}_{B}C_{\mathbf{x}}{}^{\kappa}\nabla_{\mu}({}'g_{\nu\kappa} + {}''g_{\nu\kappa})$$

$$= -B^{\mu}{}_{A}{}^{\nu}{}_{B}{}'g_{\nu\kappa}\nabla_{\mu}C_{\mathbf{x}}{}^{\kappa} - B^{\mu}{}_{A}C_{\mathbf{x}}{}^{\kappa}{}''g_{\nu\kappa}\nabla_{\mu}B^{\nu}{}_{B}$$

$$= -B^{\mu}{}_{A}C_{\kappa}\, g_{BC}\nabla_{\mu}C_{\mathbf{x}}{}^{\kappa} - B^{\mu}{}_{A}C^{\mathbf{y}}{}_{\nu}g_{\mathbf{xy}}\nabla_{\mu}B^{\nu}{}_{B} = 0$$

$$= L_{A}{}^{C}{}_{\mathbf{x}}g_{BC} - H_{AB}{}^{\mathbf{y}}g_{\mathbf{xy}} = 0 \qquad [3.2\text{-}41]$$

This proves that the **H**'s and **L**'s are related as we show directly in Section 2.3.

$$3.\ B^{\mu}{}_{A}C_{\mathbf{x}}{}^{\nu}{}_{\mathbf{y}}{}^{\kappa}\nabla_{\mu}g_{\nu\kappa} = B^{\mu}{}_{A}C_{\mathbf{x}}{}^{\nu}{}_{\mathbf{y}}{}^{\kappa}\nabla_{\mu}({}'g_{\nu\kappa} + {}''g_{\nu\kappa})$$

$$= B^{\mu}{}_{A}C_{\mathbf{x}}{}^{\nu}{}_{\mathbf{y}}{}^{\kappa}\nabla_{\mu}{}''g_{\nu\kappa} = D_{A}g_{\mathbf{xy}} = 0 \qquad [3.2\text{-}42]$$

which implies, as we also saw in Section 2.3, that

$$\mathbf{D}_{A}g_{\mathbf{xy}} = -2\,{}^{*}H_{\mathbf{xy}A} \qquad [3.2\text{-}43]$$

The following results will only be stated since their proofs are almost identical to those for equations 3.2-40 through 3.2-43.

4. $C^{\mu}{}_{\mathbf{x}}{}^{\nu}{}_{\mathbf{y}}{}^{\kappa}{}_{\mathbf{z}}\nabla_{\mu} g_{\nu\kappa} = D_{\mathbf{x}} g_{\mathbf{yz}} = 0$ [3.2-44]

5. $C^{\mu}{}_{\mathbf{x}}{}^{\nu}{}_{\mathbf{y}}B_{A}{}^{\kappa}\nabla_{\mu} g_{\nu\kappa} = {}^{*}L_{\mathbf{x}}{}^{\mathbf{z}}{}_{A} g_{\mathbf{zy}} - {}^{*}H_{\mathbf{xy}}{}^{B} g_{BA} = 0$ [3.2-45]

6. $C^{\mu}{}_{\mathbf{x}}B_{A}{}^{\nu}{}_{B}{}^{\kappa}\nabla_{\mu} g_{\nu\kappa} = D_{\mathbf{x}} g_{AB} = 0$ [3.2-46]

which implies

$$\mathfrak{D}_{\mathbf{x}} g_{AB} = -2H_{AB\mathbf{x}} \qquad [3.2\text{-}47]$$

Using the results just proved, we can proceed to develop the 2+2 decomposition of the field equations. As we saw in Section 2.3, the independent projections of the Ricci tensor can be written in terms of two+two projections of the Riemann tensor:

$$R_{AB} = B^{\mu}{}_{A}{}^{\nu}{}_{B} R_{\alpha\mu\nu}{}^{\alpha} = R_{\mathbf{x}AB}{}^{\mathbf{x}} + R_{CAB}{}^{C} \qquad [3.2\text{-}48]$$

$$R_{\mathbf{x}A} = C_{\mathbf{x}}{}^{\mu}B^{\nu}{}_{A}\, R_{\alpha\mu\nu}{}^{\alpha} = R_{\mathbf{zx}A}{}^{\mathbf{z}} + R_{C\mathbf{x}A}{}^{C} \qquad [3.2\text{-}49]$$

$$R_{\mathbf{xy}} = C_{\mathbf{x}}{}^{\mu}{}_{\mathbf{y}}{}^{\nu} R_{\alpha\mu\nu}{}^{\alpha} = R_{\mathbf{zxy}}{}^{\mathbf{z}} + R_{C\mathbf{xy}}{}^{C} \qquad [3.2\text{-}50]$$

The Ricci scalar becomes:

$$R = g^{\mu\nu}R_{\mu\nu} = g^{AB}R_{AB} + g^{\mathbf{xy}}R_{\mathbf{xy}}$$

$$= g^{AB}R_{\mathbf{x}AB}{}^{\mathbf{x}} + g^{AB}R_{CAB}{}^{C} + g^{\mathbf{xy}}R_{\mathbf{zxy}}{}^{\mathbf{z}} + g^{\mathbf{xy}}R_{C\mathbf{xy}}{}^{C} \qquad [3.2\text{-}51]$$

The first set of field equations to consider is the totally surface-projected field equations. The surface-projection of the Einstein tensor is

$$G_{AB} = R_{AB} - 1/2\ g_{AB}R \qquad [3.2\text{-}52]$$

which, in terms of the projections of the Riemann tensor, becomes

$$\begin{aligned} G_{AB} &= R_{\mathbf{x}AB}{}^{\mathbf{x}} + R_{CAB}{}^{C} \\ &\quad - 1/2\ g_{AB}(g^{CD}\ R_{\mathbf{x}CD}{}^{\mathbf{x}} + g^{CD}\ R_{ECD}{}^{E} + g^{\mathbf{xy}}R_{\mathbf{zxy}}{}^{\mathbf{z}} + g^{\mathbf{xy}}\ R_{C\mathbf{xy}}{}^{C}) \\ &= \Lambda_{A}{}^{C}{}_{B}{}^{D}(R_{\mathbf{x}CD}{}^{\mathbf{x}} + R_{ECD}{}^{E}) - 1/2\ g_{AB}(g^{\mathbf{xy}}R_{\mathbf{zxy}}{}^{\mathbf{z}} + g^{\mathbf{xy}}R_{C\mathbf{xy}}{}^{C}) \end{aligned} \qquad [3.2\text{-}53]$$

The traceless part of equation 3.2-53 forms the left-hand side of the field equation 3.2-25

$$\widetilde{G}_{AB} \equiv \Lambda_{A}{}^{C}{}_{B}{}^{D}G_{CD} = \Lambda_{A}{}^{C}{}_{B}{}^{D}(R_{\mathbf{x}CD}{}^{\mathbf{x}} + R_{ECD}{}^{E}) = \widetilde{T}_{AB} \qquad [3.2\text{-}54]$$

while the trace part of 3.2-53 is the left-hand side of equation 3.2-26

$$G \equiv g^{CD}G_{CD} = -g^{\mathbf{xy}}R_{\mathbf{zxy}}{}^{\mathbf{z}} - g^{\mathbf{xy}}\ R_{C\mathbf{xy}}{}^{C} = g^{CD}T_{CD} \qquad [3.2\text{-}55]$$

We consider the traceless equation first. The second term on the left-hand side of equation 3.2-54 vanishes identically.

[Proof:

$$\Lambda_{A}{}^{C}{}_{B}{}^{D}R_{ECD}{}^{E} = (\delta_{A}{}^{C}\ \delta_{B}{}^{D} - 1/2 g_{AB}g^{CD})\, g^{EF}\ R_{ECDF} \qquad [3.2\text{-}56]$$

Because of the symmetries of the Riemann tensor we can write R_{ECDF} in the form:

$$R_{ECDF} = \varepsilon_{EC}\varepsilon_{DF}f, \qquad [3.2\text{-}57]$$

giving

$$\Lambda_A{}^C{}_B{}^D R_{ECD}{}^E = g^{EF}\varepsilon_{EA}\varepsilon_{BF}f - 1/2 g_{AB}g^{CD}g^{EF}\varepsilon_{EC}\varepsilon_{DF}f \qquad [3.2\text{-}58]$$

Using equations 2.2-22 and 2.2-23, equation 3.2-58 becomes

$$\Lambda_A{}^C{}_B{}^D R_{ECD}{}^E = -\gamma^{-2}g_{AB}f + \gamma^{-2}g_{AB}f = 0] \qquad [3.2\text{-}59]$$

Thus, the traceless field equation takes the form

$$\tilde{G}_{AB} = \Lambda_A{}^C{}_B{}^D R_{\mathbf{x}CD}{}^{\mathbf{x}} = \tilde{T}_{AB} \qquad [3.2\text{-}60]$$

In Section 2.3, we developed an expression (equation 2.3-54) for $R_{\mathbf{x}AB}{}^{\mathbf{y}}$ in terms of the extrinsic curvature tensors. Contracting over the rigging indices, we get

$$R_{\mathbf{x}AB}{}^{\mathbf{x}} = \mathfrak{D}_{\mathbf{x}}H_{AB}{}^{\mathbf{x}} + \mathfrak{D}_A{}^*H_{\mathbf{x}}{}^{\mathbf{x}}{}_B - {}^*H_{\mathbf{x}}{}^{\mathbf{z}}{}_B{}^*H_{\mathbf{z}}{}^{\mathbf{x}}{}_A + H_{AC}{}^{\mathbf{x}}H_B{}^C{}_{\mathbf{x}} \qquad [3.2\text{-}61]$$

Inserting this equation into the left-hand side of equation 3.2-60 yields

$$\tilde{G}_{AB} = \Lambda_A{}^C{}_B{}^D[\mathfrak{D}_{\mathbf{x}}H_{CD}{}^{\mathbf{x}} + \mathfrak{D}_C{}^*H_{\mathbf{x}}{}^{\mathbf{x}}{}_D - {}^*H_{\mathbf{x}}{}^{\mathbf{z}}{}_D{}^*H_{\mathbf{z}}{}^{\mathbf{x}}{}_C + H_{CE}{}^{\mathbf{x}}L_D{}^E{}_{\mathbf{x}}] = \tilde{T}_{AB}$$
$$[3.2\text{-}62]$$

Using equation 3.2-47, we can rewrite equation 3.2-62 as

$$\tilde{G}_{AB} = -1/2\Lambda_A{}^C{}_B{}^D[g^{\mathbf{xy}}\mathfrak{D}_{\mathbf{x}}\mathfrak{D}_{\mathbf{y}}g_{CD} + \mathfrak{D}_C{}^*H_{\mathbf{x}}{}^{\mathbf{x}}{}_D$$
$$- {}^*H_{\mathbf{x}}{}^{\mathbf{z}}{}_D{}^*H_{\mathbf{z}}{}^{\mathbf{x}}{}_C + H_{CE}{}^{\mathbf{x}}H_D{}^E{}_{\mathbf{x}}] = \tilde{T}_{AB} \qquad [3.2\text{-}63]$$

We can bring the trace-removing operator inside the derivative

$$-1/2g^{\mathbf{xy}}\mathfrak{D}_{\mathbf{x}}(\Lambda_A{}^C{}_B{}^D\mathfrak{D}_{\mathbf{y}}g_{CD}) + 1/2g^{\mathbf{xy}}\mathfrak{D}_{\mathbf{y}}g_{CD}\mathfrak{D}_{\mathbf{x}}(\Lambda_A{}^C{}_B{}^D)$$
$$- 1/2\Lambda_A{}^C{}_B{}^D[\mathfrak{D}_C{}^*H_{\mathbf{x}}{}^{\mathbf{y}}{}_D - {}^*H_{\mathbf{x}}{}^{\mathbf{z}}{}_D{}^*H_{\mathbf{z}}{}^{\mathbf{y}}{}_C + H_{CE}{}^{\mathbf{y}}H_D{}^E{}_{\mathbf{x}}] = \widetilde{T}_{AB} \qquad [3.2\text{-}64]$$

We saw in Section 2.3 that $\Lambda_A{}^C{}_B{}^D\mathfrak{D}_{\mathbf{y}}g_{CD}$ can be written in terms of the conformal metric. The details are developed in Appendix B, but the final result is that the traceless surface-projected field equation has the form

$$\widetilde{G}_{CD} = -1/2\gamma g^{\mathbf{xy}}\mathfrak{D}_{\mathbf{x}}\mathfrak{D}_{\mathbf{y}}\widetilde{g}_{CD} - \Lambda^A{}_C{}^B{}_D\,[\mathfrak{D}_A{}^*H_B - {}^*H^{\mathbf{yz}}{}_B{}^*H_{\mathbf{zy}A}]$$
$$+ \gamma H^{\mathbf{x}}\widetilde{H}_{CD\mathbf{x}} + \gamma\widetilde{H}_{CE}{}^{\mathbf{x}}\widetilde{H}_D{}^E{}_{\mathbf{x}} + 1/2\gamma\widetilde{g}_{CD}\widetilde{H}_{AB}{}^{\mathbf{x}}\widetilde{H}^{AB}{}_{\mathbf{x}} \qquad [3.2\text{-}65]$$

The trace of the surface-projection of the field equations [3.2-55] can be evaluated by using equations 2.3-49 and 2.3-55 which express the desired projections of the Riemann tensor. The last term on the left-hand side of 3.2-55 is obtained from the fully contracted form of equation 2.3-55. We have, from the first contracted Gauss equation, the Ricci tensor of the rigging space

$$''R_{\mathbf{yz}} = {}''R_{\mathbf{xyz}}{}^{\mathbf{x}} = -{}^*H_{\mathbf{xz}}{}^{A*}H_{\mathbf{y}}{}^{\mathbf{x}}{}_A + {}^*H_{\mathbf{yz}}{}^{A*}H_{\mathbf{x}}{}^{\mathbf{x}}{}_A$$
$$+ 2\Omega_{\mathbf{xy}}{}^{A*}H_{\mathbf{z}}{}^{\mathbf{x}}{}_A + R_{\mathbf{xyz}}{}^{\mathbf{x}} \qquad [3.2\text{-}66]$$

and the Ricci scalar

$$''R = g^{\mathbf{yz}}\,''R_{\mathbf{yz}} = -\,g^{\mathbf{yz}*}H_{\mathbf{xz}}{}^{A*}H_{\mathbf{y}}{}^{\mathbf{x}}{}_A + g^{\mathbf{yz}*}H_{\mathbf{yz}}{}^{A*}H_{\mathbf{x}}{}^{\mathbf{x}}{}_A$$

$$+ \; 2g^{yz}\Omega_{xy}{}^{A*}H_z{}^x{}_A + g^{yz}R_{xyz}{}^x \qquad [3.2\text{-}67]$$

Thus the required projection is

$$g^{yz}R_{xyz}{}^x = g^{yz*}H_{xz}{}^{A*}H_y{}^x{}_A - g^{yz*}H_{yz}{}^{A*}H_x{}^x{}_A$$
$$- \; 2g^{yz}\Omega_{xy}{}^{A*}H_z{}^x{}_A + {}''R \qquad [3.2\text{-}68]$$

The first term on the right-hand side of 3.2-55 can be obtained from contracting equation 2.3-56

$$R_{Cxy}{}^C = \mathfrak{D}_x H_C{}^C{}_y - H_C{}^E{}_y H_E{}^C{}_x + {}^*H_{xz}{}^{C*}H_y{}^z{}_C + \mathfrak{D}_C{}^*H_{xy}{}^C \qquad [3.2\text{-}69]$$

and

$$g^{xy}R_{Cxy}{}^C = g^{xy}\mathfrak{D}_x H_C{}^C{}_y - g^{xy}H_C{}^E{}_y H_E{}^C{}_x + g^{xy*}H_{xz}{}^{C*}H_y{}^z{}_C$$
$$+ \; g^{xy}\mathfrak{D}_C{}^*H_{xy}{}^C \qquad [3.2\text{-}70]$$

Inserting 3.2-68 and 3.2-70 into 3.2-55 gives

$$g^{CD}G_{CD} = -2g^{yz*}H_{(xz)}{}^{A*}H_y{}^x{}_A + g^{yz*}H_{yz}{}^{A*}H_x{}^x{}_A$$
$$+ \; 2g^{yz}\Omega_{xy}{}^{A*}H_z{}^x{}_A - {}''R - g^{xy}\mathfrak{D}_x H_C{}^C{}_y + g^{xy}H_C{}^E{}_y H_E{}^C{}_x$$
$$- \; g^{xy}\mathfrak{D}_C{}^*H_{xy}{}^C = g^{CD}T_{CD}$$

$$= -2{}^*H_{(xz)}{}^{A*}H^{zx}{}_A + {}^*H^{A*}H_A$$
$$+ \; 2\Omega_{xy}{}^{A*}H^{yx}{}_A - {}''R - g^{xy}\mathfrak{D}_x H_C{}^C{}_y + g^{xy}H_C{}^E{}_y H_E{}^C{}_x$$
$$- \; \mathfrak{D}_C{}^*H^C + {}^*H_{xy}{}^C\mathfrak{D}_C g^{xy} \qquad [3.2\text{-}71]$$

The first and last terms cancel to give

$$G = {}^*H^{A*}H_A + 2\Omega_{xy}{}^{A*}H^{yx}{}_A - {}''R$$

$$- g^{xy}\mathcal{D}_x H_C{}^C{}_y + g^{xy} H_{ABy} H^{AB}{}_x - \mathcal{D}_C {}^*H^C \qquad [3.2\text{-}72]$$

Using $\Omega^A{}_{xy} = \varepsilon_{xy}\Omega^A$, equation 3.2-72 can be written

$$G = {}^*H^{A*}H_A + 4\rho^{-2}\Omega^A\Omega_A - {}''R$$
$$+ \gamma^{-1} g^{xy}\mathcal{D}_x\mathcal{D}_y\gamma - 1/2\gamma^{-2} g^{xy}\mathcal{D}_x\gamma\mathcal{D}_y\gamma + g^{xy}\overline{H}_{ABy}\overline{H}^{AB}{}_x$$
$$- g^{AB}\mathcal{D}_A(\rho^{-1}\mathcal{D}_B\rho) \qquad [3.2\text{-}73]$$

This equation has a second-order operator acting on γ, but also implicitly contains second-order derivatives inside the Ricci scalar of $\mathcal{T}$. We shall discuss it further in Chapters 4 and 5.

Consider next the mixed surface-rigging projections of the Einstein field equations. This projection of the Einstein tensor involves just the Ricci tensor because there are no cross terms in the metric:

$$G_{xA} = R_{xA} = R_{zxA}{}^z + R_{CxA}{}^C = T_{xA} \qquad [3.2\text{-}74]$$

The Riemann tensor projections on the left-hand side of equation 3.2-74 appear in the Codazzi equations for $\mathcal{N}$ and $\mathcal{T}$ (equations 2.3-48 and 2.3-50)

$$D_{[C}H_{A]B}{}^x = 1/2R_{CAB}{}^x \qquad [3.2\text{-}75]$$

$$D_{[u}{}^*H_{x]y}{}^A = 1/2R_{uxy}{}^A + \Omega^B{}_{ux}H_B{}^A{}_y \qquad [3.2\text{-}76]$$

Working first with equation 3.2-76, which we write in the form

$$R_{uxy}{}^A = 2D_{[u}{}^*H_{x]y}{}^A - 2\Omega^B{}_{ux}H_B{}^A{}_y, \qquad [3.2\text{-}77]$$

${}^{*}H_{\mathbf{xy}}{}^{A}$ can be written

$$ {}^{*}H_{\mathbf{xy}}{}^{A} = {}^{*}H_{[\mathbf{xy}]}{}^{A} + {}^{*}H_{(\mathbf{xy})}{}^{A} $$

$$ = -\Omega^{A}{}_{\mathbf{xy}} + {}^{*}H_{(\mathbf{xy})}{}^{A} $$

$$ = -\Omega^{A}\,\varepsilon_{\mathbf{xy}} + {}^{*}H_{(\mathbf{xy})}{}^{A} \qquad [3.2\text{-}78] $$

The first term on the right-hand side of 3.2-77 becomes

$$ 2D_{[\mathbf{u}}{}^{*}H_{\mathbf{x}]\mathbf{y}}{}^{A} = \varepsilon_{\mathbf{ux}}\varepsilon^{\mathbf{rs}}D_{\mathbf{r}}{}^{*}H_{\mathbf{sy}}{}^{A} $$

$$ = \varepsilon_{\mathbf{ux}}\varepsilon^{\mathbf{rs}}D_{\mathbf{r}}({}^{*}H_{(\mathbf{sy})}{}^{A} - \Omega^{A}\varepsilon_{\mathbf{sy}}) $$

$$ = \varepsilon_{\mathbf{ux}}\varepsilon^{\mathbf{rs}}D_{\mathbf{r}}{}^{*}H_{(\mathbf{sy})}{}^{A} - \varepsilon_{\mathbf{ux}}\varepsilon^{\mathbf{rs}}\varepsilon_{\mathbf{sy}}D_{\mathbf{r}}\Omega^{A} $$

$$ = \varepsilon_{\mathbf{ux}}\varepsilon^{\mathbf{rs}}D_{\mathbf{r}}{}^{*}H_{(\mathbf{sy})}{}^{A} + \varepsilon_{\mathbf{ux}}D_{\mathbf{y}}\Omega^{A} \qquad [3.2\text{-}79] $$

Equation 3.2-77 becomes

$$ R_{\mathbf{uxy}}{}^{A} = \varepsilon_{\mathbf{ux}}\varepsilon^{\mathbf{rs}}D_{\mathbf{r}}{}^{*}H_{(\mathbf{sy})}{}^{A} + \varepsilon_{\mathbf{ux}}(D_{\mathbf{y}}\Omega^{A} - 2\Omega^{B}H_{B}{}^{A}{}_{\mathbf{y}}) $$

$$ = \varepsilon_{\mathbf{ux}}\varepsilon^{\mathbf{rs}}D_{\mathbf{r}}{}^{*}H_{(\mathbf{sy})}{}^{A} + \varepsilon_{\mathbf{ux}}g^{AC}(D_{\mathbf{y}}\Omega_{C} - 2\Omega_{B}H_{C}{}^{B}{}_{\mathbf{y}}) $$

$$ = \varepsilon_{\mathbf{ux}}\varepsilon^{\mathbf{rs}}D_{\mathbf{r}}{}^{*}H_{(\mathbf{sy})}{}^{A} + \varepsilon_{\mathbf{ux}}g^{AC}(\mathfrak{D}_{\mathbf{y}}\Omega_{C} - \Omega_{B}H_{C}{}^{B}{}_{\mathbf{y}}) \qquad [3.2\text{-}80] $$

Using the symmetries of the Riemann tensor, equation 3.2-80 can be written

$$ R_{\mathbf{y}A\mathbf{x}}{}^{\mathbf{v}} = g^{\mathbf{vw}}\varepsilon_{\mathbf{xw}}\varepsilon^{\mathbf{rs}}D_{\mathbf{r}}{}^{*}H_{(\mathbf{sy})A} + g^{\mathbf{vw}}\varepsilon_{\mathbf{xw}}(\mathfrak{D}_{\mathbf{y}}\Omega_{A} - \Omega_{B}H_{A}{}^{B}{}_{\mathbf{y}}) \qquad [3.2\text{-}81] $$

Contracting over the rigging indices yields

$$ R_{\mathbf{y}A\mathbf{x}}{}^{\mathbf{y}} = g^{\mathbf{yw}}\varepsilon_{\mathbf{xw}}\varepsilon^{\mathbf{rs}}D_{\mathbf{r}}{}^{*}H_{(\mathbf{sy})A} + g^{\mathbf{yw}}\varepsilon_{\mathbf{xw}}(\mathfrak{D}_{\mathbf{y}}\Omega_{A} - \Omega_{B}H_{A}{}^{B}{}_{\mathbf{y}}) \qquad [3.2\text{-}82] $$

The second term on the left-hand side of 3.2-74 comes from the contracted form of 3.2-75

$$ R_{CA\mathbf{x}}{}^{C} = 1/2(D_{A}H_{\mathbf{x}} - D_{C}H_{A}{}^{C}{}_{\mathbf{x}}) \qquad [3.2\text{-}83] $$

Substituting equations 3.2-82 and 3.2-83 into equation 3.2-74 gives

$$g^{yw}\varepsilon_{xw}\varepsilon^{rs}D_r{}^*H_{(sy)A} + g^{yw}\varepsilon_{xw}(\mathfrak{D}_y\Omega_A - \Omega_B H_A{}^B{}_y)$$
$$+ 1/2(D_A H_x - D_C H_A{}^C{}_x) = T_{Ax} \qquad [3.2\text{-}84]$$

Contract both sides of this equation with $\varepsilon^{xu}g_{uz}$ to give

$$\mathfrak{D}_z\Omega_A - \Omega_B H_A{}^B{}_z + \varepsilon^{rs}D_r{}^*H_{(sz)A} + 1/2\ \varepsilon^{xu}g_{uz}(D_A H_x - D_C H_A{}^C{}_x)$$
$$= \varepsilon^{xu}g_{uz}T_{Ax} \qquad [3.2\text{-}85]$$

This is the desired result.

The last set of field equations we derive are the totally rigging-projected ones given by equation 3.2-26. We have

$$G_{zw} = R_{Azw}{}^A + R_{uzw}{}^u$$
$$- 1/2g_{zw}(g^{AB}R_{CAB}{}^C + g^{AB}R_{xAB}{}^x + g^{xy}\ R_{Axy}{}^A + g^{xy}R_{uxy}{}^u)$$
$$= T_{zw}$$

[3.2-86]

$$R_{Azw}{}^A - g_{zw}g^{xy}g^{AB}R_{xABy} + \Lambda_z{}^x{}_w{}^y R_{uxy}{}^u - 1/2g_{zw}g^{AB}R_{CAB}{}^C = T_{zw}$$

[3.2-87]

Using arguments analogous to those used in the proof of equation 3.2-19, we can drop the third term on the left-hand side to obtain:

$$R_{Azw}{}^A - g_{zw}g^{xy}g^{AB}R_{xABy} - 1/2g_{zw}g^{AB}R_{CAB}{}^C = T_{zw} \qquad [3.2\text{-}88]$$

The first two terms on the left-hand side can be combined as

$$g_{zy}g^{yx}R_{Axw}{}^A - g_{zw}g^{xy}R_{Axy}{}^A$$
$$= g_{zr}g^{yx}R_{Axs}{}^A(\delta^r{}_y\delta^s{}_w - \delta^s{}_y\delta^r{}_w) = g_{zr}g^{yx}R_{Axs}{}^A\varepsilon^{rs}\varepsilon_{yw}$$

Incorporating this expression into 3.2-88, we get

$$g_{zr}g^{yx}R_{Axs}{}^{A}\varepsilon^{rs}\varepsilon_{yw} - 1/2g_{zw}g^{AB}R_{CAB}{}^{C} = T_{zw} \qquad [3.2\text{-}89]$$

Contract with ε^{zu} and ε^{wv} to obtain

$$\rho^2 g^{us}g^{vt}R_{Ats}{}^{A} + 1/2\rho^2 g^{uv}g^{AB}R_{CAB}{}^{C} = \varepsilon^{zu}\varepsilon^{wv}T_{zw} \qquad [3.2\text{-}90]$$

If we lower the rigging indices on both sides, we obtain

$$R_{Axy}{}^{A} + 1/2g_{xy}g^{AB}R_{CAB}{}^{C} = \rho^{-2}g_{xv}g_{yu}\varepsilon^{zu}\varepsilon^{wv}T_{zw} \qquad [3.2\text{-}91]$$

The first term on the left-hand becomes, after substituting in the expression for $R_{Axy}{}^{A}$,

$$\begin{aligned} g^{AB}R_{xABy} &= g^{AB}\mathcal{D}_x H_{ABy} + g^{AB}\mathcal{D}_A{}^{*}H_{xyB} - {}^{*}g^{AB}H_x{}^{z}{}_B{}^{*}H_{zyA} + g^{AB}H_{ACy}H_B{}^{C}{}_x \\ &= \mathcal{D}_x(g^{AB}H_{ABy}) - H_{ABy}\mathcal{D}_x g^{AB} + g^{AB}\mathcal{D}_A{}^{*}H_{xyB} - {}^{*}g^{AB}H_x{}^{z}{}_B{}^{*}H_{zyA} \\ &\qquad + g^{AB}H_{ACy}H_B{}^{C}{}_x \\ &= \mathcal{D}_x H_y - 2H_{ABy}H^{AB}{}_x + g^{AB}\mathcal{D}_A{}^{*}H_{xyB} - {}^{*}g^{AB}H_x{}^{z}{}_B{}^{*}H_{zyA} \\ &\qquad + g^{AB}H_{ACy}H_B{}^{C}{}_x \\ &= -\mathcal{D}_x(\gamma^{-1}\mathcal{D}_y\gamma) - H_{ABy}H^{AB}{}_x + g^{AB}\mathcal{D}_A{}^{*}H_{xyB} - {}^{*}g^{AB}H_x{}^{z}{}_B{}^{*}H_{zyA} \\ &= -\gamma^{-1}\mathcal{D}_x\mathcal{D}_y\gamma + \gamma^{-2}\mathcal{D}_x\gamma\mathcal{D}_y\gamma - H_{ABy}H^{AB}{}_x + g^{AB}\mathcal{D}_A{}^{*}H_{xyB} \\ &\qquad - {}^{*}g^{AB}H_x{}^{z}{}_B{}^{*}H_{zyA} \end{aligned} \qquad [3.2\text{-}92]$$

The second term on the left-hand side comes from Gauss's equation

$$g^{AB}R_{CAB}{}^{C} = -H_z H^z + H^{AB}{}_z H^z{}_{AB} + {}'R \qquad [3.2\text{-}93]$$

Putting equations 3.2-92 and 3.2-93 into 3.2-91 gives the final form of the totally rigging-projected Einstein equations

$$-\gamma^{-1}\mathcal{D}_{\mathbf{x}}\mathcal{D}_{\mathbf{y}}\gamma + \gamma^{-2}\mathcal{D}_{\mathbf{x}}\gamma\mathcal{D}_{\mathbf{y}}\gamma - H_{AB\mathbf{y}}H^{AB}{}_{\mathbf{x}} + g^{AB}\mathcal{D}_{A}{}^{*}H_{\mathbf{xy}B}$$
$$- g^{AB}{}^{*}H_{\mathbf{x}}{}^{\mathbf{z}}{}_{B}{}^{*}H_{\mathbf{zy}A} + 1/2g_{\mathbf{xy}}[-H_{\mathbf{z}}H^{\mathbf{z}} + H^{AB}{}_{\mathbf{z}}H^{\mathbf{z}}{}_{AB} + {}'R] =$$
$$= \rho^{-2}g_{\mathbf{xv}}g_{\mathbf{yu}}\varepsilon^{\mathbf{zu}}\varepsilon^{\mathbf{wv}}T_{\mathbf{zw}} \qquad [3.2\text{-}94]$$

To finish this section, we shall write out the two+two form of the Bianchi identities. They are developed in Appendix B. It is convenient to define the tensor

$$S_{\mu\nu} = G_{\mu\nu} - T_{\mu\nu} \qquad [3.2\text{-}95]$$

Then the Bianchi identities are (from equations B-39 and B-40)

$$\gamma^{-1}\mathcal{D}_{\mathbf{z}}(\gamma S^{\mathbf{z}}{}_{\mathbf{x}}) + \rho^{-1}\mathcal{D}_{A}(\rho S^{A}{}_{\mathbf{x}}) + S^{A}{}_{B}H_{A}{}^{B}{}_{\mathbf{x}} = 0 \qquad [3.2\text{-}96]$$

$$\gamma^{-1}\mathcal{D}_{\mathbf{x}}(\gamma S^{\mathbf{x}}{}_{A}) + \rho^{-1}\mathcal{D}_{B}(\rho S^{B}{}_{A}) - S^{\mathbf{z}}{}_{\mathbf{x}}{}^{*}H_{\mathbf{z}}{}^{\mathbf{x}}{}_{A} = 0 \qquad [3.2\text{-}97]$$

In the following sections we shall use the forms of the Einstein equations developed above to evaluate particular initial value problems.

3.3 The Field Equations as Deformation Equations

We presented the two+two decomposition of the Einstein field equations in the last section with only minimal discussion of what quantities they determine and what initial data should be specified for their solution. This section, and the following chapters, will address these questions more fully. In addition, we shall discuss how certain qualitative features of the initial data depend on the particular initial hypersurface or hypersurfaces.

In the two+two formalism, the quantity that carries the dynamics of the field is the conformal two-geometry of a two-parameter family of spacelike two-surfaces. The set of equations that govern the evolution of this conformal two-metric, $\tilde{G}_{AB} = \tilde{T}_{AB}$, has a wave-operator as its principle part, i.e., the part which contains the highest-order derivatives. We shall refer to these equations as the dynamical equations. The interpretation of these equations as wave equations for the conformal two-metric is justified when the linearized version of the two+two formalism is discussed in the next section. We shall see that these equations reduce to a linear wave equation for the perturbations of the conformal two-metric. The appropriate initial data for these linear wave equations depends on the initial hypersurface(s): for the Cauchy problem, it is the field and its normal derivative on a single surface; for the characteristic initial value problem, it is the field on two-intersecting null hypersurfaces. We expect this behaviour to extend to the exact equations.

The remaining part of the two-metric, the conformal scale factor, appears in the principle parts of four equations, $G_{\mathbf{xy}} = T_{\mathbf{xy}}$ and $G_{AB}g^{AB} = T_{AB}g^{AB}$, although the latter equation also contains the Ricci scalar of the rigging space, and thus implicitly contains second derivatives of the rigging metric.

In this section, we shall show that the conformal scale factor is completely determined by the $G_{\mathbf{xy}} = T_{\mathbf{xy}}$ equations in a four-dimensional region if it and its first derivatives are given on a single two-surface. The equation $g^{AB}G_{AB} = g^{AB}T_{AB}$ will be used to determine one of the components of the rigging metric. The four equations $G_{A\mathbf{x}} = T_{A\mathbf{x}}$, as we shall see, are involved in the determination of ${}^*H_{\mathbf{xy}A}$.

In Section 2.2, we showed that the four-metric decomposes into mutually orthogonal parts with respect to an orthogonally rigged two-surface

$$g_{\mu\nu} = {}'g_{\mu\nu} + {}''g_{\mu\nu} \qquad [3.3\text{-}1]$$

${}'g_{\mu\nu}$ is determined by g_{AB} and $B^A{}_\mu$ since

$${}'g_{\mu\nu} = g_{AB}B^A{}_\mu{}^B{}_\nu \qquad [3.3\text{-}2]$$

while the rigging geometry is defined by

$${}'g_{\mu\nu} = g_{\mathbf{xy}}C^{\mathbf{x}}{}_\mu{}^{\mathbf{y}}{}_\nu \qquad [3.3\text{-}3]$$

Both $B^A{}_\mu$ and $C^{\mathbf{x}}{}_\mu$ are uniquely determined by $C_{\mathbf{x}}{}^\mu$, once $B_A{}^\mu$ is fixed by the duality conditions. Hence, the quantities that determine the full four-metric are the intrinsic surface geometry g_{AB}, the rigging geometry $g_{\mathbf{xy}}$, and $C_{\mathbf{x}}{}^\mu$. Clearly these 14 quantities are not independent; we can fix four of the $g_{\mathbf{xy}}$'s and $C_{\mathbf{x}}{}^\mu$'s because of the freedom to perform an arbitrary non-singular linear transformation at each point of the surface. This gives us ten independent quantities as we might expect.

There are also four available degrees of coordinate (gauge) freedom and we can use them to fix four more of the $g_{\mathbf{xy}}$'s and $C_{\mathbf{x}}{}^\mu$'s. Thus, of the eleven quantities, $g_{\mathbf{xy}}$ and $C_{\mathbf{x}}{}^\mu$, only three should be determined by the field equations.

Since the conformal metric on space-like two-surface can carry the dynamics of the gravitational field, it is important to look at the evolution of the geometry of a two-surface and how it relates to the field equations. An appropriate way to look at this evolution is by considering deformations of a two-surfaces which were defined in Section 2.1. A _deformation_ of a general geometrical object is its variation due to a _displacement_. The geometrical quantity may be displaced itself or may be a functional of something which is displaced. In Section 2.1, the displaced quantities were two-surfaces. Quantities such as the metric or extrinsic curvature of the two-surface change because of their functional dependence on the two-surface. Consider a given four-dimensional manifold with metric in which a two-surface is imbedded. A metric on the two-surface is induced from the four-metric by the pull-back of the imbedding map. As the surface is displaced, the two-metric changes solely due to the change in the imbedding. Thus the surface metric is a functional of a quantity which is displaced: the two-surface. The rigging space also evolves but in such a way as to preserve its orthogonality to the surface. Of course, we can reverse this argument and build up the four-geometry from the changing surface and rigging geometries.

Deformation of the Surface Two-Metric

We study the evolution of the two-metric off an initial two-surface by computing the Lie derivatives of the two-surface metric and extrinsic curvature in an arbitrary direction. This is analogous to that used in the 3+1-approach of Stachel [1962] and others. The essential idea of that method is to use the field equations to express the second order Lie derivative of the three-metric in terms of lower order Lie derivatives, which form part of the Cauchy data. This method is thus analogous to the Newtonian formulation of dynamics. Each higher order Lie derivative can then be expressed, using the equations of motion and their derivatives, in terms of lower order Lie deriatives: so Lie derivatives to all orders are known ,and an analytic solution is possible. In the 2+2 case, the idea works for the conformal scale factor, but not for the conformal two-metric. Data specified on a single two-surface is sufficient to propagate γ^2 throughout a four-dimensional region of space-time. The field equations for the conformal scale factor derive from the rigging projections of the Einstein tensor $G_{\mathbf{xy}}$.

However, we shall find the Newtonian approach useful when later we specify initial data for the dynamical equation on a single two-surface, rather than on a hypersurface or pair of hypersurfaces. In that case, an infinite number of functions must be specified on the initial two-surface. This method will be discussed in Chapters 4 and 5.

We begin by deriving an equation which determines the evolution of the conformal metric of an initial two-surface **N** in a direction defined by one of the deformation vector fields, **U** or **V**. These are two commuting vector fields used to foliate a four-dimensional region of space-time, as discussed earlier. The three-surfaces created by dragging **N** along the integral curves of the deformation vectors will serve as initial hypersurfaces of the evolution equations. The deformation equation for the two-metric is obtained by calculating the Lie derivative of the two-metric with respect to **V**. In the beginning of this section, **V** will be a "generic" deformation vector, so identical equations will hold for **U**.

It will simplify matters if we decompose the deformation vector into its surface and rigging projections:

$$V^{\mu} = {}'V^{\mu} + {}''V^{\mu} \qquad [3.3\text{-}4]$$

The Lie derivative of the two-surface metric is

$$\pounds_{\mathbf{V}} g_{AB} = \pounds_{'\mathbf{V}} g_{AB} + \pounds_{''\mathbf{V}} g_{AB}$$

$$= 2D_{(A}V_{B)} + V^{\mathbf{x}}\mathcal{D}_{\mathbf{x}} g_{AB}$$

$$= 2D_{(A}V_{B)} - 2V^{\mathbf{x}}H_{AB\mathbf{x}} \qquad [3.3\text{-}5]$$

Similarly, for the raised metric

$$£_{\mathbf{V}}g^{AB} = -2D^{(A}V^{B)} + 2V^{\mathbf{x}}H^{AB}{}_{\mathbf{x}} \qquad [3.3\text{-}6]$$

Hence, the Lie derivative of the conformal scale factor is related to the mean extrinsic curvature:

$$£_{\mathbf{V}}\gamma^2 = 2\gamma^2(D_AV^A - V^{\mathbf{x}}H_{\mathbf{x}}) \qquad [3.3\text{-}7]$$

Now consider the Lie derivative of $H_{AB\mathbf{x}}$. We have

$$£_{\mathbf{V}}H_{AB\mathbf{x}} = £'_{\mathbf{V}}H_{AB\mathbf{x}} + £''_{\mathbf{V}}H_{AB\mathbf{x}} \qquad [3.3\text{-}8]$$

Again, we compute the Lie derivative with respect to the surface component first:

$$£'_{\mathbf{V}}H_{AB\mathbf{x}} = V^C\mathcal{D}_CH_{AB\mathbf{x}} + H_{CB\mathbf{x}}D_AV^C + H_{AC\mathbf{x}}D_BV^C \qquad [3.3\text{-}9]$$

The second term on the right-hand side of equation 3.3-8 is:

$$£''_{\mathbf{V}}H_{AB\mathbf{x}} = V^{\mathbf{z}}\mathcal{D}_{\mathbf{z}}H_{AB\mathbf{x}} + H_{AB\mathbf{z}}D_{\mathbf{x}}V^{\mathbf{z}} \qquad [3.3\text{-}10]$$

We can use equation 2.3-76 to express $V^{\mathbf{z}}\mathcal{D}_{\mathbf{z}}H_{AB\mathbf{x}}$ in terms of projections of the Riemann tensor:

$$V^{\mathbf{z}}\mathcal{D}_{\mathbf{z}}H_{AB\mathbf{x}} = -g_{\mathbf{xy}}V^{\mathbf{z}}\mathcal{D}_A{}^*H_{\mathbf{z}}{}^{\mathbf{y}}{}_B + V^{\mathbf{z}*}H_{\mathbf{z}}{}^{\mathbf{y}}{}_A{}^*H_{\mathbf{yx}B}$$
$$- V^{\mathbf{z}}H_{CB\mathbf{x}}H_A{}^C{}_{\mathbf{z}} + V^{\mathbf{z}}R_{\mathbf{z}AB\mathbf{x}} \qquad [3.3\text{-}11]$$

Combining equations 3.3-8, 3.3-9 and 3.3-11 yields the Lie derivative of $H_{AB\mathbf{x}}$ with respect to any deformation vector

$$£_{\mathbf{V}}H_{AB\mathbf{x}} = -V^{\mathbf{z}}H_{AC\mathbf{x}}H_B{}^C{}_{\mathbf{z}} - H_{AB\mathbf{z}}D_{\mathbf{x}}V^{\mathbf{z}} + V^{\mathbf{z}*}H_{\mathbf{z}}{}^{\mathbf{y}}{}_A{}^*H_{\mathbf{yx}B}$$
$$- g_{\mathbf{xy}}V^{\mathbf{z}}\mathcal{D}_A{}^*H_{\mathbf{z}}{}^{\mathbf{y}}{}_B + V^{\mathbf{z}}R_{\mathbf{z}AB\mathbf{x}} + V^C\mathcal{D}_CH_{AB\mathbf{x}}$$
$$+ H_{CB\mathbf{x}}D_AV^C + H_{AC\mathbf{x}}D_BV^C \qquad [3.3\text{-}12]$$

Contract equation 3.3-11 which another generic deformation vector $\mathbf{U}$ which commutes with $\mathbf{V}$ ($\mathbf{U}$ may be the same as $\mathbf{V}$ or different)

$$U^{\mathbf{x}}\pounds_{\mathbf{V}}H_{AB\mathbf{x}} = - U^{\mathbf{x}}V^{\mathbf{z}}H_{AC\mathbf{x}}H_B{}^C{}_{\mathbf{z}} - U^{\mathbf{x}}H_{AB\mathbf{z}}D_{\mathbf{x}}V^{\mathbf{z}} + U^{\mathbf{x}}V^{\mathbf{z}}{}^*H_{\mathbf{z}}{}^{\mathbf{y}}{}_A{}^*H_{\mathbf{yx}B}$$
$$- g_{\mathbf{xy}}U^{\mathbf{x}}V^{\mathbf{z}}\mathcal{D}_A{}^*H_{\mathbf{z}}{}^{\mathbf{y}}{}_B + U^{\mathbf{x}}V^{\mathbf{z}}R_{\mathbf{z}AB\mathbf{x}} + U^{\mathbf{x}}V^C\mathcal{D}_C H_{AB\mathbf{x}}$$
$$+ U^{\mathbf{x}}H_{CB\mathbf{x}}D_A V^C + U^{\mathbf{x}}H_{AC\mathbf{x}}D_B V^C \qquad [3.3\text{-}13]$$

Recalling 2.1-62 and 2.1-66, this becomes

$$\pounds_{\mathbf{V}}(H_{AB\mathbf{x}}U^{\mathbf{x}}) = - U^{\mathbf{x}}V^{\mathbf{z}}H_{AC\mathbf{x}}H_B{}^C{}_{\mathbf{z}} - U^{\mathbf{x}}H_{AB\mathbf{z}}D_{\mathbf{x}}V^{\mathbf{z}} + U^{\mathbf{x}}V^{\mathbf{z}}{}^*H_{\mathbf{z}}{}^{\mathbf{y}}{}_A{}^*H_{\mathbf{yx}B}$$
$$- g_{\mathbf{xy}}\mathcal{D}_A(U^{\mathbf{x}}V^{\mathbf{z}}{}^*H_{\mathbf{z}}{}^{\mathbf{y}}{}_B) + U^{\mathbf{x}}V^{\mathbf{z}}R_{\mathbf{z}AB\mathbf{x}} + V^C\mathcal{D}_C(H_{AB\mathbf{x}}U^{\mathbf{x}})$$
$$+ U^{\mathbf{x}}H_{CB\mathbf{x}}D_A V^C + U^{\mathbf{x}}H_{AC\mathbf{x}}D_B V^C \qquad [3.3\text{-}14]$$

We can bring $g_{\mathbf{xy}}$ into the $\mathcal{D}$-derivative, to get

$$\pounds_{\mathbf{V}}(H_{AB\mathbf{x}}U^{\mathbf{x}}) = - U^{\mathbf{x}}V^{\mathbf{z}}H_{AC\mathbf{x}}H_B{}^C{}_{\mathbf{z}} - U^{\mathbf{x}}H_{AB\mathbf{z}}D_{\mathbf{x}}V^{\mathbf{z}} + U^{\mathbf{x}}V^{\mathbf{z}}{}^*H_{\mathbf{z}}{}^{\mathbf{y}}{}_A{}^*H_{\mathbf{yx}B}$$
$$- \mathcal{D}_A(U^{\mathbf{x}}V^{\mathbf{z}}{}^*H_{\mathbf{zx}B}) - 2U^{\mathbf{x}}V^{\mathbf{z}}{}^*H_{\mathbf{z}}{}^{\mathbf{y}}{}_B{}^*H_{(\mathbf{xy})A}$$
$$+ U^{\mathbf{x}}V^{\mathbf{z}}R_{\mathbf{z}AB\mathbf{x}} + V^C\mathcal{D}_C(H_{AB\mathbf{x}}U^{\mathbf{x}})$$
$$+ U^{\mathbf{x}}H_{CB\mathbf{x}}D_A V^C + U^{\mathbf{x}}H_{AC\mathbf{x}}D_B V^C \qquad [3.3\text{-}15]$$

Equation 3.3-15 is of considerable interest. No explicit reference was made in its derivation to the dimension of the subspace which is deformed, nor of the space in which it is imbedded. For the 3+1 decomposition, the dimension of the space which carries the g_{AB} is three and the dimension of the rigging space is one, so $R_{\mathbf{z}AB}{}^{\mathbf{x}}$ has only one rigging component, $R_{0AB}{}^0$. But the field equations allow us to replace this term by $R_{CAB}{}^C$ which, by Gauss's equation, can be written entirely in terms of the three-metric and the extrinsic curvature. This enables us to solve for Lie derivatives of g_{AB} of second and all higher orders, and forms the basis for an iterative solution of the field equations. In the 2+2 case, this procedure is not possible.

Equations governing the evolution of γ can be derived by taking the trace of 3.3-15 with respect to g_{AB} and bringing g_{AB} into the Lie derivative

$$\pounds_{\mathbf{V}}(H_{\mathbf{x}}U^{\mathbf{x}}) = U^{\mathbf{x}}V^{\mathbf{z}}H_{AC\mathbf{x}}H^{AC}{}_{\mathbf{z}} - U^{\mathbf{x}}H_{\mathbf{z}}D_{\mathbf{x}}V^{\mathbf{z}} + U^{\mathbf{x}}V^{\mathbf{z}\,*}H_{\mathbf{z}}{}^{\mathbf{y}}{}_{A}{}^{*}H_{\mathbf{yx}}{}^{A}$$
$$- \mathcal{D}_A(U^{\mathbf{x}}V^{\mathbf{z}\,*}H_{\mathbf{zx}}{}^{A}) - U^{\mathbf{x}}V^{\mathbf{z}\,*}H_{\mathbf{z}}{}^{\mathbf{y}A\,*}H_{\mathbf{xy}A}$$
$$- U^{\mathbf{x}}V^{\mathbf{z}\,*}H_{\mathbf{z}}{}^{\mathbf{y}A\,*}H_{\mathbf{yx}A} + g^{AB}U^{\mathbf{x}}V^{\mathbf{z}}R_{\mathbf{z}AB\mathbf{x}} + V^{C}\mathcal{D}_C(H_{\mathbf{x}}U^{\mathbf{x}})$$

[3.3-16]

Next use the field equation 3.2-91 to solve for $U^{\mathbf{x}}V^{\mathbf{z}}R_{\mathbf{z}CD\mathbf{x}}g^{CD}$

$$U^{\mathbf{x}}V^{\mathbf{z}}R_{\mathbf{z}CD\mathbf{x}}g^{CD} = U^{\mathbf{x}}V^{\mathbf{z}}\rho^{-2}g_{\mathbf{xv}}g_{\mathbf{zu}}\varepsilon^{\mathbf{ru}}\varepsilon^{\mathbf{wv}}T_{\mathbf{rw}} - 1/2U^{\mathbf{x}}V_{\mathbf{x}}R_{ACD}{}^{A}g^{CD}$$

[3.3-17]

and insert the contracted Gauss's equation into this equation to obtain

$$U^{\mathbf{x}}V^{\mathbf{z}}R_{\mathbf{z}CD\mathbf{x}}g^{CD} = U^{\mathbf{x}}V^{\mathbf{z}}\rho^{-2}g_{\mathbf{xv}}g_{\mathbf{zu}}\varepsilon^{\mathbf{ru}}\varepsilon^{\mathbf{wv}}T_{\mathbf{rw}}$$
$$- 1/2\ U^{\mathbf{x}}V_{\mathbf{x}}[{}'R + H_{AC}{}^{\mathbf{z}}H^{AC}{}_{\mathbf{z}} - H^{\mathbf{z}}H_{\mathbf{z}}] \qquad [3.3\text{-}18]$$

Finally, putting equation 3.3-18 into the right-hand side of equation 3.3-16 gives the desired result

$$\pounds_{\mathbf{V}}(U^{\mathbf{x}}H_{\mathbf{x}}) = -\mathcal{D}_A(V^{\mathbf{z}}U^{\mathbf{x}\,*}H_{\mathbf{zx}}{}^{A}) - V^{\mathbf{z}}U^{\mathbf{x}\,*}H_{\mathbf{z}}{}^{\mathbf{y}A\,*}H_{\mathbf{xy}A}$$
$$+ V^{C}\mathcal{D}_C(U^{\mathbf{x}}H_{\mathbf{x}}) - H_{\mathbf{z}}U^{\mathbf{x}}D_{\mathbf{x}}V^{\mathbf{z}} + U^{\mathbf{x}}V^{\mathbf{z}}H_{AC\mathbf{z}}H^{AC}{}_{\mathbf{x}}$$
$$- 1/2\ U^{\mathbf{x}}V_{\mathbf{x}}[{}'R + H_{AC}{}^{\mathbf{z}}H^{AC}{}_{\mathbf{z}} - H^{\mathbf{z}}H_{\mathbf{z}}]$$
$$+ U^{\mathbf{x}}V^{\mathbf{z}}\rho^{-2}g_{\mathbf{xv}}g_{\mathbf{zu}}\varepsilon^{\mathbf{ru}}\varepsilon^{\mathbf{wv}}T_{\mathbf{rw}} \qquad [3.3\text{-}19]$$

Up to now, **V** and **U** have been generic commuting deformation vector fields, which may or may not be the same. When the two deformation vectors are the same one can insert 3.3-7 into the left-hand side of 3.3-19 giving

$$-1/2 £_{\mathbf{V}}(\gamma^{-2}£_{\mathbf{V}}\gamma^{2}) = - £_{\mathbf{V}}(D_A V^A) - \mathcal{D}_A(V^{\mathbf{x}}V^{\mathbf{z}\,*}H_{\mathbf{zx}}{}^{A})$$

$$- V^{\mathbf{z}}V^{\mathbf{w}\,*}H_{\mathbf{z}}{}^{\mathbf{x}A\,*}H_{\mathbf{wx}A} + V^{C}\mathcal{D}_C(H^{\mathbf{x}}V_{\mathbf{x}})$$

$$+ V^{\mathbf{x}}V^{\mathbf{z}}H_{AC\mathbf{z}}H^{AC}{}_{\mathbf{x}} - H_{\mathbf{x}}V^{\mathbf{z}}D_{\mathbf{z}}V^{\mathbf{x}}$$

$$- 1/2 V_{\mathbf{x}}V^{\mathbf{x}}[{}'R + H_{AC}{}^{\mathbf{z}}H^{AC}{}_{\mathbf{z}} - H^{\mathbf{z}}H_{\mathbf{z}}]$$

$$+ V^{\mathbf{x}}V^{\mathbf{z}}\rho^{-2}g_{\mathbf{xv}}g_{\mathbf{zu}}\varepsilon^{\mathbf{ru}}\varepsilon^{\mathbf{wv}}T_{\mathbf{rw}} \qquad [3.3\text{-}20]$$

which is an equation for the propagation of γ along the integral curves of **V**.

Equations similar to 3.3-19 and 3.3-20 exist for

$$£_{\mathbf{U}}(V^{\mathbf{x}}H_{\mathbf{x}}) = -\mathcal{D}_A(U^{\mathbf{z}}V^{\mathbf{x}\,*}H_{\mathbf{zx}}{}^{A}) - U^{\mathbf{z}}V^{\mathbf{x}\,*}H_{\mathbf{z}}{}^{\mathbf{y}A\,*}H_{\mathbf{xy}A}$$

$$+ U^{C}\mathcal{D}_C(V^{\mathbf{x}}H_{\mathbf{x}}) - H_{\mathbf{z}}V^{\mathbf{x}}D_{\mathbf{x}}U^{\mathbf{z}} + V^{\mathbf{x}}U^{\mathbf{z}}H_{AC\mathbf{z}}H^{AC}{}_{\mathbf{x}}$$

$$- 1/2\; U^{\mathbf{x}}V_{\mathbf{x}}[{}'R + H_{AC}{}^{\mathbf{z}}H^{AC}{}_{\mathbf{z}} - H^{\mathbf{z}}H_{\mathbf{z}}]$$

$$+ V^{\mathbf{x}}U^{\mathbf{z}}\rho^{-2}g_{\mathbf{xv}}g_{\mathbf{zu}}\varepsilon^{\mathbf{ru}}\varepsilon^{\mathbf{wv}}T_{\mathbf{rw}} \qquad [3.3\text{-}21]$$

$$-1/2 £_{\mathbf{U}}(\gamma^{-2}£_{\mathbf{U}}\gamma^{2}) = - £_{\mathbf{U}}(D_A U^A) - \mathcal{D}_A(U^{\mathbf{x}}U^{\mathbf{z}\,*}H_{\mathbf{zx}}{}^{A})$$

$$- U^{\mathbf{z}}U^{\mathbf{w}\,*}H_{\mathbf{z}}{}^{\mathbf{x}A\,*}H_{\mathbf{wx}A} + U^{C}\mathcal{D}_C(H^{\mathbf{x}}U_{\mathbf{x}})$$

$$+ U^{\mathbf{x}}U^{\mathbf{z}}H_{AC\mathbf{z}}H^{AC}{}_{\mathbf{x}} - H_{\mathbf{x}}U^{\mathbf{z}}D_{\mathbf{z}}U^{\mathbf{x}}$$

$$- 1/2 U_{\mathbf{x}}U^{\mathbf{x}}[{}'R + H_{AC}{}^{\mathbf{z}}H^{AC}{}_{\mathbf{z}} - H^{\mathbf{z}}H_{\mathbf{z}}]$$

$$+ U^{\mathbf{x}}U^{\mathbf{z}}\rho^{-2}g_{\mathbf{xv}}g_{\mathbf{zu}}\varepsilon^{\mathbf{ru}}\varepsilon^{\mathbf{wv}}T_{\mathbf{rw}} \qquad [3.3\text{-}22]$$

The quantity $V^{\mathbf{x}}V^{\mathbf{z}\,*}H_{\mathbf{zx}B}$, which appears on the left-hand side of 3.3-19, can be written in the form

$$V^x V^{z*} H_{zxB} = -1/2 V^x V^z \mathfrak{D}_B g_{zx} = -1/2 \mathfrak{D}_B (V^x V^z g_{zx})$$

$V^x V^z g_{zx}$ can be interpreted as the <u>lapse function</u> of the family of two-surfaces along the deformation vector field **V**. Likewise $U^x U^z g_{zx}$ can be interpreted as the lapse function of the family of two-surfaces along the deformation vector field **U**. The lapse function gives the distance between surfaces of a foliation in a direction orthogonal to the surface. The interpretation of $V^x U^z g_{zx}$ is not obvious. The quantities U^A and V^A are the <u>shift vectors</u>, defining motions within the two-surfaces.

Equations 3.3-7 (plus a corresponding one for $\pounds_U \gamma^2$), 3.3-19, 3.3-20, 3.3-21 and 3.3-22 constitute a coupled system of equations for determining γ^2, $U^x H_x$ and $V^x H_x$ along the integral curves of either of the deformation vector fields **V** or **U** which intersect **N**. If we consider these, for the moment, as just differential equations for γ^2, $U^x H_x$ and $V^x H_x$ (i.e. if we assume that all other quantities which appear in the equation are known at all points along the integral curves of **U** and **V**), then the initial data for this set of equations is just γ^2, $U^x H_x$ and $V^x H_x$ on **N**. It is easy to show formally that γ^2 and $U^x H_x$ can be propagated along **V** by taking iterated Lie derivatives of both sides of equations 3.3-63 and 3.3-64 with respect to **V**, and by recognizing that the right-hand side of the resulting equations can be obtained from the previous iteration provided the aforementioned initial data is given. Now, at any point along the integral curve of **V**, one can start going along the integral curve of the **U**. The initial data for the second deformation is the set of quantities γ^2, and $U^x H_x$ which have already been propagated. This shows that

γ^2, $U^{\mathbf{x}}H_{\mathbf{x}}$ and $V^{\mathbf{x}}H_{\mathbf{x}}$ on $\mathbf{N}$ determine γ^2 everywhere in a four-dimensional region. Despite the fact that, in general, this system comprises four equations, only three independent field equations are involved, as one can show by a somewhat lengthy calculation.

An interesting point can be made about these equations. The next-to-last term on the right-hand side contains the scalar product of the rigging components of the deformation vectors: either $U^{\mathbf{x}}U_{\mathbf{x}}$, $V^{\mathbf{x}}V_{\mathbf{x}}$ or $U^{\mathbf{x}}V_{\mathbf{x}}$. When, for example, the hypersurface dragged out by the $\mathbf{V}$ deformation vector is null, the term $V^{\mathbf{x}}V_{\mathbf{x}}$ vanishes and the entire next-to-last term on the right-hand side disappears. The effect of this is that the Ricci scalar $'R$ of the two-surfaces, which contains surface-derivatives of γ, is no longer present. What was once a partial differential equation for γ, now is reduced to an ordinary differential equation for γ, precisely what we encountered in the analysis of the double-null initial value problem in Section 1.3.

For completeness we recall the dynamical equations for $\tilde{g}_{CD}$ which can be solved in a four dimensional region when appropriate data is given on the initial hypersurfacess. The initial data for this equation will be discussed in Chapters 4 and 5.

$$\begin{aligned} \tilde{G}_{CD} &= -1/2\gamma g^{\mathbf{xy}}\mathcal{D}_{\mathbf{x}}\mathcal{D}_{\mathbf{y}}\tilde{g}_{CD} - \Lambda^{A}{}_{C}{}^{B}{}_{D}\,[\mathcal{D}_{A}{}^{*}H_{B} - {}^{*}H^{\mathbf{yz}}{}_{B}{}^{*}H_{\mathbf{zy}A}] \\ &\quad + \gamma H^{\mathbf{x}}\tilde{H}_{CD\mathbf{x}} + \gamma\tilde{H}_{CE}{}^{\mathbf{x}}\tilde{H}_{D}{}^{E}{}_{\mathbf{x}} + 1/2\gamma\tilde{g}_{CD}\tilde{H}_{AB}{}^{\mathbf{x}}\tilde{H}^{AB}{}_{\mathbf{x}} \\ &= \tilde{T}_{CD} \end{aligned} \qquad [3.3\text{-}23]$$

<u>Evolution of the Rigging Geometry</u>

To complete our calculation of the four-dimensional metric tensor, the rigging basis vectors $C_{\mathbf{x}}^{\nu}$ and the rigging metric tensor must be determined at each space-time point.

Because the vectors $\{C_{\mathbf{x}}^{\nu}, B^{\mu}_{A}\}$ form a tetrad basis at each space-time point, the components of the deformation vectors, **V** and **U**, can be expanded in terms of this basis

$$U^{\nu} = U^{\mathbf{x}} C_{\mathbf{x}}^{\nu} + U^{A} B^{\nu}_{A}$$

$$V^{\nu} = V^{\mathbf{x}} C_{\mathbf{x}}^{\nu} + V^{A} B^{\nu}_{A} \qquad [3.3\text{-}24]$$

Rewriting these as matrix equations:

$$\begin{bmatrix} U^{\mu} - U^{A} B^{\mu}_{A} \\ V^{\mu} - V^{A} B^{\mu}_{A} \end{bmatrix} = \begin{bmatrix} U^{0} & U^{1} \\ V^{0} & V^{1} \end{bmatrix} \begin{bmatrix} C_{0}^{\mu} \\ C_{1}^{\mu} \end{bmatrix} \qquad [3.3\text{-}25]$$

shows that knowledge of U^{A} and V^{A} determines the two-plane to which the rigging basis vectors belong, and no more. Knowledge of $U^{\mathbf{x}}$ and $V^{\mathbf{x}}$, in addition, (assuming the matrix $[U^{\mathbf{x}}\ V^{\mathbf{x}}]$ is invertible) determines the rigging vectors completely. The evolution of the rigging basis vectors is therefore defined by the evolution of the eight quantities $U^{\mathbf{x}}$, $V^{\mathbf{x}}$, U^{A} and V^{A}. Clearly there are far more unknowns than equations. Besides the three components of the two-surface metric g_{AB}, there are the three components of the rigging metric $g_{\mathbf{xy}}$ and the eight components of the deformation vector fields: $U^{\mathbf{x}}$, $V^{\mathbf{x}}$, U^{A} and V^{A}. Four of these functions can be determined by using the four available degrees of coordinate freedom. We shall use this freedom to restrict some of the quantities in the set $\{g_{\mathbf{xy}}, V^{\mathbf{x}}, V^{A}, U^{\mathbf{x}}$ and $U^{A}\}$. In addition, we have the freedom to perform a 2 × 2 non-singular linear transformation (see equation 2.1-4) on the rigging basis vectors.

We shall show that the determination of the geometry of the rigging plane follows from the $G_{A\mathbf{x}} = T_{A\mathbf{x}}$ field equations. As before, we shall write these as deformation equations.

They arise from contracting equation 3.2-85 with V^{z} and U^{x}

$$\pounds_{V}\Omega_{A} = V^{C}D_{C}\Omega_{A} + \Omega_{C}D_{A}V^{C} + V^{y}\Omega_{B}H^{B}{}_{Ay} - \varepsilon^{rs}V^{y}D_{r}{}^{*}H_{(sy)}{}^{A}$$
$$+ \rho^{2}V^{y}\varepsilon_{wy}(D_{C}H_{A}{}^{Cw} - D_{A}H^{w}) + \rho^{2}\varepsilon_{wy}T_{A}{}^{w}V^{y} \quad [3.3-26]$$

$$\pounds_{U}\Omega_{A} = U^{C}D_{C}\Omega_{A} + \Omega_{C}D_{A}U^{C} + U^{y}\Omega_{B}H^{B}{}_{Ay} - \varepsilon^{rs}U^{y}D_{r}{}^{*}H_{(sy)}{}^{A}$$
$$+ \rho^{2}U^{y}\varepsilon_{wy}(D_{C}H_{A}{}^{Cw} - D_{A}H^{w}) + \rho^{2}\varepsilon_{wy}T_{A}{}^{w}U^{y} \quad [3.3-27]$$

When one of the equivalent expressions for Ω^{A} (2.1-74 through 2.1-77) is inserted into equations 3.3-26 and 3.3-27, one finds that the left-hand side contains the highest-order (second-order) derivatives of U^{A} or V^{A} off the surface while the right-hand side contains at most first-order derivatives off the surface. Equations 3.3-26 and 3.3-27 are not sufficient for solving for both U^{A} or V^{A} everywhere; two coordinate conditionsneed to be imposed as we shall see in the examples in the next section.

The last equation we consider is $G_{AB}g^{AB} = T_{AB}g^{AB}$

$$-{}''R + {}^{*}H^{A}{}^{*}H_{A} + 4\rho^{-2}\Omega^{A}\Omega_{A}$$
$$+ \gamma^{-1}g^{xy}\mathcal{D}_{x}\mathcal{D}_{y}\gamma - 1/2\gamma^{-2}g^{xy}\mathcal{D}_{x}\gamma\mathcal{D}_{y}\gamma + g^{xy}\overline{H}_{ABy}\overline{H}^{AB}{}_{x}$$
$$- g^{AB}\mathcal{D}_{A}(\rho^{-1}\mathcal{D}_{B}\rho) = T_{AB}g^{AB} \quad [3.3-28]$$

The Ricci scalar contains second derivatives off the surface of both g_{xy} and U_{x} and V_{x}. This equation cannot be written in Cauchy-Kovalevsky form for all these quantities since it is one equation for seven unknowns. Both the four degrees of rotation freedom and two degrees of gauge freedom must be used to reduce the number of unknowns to one.

Although other choices are possible, we shall discuss two important ways to use the linear transformation freedom to reduce the number of unknowns.

Case 1

The linear 2 × 2 transformation may used to simplify the relationship between the deformation vectors and the rigging vectors. For example, we could take

$$U^{\mu} = C_0{}^{\mu} + U^A B_A{}^{\mu} \qquad [3.3\text{-}29]$$

$$V^{\mu} = C_1{}^{\mu} + V^A B_A{}^{\mu} \qquad [3.3\text{-}30]$$

$U^{\mathbf{x}}$ and $V^{\mathbf{x}}$ are then constants, and therefore some of the objects of anholomicity vanish, namely $\Omega^{\mathbf{z}}{}_{\mathbf{x}A} = 0$ and $\Omega^{\mathbf{z}}{}_{\mathbf{xy}} = 0$. The $g_{\mathbf{xy}}$'s are variables and need to be determined, in part by the field equations or coordinate conditions. The rigging extrinsic curvature now takes the simple form $^{*}H_{\mathbf{xy}A} = -1/2 g_{\mathbf{xy},A}$. The Ricci scalar of the rigging space involves only second-derivatives of the three components of the rigging metric. Two coordinate conditions must then be imposed on the $g_{\mathbf{xy}}$, so that equation 3.3-28 can be solved for the remaining component.

Case 2

By applying the linear transformation 2.1-4, the components of $g_{\mathbf{xy}}$ are made to take some standard form, such as

$$\text{(A)} \qquad g_{\mathbf{xy}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad [3.3\text{-}31]$$

or

$$\text{(B)} \qquad g_{\mathbf{xy}} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \qquad [3.3\text{-}32]$$

Then $g_{\mathbf{xy},\mathbf{z}} = g_{\mathbf{xy},A} = 0$. This amounts to choosing the rigging basis vectors to be an orthonormal dyad (in case (A)) or a null dyad (case (B)). There is still the freedom of performing a one-parameter two-dimensional "Lorentz" transformation on the basis vectors which preserves the form of 3.3-31 and 3.3-32. We shall use this freedom to arbitrarily specify one of the four U^x's or V^y's. Two more coordinate conditions may then be imposed on the three remaining U^x and V^x.

For the ${}^*H_{(xy)A}$, we have

$$ {}^*H_{(xy)A} = -1/2\mathfrak{D}_A g_{xy} = -1/2\{-2\Omega^z{}_{xA}g_{zy} - 2\Omega^z{}_{yA}g_{xz}\} $$

$$ = \Omega_{yAx} + \Omega_{xAy} = 2\Omega_{(x|A|y)} \qquad [3.3\text{-}33] $$

where we define $\Omega_{xAy} = \Omega^z{}_{xA}g_{zy}$.

In Case 2, the rigging connection takes the form (from equation 2.2-46)

$$ {}''\Gamma^x{}_{yz} = g^{xw}\{\Omega_{ywz} - \Omega_{wzy} + \Omega_{zyw}\} \qquad [3.3\text{-}34] $$

It follows that

$$ {}''\Gamma^x{}_{xz} = -2\Omega^x{}_{xz} \qquad [3.3\text{-}35] $$

$$ {}''\Gamma^x{}_{zx} = 0 \qquad [3.3\text{-}36] $$

$$ g^{yz}\,{}''\Gamma^x{}_{yz} = 2g^{xz}\Omega^y{}_{yz} \qquad [3.3\text{-}37] $$

Thus, the Ricci tensor of the rigging space is

$$ {}''R = 4g^{xw}\partial_x\Omega_z{}^z{}_w + 4g^{xw}\Omega^y{}_{yx}\Omega^s{}_{sw} - g^{yw}\Gamma^x{}_{sy}\Gamma^s{}_{xw} $$

$$ + 4g^{yw}\Omega^A{}_{xy}\Omega^z{}_{wA} \qquad [3.3\text{-}38] $$

which contains second derivatives off the surface of U^x and V^x (using 2.1-81 for $\Omega^x{}_{sy}$) but obviously no derivatives of g_{xy}. When the appropriate gauge conditions are applied, the Ricci scalar is a second-order differential operator acting on the one remaining dynamical U^x or V^x.

In the next section, we shall develop examples using the linearized version of the field equations presented in this section.

CHAPTER IV

LINEARIZED GRAVITY IN THE TWO+TWO FORMALISM

4.1 The Linearized Field Equations

A simple way of understanding the geometrical content of the field equations developed in the preceding section is to study their linearized versions. This will help us see more clearly which geometrical quantities are determined by each equation and what initial data must be specified for their solution. The linearized theory is constructed by perturbing each of the metric coefficients by an infinitesimal amount from a known exact solution; in our case, the exact solution is Minkowski space. We pick a background foliation and fibration of Minkowksi space to which to refer the perturbations. All the geometric quantities which we have defined: two-metric, rigging metric, extrinsic curvature tensors, rigging vectors, anholonomic objects and components of the deformation vectors, need to be evaluated in the background. The field equations will govern the perturbations of these quantities, which are considered as fields on the background space-time.

Our first task is to pick an appropriate foliation for Minkowski space-time and a pair of deformation vectors. There are many ways to foliate flat Minkowski space-time into a two-parameter family of two-surfaces, for example, a family of flat two-surfaces, or of two-spheres; still another way is to use a family of two-surfaces with no special metrical properties at all. Our choice will be based solely upon the simplicity of the resulting field equations. We then analyze the perturbation of the rigging vectors, connecting quantities and anholonomic objects. Next, perturbations for the metric tensor projections, extrinsic curvature tensors and curvature scalars will be developed. This will actually be done for a general background space. Then, the linearized field equations are obtained by omitting those terms in the exact equations which are quadratic or higher in the perturbation terms. At this point we shall introduce the simplifying assumption of a flat foliation: all zeroth-order terms vanish in the field equations.

Three examples will be considered in the following sections: the double-null initial value problem worked out in two different gauges, and the Cauchy problem. The first double-null problem we investigate is the linearized version of Sachs' problem discussed in Section 1.3 with the linearized version of his coordinate conditions. This formulation leads to simple equations for the conformal scale factor and rigging anholonomic object but the dynamical equations are coupled to the other parts of the field. In the second version of the double-null problem, the gauge conditions chosen decouple the dynamical equations from the rest of the field equations, but the remaining equations become somewhat more complicated. The third example considers the Cauchy problem using gauge conditions similar to those of the first example.

The simplest foliations to use in Minkowski space have high symmetry. We shall consider a two-parameter family of two-surfaces which are surfaces of transitivity of a pair of Killing vectors X^{μ} and Y^{μ}. Examples of foliations obeying this condition are families of flat two-planes, two-spheres and two-cylinders. The projections of the Killing equationsfor the two vector fields are

$$D_{(A}X_{B)} = D_{(A}Y_{B)} = 0 \qquad [4.1\text{-}1]$$

$$\mathfrak{D}_{\mathbf{x}}X^{B} = \mathfrak{D}_{\mathbf{x}}Y^{B} = 0 \qquad [4.1\text{-}2]$$

$$X^{A*}H_{(\mathbf{xy})A} = Y^{A*}H_{(\mathbf{xy})A} = 0 \qquad [4.1\text{-}3]$$

Equation 4.1-1 shows that X_B and Y_B are Killing vectors with respect to the two-geometry of the imbedded two-surfaces. Equation 4.1-3 implies that

$$^{*}H_{(\mathbf{xy})A} = 0 \qquad [4.1\text{-}4]$$

This means that the rigging spaces are totally geodesic.

In Minkowski space, it is possible to pick the deformation vector fields, **U** and **V**, to be a pair of commuting Killing vector fields orthogonal to the members of the background foliation

$$U_B = U^B = V_B = V^B = 0 \qquad [4.1\text{-}5]$$

We need only pick them orthogonal to a single member of the family of two-surfaces; commutativity and Killing vector conditions insure that they remain orthogonal. Then the rigging planes are holonomic surfaces and we have

$$^{*}H_{[\mathbf{xy}]A} = 0 \qquad [4.1\text{-}6]$$

Of the possible families of two-surfaces to use, we pick flat two-planes, since the extrinsic curvature of each two-plane vanishes which considerably simplifies the linearized field equations to be developed

$$H_{AB\mathbf{x}} = 0 \qquad [4.1\text{-}7]$$

We take equations 4.1-4, 4.1-5 and 4.1-6 as the basic conditions on the unperturbed foliation.

For families of flat two-surfaces, we can further require that the pair of deformation vector fields be <u>translations</u> with respect to the flat space-time. A translation **U** satisfies

$$\nabla_\mu U_\nu = 0 \qquad [4.1\text{-}8]$$

and is automatically a Killing vector. Two translations **U** and **V** necessarily commute since

$$U^{\mu}\nabla_{\mu}V_{\nu} - V^{\mu}\nabla_{\mu}U_{\nu} = 0 \qquad [4.1\text{-}9]$$

[N.B.: A carat (^) above a quantity will denote a quantity belonging to the perturbed space-time while a quantity without a carat will refer to the background Minkowski space-time].

In terms of the 2+2 breakup, the translation property becomes

$$D_{\mathbf{x}}U_{\mathbf{y}} = 0 \qquad [4.1\text{-}10]$$

$$D_{A}U_{\mathbf{x}} = 0, \qquad [4.1\text{-}11]$$

with a similar set of equations holding for **V**. Due to the high degree of symmetry we have imposed, the rigging components of the deformation vectors are constants.

We assume the usual form of the Minkowski metric

$$\eta_{\mu\nu} \overset{*}{=} \text{diagonal } (-1,1,1,1) \qquad [4.1\text{-}12]$$

Letting $C_{\mathbf{x}}{}^{\mu}$ and $B_{A}{}^{\mu}$ be the connecting quantities for the background foliation, we define the unperturbed rigging metric

$$\eta_{\mathbf{xy}} = \eta_{\mu\nu}C_{\mathbf{x}}{}^{\mu}{}_{\mathbf{y}}{}^{\nu} \qquad [4.1\text{-}13]$$

and two-surface metric

$$\eta_{AB} = \eta_{\mu\nu}B_{A}{}^{\mu}{}_{B}{}^{\nu} \qquad [4.1\text{-}14]$$

The perturbations of the connecting quantities can be considered first independently of the metric perturbation. For any rigging perturbation which preserves the family of two-surfaces, we have

$$\hat{B}^{\mu}{}_{A} = B^{\mu}{}_{A} \qquad [4.1\text{-}15]$$

$$\hat{C}_{\mathbf{x}}{}^{\mu} = C_{\mathbf{x}}{}^{\mu} - c_{\mathbf{x}}{}^{\mathbf{y}} C_{\mathbf{y}}{}^{\mu} - b^{A}{}_{\mathbf{x}} B_{A}{}^{\mu} \qquad [4.1\text{-}16]$$

$$\hat{C}^{\mathbf{x}}{}_{\mu} = C^{\mathbf{x}}{}_{\mu} + c_{\mathbf{y}}{}^{\mathbf{x}} C^{\mathbf{y}}{}_{\mu} \qquad [4.1\text{-}17]$$

$$\hat{B}^{A}{}_{\mu} = B^{A}{}_{\mu} + b^{A}{}_{\mathbf{x}} C^{\mathbf{x}}{}_{\mu} \qquad [4.1\text{-}18]$$

They have the same form as the variations in Section 3.2. Below, we shall relate these quantities to the metric perturbation by the simple requirement that the perturbed rigging vectors in the perturbed space-time be orthogonal to each two-surface just as the unperturbed rigging vectors were in the unperturbed space-time.

Because the rigging vectors are changed, any quantity which depends on them is changed. For an arbitrary four-vector s^{μ}, we have

$$\hat{s}^{\mathbf{x}} = \hat{C}^{\mathbf{x}}{}_{\mu} s^{\mu} = s^{\mathbf{x}} + c^{\mathbf{x}}{}_{\mathbf{y}} s^{\mathbf{y}} \qquad [4.1\text{-}19]$$

$$\hat{s}^{A} = \hat{B}^{A}{}_{\mu} s^{\mu} = s^{A} + b^{A}{}_{\mathbf{x}} s^{\mathbf{x}} \qquad [4.1\text{-}20]$$

For a covector, we have

$$\hat{w}_{\mathbf{x}} = \hat{C}_{\mathbf{x}}{}^{\mu} w_{\mu} = w_{\mathbf{x}} - c^{\mathbf{y}}{}_{\mathbf{x}} w_{\mathbf{y}} - b^{A}{}_{\mathbf{y}} w_{A} \qquad [4.1\text{-}21]$$

$$\hat{w}_{A} = \hat{B}_{A}{}^{\mu} w_{\mu} = w_{A} \qquad [4.1\text{-}22]$$

For the original metric tensor in the perturbed basis, we have

$$\hat{\eta}^{\mathbf{xy}} \equiv \hat{C}^{\mathbf{x}}{}_{\mu}{}^{\mathbf{y}}{}_{\nu} \eta^{\mu\nu} = \eta^{\mathbf{xy}} + 2c^{(\mathbf{xy})} \qquad [4.1\text{-}23]$$

$$\hat{\eta}_{\mathbf{xy}} \equiv \hat{C}_{\mathbf{x}}{}^{\mu}{}_{\mathbf{y}}{}^{\nu} \eta_{\mu\nu} = \eta_{\mathbf{xy}} - 2c_{(\mathbf{xy})} \qquad [4.1\text{-}24]$$

where $c_{\mathbf{xy}} \equiv c^{\mathbf{z}}{}_{\mathbf{y}} \eta_{\mathbf{xz}}$ and $c^{\mathbf{xy}} \equiv c^{\mathbf{x}}{}_{\mathbf{z}} \eta^{\mathbf{yz}}$

$$\hat{\eta}^{\mathbf{x}A} \equiv \hat{C}^{\mathbf{x}}{}_{\mu} \hat{B}^{A}{}_{\nu} \eta^{\mu\nu} = b^{A\mathbf{x}} \qquad [4.1\text{-}25]$$

$$\hat{\eta}_{\mathbf{x}A} \equiv \hat{C}_{\mathbf{x}}{}^{\mu} \hat{B}_A{}^{\nu}\eta_{\mu\nu} = -b_{A\mathbf{x}} \quad [4.1\text{-}26]$$

$$\hat{\eta}^{AB} \equiv \hat{B}^A{}_{\mu}{}^B{}_{\nu}\eta^{\mu\nu} = \eta^{AB} \quad [4.1\text{-}27]$$

$$\hat{\eta}_{AB} \equiv \hat{B}_A{}^{\mu}{}_B{}^{\nu}\eta_{\mu\nu} = \eta_{AB} \quad [4.1\text{-}28]$$

In addition,

$$\hat{\eta}^2 = \eta^2 \quad [4.1\text{-}29]$$

$$*\hat{\eta}^2 = *\eta^2 - 2c^{\mathbf{x}}{}_{\mathbf{x}} \quad [4.1\text{-}30]$$

where $-\eta^2$ and $-*\eta^2$ are the determinants of η_{AB} and $\eta_{\mathbf{xz}}$, respectively. ($*\eta^2$ corresponds to ρ^2 defined in Section 2.2)

The perturbations of the anholonomic objects follows from the perturbations of the basis vectors. One can show, after some calculation, that

$$\hat{\Omega}^C{}_{AB} = 0 \quad [4.1\text{-}31]$$

$$\hat{\Omega}^{\mathbf{x}}{}_{AB} = 0 \quad [4.1\text{-}32]$$

$$\begin{aligned}\hat{\Omega}^{\mathbf{x}}{}_{\mathbf{y}A} &= \Omega^{\mathbf{x}}{}_{\mathbf{y}A} - c^{\mathbf{z}}{}_{\mathbf{y}}\Omega^{\mathbf{x}}{}_{\mathbf{z}A} + c^{\mathbf{x}}{}_{\mathbf{w}}\Omega^{\mathbf{w}}{}_{\mathbf{y}A} - 1/2\partial_A c^{\mathbf{x}}{}_{\mathbf{y}} \\ &= \Omega^{\mathbf{x}}{}_{\mathbf{y}A} - 1/2\mathcal{D}_A c^{\mathbf{x}}{}_{\mathbf{y}} \quad [4.1\text{-}33]\end{aligned}$$

$$\hat{\Omega}^A{}_{B\mathbf{x}} = \Omega^A{}_{B\mathbf{x}} - c_{\mathbf{x}}{}^{\mathbf{y}}\Omega^A{}_{B\mathbf{y}} + b^A{}_{\mathbf{w}}\Omega^{\mathbf{w}}{}_{B\mathbf{x}} + 1/2\partial_B b^A{}_{\mathbf{x}} \quad [4.1\text{-}34]$$

$$\hat{\Omega}^A{}_{\mathbf{xy}} = \Omega^A{}_{\mathbf{xy}} - c^{\mathbf{z}}{}_{\mathbf{y}}\Omega^A{}_{\mathbf{xz}} - c^{\mathbf{z}}{}_{\mathbf{x}}\Omega^A{}_{\mathbf{zy}} + 1/2\mathcal{D}_{[\mathbf{x}}b^A{}_{\mathbf{y}]} \quad [4.1\text{-}35]$$

$$\begin{aligned}\hat{\Omega}^{\mathbf{z}}{}_{\mathbf{xy}} &= \Omega^{\mathbf{z}}{}_{\mathbf{xy}} - (b^B{}_{\mathbf{y}}\Omega^{\mathbf{z}}{}_{\mathbf{x}B} - b^B{}_{\mathbf{x}}\Omega^{\mathbf{z}}{}_{\mathbf{y}B}) \\ &\quad - (c_{\mathbf{y}}{}^{\mathbf{s}}\Omega^{\mathbf{z}}{}_{\mathbf{xs}} - c_{\mathbf{x}}{}^{\mathbf{s}}\Omega^{\mathbf{z}}{}_{\mathbf{ys}}) - 1/2[\partial_{\mathbf{x}}c^{\mathbf{z}}{}_{\mathbf{y}} - \partial_{\mathbf{y}}c^{\mathbf{z}}{}_{\mathbf{x}}]\end{aligned} \quad [4.1\text{-}36]$$

The changes in the $\mathcal{D}$-derivative follows from these results

$$\hat{\mathcal{D}}_{\mathbf{x}}\hat{w}_A = \mathcal{D}_{\mathbf{x}}w_A - c_{\mathbf{x}}{}^{\mathbf{y}}\mathcal{D}_{\mathbf{y}}w_A - \mathcal{D}_A(w_B b^B{}_{\mathbf{x}}) \quad [4.1\text{-}37]$$

$$\hat{\mathcal{D}}_A \hat{w}_{\mathbf{x}} = \mathcal{D}_A w_{\mathbf{x}} - c_{\mathbf{x}}{}^{\mathbf{y}}\mathcal{D}_A w_{\mathbf{y}} - \mathcal{D}_A (w_B b^B{}_{\mathbf{x}}) \qquad [4.1\text{-}38]$$

$$\hat{\mathcal{D}}_A \hat{s}^{\mathbf{x}} = \mathcal{D}_A s^{\mathbf{x}} - c_{\mathbf{y}}{}^{\mathbf{x}}\mathcal{D}_A s^{\mathbf{y}} + s^B \mathcal{D}_B b^A{}_{\mathbf{x}}$$

$$- b^B{}_{\mathbf{x}} \mathcal{D}_B s^A - \mathcal{D}_{\mathbf{x}} (b^A{}_{\mathbf{z}} s^{\mathbf{z}}) \qquad [4.1\text{-}39]$$

$$\hat{\mathcal{D}}_{\mathbf{x}} \hat{s}^A = \mathcal{D}_{\mathbf{x}} s^A + \mathcal{D}_{\mathbf{x}} (b_{\mathbf{y}}{}^A s^{\mathbf{y}}) - c_{\mathbf{x}}{}^{\mathbf{y}} \mathcal{D}_{\mathbf{y}} s^A$$

$$+ s^B \mathcal{D}_B b_{\mathbf{x}}{}^A - b_{\mathbf{x}}{}^B \mathcal{D}_B s^A \qquad [4.1\text{-}40]$$

The perturbation of the rigging components of the deformation vectors follows from equations 4.1-19 and 4.1-20. If,

$$\hat{U}^{\mathbf{x}} = U^{\mathbf{x}} + u^{\mathbf{x}} \qquad [4.1\text{-}41]$$

$$\hat{V}^{\mathbf{x}} = V^{\mathbf{x}} + v^{\mathbf{x}}, \qquad [4.1\text{-}42]$$

where $u^{\mathbf{x}}$ and $v^{\mathbf{x}}$ represent the first-order perturbation terms, then

$$u^{\mathbf{y}} = c^{\mathbf{x}}{}_{\mathbf{y}} U^{\mathbf{y}} \qquad [4.1\text{-}43]$$

$$v^{\mathbf{y}} = c^{\mathbf{x}}{}_{\mathbf{y}} V^{\mathbf{y}} \qquad [4.1\text{-}44]$$

(We shall adopt the notational convention that most quantities will be represented by <u>uppercase</u> letters and their perturbations by their <u>lowercase</u> equivalents. We do not apply this rule to the metric tensor, its projections or perturbations.)

For the surface component of the rigging vector,

$$\hat{U}^A = U^A + u^A \qquad [4.1\text{-}45]$$

$$\hat{V}^A = V^A + v^A, \qquad [4.1\text{-}46]$$

where u^A and v^A are given by

$$u^A = b^A{}_{\mathbf{y}} U^{\mathbf{y}} \qquad [4.1\text{-}47]$$

$$v^A = b^A{}_{\mathbf{y}} V^{\mathbf{y}} \qquad [4.1\text{-}48]$$

In our case, by assumption, $U^A = V^A = 0$, then $\hat{U}^A$ and $\hat{V}^A$ are entirely first-order in the perturbation.

The perturbation of the space-time metric has the general form

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \chi_{\mu\nu} \qquad [4.1\text{-}49]$$

$$\hat{g}^{\mu\nu} = \eta^{\mu\nu} - \chi^{\mu\nu} , \qquad [4.1\text{-}50]$$

with the field $\chi_{\mu\nu}$ being the infinitesimal perturbation of the Minkowski metric tensor.

We now consider what happens to various geometrical quantities as a result of the metric perturbation.

The rigging projections of the perturbed metric in the perturbed basis becomes

$$\hat{g}_{\mathbf{xy}} = \eta_{\mathbf{xy}} - 2c_{(\mathbf{xy})} + \chi_{\mathbf{xy}} \qquad [4.1\text{-}51]$$

$$\hat{g}^{\mathbf{xy}} = \eta^{\mathbf{xy}} + 2c^{(\mathbf{xy})} - \chi^{\mathbf{xy}} \qquad [4.1\text{-}52]$$

Requiring that the perturbed rigging vectors remain orthogonal to the two-surfaces relates $\chi_{A\mathbf{x}}$ to $b_{A\mathbf{x}}$:

$$\hat{g}_{A\mathbf{x}} = (\eta_{\mu\nu} + \chi_{\mu\nu}) B^{\mu}{}_{A} (C_{\mathbf{x}}{}^{\nu} + \delta C_{\mathbf{x}}{}^{\nu})$$

$$= \eta_{\mu\nu} B^{\mu}{}_{A} \delta C_{\mathbf{x}}{}^{\nu} + \chi_{A\mathbf{x}} = 0$$

$$= \eta_{\mu\nu} B^{\mu}{}_{A} (-b^{B}{}_{\mathbf{x}} B_{B}{}^{\nu}) + \chi_{A\mathbf{x}} = 0 \qquad [4.1\text{-}53]$$

so that

$$\chi_{A\mathbf{x}} = \eta_{AB} b^{B}{}_{\mathbf{x}} = b_{A\mathbf{x}} \qquad [4.1\text{-}54]$$

Similarly,

$$\chi^{A\mathbf{x}} = b^{A\mathbf{x}} \qquad [4.1\text{-}55]$$

These results show that the perturbed shift vectors are related to the perturbed metric projections, so we may write

$$u^A = \chi^A{}_{\mathbf{x}} U^{\mathbf{x}}$$

$$v^A = \chi^A{}_{\mathbf{x}} V^{\mathbf{x}} \qquad [4.1\text{-}56]$$

The surface projection of the perturbed metric is

$$\hat{g}_{AB} = B_A{}^\nu{}_B{}^\nu(\eta_{\mu\nu} + \chi_{\mu\nu}) = \eta_{AB} + \chi_{AB} \qquad [4.1\text{-}57]$$

$$\hat{g}^{AB} = \eta^{AB} - \chi^{AB} \qquad [4.1\text{-}58]$$

The perturbation of the conformal scale factor is given by

$$\hat{\gamma}^2 = |\hat{g}_{AB}| = 1/2\varepsilon^{AC}\varepsilon^{BD}[\eta_{AB} + \chi_{AB}]\ [\eta_{CD} + \chi_{CD}]$$

$$= 1/2\varepsilon^{AC}\varepsilon^{BD}\eta_{AB}\eta_{CD} + 1/2\varepsilon^{AC}\varepsilon^{BD}\chi_{AB}\eta_{CD} + 1/2\varepsilon^{AC}\varepsilon^{BD}\eta_{AB}\chi_{CD}$$

$$= \eta^2 + 1/2\eta^2\varepsilon^A{}_D\varepsilon^{BD}\chi_{AB} + 1/2\eta^2\varepsilon_B{}^C\varepsilon^{BD}\chi_{CD}$$

$$= \eta^2 + 1/2\eta^2\eta^{AB}\chi_{AB} + 1/2\eta^2\eta^{CD}\chi_{CD}$$

$$= \eta^2(1 + \chi) \qquad [4.1\text{-}59]$$

where $\chi \equiv \eta^{CD}\chi_{CD}$

$$\hat{\gamma}^{-2} = \eta^{-2}(1 - \chi) \qquad [4.1\text{-}60]$$

We also have for the determinant of the perturbed rigging metric

$$\hat{\rho}^2 = {}^*\eta^2 - 2c^{\mathbf{z}}{}_{\mathbf{z}} + {}^*\chi \qquad [4.1\text{-}61]$$

where ${}^*\chi \equiv \eta^{\mathbf{xy}}\chi_{\mathbf{xy}}$

The perturbation of the conformal two-metric is

$$\tilde{g}_{AB} = \eta^{-1}(1 - 1/2\chi)[\eta_{AB} + \chi_{AB}] = \eta^{-1}(\eta_{AB} + \chi_{AB} - 1/2\chi\,\eta_{AB}) \qquad [4.1\text{-}62]$$

(For typographic reasons, we omit the carat over the perturbed conformal two-metric)

If we define the traceless two-tensor density,

$$\tilde{\chi}_{AB} = \eta^{-1}(\chi_{AB} - 1/2\chi\eta_{AB}) \qquad [4.1\text{-}63]$$

satisfying $g^{AB}\tilde{\chi}_{AB} = 0$, then the perturbation of the conformal two-metric can be written

$$\tilde{g}_{AB} = \eta^{-1}\eta_{AB} + \tilde{\chi}_{AB} \qquad [4.1\text{-}64]$$

The perturbed Christoffel symbols of the two-surfaces are given by

$$\hat{\Gamma}^{D}{}_{BC} = 1/2\,\hat{g}^{DE}\{-\hat{g}_{BC,E} + \hat{g}_{EB,C} + \hat{g}_{CE,B}\}$$
$$= \Gamma^{D}{}_{BC} + 1/2\eta^{DE}\{-\chi_{BC:E} + \chi_{EB:C} + \chi_{CE:B}\} \qquad [4.1\text{-}65]$$

where $\Gamma^{D}{}_{BC}$ denotes the affine connection of the background two-metric and the colon (':') denotes covariant derivative with respect to the background metric.

The perturbed Ricci scalar of the two-surface takes the form

$$'\hat{R} = (1-\chi)\,'R - g^{AB}\chi_{:AB} + \chi^{AB}{}_{:AB} \qquad [4.1\text{-}66]$$

In terms of the conformal perturbation, the perturbed Ricci scalar is

$$'\hat{R} = (1-\chi)\,'R - 1/2\eta^{AB}\chi_{:AB} + \tilde{\chi}^{AB}{}_{:AB} \qquad [4.1\text{-}67]$$

The determination of the perturbed affine connection and Ricci scalar of the rigging space is far more complicated since the perturbation changes the orientation of the rigging space as well as its intrinsic metric. The details have been worked out by Smallwood [1980] in the context of a variational principle. We use his results

$$''\hat{\Gamma}^{\mathbf{x}}{}_{\mathbf{yz}} = ''\Gamma^{\mathbf{x}}{}_{\mathbf{yz}} + 1/2\eta^{\mathbf{xw}}\{-(\chi_{\mathbf{yz}} - c_{\mathbf{yz}})_{:\mathbf{w}} + (\chi_{\mathbf{wy}} - c_{\mathbf{wy}})\chi_{:\mathbf{z}} + (\chi_{\mathbf{zw}} - c_{\mathbf{zw}})_{:\mathbf{y}}\} \qquad [4.1\text{-}68]$$

$$''\hat{R} = (1-{}^{*}\chi)\,''R - \eta^{\mathbf{xy}*}\chi_{:\mathbf{xy}} + \chi^{\mathbf{xy}}{}_{:\mathbf{xy}} + 2b^{A}{}_{\mathbf{x}}(D^{\mathbf{x}*}H_{A} - D^{\mathbf{x}*}H^{\mathbf{z}}{}_{\mathbf{x}A}) + 2(D^{\mathbf{x}}(b^{A}{}_{\mathbf{x}}{}^{*}H_{A} - b^{A}{}_{\mathbf{x}}{}^{*}H^{\mathbf{z}}{}_{\mathbf{x}A}) \qquad [4.1\text{-}69]$$

The perturbed extrinsic curvature is obtained by applying the perturbed $\mathfrak{D}$-derivative to g_{AB}. We have thus

$$\hat{\mathfrak{D}}_{\mathbf{x}}\hat{g}_{AB} = \mathfrak{D}_{\mathbf{x}}\eta_{AB} + \mathfrak{D}_{\mathbf{x}}\chi_{AB} - c_{\mathbf{x}}{}^{\mathbf{y}}\mathfrak{D}_{\mathbf{y}}\eta_{AB} - 2\mathfrak{D}_{(A}b_{B)\mathbf{x}}, \qquad [4.1\text{-}70]$$

which, when the definition of the extrinsic curvature is used, becomes

$$\hat{H}_{AB\mathbf{x}} = H_{AB\mathbf{x}} - 1/2\mathfrak{D}_{\mathbf{x}}\chi_{AB} + 1/2c_{\mathbf{x}}{}^{\mathbf{y}}\mathfrak{D}_{\mathbf{y}}\eta_{AB} + \mathfrak{D}_{(A}b_{B)\mathbf{x}}$$

$$\hat{H}_{AB\mathbf{x}} = H_{AB\mathbf{x}} + h_{(AB)\mathbf{x}}, \qquad [4.1\text{-}71]$$

where

$$h_{(AB)\mathbf{x}} \equiv -1/2\mathcal{D}_{\mathbf{x}}\chi_{AB} + 1/2c_{\mathbf{x}}{}^{\mathbf{y}}\mathcal{D}_{\mathbf{y}}\eta_{AB} + \mathcal{D}_{(A}b_{B)\mathbf{x}} \qquad [4.1\text{-}72]$$

The perturbed mean extrinsic curvature is defined by

$$\hat{H}_{\mathbf{x}} = \hat{g}^{AB}\hat{H}_{AB\mathbf{x}} = (\eta^{AB} - \chi^{AB})(H_{AB\mathbf{x}} + h_{AB\mathbf{x}})$$

$$= \eta^{AB}H_{AB\mathbf{x}} - \chi^{AB}H_{AB\mathbf{x}} + \eta^{AB}h_{AB\mathbf{x}} \qquad [4.1\text{-}73]$$

$$= H_{\mathbf{x}} - \chi^{AB}H_{AB\mathbf{x}} + \eta^{AB}h_{AB\mathbf{x}} \qquad [4.1\text{-}74]$$

Inserting $h_{AB\mathbf{x}}$ from equation 4.1-72, we obtain

$$\hat{H}_{\mathbf{x}} = H_{\mathbf{x}} - \chi^{AB}H_{AB\mathbf{x}} + \eta^{AB}(-1/2\mathcal{D}_{\mathbf{x}}\chi_{AB} + 1/2c_{\mathbf{x}}{}^{\mathbf{y}}\mathcal{D}_{\mathbf{y}}\eta_{AB} + \mathcal{D}_{(A}b_{B)\mathbf{x}})$$

$$= H_{\mathbf{x}} - 1/2\mathcal{D}_{\mathbf{x}}\chi + 1/2c_{\mathbf{x}}{}^{\mathbf{y}}H_{\mathbf{y}} + \mathcal{D}_{A}b^{A}{}_{\mathbf{x}}$$

$$= H_{\mathbf{x}} + h_{\mathbf{x}}, \qquad [4.1\text{-}75]$$

where

$$h_{\mathbf{x}} \equiv -1/2\mathcal{D}_{\mathbf{x}}\chi + 1/2c_{\mathbf{x}}{}^{\mathbf{y}}H_{\mathbf{y}} + \mathcal{D}_{A}b^{A}{}_{\mathbf{x}} \qquad [4.1\text{-}76]$$

From the definition of the conformal extrinsic curvature, $\tilde{H}_{(AB)\mathbf{x}}$, from Section 2.3, we have its linearized form as

$$\tilde{H}_{AB\mathbf{x}} = \eta^{-1}[1 - 1/2\chi]\,\overline{H}_{AB\mathbf{x}} + \eta^{-1}\overline{h}_{AB\mathbf{x}} - 1/2\eta^{-1}\overline{\chi}_{AB}H_{\mathbf{x}}$$

$$-1/4\eta^{-1}\eta_{AB}\chi H_{\mathbf{x}}, \qquad [4.1\text{-}77]$$

where the bar over a quantity denotes removal of its trace with respect to the background metric. We then have

$$\tilde{H}_{AB\mathbf{x}} = \tilde{H}_{AB\mathbf{x}} + \tilde{h}_{AB\mathbf{x}}$$

where

$$\tilde{h}_{AB\mathbf{x}} \equiv \eta^{-1}\overline{h}_{AB\mathbf{x}} - \eta^{-1}1/2\chi\,\tilde{H}_{AB\mathbf{x}} - 1/2\eta^{-1}\overline{\chi}_{AB}H_{\mathbf{x}} - 1/4\eta^{-1}\eta_{AB}\chi H_{\mathbf{x}}$$

[4.1-78]

The traceless quantity $\overline{h}_{(AB)\mathbf{x}}$ is given by

$$\overline{h}_{(AB)\mathbf{x}} \equiv - 1/2\Lambda^{C}{}_{A}{}^{D}{}_{B}\mathfrak{D}_{\mathbf{x}}\chi_{CD} + 1/2c_{\mathbf{x}}{}^{\mathbf{y}}\Lambda^{C}{}_{A}{}^{D}{}_{B}\mathfrak{D}_{\mathbf{y}}\eta_{CD} + \Lambda^{C}{}_{A}{}^{D}{}_{B}\mathfrak{D}_{(C}b_{D)\mathbf{x}}$$

$$= - 1/2\Lambda^{C}{}_{A}{}^{D}{}_{B}\mathfrak{D}_{\mathbf{x}}\chi_{CD} - c_{\mathbf{x}}{}^{\mathbf{y}}\overline{H}_{AB\mathbf{y}} + \Lambda^{C}{}_{A}{}^{D}{}_{B}\mathfrak{D}_{(C}b_{D)\mathbf{x}}$$

$$= - 1/2\mathfrak{D}_{\mathbf{x}}\overline{\chi}_{CD} + 1/2\chi_{CD}\mathfrak{D}_{\mathbf{x}}\Lambda^{C}{}_{A}{}^{D}{}_{B} - c_{\mathbf{x}}{}^{\mathbf{y}}H_{AB\mathbf{y}} + \Lambda^{C}{}_{A}{}^{D}{}_{B}\mathfrak{D}_{(C}b_{D)\mathbf{x}}$$

$$= - 1/2\mathfrak{D}_{\mathbf{x}}\overline{\chi}_{AB} + 1/2\chi H_{AB\mathbf{x}} - 1/2\chi_{CD}H^{CD}{}_{\mathbf{x}}g_{AB}$$

$$- c_{\mathbf{x}}{}^{\mathbf{y}}\overline{H}_{AB\mathbf{y}} + \Lambda^{C}{}_{A}{}^{D}{}_{B}\mathfrak{D}_{(C}b_{D)\mathbf{x}} \qquad [4.1\text{-}79]$$

Inserting this into 4.1-78, gives, after some calculation,

$$\widetilde{h}_{AB\mathbf{x}} \equiv -1/2\mathfrak{D}_{\mathbf{x}}\widetilde{\chi}_{AB} - 1/2\eta^{-1}\widetilde{\chi}_{CD}\widetilde{H}^{CD}{}_{\mathbf{x}}\eta_{AB} - 1/4\eta^{-1}\chi H_{\mathbf{x}}\eta_{AB}$$

$$- c_{\mathbf{x}}{}^{\mathbf{y}}\widetilde{H}_{AB\mathbf{y}} + \eta^{-1}\Lambda^{C}{}_{A}{}^{D}{}_{B}\mathfrak{D}_{(C}b_{D)\mathbf{x}} \qquad [4.1\text{-}80]$$

For the rigging space, the perturbed extrinsic curvature tensor is

$$^{\star}\hat{H}_{(\mathbf{xy})A} \equiv -1/2\,\hat{\mathfrak{D}}_{A}\,\hat{g}_{\mathbf{xy}} = {}^{\star}H_{(\mathbf{xy})A} - 1/2\mathfrak{D}_{A}\chi_{\mathbf{xy}}$$

$$- 1/2c_{\mathbf{x}}{}^{\mathbf{z}}{}^{\star}H_{(\mathbf{zy})A} - 1/2c_{\mathbf{y}}{}^{\mathbf{z}}{}^{\star}H_{(\mathbf{xz})A}$$

$$= {}^{\star}H_{(\mathbf{xy})A} + {}^{\star}h_{\mathbf{xy}A}, \qquad [4.1\text{-}81]$$

where $^{\star}h_{\mathbf{xy}} = - 1/2\mathfrak{D}_{A}\chi_{\mathbf{xy}} - 1/2c_{\mathbf{x}}{}^{\mathbf{z}}{}^{\star}H_{(\mathbf{zy})A} - 1/2c_{\mathbf{y}}{}^{\mathbf{z}}{}^{\star}H_{(\mathbf{xz})A}$

[4.1-82]

and

$$^{\star}\hat{H}_{A} = {}^{\star}H_{A} - 1/2\mathfrak{D}_{A}{}^{\star}\chi$$

$$= {}^{\star}H_{A} + {}^{\star}h_{A}, \qquad [4.1\text{-}83]$$

where $^{\star}h_{A} = -1/2\mathfrak{D}_{A}{}^{\star}\chi$

For the antisymmetric part of $^{\star}\hat{H}_{\mathbf{xy}A}$, we have, from equation

4.1-35,

$$^{*}\hat{H}_{[\mathbf{xy}]A} = -\hat{\Omega}_{\mathbf{xy}A} = -(\Omega_{\mathbf{xy}A} + \omega^{A}{}_{\mathbf{xy}})$$

where

$$\omega^{A}{}_{\mathbf{xy}} \equiv - c^{\mathbf{z}}{}_{\mathbf{y}}\Omega^{A}{}_{\mathbf{xz}} - c^{\mathbf{z}}{}_{\mathbf{x}}\Omega^{A}{}_{\mathbf{zy}} + 1/2\mathcal{D}_{[\mathbf{x}}b^{A}{}_{\mathbf{y}]} \qquad [4.1\text{-}84]$$

We introduce the vector density ω^A by

$$\omega^{A}{}_{\mathbf{xy}} = \varepsilon_{\mathbf{xy}}\omega^{A} \text{ where } \omega^{A} \equiv 1/2\varepsilon^{\mathbf{xy}}\omega^{A}{}_{\mathbf{xy}} \qquad [4.1\text{-}85]$$

[Note we violate our notation convention and represent the symmetric part of the perturbation of the rigging extrinsic curvatures by $^{*}h_{\mathbf{xy}A}$, using $\omega_{\mathbf{xy}A}$ for the antisymmetric part].

In order to obtain the linearized field equations, we just omit those terms in the exact field equations in Section 3.3 which are quadratic or higher in the perturbation. The resulting equations are particularly simple if all the zero order extrinsic curvature terms vanish, which is true if the foliation consists of flat two-surfaces. In that case we have

$$\hat{H}_{AB\mathbf{x}} = h_{AB\mathbf{x}} = -1/2\mathcal{D}_{\mathbf{x}}\chi_{AB} + \mathcal{D}_{(A}b_{B)\mathbf{x}} \qquad [4.1\text{-}86]$$

$$\hat{H}_{\mathbf{x}} = h_{\mathbf{x}} = - 1/2\mathcal{D}_{\mathbf{x}}\chi + \mathcal{D}_{A}b^{A}{}_{\mathbf{x}} \qquad [4.1\text{-}87]$$

$$\tilde{H}_{AB\mathbf{x}} = h_{AB\mathbf{x}} = -1/2\mathcal{D}_{\mathbf{x}}\tilde{\chi}_{AB} + \eta^{-1}\Lambda^{C}{}_{A}{}^{D}{}_{B}\mathcal{D}_{(C}b_{D)\mathbf{x}} \qquad [4.1\text{-}88]$$

$$^{*}\hat{H}_{(\mathbf{xy})A} = {}^{*}h_{\mathbf{xy}A} = -1/2\mathcal{D}_{A}\chi_{\mathbf{xy}} \qquad [4.1\text{-}89]$$

$$^{*}\hat{H}_{A} = {}^{*}h_{A} = -1/2\mathcal{D}_{A}{}^{*}\chi \qquad [4.1\text{-}90]$$

$$\hat{\Omega}^{A}{}_{\mathbf{xy}} = \omega^{A}{}_{\mathbf{xy}} = 1/2\mathcal{D}_{[\mathbf{x}}b^{A}{}_{\mathbf{y}]} \qquad [4.1\text{-}91]$$

$$'\hat{R} = -1/2\eta^{AB}\chi_{:AB} + \tilde{\chi}^{AB}{}_{:AB} \qquad [4.1\text{-}92]$$

$$''\hat{R} = -\eta^{\mathbf{xy}}{}^{*}\chi_{:\mathbf{xy}} + \chi^{\mathbf{xy}}{}_{:\mathbf{xy}} \qquad [4.1\text{-}93]$$

When these expressions are inserted into the linearized field equations, we arrive at (using the stress-energy tensor

$T_{\mu\nu}$ defined with respect to the background metric and assumed to be of first order in the perturbation.).

The <u>rigging-projected</u> field equations:

$$\text{I)}\quad \pounds_{\mathbf{U}}(U_{\mathbf{x}}h^{\mathbf{x}}) = -\mathcal{D}_A(U^{\mathbf{x}}U^{\mathbf{z}}{}^*h_{\mathbf{zx}}{}^A) - 1/2U^{\mathbf{x}}U_{\mathbf{x}}(-1/2\eta^{AB}\chi_{:AB} + \tilde{\chi}^{AB}{}_{:AB})$$
$$+ U^{\mathbf{x}}U^{\mathbf{z}}\eta_{\mathbf{xv}}\eta_{\mathbf{zu}}\varepsilon^{\mathbf{ru}}\varepsilon^{\mathbf{wv}}T_{\mathbf{rw}} \qquad [4.1-94]$$

$$\text{II)}\quad \pounds_{\mathbf{V}}(V_{\mathbf{x}}h^{\mathbf{x}}) = -\mathcal{D}_A(V^{\mathbf{x}}V^{\mathbf{z}}{}^*h_{\mathbf{zx}}{}^A) - 1/2V^{\mathbf{x}}V_{\mathbf{x}}(-1/2\eta^{AB}\chi_{:AB} + \tilde{\chi}^{AB}{}_{:AB})$$
$$+ V^{\mathbf{x}}V^{\mathbf{z}}\eta_{\mathbf{xv}}\eta_{\mathbf{zu}}\varepsilon^{\mathbf{ru}}\varepsilon^{\mathbf{wv}}T_{\mathbf{rw}} \qquad [4.1-95]$$

$$\text{III)}\quad \pounds_{\mathbf{V}}(U_{\mathbf{x}}h^{\mathbf{x}}) = -\mathcal{D}_A(U^{\mathbf{x}}V^{\mathbf{z}}{}^*h_{\mathbf{zx}}{}^A + U^{\mathbf{x}}V^{\mathbf{z}}\omega_{\mathbf{zx}}{}^A)$$
$$- 1/2U^{\mathbf{x}}V_{\mathbf{x}}(-1/2\eta^{AB}\chi_{:AB} + \tilde{\chi}^{AB}{}_{:AB}) + V^{\mathbf{x}}U^{\mathbf{z}}\eta_{\mathbf{xv}}\eta_{\mathbf{zu}}\varepsilon^{\mathbf{ru}}\varepsilon^{\mathbf{wv}}T_{\mathbf{rw}}$$
$$[4.1-96]$$

with the auxilliary equations

$$\pounds_{\mathbf{V}}\chi = 2(D_A v^A - V^{\mathbf{x}}h_{\mathbf{x}}) \qquad [4.1-97]$$

$$\pounds_{\mathbf{U}}\chi = 2(D_A u^A - U^{\mathbf{x}}h_{\mathbf{x}}) \qquad [4.1-98]$$

The <u>surface-rigging</u> projected field equations are

$$\text{IV)}\quad \pounds_{\mathbf{U}}\omega_A = -\varepsilon^{\mathbf{rs}}U^{\mathbf{y}}D_{\mathbf{r}}{}^*h_{\mathbf{sy}A} + U^{\mathbf{y}}\varepsilon_{\mathbf{wy}}(D_C h_A{}^{C\mathbf{w}} - D_A h^{\mathbf{w}})$$
$$+ {}^*\eta^2\varepsilon_{\mathbf{wy}}T_A{}^{\mathbf{w}}U^{\mathbf{y}} \qquad [4.1-99]$$

$$\text{V)}\quad \pounds_{\mathbf{V}}\omega_A = -\varepsilon^{\mathbf{rs}}V^{\mathbf{y}}D_{\mathbf{r}}{}^*h_{\mathbf{sy}A} + V^{\mathbf{y}}\varepsilon_{\mathbf{wy}}(D_C h_A{}^{C\mathbf{w}} - D_A h^{\mathbf{w}})$$
$$+ {}^*\eta^2\varepsilon_{\mathbf{wy}}T_A{}^{\mathbf{w}}V^{\mathbf{y}} \qquad [4.1-100]$$

together with the auxilliary equations

$$U^{\mathbf{x}}\mathcal{D}_{\mathbf{x}}v^B - V^{\mathbf{x}}\mathcal{D}_{\mathbf{x}}u^B = 2U^{\mathbf{x}}V^{\mathbf{y}}\omega^B{}_{\mathbf{xy}}$$
$$= 2U^{\mathbf{x}}V^{\mathbf{y}}\varepsilon_{\mathbf{xy}}\omega^B = 2\xi\omega^B \qquad [4.1-101]$$

where $\xi \equiv U^{\mathbf{x}}V^{\mathbf{y}}\varepsilon_{\mathbf{xy}}$

which is obtained by contracting 4.1-91 with $U^{\mathbf{x}}$ and $V^{\mathbf{y}}$ and using equations 4.1-47 and 4.1-48.

Equation 4.1-101 may also be written

$$\pounds_{\mathbf{U}}v^{B} - \pounds_{\mathbf{V}}u^{B} = 2\xi\omega^{B} \qquad [4.1\text{-}102]$$

The trace of the <u>surface-projected</u> field equations is

$$\text{VI)} \quad -1/4\eta^{\mathbf{xy}}\mathcal{D}_{\mathbf{x}}\mathcal{D}_{\mathbf{y}}\chi - \eta^{AB}\mathcal{D}_{A}{}^{*}h_{B} - {}''\hat{R} = T_{AB}g^{AB} \qquad [4.1\text{-}103]$$

The <u>traceless surface-projection</u> of the field equations

$$\text{VII)} \quad -1/2\eta^{\mathbf{xy}}\mathcal{D}_{\mathbf{x}}\mathcal{D}_{\mathbf{y}}\tilde{\chi}_{CD} + 1/2\eta^{\mathbf{xy}}\mathcal{D}_{\mathbf{x}}\Lambda^{A}{}_{C}{}^{B}{}_{D}\mathcal{D}_{(A}b_{B)\mathbf{y}} - \Lambda^{A}{}_{C}{}^{B}{}_{D}\mathcal{D}_{A}{}^{*}h_{B}$$
$$= \tilde{T}_{AB} \qquad [4.1\text{-}104]$$

With our choice of foliation, $\mathcal{D}_{\mathbf{x}}$ and $\mathcal{D}_{A}$ commute (see Schouten [1954]) and since $\mathcal{D}_{\mathbf{x}}\Lambda^{A}{}_{C}{}^{B}{}_{D} = 0$, we can write this equation as

$$-1/2\eta^{\mathbf{xy}}\mathcal{D}_{\mathbf{x}}\mathcal{D}_{\mathbf{y}}\tilde{\chi}_{CD} + 1/2\Lambda^{A}{}_{C}{}^{B}{}_{D}\mathcal{D}_{(A}\mathcal{D}_{\mathbf{x}}b_{B)}{}^{\mathbf{x}} - \Lambda^{A}{}_{C}{}^{B}{}_{D}\mathcal{D}_{A}{}^{*}h_{B} = 0$$
$$[4.1\text{-}105]$$

[N.B. When we propose integration schemes for the field equations, we shall refer to them by Roman numerals.]

The gauge group of the linearized field equations is the set of linear coordinate transformations

$$\chi_{\mu\nu} \rightarrow {}^{\#}\chi_{\mu\nu} = \chi_{\mu\nu} + 2\kappa_{(\mu:\nu)}, \qquad [4.1\text{-}106]$$

where κ_{μ} is an arbitrary infinitesimal vector field.

Next we consider various possible integration schemes. The quantities for which the field equations are to be solved are

$\tilde{\chi}_{AB}$, χ, $c_{\mathbf{xy}}$, $\chi_{\mathbf{xy}}$, and $b^{A}{}_{\mathbf{x}}$.

4.2 The Linearized Sachs' Double-Null Initial Value Problem

We consider first the linearized version of the vacuum double-null initial value problem. In this example, the two deformation vectors **U** and **V** are chosen to be null vectors with respect to the Minkowski metric and are orthogonal to an initial space-like two-surface $\mathbf{N}_0$ which is flat. The initial

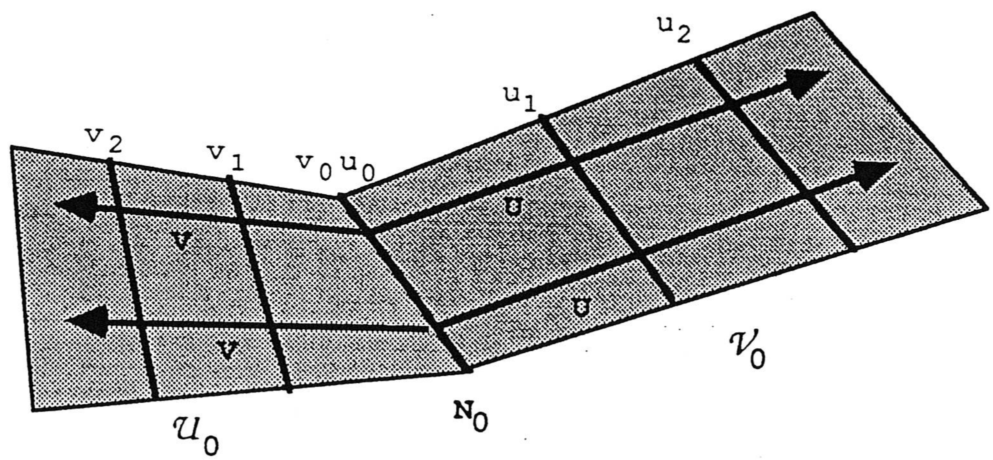

Figure 4-1 Linearized Double-Null Initial Value Problem

null hypersurfaces on which data is to be set are created by dragging $\mathbf{N}_0$ along **U** and **V**, respectively. The two hypersurfaces thus formed are labelled $\mathcal{V}_0$ and $\mathcal{U}_0$, respectively (see Figure 4-1). If we parametrize the vector fields **U** and **V** by u and v, respectively, then the hypersurface given by u=0 corresponds to $\mathcal{U}_0$ and the hypersurface given by v=0 to $\mathcal{V}_0$. $\mathcal{U}_u$ and $\mathcal{V}_v$ represent surfaces of constant u and v, respectively. The null deformation vectors satisfy (recalling that $U^A = V^A = 0$)

$$U^{\mu}U_{\mu} = U^{\mathbf{x}}U_{\mathbf{x}} = 0 \qquad [4.2\text{-}1]$$

$$V^{\mu}V_{\mu} = V^{\mathbf{x}}V_{\mathbf{x}} = 0 \qquad [4.2\text{-}2]$$

and without loss of generality, we can set

$$U^{\mu}V_{\mu} = U^{\mathbf{x}}V_{\mathbf{x}} = -1 \qquad [4.2\text{-}3]$$

We choose the rigging basis $C_{\mathbf{x}}{}^{\mu}$ so that $\eta_{\mathbf{xy}}$ is a constant matrix of the form given by equation 3.3-32.

Since, as we saw in Section 3.3, we are free to perform any non-singular linear transformation on the rigging vectors, we shall perturb the rigging vectors in such a way as to preserve their metrical properties. i.e, the perturbed rigging metric shall also be of the form 3.3-32. Then, from equation 4.1-51, we have the following relation between the perturbed metric tensor and the $c_{\mathbf{x}}{}^{\mathbf{y}}$

$$c_{\mathbf{xy}} + c_{\mathbf{yx}} = 2c_{(\mathbf{xy})} = \chi_{\mathbf{xy}} \qquad [4.2\text{-}4]$$

and for the trace of the rigging perturbation

$$^{*}\chi = 2c^{\mathbf{x}}{}_{\mathbf{x}} \qquad [4.2\text{-}5]$$

Condition 4.2-4 fixes three of the four components of $c_{\mathbf{xy}}$. It represents no restriction on the space-time metric but only on the choice of basis vectors spanning the rigging space.

We still have the freedom to perform a one-dimensional "Lorentz" transformation, i.e., a transformation which preserves the form of $\eta_{\mathbf{xy}}$. Letting $L^{\mathbf{x}}{}_{\mathbf{y}}$ be such an infinitesimal Lorentz transformation

$$L^{\mathbf{x}}{}_{\mathbf{y}} = \delta^{\mathbf{x}}{}_{\mathbf{y}} + e^{\mathbf{x}}{}_{\mathbf{y}} \qquad [4.2\text{-}6]$$

where $e^{x}{}_{y}$ is an arbitrary infinitesimal quantity which satisfies

$$e_{xy} + e_{yx} = 0 \qquad [4.2\text{-}7]$$

There is only one free-parameter in e_{xy}. Under such a transformation, $c^{x}{}_{y}$ transforms as

$$c^{x}{}_{y} \rightarrow c^{x}{}_{y} + e^{x}{}_{y} \qquad [4.2\text{-}8]$$

e_{xy} can be chosen to arbitrarily fix the fourth $c^{x}{}_{y}$.

At this point, we need to establish several identities which are easily proved and are useful in subsequent calculations. They rely on the fact that V^{x} and U^{x} span the rigging space. They are

$$\eta^{xy} = 2U^{(x}V^{y)} \qquad [4.2\text{-}9]$$

$$U^{x}\varepsilon_{xy} = -\xi U_{y} \qquad [4.2\text{-}10]$$

$$V^{x}\varepsilon_{xy} = \xi V_{y} \qquad [4.2\text{-}11]$$

$$U_{x}\varepsilon^{xy} = \xi\eta^{2}U^{y} \qquad [4.2\text{-}12]$$

$$V_{x}\varepsilon^{xy} = -\xi\eta^{2}V^{y} \qquad [4.2\text{-}13]$$

and

$$\chi_{xy} = \chi_{VV}U_{x}U_{y} + 2\chi_{UV}U_{(x}V_{y)} + \chi_{UU}V_{x}V_{y} \qquad [4.2\text{-}14]$$

where

$$\chi_{UU} = \chi_{xy}U^{x}U^{y}$$

$$\chi_{VV} = \chi_{xy}V^{x}V^{y}$$

$$\chi_{UV} = \chi_{xy}U^{x}V^{y}$$

We shall use the gauge freedom to impose the conditions: that the surfaces to which **U** and **V** are tangent remain null surfaces with respect to the perturbed metric. This can be expressed in terms of the perturbed metric components by $\chi_{UU} = 0$ and $\chi_{VV} = 0$:

$$\hat{U}^{\mathbf{x}}\hat{U}^{\mathbf{y}}\hat{g}_{\mathbf{xy}} = 0 = (U^{\mathbf{x}} + U^{\mathbf{z}}c^{\mathbf{x}}{}_{\mathbf{z}})(U^{\mathbf{y}} + U^{\mathbf{w}}c^{\mathbf{y}}{}_{\mathbf{w}})(\eta_{\mathbf{xy}} - c_{\mathbf{xy}} - c_{\mathbf{yx}} + \chi_{\mathbf{xy}})$$

$$= (U^{\mathbf{x}}U^{\mathbf{y}} + U^{\mathbf{y}}U^{\mathbf{z}}c^{\mathbf{x}}{}_{\mathbf{z}} + U^{\mathbf{x}}U^{\mathbf{w}}c^{\mathbf{y}}{}_{\mathbf{w}})(\eta_{\mathbf{xy}} - c_{\mathbf{xy}} - c_{\mathbf{yx}} + \chi_{\mathbf{xy}})$$

$$= (U^{\mathbf{x}}U^{\mathbf{y}} + U^{\mathbf{y}}U^{\mathbf{z}}c^{\mathbf{x}}{}_{\mathbf{z}} + U^{\mathbf{x}}U^{\mathbf{w}}c^{\mathbf{y}}{}_{\mathbf{w}})\eta_{\mathbf{xy}}$$

$$+ U^{\mathbf{x}}U^{\mathbf{y}}(-c_{\mathbf{xy}} - c_{\mathbf{yx}} + \chi_{\mathbf{xy}})$$

$$= U^{\mathbf{x}}U^{\mathbf{y}}\chi_{\mathbf{xy}} = \chi_{UU} = 0; \qquad [4.2\text{-}15]$$

Similarly,

$$\hat{V}^{\mathbf{x}}\hat{V}^{\mathbf{y}}\hat{g}_{\mathbf{xy}} = V^{\mathbf{x}}V^{\mathbf{y}}\chi_{\mathbf{xy}} = \chi_{VV} = 0 \qquad [4.2\text{-}16]$$

To show that condition may 4.2-15 may be imposed, project the gauge transformation equations 4.1-106 into the rigging direction and contract with $U^{\mathbf{x}}U^{\mathbf{y}}$. We seek a transformation such that $U^{\mathbf{x}}U^{\mathbf{y}\#}\chi_{\mathbf{xy}}$ vanishes

$$U^{\mathbf{x}}U^{\mathbf{y}}\chi_{\mathbf{xy}} \rightarrow U^{\mathbf{x}}U^{\mathbf{y}\#}\chi_{\mathbf{xy}} = 0 = U^{\mathbf{x}}U^{\mathbf{y}}\chi_{\mathbf{xy}} + U^{\mathbf{x}}U^{\mathbf{y}}D_{\mathbf{x}}\kappa_{\mathbf{y}} \qquad [4.2\text{-}17]$$

Since $D_{\mathbf{x}}U^{\mathbf{y}} = 0$, equation 4.2-122 takes the form

$$U^{\mathbf{x}}D_{\mathbf{x}}(U^{\mathbf{y}}\kappa_{\mathbf{y}}) = -U^{\mathbf{x}}U^{\mathbf{y}}\chi_{\mathbf{xy}} = -\chi_{UU}$$

or

$$\pounds_{\mathbf{U}}(\kappa_U) = -\chi_{UU} \qquad [4.2\text{-}18]$$

where $\kappa_U \equiv U^{\mathbf{y}}\kappa_{\mathbf{y}}$. A solution to 4.2-18 is given by

$$\kappa_U = -\int\chi_{UU}du + F_0(v,x^A), \qquad [4.2\text{-}19]$$

where $F_0(v,x^A)$ is an arbitrary function of the parameter <u>v</u> and the two-surface coordinates. With κ_U of the form 4.2-19, the gauge condition 4.2-15 holds.

Similarly, if we choose

$$\kappa_V \equiv U^{\mathbf{y}}\kappa_{\mathbf{y}} = -\int\chi_{VV}dv + F_1(u,x^A) \qquad [4.2\text{-}20]$$

where is an arbitrary function of three-parameters, then the gauge condition 4.2-16 holds.

Equations 4.2-15 and 4.2-16 also imply that

$$U^{\mathbf{x}}U^{\mathbf{y}*}h_{\mathbf{xy}A} = -1/2\mathfrak{D}_A(U^{\mathbf{x}}U^{\mathbf{y}}\chi_{\mathbf{xy}}) = 0 \qquad [4.2\text{-}21]$$

$$V^{\mathbf{x}}V^{\mathbf{y}*}h_{\mathbf{xy}A} = -1/2\mathfrak{D}_A(V^{\mathbf{x}}V^{\mathbf{y}}\chi_{\mathbf{xy}}) = 0 \qquad [4.2\text{-}22]$$

The two arbitrary functions of three-variables F_0 and F_1 can be picked to set

$$\chi_{UV} \equiv U^{\mathbf{x}}V^{\mathbf{y}}\chi_{\mathbf{xy}} = 0 \text{ on } \mathcal{U}_0 \text{ and } \mathcal{V}_0, \qquad [4.2\text{-}23]$$

which also implies that

$$U^{\mathbf{x}}V^{\mathbf{y}*}h_{\mathbf{xy}}{}^{A} = 0 \text{ on } \mathcal{U}_0 \text{ and } \mathcal{V}_0 \qquad [4.2\text{-}24]$$

The remaining two four-dimensional coordinate conditions can be used to require that one of the deformation vectors, say $\mathbf{U}$, remain orthogonal to the members of the family of two-surfaces, i.e. that the perturbation u^A vanish. Hence

$$\hat{U}_A = U^{\mu}B_A{}^{\nu}\hat{g}_{\mu\nu} = 0 = U^{\mu}B_A{}^{\nu}(\eta_{\mu\nu} + \chi_{\mu\nu}) = \chi_{AU} = U^{\mathbf{x}}b_{A\mathbf{x}} \qquad [4.2\text{-}25]$$

where $\chi_{AU} \equiv U^{\mathbf{x}}\chi_{A\mathbf{x}}$

It is not hard to show that equation 4.2-25, together with the other gauge conditions on $U^{\mathbf{x}}$ and $V^{\mathbf{x}}$, implies that $b_{A\mathbf{x}}$ has the form

$$b_{A\mathbf{x}} = \chi_{A\mathbf{x}} = -v_A U_{\mathbf{x}} \qquad [4.2\text{-}26]$$

This equation relates the shift vectors to the metric perturbation.

The coordinate condition 4.2-25 can be imposed by using the remaining gauge transformation freedom in choosing κ_A

$$\chi_{A\mathbf{y}}U^{\mathbf{y}} \rightarrow {}^{\#}\chi_{A\mathbf{y}}U^{\mathbf{y}} = 0 = \chi_{A\mathbf{y}}U^{\mathbf{y}} + 2U^{\mathbf{y}}D_{(A}\kappa_{\mathbf{y})} \qquad [4.2\text{-}27]$$

which can be expressed as

$$U^{\mathbf{y}}D_{\mathbf{y}}\kappa_A = -\chi_{A\mathbf{y}}U^{\mathbf{y}} - D_A(\kappa_{\mathbf{y}}U^{\mathbf{y}})$$

or, since the zeroth-order extrinsic curvature tensors vanish, as a Lie derivative

$$\pounds_{\mathbf{U}}\kappa_A = -\chi_{AU} - D_A(\kappa_U). \qquad [4.2\text{-}28]$$

Equation 4.2-28 has the solution

$$\kappa_A = -\int[\chi_{AU} - D_A(\kappa_U)]\,du + F_A(v, x^B) \qquad [4.2\text{-}29]$$

where F_A are an arbitrary pair of functions of three parameters, which can be chosen so that

$$v_A = \chi_{AV} = 0 \text{ on } \mathcal{U}_0 \qquad [4.2\text{-}30]$$

Summarizing the gauge conditions, they are

$$\chi_{UU} = 0$$

$$\chi_{VV} = 0$$

$$u_A = \chi_{AU} = 0$$

$$\chi_{UV} = 0 \text{ on } \mathcal{U}_0 \text{ and } \mathcal{V}_0,$$

$$v_A = \chi_{AV} = 0 \text{ on } \mathcal{U}_0$$

These gauge conditions are equivalent to the coordinate conditions imposed by Sachs in his formulation of the double-null initial value problem discussed in Section 1.3.

We can now compute the linearized Ricci scalar of the rigging space. Imposing the gauge conditions on equation 4.2-14, gives

$$\chi_{\mathbf{xy}} = 2\chi_{UV}U_{(\mathbf{x}}V_{\mathbf{y})} \qquad [4.2\text{-}31]$$

$$^{*}\chi = -2\chi_{UV} \qquad [4.2\text{-}32]$$

and if this is used in equation 4.1-93, we obtain

$$"\hat{R} = 2\eta^{xy}\chi_{UV:xy} + 2U^{(x}V^{y)}\chi_{UV:xy}$$

$$= 4U^{x}D_{x}(V^{y}D_{y}\chi_{UV}) = 4£_{U}£_{V}\chi_{UV} \qquad [4.2\text{-}33]$$

After incorporating the coordinate conditions, the <u>rigging-projections</u> of the field equations become

I) $£_{U}(U_{x}h^{x}) = \xi^{2}\eta^{4}T_{UU}$ where $T_{UU} \equiv T_{xy}U^{x}U^{y}$ [4.2-34]

with

$$£_{U}\chi = -2U^{x}h_{x} \qquad [4.2\text{-}35]$$

which implies

$$£^{2}_{U}\chi = -2\xi^{2}\eta^{4}T_{UU} \qquad [4.2\text{-}36]$$

II) $£_{V}(V_{x}h^{x}) = \xi^{2}\eta^{4}T_{VV}$ where $T_{VV} \equiv T_{xy}V^{x}V^{y}$ [4.2-37]

$$£_{V}\chi = 2(D_{A}v^{A} - V^{x}h_{x}) \qquad [4.2\text{-}38]$$

which implies

$$£^{2}_{V}\chi - 2£_{V}(D_{A}v^{A}) = -2\xi^{2}\eta^{4}T_{VV} \qquad [4.2\text{-}39]$$

and

III) $£_{V}(U_{x}h^{x}) = -\mathcal{D}_{A}(V^{x}U^{z*}h_{xz}{}^{A} + V^{x}U^{z}\omega_{xz}{}^{A})$

$$- 1/2V^{x}U_{x}(-1/2\eta^{AB}\chi_{:AB} + \tilde{\chi}^{AB}{}_{:AB}) - \xi^{2}\eta^{4}T_{VU},$$

where $T_{VU} \equiv T_{xy}V^{x}U^{y}$; [4.2-40]

which takes the form

$$£_{V}(U_{x}h^{x}) = -1/2\mathcal{D}_{A}\mathcal{D}^{A}\chi_{UV} + \mathcal{D}_{A}\omega^{A}$$

$$-1/2V^{x}U_{x}(-1/2\eta^{AB}\chi_{:AB} + \tilde{\chi}^{AB}{}_{:AB}) - \xi^{2}\eta^{4}T_{VU} \qquad [4.2\text{-}41]$$

or equivalently

$$£_{V}£_{U}\chi = \mathcal{D}_{A}\mathcal{D}^{A}\chi_{UV} - 2\mathcal{D}_{A}\omega^{A}$$

$$+ V^x U_x(-1/2\eta^{AB}\chi_{:AB} + \tilde{\chi}^{AB}{}_{:AB}) + 2\xi^2\eta^4 T_{VU} \qquad [4.2\text{-}42]$$

The <u>surface-rigging</u> projections are:

IV) $$(2\xi)^{-1}\pounds_U(\pounds_U v^A) = -\varepsilon^{rs}U^y D_r{}^*h_{sy}{}^A + U^y\varepsilon_{wy}(D_C h^{ACw} - D^A h^w)$$

$$+ \xi\eta^2 T_{AU} \qquad [4.2\text{-}43]$$

with $T_{AU} \equiv T_{Ax}U^x$

V) $$(2\xi)^{-1}\pounds_V(\pounds_U v^A) = -\varepsilon^{rs}V^y D_r{}^*h_{sy}{}^A + V^y\varepsilon_{wy}(D_C h^{ACw} - D^A h^w)$$

$$- \xi\eta^2 T_{AV} \qquad [4.2\text{-}44]$$

where $T_{AV} \equiv T_{Ax}V^x$

where we have used the gauge conditions to put equation 4.1-101 in the form

$$\omega^A = (2\xi)^{-1}(\pounds_U v^A), \qquad [4.2\text{-}45]$$

and substituted thie into 4.1-99 and 4.1-100.

Using the identities 4.1-10 through 4.2-13 and 4.2-31, the right-hand sides of equations 4.2-43 and 4.2-44 can be simplified. For example, because $D_r U^y = 0$ and $D_A U^y = 0$, in equation 4.2-43 we can bring U^y into the covariant derivatives to get

$$(2\xi)^{-1}\pounds_U(\pounds_U v^A) = -D_r(\varepsilon^{rs}U^{y*}h_{sy}{}^A) + D_C(U^y\varepsilon_{wy}h^{ACw}) - D^A(U^y\varepsilon_{wy}h^w) \qquad [4.2\text{-}46]$$

After some further manipulations, we obtain

IV) $$(2\xi)^{-1}\pounds^2_U v^A = U^r D_r\mathcal{D}^A\chi_{UV} + D_C(U^w h^{AC}{}_w) - D^A(U^w h_w)$$

$$+ \xi\eta^2 T_{AU} \qquad [4.2\text{-}47]$$

and

V) $$(2\xi)^{-1}\pounds_V\pounds_U v^A = -V^r D_r\mathcal{D}^A\chi_{UV} - D_C(V^w h^{AC}{}_w) + D^A(V^w h_w)$$

$$- \xi \eta^2 T_{AV} \qquad [4.2\text{-}48]$$

The trace of the surface-projection of the field equations becomes

$$\text{VI)} \quad -1/2 \pounds_{\mathbf{U}} \pounds_{\mathbf{V}} \chi - \eta^{AB} \mathcal{D}_A \mathcal{D}_B \chi_{UV} + 4 \pounds_{\mathbf{U}} \pounds_{\mathbf{V}} \chi_{UV} = T_{AB} g^{AB} \qquad [4.2\text{-}49]$$

The traceless surface-projection or dynamical field equations becomes

$$\text{VII)} \quad -\pounds_{\mathbf{U}} \pounds_{\mathbf{V}} \tilde{\chi}_{CD} + 1/2 \Lambda^A{}_C{}^B{}_D \mathcal{D}_{(A} \pounds_{\mathbf{U}} v_{B)} - \Lambda^A{}_C{}^B{}_D \mathcal{D}_A \mathcal{D}_B \chi_{UV} = \tilde{T}_{CD}$$

[4.2-50]

We can now outline an integration scheme for this system of equations with the following initial data:

- Three quantities on an initial two-surface $\mathbf{N}_0$: $\chi, h_{\mathbf{x}}$
- A two-vector on $\mathbf{N}_0$: $\omega^A = \pounds_{\mathbf{U}} v^A$
- A one-parameter family of symmetric traceless two-tensors $\tilde{\chi}_{AB}$ on $\mathcal{U}_0$ which is the linear form of the conformal to structure on $\mathcal{U}_0$.
- A one-parameter family of symmetric traceless two-tensors $\tilde{\chi}_{AB}$ on $\mathcal{V}_0$ which is the linear form of the conformal two-structure on $\mathcal{V}_0$.
- The three- and four-dimensional gauge conditions which we discussed above.

The integration scheme proceeds as follows:

1) Integrate equation (II) for χ on $\mathcal{U}_0$ in terms of $[\chi]_0$ and $[V^{\mathbf{x}} h_{\mathbf{x}}]_0 = -1/2 \partial_V \chi$ on $\mathbf{N}_0$. This has the general solution

$$\chi = [V^{x} h_{x}]_0 v + [\chi]_0 \qquad [4.2\text{-}51]$$

2) Integrate equation (V) on $\mathcal{U}_0$ for $\omega^A = 1/2 \pounds_U v^A$ in terms of χ on $\mathcal{U}_0$ (which is known from the previous step) and $\tilde{\chi}^{AB}$ on $\mathcal{U}_0$, which represents specifiable initial data, and the initial value $[\omega^A]_0$ on $\mathbf{N}_0$. Note that χ_{UV} vanishes on $\mathcal{U}_0$. The solution has the general form

$$\pounds_U v^A = \int \xi [D_C(\pounds_V \tilde{\chi}^{AC}) - D^A(\pounds_V \chi)] dv + [\xi \pounds_U v^A]_0 \qquad [4.2\text{-}52]$$

3) Integrate (III) on $\mathcal{U}_0$ for $U^{x} h_{x} = -1/2 \pounds_U \chi$ in terms of $\pounds_U v^A$, $\tilde{\chi}^{AC}$ and χ on $\mathcal{U}_0$ (which are known from Steps 1 and 2) and $[U^{x} h_{x}]_0$ on $\mathbf{N}_0$ (χ_{UV} vanishes on $\mathcal{U}_0$). The solution has the form

$$U_{x} h^{x} = \int [\xi \mathcal{D}_A(\omega^A) + 1/4 \eta^{AB} \chi_{,AB} - 1/2 \tilde{\chi}^{AB}{}_{,AB}] dv + [U_{x} h^{x}]_0 \qquad [4.2\text{-}53]$$

4) Integrate (I) along the trajectories of $\mathbf{U}$ for χ everywhere in terms of χ and $U_{x} h^{x}$ on $\mathcal{U}_0$ (which are known from steps 1 and 3).

$$\chi = [U^{x} h_{x}]_0 u + [\chi]_0 \qquad [4.2\text{-}54]$$

5) Integrate (VI) for χ_{UV}, since χ is now known everywhere. The initial data for this null wave equation is χ_{UV} on $\mathcal{U}_0$ and $\mathcal{V}_0$ which is zero by virtue of the gauge freedom.

6) Integrate (IV) and (VII) together since they are coupled. This can be done iteratively, using (VII) to

calculate $\pounds_{\mathbf{V}}\tilde{\chi}_{AC}$ on the next infinitesimally close null hypersurface $\mathcal{U}_1$ assuming $\pounds_{\mathbf{U}}v^A$ is known on $\mathcal{U}_0$, which is true from step 2. With $\pounds_{\mathbf{V}}\tilde{\chi}_{AC}$ now known on $\mathcal{U}_1$, $\tilde{\chi}_{AC}$ is determined on $\mathcal{U}_1$ from its values on $\mathcal{V}_0$ which are part of the specifiable data. Equation (IV) then gives $\pounds_{\mathbf{U}}v^A$ on $\mathcal{U}_1$ and the first iteration is complete. These steps may be repeated as many times as necessary to propagate $\tilde{\chi}_{AC}$ and $\pounds_{\mathbf{U}}v^A$ (and hence v^A).

4.3 The Linearized Double-Null Initial Value Problem in the Decoupling Gauge

The gauge conditions used in the previous example were the linearized counterpart of those used by Sachs in the exact double-null initial value problem (Section 1.3). In this section, we show that gauge conditions may be chosen so as to decouple the dynamics of the linearized conformal two-metric from the remaining components of the metric field. This enables one to solve the dynamical equations.

The gauge conditions we shall adopt are

$$^{*}\chi = \eta^{\mathbf{xy}}\chi_{\mathbf{xy}} = 0 \qquad [4.3\text{-}1]$$

$$\chi_{UU} = U^{\mathbf{x}}U^{\mathbf{y}}\chi_{\mathbf{xy}} = 0 \qquad [4.3\text{-}2]$$

$$\eta^{\mathbf{xy}}\mathbf{D}_{\mathbf{x}}\chi^{A}{}_{\mathbf{y}} = 0 \qquad [4.3\text{-}3]$$

We shall show how they are implemented below. Condition 4.3-2 has the same interpretation it did in the previous example: the hypersurfaces $\{\mathcal{V}_v\}$ tangent to the vector field $\mathbf{U}$ are null surfaces. This is no longer true of the hypersurfaces $\{\mathcal{U}_u\}$ tangent to $\mathbf{V}$ except for $\mathcal{U}_0$. This example deserves to be called "double-null" by virtue of the null character of both initial hypersurfaces, this condition is not maintained for the $\{\mathcal{U}_u\}$-family.

Since the null rigging metric still has the form 4.2-9, the gauge condition 4.3-1 can be re-written

$$\chi_{UV} = 2U^{\mathbf{x}}V^{\mathbf{y}}\chi_{\mathbf{xy}} = 0 \qquad [4.3\text{-}4]$$

Instead of the form given by equation 4.2-31, $\chi_{\mathbf{xy}}$ now takes the form

$$\chi_{\mathbf{xy}} = \chi_{VV}U^{\mathbf{x}}U^{\mathbf{y}} \qquad [4.3\text{-}5]$$

The four quantities $c_{\mathbf{xy}}$ are fixed the same way they were in the previous example.

Also using the form 4.2-9 of the rigging metric tensor, the gauge condition 4.3-3 can be rewritten as

$$£_{\mathbf{U}}v^A + £_{\mathbf{V}}u^A = 0 \qquad [4.3\text{-}6]$$

We now proceed to show how these coordinate conditions can be implemented. Since coordinate condition 4.3-2 is the same as one adopted in the previous example, we pick a function κ_U such that

$$\kappa_U \equiv \kappa_{\mathbf{x}}U^{\mathbf{x}} = -\int\chi_{UU}du + F_0(v, x^A) \qquad [4.3\text{-}7]$$

with $F_0(v,x^A)$ an arbitrary function of three parameters.

The condition 4.3-4 is imposed by solving

$$\#\chi_{UV} = £_{\mathbf{U}}\kappa_V + £_{\mathbf{V}}\kappa_U + \chi_{UV} = 0 \qquad [4.3\text{-}8]$$

which has the general solution

$$\kappa_V = -\int\chi_{UV}du - \int[£_{\mathbf{V}}\int\chi_{UU}du]du' + \int£_{\mathbf{V}}F_0(v,x^A)du + F_1(v,x^A), \qquad [4.3\text{-}9]$$

with $F_1(v,x^A)$ another arbitrary function of three parameters. Under these gauge transformations induced by $\kappa_{\mathbf{x}}$, the value of χ_{VV}, the remaining projection of $\chi_{\mathbf{xy}}$, changes by $2£_{\mathbf{V}}\kappa_V$. We can use the arbitrariness of F_0 and F_1 to set

$$\chi_{VV} = £_{\mathbf{U}}\chi_{VV} = 0 \text{ on } \mathcal{U}_0 \qquad [4.3\text{-}10]$$

The last of the gauge conditions (4.3-3) can be imposed by projecting 4.1-106 into both the rigging and surface, giving

$$\#\chi_{A\mathbf{x}} = \chi_{A\mathbf{x}} + \mathbf{D}_A\kappa_{\mathbf{x}} + D_{\mathbf{x}}\kappa_A = 0 \qquad [4.3\text{-}11]$$

Taking the divergence this equation and re-arranging terms, we get

$$\eta^{xy}D_x D_y \kappa_A = - D_x \chi_A{}^x - \mathcal{D}_A D_x \kappa^x$$

and using the form of the metric tensor in terms of U^x and V^x

$$2\pounds_U \pounds_V \kappa_A = - \mathcal{D}_A (D_x \kappa^x) - \mathcal{D}_x \chi_A{}^x \qquad [4.3\text{-}12]$$

which is the two-dimensional inhomogeneous wave equation for κ_A and can hence be solved uniquely up to an arbitrary solution of the homogeneous equation. We can use this remaining freedom to set $v^A = 0$ on $\mathcal{U}_0$ and $u^A = 0$ on $\mathcal{V}_0$.

With these gauge conditions, the perturbed Riemann tensor of the rigging space takes the form

$$"\hat{R} = \pounds^2{}_U \chi_{VV} \qquad [4.3\text{-}13]$$

The <u>rigging-projections</u> of the vacuum field equations in this gauge become

I) $$\pounds_U (U_x h^x) = 0 \qquad [4.3\text{-}14]$$

with

$$\pounds_U \chi = 2(D_A u^A - U^x h_x) \qquad [4.3\text{-}15]$$

II) $$\pounds_V (V_x h^x) = 1/2 \mathcal{D}_A \mathcal{D}^A \chi_{VV} \qquad [4.3\text{-}16]$$

with

$$\pounds_V \chi = 2(D_A v^A - V^x h_x) \qquad [4.3\text{-}17]$$

III) $$\pounds_V (U_x h^x) = 1/2 \mathcal{D}_A \omega^A + 1/2(-1/2\eta^{AB}\chi_{:AB} + \tilde{\chi}^{AB}{}_{:AB}) \qquad [4.3\text{-}18]$$

For ω^B, we have

$$\pounds_{\mathbf{U}} v^B - \pounds_{\mathbf{V}} u^B = 2\xi\omega^B; \qquad [4.3\text{-}19]$$

but the gauge condition

$$\pounds_{\mathbf{U}} v^B + \pounds_{\mathbf{V}} u^B = 0 \qquad [4.3\text{-}20]$$

lets us solve for ω^B as

$$\omega^B = \xi^{-1}\pounds_{\mathbf{U}} v^B = -\xi^{-1}\pounds_{\mathbf{V}} u^B \qquad [4.3\text{-}178]$$

The <u>surface-rigging</u> projected field equations become

IV) $\pounds^2{}_{\mathbf{U}} v_A = 2D_C(h_A{}^{C\mathbf{x}}U_{\mathbf{x}}) - 2D_A(h^{\mathbf{x}}U_{\mathbf{x}})$ [4.3-20]

V) $\pounds_{\mathbf{V}}\pounds_{\mathbf{U}} v_A = -V^{\mathbf{y}}D_{\mathbf{y}}D_A\chi_{VV} - 2D_C(h_A{}^{C\mathbf{x}}V_{\mathbf{x}}) - 2D_A(h^{\mathbf{x}}V_{\mathbf{x}})$

[4.3-21]

The trace of the <u>surface-projection</u> equations becomes

VI) $2\pounds^2{}_{\mathbf{U}}\chi_{VV} - \pounds_{\mathbf{U}}\pounds_{\mathbf{V}}\chi = 0$ [4.3-22]

The <u>dynamical equation</u> or <u>traceless-surface-projection</u> equations is simply

VII) $\pounds_{\mathbf{U}}\pounds_{\mathbf{V}}\tilde{\chi}_{AB} = 0$ [4.3-23]

We now consider an integration scheme for this set of equations using the same set of initial data that was used in the Sach's double null initial value problem. The scheme proceeds as follows:

1) Integrate the null wave equations (VII) for $\tilde{\chi}_{AB}$ in a four-dimensional region in terms of $\tilde{\chi}_{AB}$ on $\mathcal{U}_0$ and $\mathcal{V}_0$. In this gauge, the linearized conformal two-structure, the dynamical part of the field, is propagated independently of the other parts. The most general solution to (VII) has the form

$$\tilde{\chi}_{AB} = \zeta_{AB}(u,x^A) + \psi_{AB}(v,x^A)$$

where $\zeta_{AB}(u,x^A)$ and $\psi_{AB}(v,x^A)$ are arbitrary functions.

2) Solve (II) on $\mathcal{U}_0$ for χ in terms of $[\chi]_0$ and $[V^{\mathbf{x}}h_{\mathbf{x}}]_0$ on $\mathbf{N}_0$ recalling that v^A and χ_{VV} vanish on $\mathcal{U}_0$.

3) Solve (V) on $\mathcal{U}_0$ for $\pounds_{\mathbf{U}}v^A$ in terms of χ_{VV} (which vanishes), $\tilde{\chi}_{AB}$ (part of the specifiable initial data) and χ (from step 2) on $\mathcal{U}_0$ and $[\pounds_{\mathbf{U}}v^A]_0$ on $\mathbf{N}_0$.

4) Solve (III) for $U^{\mathbf{x}}h_{\mathbf{x}}$ on $\mathcal{U}_0$ in terms of χ, $\tilde{\chi}_{AB}$ and ω^A on $\mathcal{U}_0$, and $[U^{\mathbf{x}}h_{\mathbf{x}}]_0$ on $\mathbf{N}_0$.

5) Solve the coupled system (I), (IV), and (VI), using the gauge equation

$$\pounds_{\mathbf{V}}u^B = -\pounds_{\mathbf{U}}v^B,$$

iteratively for χ, χ_{VV} and v^A everywhere in terms of χ, $U^{\mathbf{x}}h_{\mathbf{x}}$, χ_{VV}(which vanishes), $\pounds_{\mathbf{U}}\chi_{VV}$ (which vanishes), v^A (which vanishes) and $\pounds_{\mathbf{U}}v^A$ on $\mathcal{U}_0$ and u^B on $\mathcal{V}_0$.

Generalized Conformal Two-structure.

Only $\tilde{\chi}_{AB}$ is specified on $\mathcal{U}_0$ and $\mathcal{V}_0$ while the rest of the initial data is given on $\mathbf{N}_0$. One may wonder whether it is possible to specify all the initial data on a single two-surface. We now outline how this can be done for $\tilde{\chi}_{AB}$. Knowledge of $\tilde{\chi}_{AB}$ on $\mathcal{U}_0$ is formally equivalent to knowledge of the Lie derivatives to all orders of $\tilde{\chi}_{AB}$ with respect to $\mathbf{V}$ on $\mathbf{N}_0$; $\tilde{\chi}_{AB}$ on $\mathcal{U}_0$ can then be written in a Taylor series expansion.

Now the Lie derivative of $\tilde{\chi}_{AB}$ on $\mathcal{U}_0$ can be written

$$\pounds_{\mathbf{V}}\tilde{\chi}_{AB} = V^{\mathbf{x}}\mathcal{D}_{\mathbf{x}}\tilde{\chi}_{AB} \qquad [4.3\text{-}24]$$

The 2nd order Lie derivative is

$$\pounds^2{}_{\mathbf{V}}\tilde{\chi}_{AB} = V^{\mathbf{x}_1}V^{\mathbf{x}_2}\mathcal{D}_{\mathbf{x}_1}\mathcal{D}_{\mathbf{x}_2}\tilde{\chi}_{AB} \qquad [4.3\text{-}25]$$

For the nth-order Lie derivative we have

$$\pounds^n{}_{\mathbf{V}}\tilde{\chi}_{AB} = V^{\mathbf{x}_1}V^{\mathbf{x}_2}\ldots V^{\mathbf{x}_n}\mathcal{D}_{\mathbf{x}_1}\mathcal{D}_{\mathbf{x}_2}\ldots\mathcal{D}_{\mathbf{x}_n}\tilde{\chi}_{AB} \qquad [4.3\text{-}26]$$

The nth-Lie derivative with respect to $\mathbf{V}$ can thus be written entirely in terms of the projection of the totally symmetric part $\mathcal{D}_{(\mathbf{x}_1}\mathcal{D}_{\mathbf{x}_2}\ldots\mathcal{D}_{\mathbf{x}_n)}\tilde{\chi}_{AB}$ into the $\mathbf{V}$ direction. However, because the dynamical field equation takes the form

$$\eta^{\mathbf{xy}}\mathcal{D}_{\mathbf{x}}\mathcal{D}_{\mathbf{y}}\tilde{\chi}_{AB} = 0 \qquad [4.3\text{-}27]$$

all the $\mathcal{D}_{(\mathbf{x}_1}\mathcal{D}_{\mathbf{x}_2}\ldots\mathcal{D}_{\mathbf{x}_n)}\tilde{\chi}_{AB}$ must be completely traceless (with respect to the rigging indices) in order to generate a solution to the field equations.

Similarly, to prescribe $\tilde{\chi}_{AB}$ on $\mathcal{V}_0$, one needs the projection of the completely traceless part (also with respect to the rigging indices) of $\mathcal{D}_{(\mathbf{x}_1}\mathcal{D}_{\mathbf{x}_2}\ldots\mathcal{D}_{\mathbf{x}_n)}\tilde{\chi}_{AB}$ in the $\mathbf{U}$ direction. Thus, formally at least, the initial data for the dynamical equations consists of a denumerably infinite set of covariant totally-symmetric traceless tensor fields on $\mathbf{N}_0$, each with a pair of indices on $\mathbf{N}_0$, and all the remaining indices in the rigging space. We denote by <u>generalized conformal two-structure</u>, the specification of such a set.

For a pair of null deformation vector fields, the only non-vanishing projections of $\mathfrak{D}_{(x_1}\mathfrak{D}_{x_2}\ldots\mathfrak{D}_{x_n)}\tilde{\chi}_{AB}$ are

$$U^{x_1}U^{x_2}\ldots U^{x_n}\mathfrak{D}_{(x_1}\mathfrak{D}_{x_2}\ldots\mathfrak{D}_{x_n)}\tilde{\chi}_{AB}$$

$$V^{x_1}V^{x_2}\ldots V^{x_n}\mathfrak{D}_{(x_1}\mathfrak{D}_{x_2}\ldots\mathfrak{D}_{x_n)}\tilde{\chi}_{AB}$$

Any mixed projections will necessarily vanish because $U^{(x}V^{y)}$ is proportional to η^{xy}.

We show at the end of Chapter 5 that, to any order, a totally symmetric traceless tensor in a two-dimensional space has only two independent components; thus each component of the genralized conformal two-structure has two independent rigging components and two independent surface components.

4.4 The Linearized Cauchy Problem

The deformation vectors are **U** and **V**, but now **U** is a unit time-like vector and **V** is a unit space-like vector with respect to the Minkowski metric. They are orthogonal to each other and to members of our family of flat two-surfaces. The initial space-like hypersurface $\mathbf{S}_0$ for the Cauchy problem arises by dragging the initial two-surface $\mathbf{N}_0$ along the space-like vector field **V**.

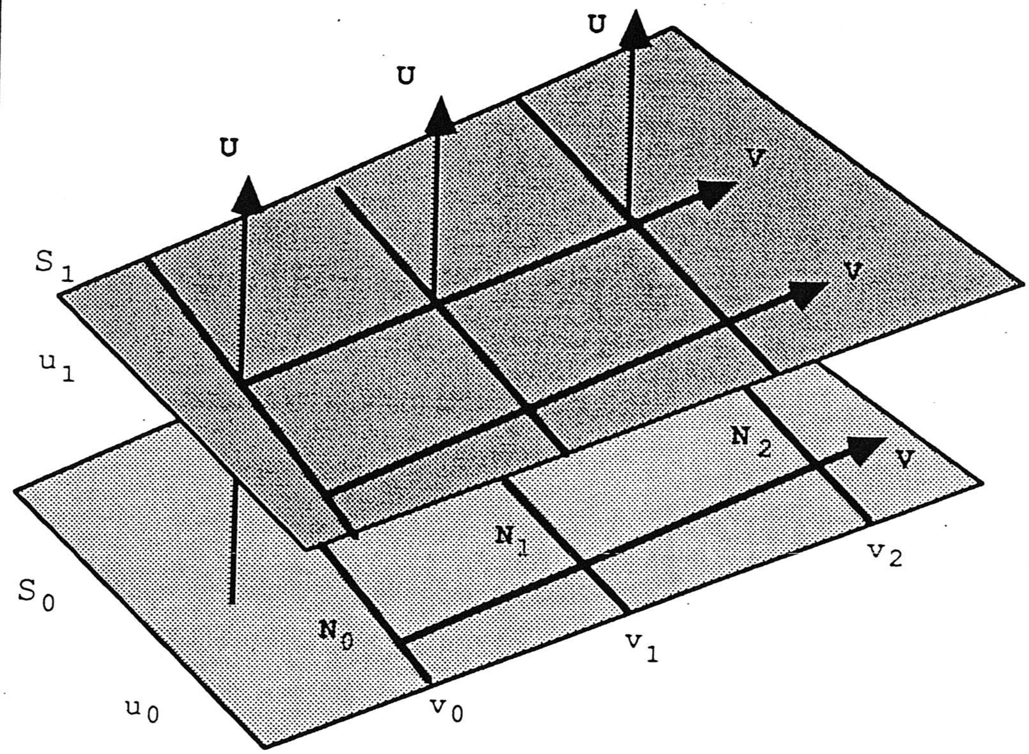

Figure 4-2 The Linearized Cauchy Problem

The deformation vectors **U** and **V** thus satisfy

$$U^A = V^A = 0 \qquad [4.4\text{-}1]$$

$$U^{\mu}U_{\mu} = U^{\mathbf{x}}U_{\mathbf{x}} = -1 \qquad [4.4\text{-}2]$$

$$U^{\mu}V_{\mu} = U^{\mathbf{x}}V_{\mathbf{x}} = 0 \qquad [4.4\text{-}3]$$

$$V^{\mu}V_{\mu} = V^{\mathbf{x}}V_{\mathbf{x}} = 1 \qquad [4.4\text{-}4]$$

The unperturbed rigging basis consists of a pair of orthogonal vectors $C_{\mathbf{x}}{}^{\mu}$ such that $\eta_{\mathbf{xy}}$ takes the form 3.3-31. Under the perturbation, we assume the rigging vectors maintain the same orthogonality conditions as the unperturbed ones. This, plus the freedom of the "Lorentz" transformation, determines $c_{\mathbf{xy}}$ as in the double-null case.

Just as in the double-null case we adopt gauge condition on **U** and **V**, this time that **U** remains a unit timelike vector field orthogonal to the family of three-surfaces to which **V** is tangent. These conditions can be expressed as

$$\hat{U}^{\mathbf{x}}\hat{U}^{\mathbf{y}}\hat{g}_{\mathbf{xy}} = -1 \qquad [4.4\text{-}5]$$

$$\hat{U}^{\mathbf{x}}\hat{V}^{\mathbf{y}}\hat{g}_{\mathbf{xy}} = 0 \qquad [4.4\text{-}6]$$

$$\hat{U}^{A} = 0 \qquad [4.4\text{-}7]$$

It is straightforward to show, as we have done in Section 4.2, that these conditions are, in terms of the metric perturbation

$$U^{\mathbf{x}}U^{\mathbf{y}}\chi_{\mathbf{xy}} = \chi_{UU} = 0 \qquad [4.4\text{-}8]$$

$$U^{\mathbf{x}}V^{\mathbf{y}}\chi_{\mathbf{xy}} = \chi_{UV} = 0 \qquad [4.4\text{-}9]$$

$$\chi_{A\mathbf{x}}U^{\mathbf{x}} = \chi_{AU} = 0 \qquad [4.4\text{-}10]$$

This set of gauge conditions is the linear analogue of geodesic normal coordinates $ds^2 = dt^2 - g_{ij}dx^i dx^j$.

Equations 4.4-8 and 4.4-9 also imply that

$$U^{\mathbf{x}}U^{\mathbf{y}*}h_{\mathbf{xy}A} = -1/2\mathfrak{D}_A\chi_{UU} = 0 \qquad [4.4\text{-}11]$$

$$U^{\mathbf{x}}V^{\mathbf{y}*}h_{\mathbf{xy}A} = -1/2\mathfrak{D}_A\chi_{UV} = 0 \qquad [4.4\text{-}12]$$

The unperturbed metric can be written in the form

$$\eta^{\mathbf{xy}} = -U^{\mathbf{x}}U^{\mathbf{y}} + V^{\mathbf{x}}V^{\mathbf{y}} \qquad [4.4\text{-}13]$$

while $\chi_{\mathbf{xy}}$ can be written in terms of

$$\chi_{\mathbf{xy}} = \chi_{VV}V_{\mathbf{x}}V_{\mathbf{y}} \qquad [4.4\text{-}14]$$

$$^*\chi = \chi_{VV} \qquad [4.4\text{-}15]$$

The following relations can be easily shown to hold

$$U^{\mathbf{x}}\varepsilon_{\mathbf{xy}} = \xi V_{\mathbf{y}} \qquad [4.4\text{-}16]$$

$$V^{\mathbf{x}}\varepsilon_{\mathbf{xy}} = \xi U_{\mathbf{y}} \qquad [4.4\text{-}17]$$

$$U_{\mathbf{x}}\varepsilon^{\mathbf{xy}} = -\xi\eta^2 V_{\mathbf{y}} \qquad [4.4\text{-}18]$$

$$V_{\mathbf{x}}\varepsilon^{\mathbf{xy}} = -\xi\eta^2 U_{\mathbf{y}} \qquad [4.4\text{-}19]$$

The proofs of how the gauge conditions are implemented are similar to those developed in earier sections and are left to the reader. The remaining three-dimensional gauge conditions are

$$\chi_{VV} = 0 \quad \text{on } \mathbf{S}_0 \qquad [4.4\text{-}20]$$

$$\pounds_{\mathbf{U}}\chi_{VV} = 0 \text{ on } \mathbf{S}_0 \qquad [4.4\text{-}21]$$

$$v^A = \chi^A{}_{\mathbf{x}}V^{\mathbf{x}} = 0 \text{ on } \mathbf{S}_0 \qquad [4.4\text{-}22]$$

Using the above gauge conditions, the field equations (with sources) are as follows:

Rigging Projections

I) $\pounds_{\mathbf{U}}(U^{\mathbf{x}}h_{\mathbf{x}}) = 1/2\{-1/2\eta^{AB}\chi_{:AB} + \tilde{\chi}^{AB}{}_{:AB}\} + \xi^2\eta^4 T_{VV},$ [4.4-23]

which together with

$$\pounds_{\mathbf{U}}\chi = -2U^{\mathbf{x}}h_{\mathbf{x}}, \qquad [4.4\text{-}24]$$

yields

$$\pounds^2{}_{\mathbf{U}}\chi - 1/2\eta^{AB}\chi_{:AB} = -\tilde{\chi}^{AB}{}_{:AB} - 2\xi^2\eta^4 T_{VV} \qquad [4.4\text{-}25]$$

II) $\pounds_{\mathbf{V}}(V_{\mathbf{x}}h^{\mathbf{x}}) = -1/2\{-1/2\eta^{AB}\chi_{:AB} + \tilde{\chi}^{AB}{}_{:AB}\} + \xi^2\eta^4 T_{UU},$ [4.4-26]

and

$$\pounds_{\mathbf{V}}\chi = -2D_A v^A - 2V^{\mathbf{x}}h_{\mathbf{x}}, \qquad [4.4\text{-}27]$$

which implies that

II) $\pounds^2{}_{\mathbf{V}}\chi + 1/2\eta^{AB}\chi_{:AB} = -\pounds_{\mathbf{V}}D_A v^A + \tilde{\chi}^{AB}{}_{:AB} - 2\xi^2\eta^4 T_{UU}$ [4.4-28]

which is an elliptic equation for χ. This shows clearly how the conformal scale factor is coupled to the energy of the source of the field.

III) $\pounds_{\mathbf{V}}\pounds_{\mathbf{U}}\chi = -\xi^2\eta^4 T_{UV}$ [4.4-29]

The surface-rigging projections of the field equations are

IV) $1/2\pounds^2{}_{\mathbf{U}}v^A = -\varepsilon^{\mathbf{rs}}U^{\mathbf{y}}D_{\mathbf{r}}{}^*h_{\mathbf{sy}}{}^A$

$$+ U^{\mathbf{y}}\varepsilon_{\mathbf{wy}}(D_C h^{AC\mathbf{w}} - D^A h^{\mathbf{w}}) - \xi\eta^2 T^A{}_V \qquad [4.4\text{-}30]$$

V) $1/2\pounds_{\mathbf{V}}\pounds_{\mathbf{U}}v^A = -\varepsilon^{\mathbf{rs}}V^{\mathbf{y}}D_{\mathbf{r}}{}^*h_{\mathbf{sy}}{}^A$

$$+ V^{\mathbf{y}}\varepsilon_{\mathbf{wy}}(D_C h^{AC\mathbf{w}} - D^A h^{\mathbf{w}}) - \xi\eta^2 T^A{}_U \qquad [4.4\text{-}31]$$

which become

IV) $1/2£^2_{\mathbf{U}}v^A = D_C(V^{\mathbf{x}}h^{AC}_{\mathbf{x}}) - D^A(V^{\mathbf{x}}h_{\mathbf{x}}) - \xi\eta^2T^A{}_V$ [4.4-32]

V) $1/2£_{\mathbf{V}}£_{\mathbf{U}}v^A = 1/2\eta^{AB}U^{\mathbf{x}}D_{\mathbf{x}}\mathbf{D}_B\chi_{VV} + D_C(U^{\mathbf{x}}h^{AC}_{\mathbf{x}})$

$- D^A(U^{\mathbf{x}}h_{\mathbf{x}}) - \xi\eta^2T^A{}_U$ [4.4-33]

Since χ_{VV} vanishes on $\mathbf{S}_0$, this last equation can be readily solved for $£_{\mathbf{U}}v^A$ in terms of $£_{\mathbf{U}}v^A$ on $\mathbf{N}_0$ and $U^{\mathbf{x}}h^{AC}_{\mathbf{x}}$ and $U^{\mathbf{x}}h_{\mathbf{x}}$ on $\mathbf{S}_0$, which are already known.

One can show that the linearized rigging-space Ricci scalar for the Cauchy problem is

$$''\hat{R} = £^2_{\mathbf{U}}\chi_{VV}$$ [4.4-34]

The trace of the <u>surface-projection</u> of the field equations becomes

VI) $-£^2_{\mathbf{U}}\chi + £^2_{\mathbf{V}}\chi - \eta^{AB}D_AD_B\chi_{VV} + £^2_{\mathbf{U}}\chi_{VV} = T_{AB}g^{AB}$ [4.4-35]

The <u>traceless surface-projection</u> or <u>dynamical</u> field equations becomes

VII) $£^2_{\mathbf{U}}\tilde{\chi}_{AB} - £^2_{\mathbf{V}}\tilde{\chi}_{AB} - 1/2\Lambda^C{}_A{}^D{}_B£_{\mathbf{V}}(D_Cv_D)$

$- \Lambda^C{}_A{}^D{}_BD_CD_D\chi_{VV} = \tilde{T}_{AB}$ [4.4-36]

We can now work out an integration scheme for these equations in vacuum with the following initial data:

- A scalar quantity on a closed two-surface on $\mathbf{S}_0$: χ

  which is formally equivalent to knowing χ and $V^{\mathbf{x}}h_{\mathbf{x}}$ on two infinitesimally neighboring two-surfaces in $\mathbf{S}_0$.

Another quantity on the initial two-surface $\mathbf{N}_0$: $U^{\mathbf{x}}h_{\mathbf{x}}$

- A two-vector on $\mathbf{N}_0$: $\omega^A = £_{\mathbf{U}}v^A$
- The linearized conformal two-structure and its velocity; i.e. the families of symmetric traceless two-tensors on $\mathbf{S}_0$: $\tilde{\chi}_{AB}$, $£_{\mathbf{U}}\tilde{\chi}_{AB}$.

1) Integrate the elliptic equation (II) for χ on $\mathbf{S}_0$ in terms of χ on a closed three-surface such as one that might be sandwiched between two two-planes $\mathbf{N}_0$ and $\mathbf{N}_1$. This is formally equivalent to knowing χ and $V^{\mathbf{x}}h_{\mathbf{x}}$ on a single two-surface $\mathbf{N}_0$. Note that this equation is one of the usual constraint equations: χ cannot be given arbitrarily on the initial surface.
2) Integrate (II) on $\mathbf{S}_0$ for $U^{\mathbf{x}}h_{\mathbf{x}}$ on $\mathbf{S}_0$ in terms $U^{\mathbf{x}}h_{\mathbf{x}}$ on $\mathbf{N}_0$. This is another of one of the constraint equations: $U^{\mathbf{x}}h_{\mathbf{x}}$ cannot be specified arbitrarily on the Cauchy surface.
3) Integrate (V) on $\mathbf{S}_0$ for ω^A in terms of ω^A on $\mathbf{N}_0$, $£_{\mathbf{U}}\chi_{VV}$ on $\mathbf{S}_0$ (which vanishes due to the three-dimensional gauge condition) and $£_{\mathbf{U}}\chi$ on $\mathbf{S}_0$ (which is obtained from Step 2 above) and $£_{\mathbf{U}}\tilde{\chi}^{AB}$ on $\mathbf{S}_0$ [which represents freely specifiable initial data]. This set comprises two equations which are also constraint equations.
4) Iteratively solve equations (I), (IV), (VI) and (VII) for χ, v^A, χ_{VV} and $\tilde{\chi}^{AC}$:

- Given χ, $U^{x}H_{x}$, v^{A}, χ_{VV} as well as $\tilde{\chi}_{AB}$ and $\pounds_{U}\tilde{\chi}^{AB}$ on S_0, solve (I) and (VII) for $\pounds^2_{U}\chi$ and $\pounds^2_{U}\tilde{\chi}^{AB}$ on S_0 which is equivalent to knowing χ, $\pounds_{U}\chi$, $\tilde{\chi}^{AB}$ and $\pounds_{U}\tilde{\chi}^{AB}$ on the next infinitesimally close space-like hypersurface S_1.
- Given v^{A}, $\pounds_{U}v^{A}$, χ and $\tilde{\chi}^{AC}$ on S_0, solve (IV) for $\pounds^2_{U}v^{A}$, on S_0. This gives us v^{A} and $\pounds_{U}v^{A}$ on S_1.
- Given χ_{VV}, $\pounds_{U}\chi_{VV}$ (which vanish) and $\pounds_{U}\chi$ on S_0, solve (VI) for $\pounds^2_{U}\chi_{VV}$ on S_0. This is equivalent to knowing χ_{VV} and $\pounds_{U}\chi_{VV}$ on S_1.

All quantities which are needed to repeat the sequence of steps are now known on S_1 and can be used to push the solution on to S_2 and so forth.

In the Cauchy case one can also formally specify all initial data on a single two-surface. Knowledge of $\tilde{\chi}_{AB}$ on S_0 is obtained in a formal sense if the Lie derivatives of $\tilde{\chi}_{AB}$ with respect to $\mathbf{V}$ are known on $\mathbf{N}_0$. Again, the n^{th}-order Lie derivative of $\tilde{\chi}_{AB}$ with respect to $\mathbf{V}$ can be written in the form

$$\pounds^{n}_{\mathbf{V}}\tilde{\chi}_{AB} = V^{x_1}V^{x_2}\cdots V^{x_n}\mathcal{D}_{x_1}\mathcal{D}_{x_2}\ldots\mathcal{D}_{x_n}\tilde{\chi}_{AB} \qquad [4.4\text{-}37]$$

and is written out entirely in terms of the projection of the totally symmetric part $\mathcal{D}_{(\mathbf{x}_1}\mathcal{D}_{\mathbf{x}_2}\ldots\mathcal{D}_{\mathbf{x}_n)}\tilde{\chi}_{AB}$ into the **V** direction.

Similarly, knowledge of $\pounds_{\mathbf{U}}\tilde{\chi}_{AB}$ on $\mathbf{S}_0$ is obtained in a formal sense if the Lie derivatives to all orders of $\pounds_{\mathbf{U}}\tilde{\chi}_{AB}$ with respect to **V** are known on $\mathbf{N}_0$.

To first order, we have

$$\pounds_{\mathbf{V}}\pounds_{\mathbf{U}}\tilde{\chi}_{AB} = V^{\mathbf{x}}U^{\mathbf{y}}\mathcal{D}_{\mathbf{x}}\mathcal{D}_{\mathbf{y}}\tilde{\chi}_{AB}$$

To 2nd-order, one can show

$$\pounds^2_{\mathbf{V}}\pounds_{\mathbf{U}}\tilde{\chi}_{AB} = V^{\mathbf{x}_1}V^{\mathbf{x}_2}\, U^{\mathbf{y}}\mathcal{D}_{\mathbf{x}_1}\mathcal{D}_{\mathbf{x}_2}\mathcal{D}_{\mathbf{y}}\tilde{\chi}_{AB}$$

To nth-order

$$\pounds^n_{\mathbf{V}}\pounds_{\mathbf{U}}\tilde{\chi}_{AB} = V^{\mathbf{x}_1}V^{\mathbf{x}_2}\cdots V^{\mathbf{x}_n}U^{\mathbf{y}}\mathcal{D}_{\mathbf{x}_1}\mathcal{D}_{\mathbf{x}_2}\ldots\mathcal{D}_{\mathbf{x}_n}\mathcal{D}_{\mathbf{y}}\tilde{\chi}_{AB} \qquad [4.4\text{-}38]$$

We can symmetrize this expression with respect to the derivative indices since the commutator of two $\mathcal{D}$-derivatives is of one lower order and is therefore known from the expression for the next previous derivative. Thus, the nth-Lie derivative of $\pounds_{\mathbf{U}}\tilde{\chi}_{AB}$ can be written out entirely in terms of the projection of the totally symmetric part $\mathcal{D}_{(\mathbf{x}_1}\mathcal{D}_{\mathbf{x}_2}\ldots\mathcal{D}_{\mathbf{x}_n)}\tilde{\chi}_{AB}$ once into the **U** direction and n-1 times into the **V** direction. Again, because the dynamic field equations take the form

$$\eta^{\mathbf{xy}}\mathcal{D}_{\mathbf{x}}\mathcal{D}_{\mathbf{y}}\tilde{\chi}_{CD} = 0 \text{ on } \mathbf{S}_0.$$

only the completely traceless part of $\mathcal{D}_{(\mathbf{x}_1}\mathcal{D}_{\mathbf{x}_2}\ldots\mathcal{D}_{\mathbf{x}_n)}\tilde{\chi}_{AB}$ is non-vanishing for a solution to the field equations. Thus, as in the double null case, specification of the initial data for the dynamical parts of the gravitational field can be accomplished by specifying a denumerably infinite set of traceless symmetric tensors (traceless with respect to the rigging space) on $\mathbf{N}_0$, which also are traceless and symmetric with respect to their two-surface indices, the generalized conformal two-structure. By this procedure we have formally unified the characteristic and Cauchy initial value problems: all initial data can be given on a single two-surface and are the same sets of quantities for both problems.

CHAPTER 5

THE CHARACTERISTIC INITIAL VALUE PROBLEM

In this chapter, we shall develop the field equations for the two+two covariant formulation of the double-null problem presented in Section 1.3, whose linearized version was discussed in Section 4.2. Data will be set on a pair of null surfaces just as in that case, but the exact field equations will be written out. The initial hypersurfaces, $\mathcal{U}_0$ and $\mathcal{V}_0$, upon which data will be set are gotten from dragging a single two-surface $\mathbf{N}_0$ along the two-deformation vector fields **V** and **U**, respectively. Again the rigging basis vectors are chosen so that the rigging metric takes the null dyad form given in 3.3-32.

As in the linearized case, conditions are to be imposed on the deformation vectors everywhere which make them tangent to null surfaces, i.e.

$$g_{\mathbf{xy}}U^{\mathbf{x}}U^{\mathbf{y}} = 0 \qquad [5\text{-}1]$$

$$g_{\mathbf{xy}}V^{\mathbf{x}}V^{\mathbf{y}} = 0 \qquad [5\text{-}2]$$

everywhere. These conditions correspond to equations 4.2-15 and 4.2-16 of the linearized case. A three dimensional coordinate condition

$$g_{\mathbf{xy}}U^{\mathbf{x}}V^{\mathbf{y}} = -1 \qquad [5\text{-}3]$$

will be adopted on $\mathcal{U}_0$ and $\mathcal{V}_0$. If we define the scalar quantity q by

$$e^{q} = -g_{\mathbf{xy}}U^{\mathbf{x}}V^{\mathbf{y}} \qquad [5\text{-}4]$$

then $q = 0$ on $\mathcal{U}_0$ and $\mathcal{V}_0$.

As in the linearized case, we adopt the two conditions

$$U^{A} = 0 \qquad [5\text{-}5]$$

which say that **U** is orthogonal to the members of the family of two-surfaces which foliate a region of space-time. Another three-dimensional condition to be imposed is

$$V^A = 0 \quad \text{on } \mathcal{U}_0. \qquad [5\text{-}6]$$

A useful expression for $g_{\mathbf{xy}}$ can be found in terms of the deformation vector fields

$$g^{\mathbf{xy}} = -2e^{-q}U^{(\mathbf{x}}V^{\mathbf{y})} \qquad [5\text{-}7]$$

We now consider the exact vacuum field equations, given in Section 3.3.

The <u>rigging projections</u> of the field equations are

$$\text{I)} \quad -1/2£_U(\gamma^{-2}£_U\gamma^2) = -U^{\mathbf{z}}U^{\mathbf{w}}{}^*H_{\mathbf{z}}{}^{\mathbf{x}A}{}^*H_{\mathbf{wx}A} + U^{\mathbf{x}}U^{\mathbf{z}}H_{AC\mathbf{z}}H^{AC}{}_{\mathbf{x}} + H^{\mathbf{x}}U^{\mathbf{z}}D_{\mathbf{z}}U_{\mathbf{x}} \qquad [5\text{-}8]$$

$$\text{II)} \quad -1/2£_V(\gamma^{-2}£_V\gamma^2) = -£_V(D_AV^A) + V^C\mathfrak{D}_C(H^{\mathbf{x}}V_{\mathbf{x}}) - V^{\mathbf{z}}V^{\mathbf{w}}\,{}^*H_{\mathbf{z}}{}^{\mathbf{x}A}{}^*H_{\mathbf{wx}A} + V^{\mathbf{x}}V^{\mathbf{z}}H_{AC\mathbf{z}}H^{AC}{}_{\mathbf{x}} + H^{\mathbf{x}}V^{\mathbf{z}}D_{\mathbf{z}}V_{\mathbf{x}} \qquad [5\text{-}9]$$

We first show that the terms $U^{\mathbf{z}}U^{\mathbf{w}}{}^*H_{\mathbf{z}}{}^{\mathbf{x}A}{}^*H_{\mathbf{wx}A}$ in equation 5-8 and $V^{\mathbf{z}}V^{\mathbf{w}}{}^*H_{\mathbf{z}}{}^{\mathbf{x}A}{}^*H_{\mathbf{wx}A}$ in 5-9 vanish. In terms of unknown coefficients A^A and B^A, $U^{\mathbf{z}}{}^*H_{\mathbf{z}}{}^{\mathbf{x}A}$ takes the general form

$$U^{\mathbf{z}}{}^*H_{\mathbf{z}}{}^{\mathbf{x}A} = A^AU^{\mathbf{x}} + B^AV^{\mathbf{x}} \qquad [5\text{-}10]$$

and since $U^{\mathbf{x}}$ is null, we have

$$U^{\mathbf{z}}U^{\mathbf{x}}{}^*H_{\mathbf{zx}}{}^A = B^AV^{\mathbf{x}}U_{\mathbf{x}} \qquad [5\text{-}11]$$

Only the symmetric part of ${}^*H_{\mathbf{zx}}{}^A$ enters 5-11 so we can write the left-hand side of 5-11 as

$$U^{\mathbf{z}}U^{\mathbf{x}}\mathfrak{D}^Ag_{\mathbf{zx}} = -2B^AV^{\mathbf{x}}U_{\mathbf{x}} \qquad [5\text{-}12]$$

Using the properties of deformation vectors ($\mathfrak{D}_AU^{\mathbf{z}} = \mathfrak{D}_AU^{\mathbf{z}} = 0$), we see that $U^{\mathbf{x}}$ can be brought inside the derivative, giving

$$\mathfrak{D}^A(U^{\mathbf{z}}U^{\mathbf{x}}g_{\mathbf{zx}}) = 0 = -2B^AV^{\mathbf{x}}U_{\mathbf{x}} \qquad [5\text{-}13]$$

which vanishes due to the adopted condition 5-1. Equation 5-13 implies that $B^A = 0$ and hence

$$U^{\mathbf{z}}{}^{*}H_{\mathbf{z}}{}^{\mathbf{x}A} = A^A U^{\mathbf{x}} \qquad [5\text{-}14]$$

The exact form of A^A is not needed since equation 5-14 shows that $U^{\mathbf{z}}U^{\mathbf{w}}{}^{*}H_{\mathbf{z}}{}^{\mathbf{x}A}{}^{*}H_{\mathbf{wx}A}$ is proportional to $U^{\mathbf{z}}U^{\mathbf{x}}g_{\mathbf{zx}}$, which vanishes. A similar argument holds for $V^{\mathbf{z}}V^{\mathbf{w}}{}^{*}H_{\mathbf{z}}{}^{\mathbf{x}A}{}^{*}H_{\mathbf{wx}A}$.

The field equations 5-8 and 5-9 become

$$\text{I)} \quad -1/2\pounds_{\mathbf{U}}(\gamma^{-2}\pounds_{\mathbf{U}}\gamma^2) = U^{\mathbf{x}}U^{\mathbf{z}}H_{AC\mathbf{z}}H^{AC}{}_{\mathbf{x}} + H^{\mathbf{x}}U^{\mathbf{z}}D_{\mathbf{z}}U_{\mathbf{x}} \qquad [5\text{-}15]$$

$$\text{II)} \quad -1/2\pounds_{\mathbf{V}}(\gamma^{-2}\pounds_{\mathbf{V}}g^2) = -\pounds_{\mathbf{V}}(D_A v^A) + V^C\mathfrak{D}_C(H^{\mathbf{x}}V_{\mathbf{x}})$$
$$+ V^{\mathbf{x}}V^{\mathbf{z}}H_{AC\mathbf{z}}H^{AC}{}_{\mathbf{x}} + H^{\mathbf{x}}V^{\mathbf{z}}D_{\mathbf{z}}V_{\mathbf{x}} \qquad [5\text{-}16]$$

$H_{AC\mathbf{z}}$ implicitly contains derivatives of both γ and $\tilde{g}_{AC}$, which makes equations 5-15 and 5-16 non-linear in γ.

The quantity $U^{\mathbf{x}}V^{\mathbf{y}}{}^{*}H_{\mathbf{xy}}{}^{A}$ appears frequently in later analysis, so we will rework this expression:

$$U^{\mathbf{x}}V^{\mathbf{y}}{}^{*}H_{\mathbf{xy}A} = U^{\mathbf{x}}V^{\mathbf{y}}{}^{*}H_{(\mathbf{xy})A} + U^{\mathbf{x}}V^{\mathbf{y}}{}^{*}H_{[\mathbf{xy}]A}$$
$$= -1/2\mathfrak{D}_A(U^{\mathbf{x}}V_{\mathbf{x}}) - U^{\mathbf{x}}V^{\mathbf{y}}\Omega_{\mathbf{xy}A}$$
$$= 1/2e^q\mathfrak{D}_A q - \xi\Omega_A \qquad [5\text{-}17]$$

(we recall the definition of the scalar density $\xi = U^{\mathbf{x}}V^{\mathbf{y}}\varepsilon_{\mathbf{xy}}$) Similarly, noting ${}^{*}H_{\mathbf{yx}A}$ is not symmetric,

$$U^{\mathbf{x}}V^{\mathbf{y}}{}^{*}H_{\mathbf{yx}A} = 1/2e^q\mathfrak{D}_A q + \xi\Omega_A \qquad [5\text{-}18]$$

The last term on the right hand side of equation 5-15 contains the term $D_{\mathbf{y}}U_{\mathbf{x}}$ which we can also evaluate further. Expanding it in the form

$$D_{\mathbf{y}}U_{\mathbf{x}} = AU_{\mathbf{x}}U_{\mathbf{y}} + BU_{\mathbf{x}}V_{\mathbf{y}} + CU_{\mathbf{y}}V_{\mathbf{x}} + DV_{\mathbf{x}}V_{\mathbf{y}}, \qquad [5\text{-}19]$$

and then contracting 5-19 with U^x gives

$$U^x D_y U_x = 0 = CU_y(U^x V_x) + DV_y(U^x V_x) \qquad [5\text{-}20]$$

which implies that C = D = 0. Hence

$$D_y U_x = AU_x U_y + BU_x V_y \qquad [5\text{-}21]$$

Contracting 5-21 with V^y gives

$$V^y D_y U_x = AU_x U_y \qquad [5\text{-}22]$$

Since the Lie derivatives with respect to each other of the rigging components of the two deformation vectors vanish (see Section 2.1), we can write the left-hand side of 5-22 as

$$U^y D_y V_x = AU_x U_y \qquad [5\text{-}23]$$

Contracting this with V^x makes the left-hand side vanish, so we have A = 0. Thus,

$$D_y U_x = BU_x V_y \qquad [5\text{-}24]$$

We can finally solve for B:

$$B = V^x U^y D_y U_x / U_z V^z = [V^y D_y (V^x U_x) - U_x U^y D_y V^x]/(U_z V^z)$$
$$= V^y D_y (V^x U_x)/(U_z V^z) = \pounds_{\mathbf{U}} q \qquad [5\text{-}25]$$

Thus,

$$D_y U_x = \pounds_{\mathbf{U}} q \; U_x V_y \qquad [5\text{-}26]$$

Similarly,

$$D_y V_x = \pounds_{\mathbf{V}} q \; V_x U_y \qquad [5\text{-}27]$$

Two results that easily follow from equations 5-26 and 5-27 are

$$V^y D_y U_x = 0 \qquad [5\text{-}28]$$

$$U^{\mathbf{y}}D_{\mathbf{y}}V_{\mathbf{x}} = 0 \quad [5\text{-}29]$$

When all the above results are taken into account, the rigging projections of the field equations become

$$\text{I)} \quad -1/2\pounds_{\mathbf{U}}(\gamma^{-2}\pounds_{\mathbf{U}}\gamma^2) = U^{\mathbf{x}}U^{\mathbf{z}}H_{AC\mathbf{z}}H^{AC}{}_{\mathbf{x}} + U_{\mathbf{x}}H^{\mathbf{x}}\pounds_{\mathbf{U}}q \quad [5\text{-}30]$$

$$\text{II)} \quad -1/2\pounds_{\mathbf{V}}(-1/2\gamma^{-2}\pounds_{\mathbf{V}}\gamma^2) = -\pounds_{\mathbf{V}}(D_A v^A) + V^C\mathcal{D}_C(H^{\mathbf{x}}V_{\mathbf{x}})$$
$$+ V^{\mathbf{x}}V^{\mathbf{z}}H_{AC\mathbf{z}}H^{AC}{}_{\mathbf{x}} + V_{\mathbf{x}}H^{\mathbf{x}}\pounds_{\mathbf{V}}q, \quad [5\text{-}31]$$

with

$$U^{\mathbf{x}}H_{\mathbf{x}} = -1/2\gamma^{-2}\pounds_{\mathbf{U}}\gamma^2 \quad [5\text{-}32]$$

and

$$V^{\mathbf{x}}H_{\mathbf{x}} = D_A V^A - 1/2\ \gamma^{-2}\pounds_{\mathbf{V}}\gamma^2 \quad [5\text{-}33]$$

On $\mathcal{V}_0$, field equation 5-30 becomes

$$-1/2\pounds_{\mathbf{U}}(\gamma^{-2}\pounds_{\mathbf{U}}\gamma^2) = U^{\mathbf{x}}U^{\mathbf{z}}H_{AC\mathbf{z}}H^{AC}{}_{\mathbf{x}} \quad [5\text{-}34]$$

while on $\mathcal{U}_0$, equation 5-31 becomes

$$-1/2\pounds_{\mathbf{V}}(\gamma^{-2}\pounds_{\mathbf{V}}\gamma^2) = V^{\mathbf{x}}V^{\mathbf{z}}H_{AC\mathbf{z}}H^{AC}{}_{\mathbf{x}} \quad [5\text{-}35]$$

The last rigging projection equation is

$$\text{III)} \quad \pounds_{\mathbf{V}}(U_{\mathbf{x}}H^{\mathbf{x}}) = -\mathcal{D}_A(U^{\mathbf{x}}V^{\mathbf{z}*}H_{\mathbf{zx}}{}^A) - V^{\mathbf{z}}U^{\mathbf{w}*}H_{\mathbf{z}}{}^{\mathbf{x}A*}H_{\mathbf{wx}A}$$
$$+ V^C\mathcal{D}_C(H^{\mathbf{x}}U_{\mathbf{x}}) + U^{\mathbf{x}}V^{\mathbf{z}}H_{AC\mathbf{z}}H^{AC}{}_{\mathbf{x}} + H^{\mathbf{x}}V^{\mathbf{z}}D_{\mathbf{z}}U_{\mathbf{x}}$$
$$-1/2\ U_{\mathbf{x}}V^{\mathbf{x}}[{}'R + H_{AC}{}^{\mathbf{z}}H^{AC}{}_{\mathbf{z}} - H^{\mathbf{z}}H_{\mathbf{z}}] \quad [5\text{-}36]$$

which can be rewritten using 5-17 as

$$\pounds(U_{\mathbf{x}}H^{\mathbf{x}}) = 1/2\mathcal{D}_A\mathcal{D}^A(U^{\mathbf{x}}V_{\mathbf{x}}) - D_A(\xi\Omega^A) - V^{\mathbf{z}}U^{\mathbf{w}*}H_{\mathbf{z}}{}^{\mathbf{x}A*}H_{\mathbf{wx}A}$$
$$+ V^C\mathcal{D}_C(H^{\mathbf{x}}U_{\mathbf{x}}) + U^{\mathbf{x}}V^{\mathbf{y}}\Lambda^{\mathbf{z}}{}_{\mathbf{x}}{}^{\mathbf{w}}{}_{\mathbf{y}}H_{AC\mathbf{z}}H^{AC}{}_{\mathbf{x}} + H^{\mathbf{x}}V^{\mathbf{z}}D_{\mathbf{z}}U_{\mathbf{x}}$$
$$-1/2\ U_{\mathbf{x}}V^{\mathbf{x}}[{}'R - H^{\mathbf{z}}H_{\mathbf{z}}] \quad [5\text{-}37]$$

The last term on the second line of this equation vanishes due to equation 5-28. Since g^{xy} is proportional to $U^{(x}V^{y)}$, $U^xV^y\Lambda^z{}_x{}^w{}_y H_{ACz}H^{AC}{}_x$ is also zero.

The term $V^zU^{w*}H_z{}^{xA*}H_{wxA}$ can be re-written, using 5-17, as:

$$V^zU^{w*}H_z{}^{xA*}H_{wxA}$$

$$= (-1/2\mathcal{D}^A(U^xV_x) + \xi\Omega^A)(-1/2\mathcal{D}_A(U^xV_x) - \xi\Omega_A)/(U^zV_z)$$

$$= [1/4\mathcal{D}^A(U^xV_x)\mathcal{D}_A(U^xV_x) - \xi^2\Omega^A\Omega_A]/(U^zV_z) \qquad [5\text{-}38]$$

Equation 5-37 thus becomes

$$\pounds_V(U_xH^x) = 1/2\mathcal{D}_A\mathcal{D}^A(U^xV_x) - \mathcal{D}_A(\xi\Omega^A)$$
$$- [1/4\mathcal{D}^A(U^xV_x)\mathcal{D}_A(U^xV_x) - \xi^2\Omega^A\Omega_A]/(U^zV_z)$$
$$+ V^C\mathcal{D}_C(H^xU_x) - 1/2U_xV^x['R - H^zH_z] \qquad [5\text{-}39]$$

Since

$$H^zH_z = -2e^{-q}U^{(x}V^{y)}H_xH_y = -e^{-q}(U^xH_x)(V^yH_y),$$

we can re-write 5-39 as:

III) $$\pounds_V(U_xH^x) = 1/2\mathcal{D}_A\mathcal{D}^A(U^xV_x) - \mathcal{D}_A(\xi\Omega^A)$$
$$- [1/4\mathcal{D}^A(U^xV_x)\mathcal{D}_A(U^xV_x) - \xi^2\Omega^A\Omega_A]/(U^zV_z)$$
$$+ V^C\mathcal{D}_C(H^xU_x) - 1/2U_xV^x['R + e^{-q}(U^xH_x)(V^yH_y)]$$

[5-40]

On $\mathcal{U}_0$, where $q = 0$ and $V^A = 0$, equation 5-40 reduces to

$$\pounds_V(U_xH^x) = -\mathcal{D}_A(\xi\Omega^A) - \xi^2\Omega^A\Omega_A + 1/2['R + (U^xH_x)(V^yH_y)]$$

[5-41]

The <u>surface-rigging</u> projections of the field equations are

IV) $$£_{\mathbf{U}}\Omega_A = U^{\mathbf{y}}\Omega_B H^B{}_{A\mathbf{y}} - \varepsilon^{\mathbf{rs}}U^{\mathbf{y}}D_{\mathbf{r}}H_{(\mathbf{sy})A} + \rho^2 U^{\mathbf{y}}\varepsilon_{\mathbf{wy}}(D_C H_A{}^{C\mathbf{w}} - D_A H^{\mathbf{w}})$$ [5-42]

V) $$£_{\mathbf{V}}\Omega_A = V^C D_C\Omega_A + \Omega_C D_A V^C + V^{\mathbf{y}}\Omega_B H^B{}_{A\mathbf{y}} - \varepsilon^{\mathbf{rs}}V^{\mathbf{y}}D_{\mathbf{r}}H_{(\mathbf{sy})A} + V^{\mathbf{y}}\varepsilon_{\mathbf{wy}}(D_C H_A{}^{C\mathbf{w}} - D_A H^{\mathbf{w}}),$$ [5-43]

with

$$£_{\mathbf{U}}V^A = 2U^{\mathbf{x}}V^{\mathbf{y}}\Omega^A{}_{\mathbf{xy}} = 2U^{\mathbf{x}}V^{\mathbf{y}}\varepsilon_{\mathbf{xy}}\Omega^A = 2\xi\Omega^A,$$

or

$$\Omega^A = 1/(2\xi)£_{\mathbf{U}}V^A$$ [5-44]

After some manipulation, equations 5-42 and 5-43 become

IV) $$£_{\mathbf{U}}\Omega_A = U^{\mathbf{y}}\Omega_B H^B{}_{A\mathbf{y}} - 1/2U^{\mathbf{y}}D_{\mathbf{y}}\mathcal{D}_A(U^{\mathbf{x}}V_{\mathbf{x}}) + D_C(U^{\mathbf{x}}H_A{}^C{}_{\mathbf{x}}) - D_A(U^{\mathbf{x}}H_{\mathbf{x}})$$ [5-45]

V) $$£_{\mathbf{V}}\Omega_A = V^C D_C\Omega_A + \Omega_C D_A V^C + V^{\mathbf{y}}\Omega_B H^B{}_{A\mathbf{y}} + 1/2V^{\mathbf{y}}D_{\mathbf{y}}D_A(U^{\mathbf{x}}V_{\mathbf{x}}) - D_C(V^{\mathbf{x}}H_A{}^C{}_{\mathbf{x}}) + D_A(V^{\mathbf{x}}H_{\mathbf{x}})$$ [5-46]

On $\mathcal{U}_0$, equation 5-46 becomes

V) $$£_{\mathbf{V}}\Omega_A = V^{\mathbf{y}}\Omega_B H^B{}_{A\mathbf{y}} - D_C(V^{\mathbf{x}}H_A{}^C{}_{\mathbf{x}}) + D_A(V^{\mathbf{x}}H_{\mathbf{x}})$$ [5-47]

The trace of the <u>surface-projection</u> equation is

VI) $$-1/2g^{\mathbf{xy}}\mathcal{D}_{\mathbf{x}}\mathcal{D}_{\mathbf{y}}\ln\gamma - \mathcal{D}_A {}^*H^A + H_{AC}{}^{\mathbf{x}}H^{AC}{}_{\mathbf{x}} - {}''R + 4\Omega^A\Omega_A = 0$$ [5-48]

The principle part of the rigging Ricci scalar on the right-hand side can be shown to be

$$''R = -4£_{\mathbf{U}}£_{\mathbf{V}}e^q$$

The dynamical equation is

$$\text{VII)}\quad \tilde{G}_{CD} = -1/2\ \gamma g^{xy}\mathcal{D}_x\mathcal{D}_y\tilde{g}_{CD} - \Lambda^A{}_C{}^B{}_D\ [\mathcal{D}_A{}^*H_B - {}^*H^{yz}{}_B{}^*H_{zyA}] - \gamma H^x\tilde{H}_{CDx} + \gamma\tilde{H}_{CE}{}^x\tilde{H}_D{}^E{}_x + 1/2\gamma\tilde{g}_{CD}\tilde{H}_{AB}{}^x\tilde{H}^{AB}{}_x \qquad [5\text{-}49]$$

We can now present a formal integration scheme for this set of equations, which is essentially the same as that provided in Section 4.2 for the Sach's double-null initial value problem. The initial data will be:

- γ, H_x, Ω^A on $\mathbf{N}_0$.
- $\tilde{g}_{CD}$ on $\mathcal{U}_0$ and $\mathcal{V}_0$.

The integration scheme is as follows:

1) The initial data implies that V^xH_{ABx} and U^xH_{ABx} are known on $\mathcal{U}_0$ and $\mathcal{V}_0$, respectively.

2) Integrate (II) on $\mathcal{U}_0$ for γ in terms of $\tilde{g}_{CD}$ on $\mathcal{U}_0$ and the constants of integration $[\gamma]$ and $[V^xH_x]_0$ on $\mathbf{N}_0$.

3) Integrate (V) on $\mathcal{U}_0$ for Ω_A in terms of γ, $\tilde{g}_{CD}$, and V^A on $\mathcal{U}_0$ which are known from either: previous steps (γ), initial data ($\tilde{g}_{CD}$) or gauge conditions (V^A) and the integration constant $[\Omega_A]_0$ on $\mathbf{N}_0$.

4) Integrate (III) on $\mathcal{U}_0$ for U^xH_x in terms of γ, $\tilde{g}_{CD}$, and V^A on $\mathcal{U}_0$ and the integration constant $[U^xH_x]_0$ on $\mathbf{N}_0$.

5) At this point, γ, U^xH_x, Ω^A are known on $\mathcal{U}_0$. We must propagate them, as well as $\tilde{g}_{CD}$ and q, off the surface.
   - The principle part of (VII) is

$-1/2\gamma £_{\mathbf{U}} £_{\mathbf{V}} \tilde{g}_{CD}$

so that knowledge of $\tilde{g}_{CD}$ on $\mathcal{U}_0$ determines $£_{\mathbf{V}} \tilde{g}_{CD}$ on the next infinitesimally close hypersurface $\mathcal{U}_1$. It also determines $£_{\mathbf{U}} \tilde{g}_{CD}$ on $\mathcal{U}_0$. $\tilde{g}_{CD}$ itself can be determined on $\mathcal{U}_1$ since $\tilde{g}_{CD}$ is known on $\mathcal{V}_0$.

- Use (VI) on $\mathcal{U}_0$ to give $£_{\mathbf{U}} £_{\mathbf{V}} q$ on $\mathcal{U}_0$ if which is equivalent to knowing $£_{\mathbf{V}} q$ on $\mathcal{U}_1$.as well as $£_{\mathbf{U}} q$ on $\mathcal{U}_0$. Then q is known on $\mathcal{U}_1$ from q given on $\mathcal{V}_0$.
- Use (I) on $\mathcal{U}_0$ to give γ and $U^{\mathbf{x}} H_{\mathbf{x}}$ on $\mathcal{U}_0$.
- Use the coupled system (IV) and the expression $\Omega^A = (2\xi)^{-1} £_{\mathbf{U}} V^A$ to give Ω^A and V^A on $\mathcal{U}_1$.
- Now all quantities are known on $\mathcal{U}_1$ and the process can be repeated.

In order to formally replace the hypersurface initial data on $\mathcal{U}_0$ and $\mathcal{V}_0$ by initial data on $\mathbf{N}_0$, we proceed by expanding $\tilde{g}_{AB}$ on $\mathcal{U}_0$ in a Taylor series as we did for the linearized version. The Taylor series involves Lie derivatives of $\tilde{g}_{AB}$ to all orders. We have on $\mathcal{U}_0$, using

$$£_{\mathbf{V}} \tilde{g}_{AB} = V^{\mathbf{x}} \mathcal{D}_{\mathbf{x}} \tilde{g}_{AB} \qquad [5\text{-}50]$$

$$£^2_{\mathbf{V}} \tilde{g}_{AB} = V^{\mathbf{x}} \mathcal{D}_{\mathbf{x}} (V^{\mathbf{z}} \mathcal{D}_{\mathbf{z}} \tilde{g}_{AB}) = V^{\mathbf{x}} V^{\mathbf{z}} \mathcal{D}_{\mathbf{x}} \mathcal{D}_{\mathbf{z}} \tilde{g}_{AB} + \mathcal{D}_{\mathbf{z}} \tilde{g}_{AB} V^{\mathbf{x}} \mathcal{D}_{\mathbf{x}} V^{\mathbf{z}}$$

$$= V^{\mathbf{x}} V^{\mathbf{z}} \mathcal{D}_{\mathbf{x}} \mathcal{D}_{\mathbf{z}} \tilde{g}_{AB} \qquad [5\text{-}51]$$

since the last term on the right-hand side vanishes on $\mathcal{U}_0$. To all orders, we have

$$\pounds^n_{\mathbf{V}} \tilde{g}_{AB} = V^{y_1} V^{y_2} \cdots V^{y_n} \mathfrak{D}_{y_1} \mathfrak{D}_{y_2} \mathfrak{D}_{y_3} \ldots \mathfrak{D}_{y_n} \tilde{g}_{AB} \qquad [5\text{-}52]$$

Again, the right-hand side involves only the completely symmetric part of $\mathfrak{D}_{(y_1} \mathfrak{D}_{y_2} \mathfrak{D}_{y_3} \ldots \mathfrak{D}_{y_n)} \tilde{g}_{AB}$:

$$\pounds^n_{\mathbf{V}} \tilde{g}_{AB} = V^{y_1} V^{y_2} \cdots V^{y_n} \mathfrak{D}_{(y_1} \mathfrak{D}_{y_2} \mathfrak{D}_{y_3} \ldots \mathfrak{D}_{y_n)} \tilde{g}_{AB} \qquad [5\text{-}53]$$

Furthermore, all trace-terms on the right-hand side of 5-53 are determined in terms of lower order $\mathfrak{D}$-derivatives by using the field equations, which are assumed to hold in a four-dimensional neighborhood of $\mathbf{N}_0$. Simlar arguments hold for the initial data on $\mathcal{V}_0$. Thus, the two-dimensional initial data are the totally symmetric traceless parts (traceless with respect to the rigging indices) of the set of $\mathfrak{D}$-derivatives

$$\mathfrak{D}_{(y_1} \mathfrak{D}_{\mathbf{x})} \tilde{g}_{AB}$$

$$\mathfrak{D}_{(y_1} \mathfrak{D}_{y_2} \mathfrak{D}_{\mathbf{x})} \tilde{g}_{AB}$$

$$\mathfrak{D}_{(y_1} \mathfrak{D}_{y_2} \mathfrak{D}_{y_3} \mathfrak{D}_{\mathbf{x})} \tilde{g}_{AB} \qquad [5\text{-}54]$$

etc.

We have just shown that defining the conformal two-structure on a pair of null hypersurface is formally equivalent to prescribing a denumerably infinite set of covariant totally-symmetric traceless tensors (traceless and symmetric with respect to the rigging indices) on a single two-surface. Just as in the linearized case, we denote by <u>generalized conformal two-structure</u>, such a set of tensors.

We now show that, for any order $\underline{m}$, there are only two independent totally symmetric traceless tensors: they span a two-dimensional space.

Proof: For any totally symmetric tensor of order $\underline{m}$ in a two-dimensional space, there are $\underline{m+1}$ independent components (we disregard any $\mathcal{N}$-indices for the time being). Thus the set of totally symmetric tensors spans a space of dimension $\underline{m+1}$. A totally traceless tensor, for example $Z_{\mathbf{x}_1\mathbf{x}_2...\mathbf{x}_m}$, satisfies $\underline{m-1}$ relations

$$g^{\mathbf{x}_i\mathbf{x}_j}Z_{\mathbf{x}_1\mathbf{x}_2...\mathbf{x}_i\mathbf{x}_j...\mathbf{x}_m} = 0 \qquad [5\text{-}55]$$

leaving just two independent components. ♦

The initial data set on a single two-surface $\mathbf{N}_0$ to formally define a solution to the double-null initial value problem is:

- The conformal scale factor γ, the mean extrinsic curvatures $H_{\mathbf{x}}$, and the anholonomic object Ω^A
- The infinite set of totally-symmetric traceless tensors $(m=1,\infty)$

  $$Z^{(m)}{}_{\mathbf{x}_1\mathbf{x}_2...\mathbf{x}_i\mathbf{x}_j...\mathbf{x}_m AB}$$

  which forms the generalized conformal two-structure.

CHAPTER VI

CONCLUSIONS

The main result of this dissertation has been to develop a covariant two+two formalism for the gravitational field and apply it to various initial value problems. Space-time was foliated by a two-parameter family of two-surfaces which arises by dragging a single space-like two-surface along a pair of commuting vector fields. Then, each member of the family was orthogonally rigged by a pair of vectors spanning a time-like plane. All geometrical quantities, including the field equations themselves, were orthogonally decomposed by projecting them along surface and rigging directions.

The field equations, derived using a Palatini variational principle, break up into several sets. The first set consists of the three field equations projected into the rigging space. These equations involve all second-order Lie derivatives of the conformal scale factor with respect to the deformation vector fields. From an analysis of various initial value problems, it was shown that these equations completely determine the conformal scale factor in a four-dimensional region when its value on a single two-surface is given along with the mean extrinsic curvature of the surface. This is the origin of two of the contraint equations that arise in the Cauchy problem. Starting from an initial two-surface, one can use the field equations to propagate the scale factor and mean curvatures along one of the deformation vector fields, thus defining it on a single hypersurface. These quantities would be overspecified if we prescribed them arbitrarily on the hypersurface.

The second set consists of four mixed surface-rigging projections of the field equations. They can be interpreted as determining the evolution of the anholonomic object formed by the rigging vectors projected into the two-surface. Two of the mixed projection equations determine the anholonomic object on a single hypersurface when its value on a single two-surface is given. The solution of this equation would also be overdetermined if we tried to prescribe it everywhere and thus is the origin of the remaining two constraint equations.

The third set of equations is the trace of the surface projections of the field equations and determines the Ricci scalar of the rigging plane. When the gauge conditions are imposed, the Ricci scalar becomes a second-order partial differential equation for the one remaining component of the rigging metric or the one remaining rigging component of the deformation vector fields as discussed in Section 3.3.

The fourth set, the dynamical equations, determines the conformal two-metric, which, in our formalism, carries the dynamics of the field. The form of the dynamical equations resembles a wave equation, and indeed, in the linearized case reduces to a wave equation in Minkowski space. A gauge was found which decouples the linearized dynamical equation from the remaining field equations.

An appealling feature of this formalism is that kinematical conditions on the deformation vector fields can be introduced naturally in a covariant form. This lets us see clearly the geometric meaning of various coordinate conditions which arise when a coordinate system is adapted to these vector fields.

We indicated how to interpret the conformal scale factor as the cross-sectional area of a bundle of light rays. The conformal two-metric was interpreted in terms of the distortion of the shape of such a bundle.

A new feature of this work is the specification of initial data on a single two-surface, such that the Einstein field equations propagate them into a four-dimensional region. From an analysis of the linearized theory, one sees that this initial data can be used to generate the data for both Cauchy and characteristic initial value problems. Besides the conformal scale factor and mean extrinsic curvatures of the two-surface and the rigging anholonomic object, one has to specify the generalized conformal two-structure: a denumerably-infinite set of totally-symmetric tensor fields that are traceless with respect to rigging indices. Each member of the set also has a pair of surface indices with respect to which it is traceless and symmetric.

The well-posedness of such an initial value problem, with all initial data set on a single two-surface needs, to be addressed. Existence and uniqueness are easily proved for analytic solutions for analytic initial data. For characteristic initial value problems, we can use the results of Müller zum Hagen and Seifert [1977], which apply more generally, to show stability as well. Thus, when the generalized conformal two-structure is projected along null directions, the problem is well-posed for analytic solutions and initial data. Can this result be extended to the case when the two deformation vector fields are space-like and time-like, respectively, i.e., the Cauchy problem, or even for generic vector fields?

Much of the interpretation of the field equations came out of analyzing the linearized field equations and work remains to be done in that case. A more interesting linearized case would use spherical two-surfaces, rather than flat ones, in the background space-time. This would allow us a simpler analysis of the field surrounding a bounded source. Is there still a gauge analogous to the decoupling gauge. Furthermore, what is the counterpart of the decoupling gauge in the exact theory?

A possible application of the two+two formalism would be to consider space-times with symmetries. At least two interesting classes of such space-times can be identified: the first is where there exists a Killing vector field tangent to the members of the family of two-surfaces. One example is the family of axisymmetric solutions (with one Killing vector), with the special case of cylindrical solutions (with two Killing vectors). Another important example is space-times which have a time-like Killing vector field (stationary solutions).

The conformal two-structure approach may be helpful in quantizing the gravitational field. Stachel [1984a] has considered the quantization of the Klein-Gordon scalar field in Minkowski space from a two+two point of view. He shows how the covariant commutation relations between the field operators at two points generate an infinite set of commutation relations between the derivatives of the field operators on a single two-surface. These commutation relations involve an infinite set of totally-symmetric traceless field operators on the two-surface. Since the generalized conformal two-structure may be given without constraints and since the dynamical equations for them look very much like a wave equation, one should investigate whether the quantization of general relativity can preceed along these lines, interpreting the generalized conformal two-structure as a set of field operators on two-surfaces obeying appropriate commutation relations between themselves.

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APPENDIX A

MANIFOLDS AND GEOMETRY

This appendix is meant as a brief review of the differential geometry of manifolds particularly those aspects pertaining to the imbedding of manifolds in higher dimensional manifolds. Most of the definitions can be found in works which develop the mathematical foundations of general relativity such as Hawking and Ellis [1973].

A manifold is essentially a topological space that is locally Euclidean meaning each point has a neighborhood that can be mapped 1:1 into an open neighborhood of R^n. Such a map is called a coordinate system. This permits the process of differentiation to be defined but does not single out any particular coordinatization as preferred. These ideas will be made more precise below.

<u>euclidean spaces</u>

Let R^n denote the Euclidean space of n-dimensions which consists of the set of all n-tuples $\mathbf{x} = (x^1, x^2, \ldots, x^n)$ where $-\infty < x^i < \infty$. The set of open balls

$$B_a(y^i) \equiv \{\mathbf{x} \mid \Sigma^n{}_{i=1}(x^i - y^i)^2 < a^2\}$$

forms a basis for the usual topology on R^n by which the continuity of mappings from R^n into R^m can be defined.

A map

$$\Phi : O \to \tilde{O}$$

(where O, $\tilde{O}$ are open sets of R^n, R^m, respectively), is of differentiability class C^r if $(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^m)$ and all their derivatives up to and including the r^{th}-order are continuous functions of $(x^1, x^2, \ldots, x^n)$.

charts and atlases

Let **S** be a subset of **M**. A chart $(\mathcal{V}, \Phi)$ on **M** is an open set $\mathcal{V}$ of **M** together with a homeomorphism Φ from a subset $\mathcal{V} \subset \mathbf{S}$ into an open set $\mathcal{O} \subset R^n$. A chart is a local coordinate neighborhood for each of its points. The point of R^n which is the image of a point $x \in \mathbf{M}$ is called the coordinate of x. An atlas $\mathcal{A}$ is a denumerable family of charts $\{(\mathcal{V}_i, \Phi_i)\}$ with the following properties:

a. **S** is contained in the union of all charts belonging to the family.

$$\mathbf{S} \subset \cup_i \mathcal{V}_i$$

b. When two charts overlap, i.e. $\mathcal{V}_{ik} = \mathcal{V}_i \cap \mathcal{V}_k \neq \emptyset$, the composite map

$$\Phi_{ik} = \Phi_i \circ \Phi_k^{-1} \colon \Phi_k(\mathcal{V}_{ik}) \to \Phi_i(\mathcal{V}_{ik})$$

is a bijection and Φ_{ik} and Φ_{ik}^{-1} are C^r. This is the requirement that Φ_{ik} be a diffeomorphism. The local coordinates of a point in $\mathcal{V}_{ik}$ are C^r functions of any other coordinate systems in the same atlas.

differentiable structures

Two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent if $\mathcal{A}_1 \cup \mathcal{A}_2$ is an atlas. All atlases which are equivalent form an equivalence class. A differentiable structure, denoted $\mathcal{D}$, is an equivalence class of atlases on **S**. There may be more than one differentiable structure on **S**. The union of all atlases forms the <u>complete</u> or <u>maximal</u> atlas.

<u>differential manifolds</u>

A differential manifold **M** is a pair (**S**,$\mathcal{D}$) where **S** is a set and $\mathcal{D}$ is a differentiable structure.

<u>orientable manifolds</u>

When two charts belonging to an atlas overlap, the Jacobian of the coordinate transformation is either positive or negative. If there is a subatlas of the atlas, such that the Jacobian is positive in every overlap, then the manifold is said to be orientable.

<u>time-orientable manifolds</u>

If it is possible to make a continuous distinction between "past" and "future" pointing spacelike vectors over **M**, then **M** is said to be <u>time orientable</u> [this presupposes a metrical or at least a conformal structure on **M**]. If **M** is time orientable, then there exists a non-unique, smooth, nowhere vanishing time-like vector field **t** on **M** (see, for example, Wald [1984]).

<u>tangent vectors and spaces</u>

A <u>tangent vector</u> to a differentiable manifold **M** at a point x∈**M** is a linear function from the space of functions defined and differentiable on some neighborhood of x, which satisfies Leibnitz rule. The space $T_x\mathbf{M}$ of tangent vectors at a point x forms a linear vector space called the tangent space. The set of all pairs (x,**V**) where x is a point of **M** and **V** is a vector of $T_x\mathbf{M}$ can be given a fibre bundle structure. The resulting bundle is the <u>tangent bundle</u>.

vector fields

A vector field is a continuous assignment of a member of $T_x\mathbf{M}$ to each point $x \in \mathbf{M}$.

cotangent spaces and covariant vectors

The dual space V^* of a vector space V is the collection of all linear, real-valued functions on V. When the V is the tangent space at a point $x \in \mathbf{M}$, V is called the cotangent space and is denoted $T^*{}_x\mathbf{M}$. The members $T^*{}_x\mathbf{M}$ of are called covariant vectors at x. A covariant vector field is the continuous assignment of a member of $T^*{}_x\mathbf{M}$ at each point $x \in \mathbf{M}$.

maps between manifolds and diffeomorphisms

Let **N** and **M** be two differentiable manifolds of dimension n and m, $m \geq n$, respectively and let f be a map from **N** into **M**.

$$f: \mathbf{N} \rightarrow \mathbf{M}$$

f is differentiable if its representation in terms of coordinates is differentiable. That is, if $(\mathcal{U},\phi)$ and $(\mathcal{V},\psi)$ are local charts of **N** and **M** respectively, then $\psi \circ f \circ \psi^{-1}$ represents f in local charts:

$$\psi \circ f \circ \psi^{-1}: R^n \rightarrow R^m$$

If this map is differentiable (the coordinates of **M** are differentiable functions of the coordinates of **N**), then f is differentiable. When, in addition, f is a bijection and f^{-1} is differentiable, f is a diffeomorphism. This defines diffeomorphism of a mapping in terms of diffeomorphism of coordinate transformations.

pull-back of functions

If g is a function on $\mathbf{M}$ ($g: \mathbf{M} \rightarrow R$), the pull-back of g to $\mathbf{N}$ by f is denoted by f^*g and is defined by $f^*g(x) = g \circ f(x) = g(f(x))$ where $x \in \mathbf{N}$.

push-forward mappings

In a natural way, the manifold map carries along the tangent vectors from $\mathbf{N}$ into $\mathbf{M}$. If $\mathbf{v}$ is a tangent vector at the point $p \in \mathbf{N}$, i.e. $\mathbf{v} \in T_p\mathbf{N}$, then $f_*\mathbf{v}$ is a tangent vector at the point $x=f(p) \in \mathbf{M}$ defined by

$$(f_*\mathbf{v})h = \mathbf{v}(h \circ f)$$

where $h:\mathbf{N} \rightarrow \mathbf{R}$ is a smooth function. We can extend the action of this map to arbitrary contravariant tensors on N because any such tensor can be written as the sum of exterior products of contravariant vectors.

pull-back mappings

The pull-back map carries covariant vectors from $\mathbf{M}$ back to $\mathbf{N}$. We define the map

$$f^*: T^*_{f(p)}\mathbf{M} \rightarrow T^*_p\mathbf{N}$$

by requiring for all $\mathbf{v} \in T_p\mathbf{N}$ and for all covectors $\mathbf{w} \in T^*_{f(p)}\mathbf{M}$

$$(f^*\mathbf{w})\mathbf{v} = \mathbf{w}(f_*\mathbf{v})$$

We can likewise extend the action of this map to arbitrary covariant tensors.

curves

An example of a manifold mapping is a curve. A curve γ is a differentiable map from an open portion of the real line R into **M**

$$\gamma: I \to \mathbf{M}$$

where $I = (a,b) \subset R$.

tangent vector to a curve

Denote by γ_{*s} the tangent vector to a curve at s.

integral curves of vector fields

A curve is an integral curve of a vector field **v** if its tangent vector at each point equals the vector field

$$\gamma_{*s} = \mathbf{v}_{\gamma(s)}$$

or

$$d\gamma(s)/ds = \mathbf{v}(\gamma(s))$$

The local existence of integral curves is guaranteed by existence theorems for solutions of systems of ordinary differential equations (see Greenspan [1960], p.85). The parametrization of the curve is defined up to a choice of origin.

vector fields as generators of infinitesimal mappings

A smooth vector field **v** and its integral curves γ define a one-parameter family of diffeomorphism

$$\sigma_s: \mathbf{M} \to \mathbf{M}$$

gotten by mapping each point p of **M** into another point which is located a parameter distance s along the integral curve of

v passing through p. An inverse map $\sigma^{-1}{}_s$: **M** $\rightarrow$ **M** also exists.

<u>Lie derivatives</u>

Let **v** and **x** be vector fields on **M** and **s** be a covector field on **M**. The Lie derivatives of **x** and **w** can be defined in terms of the push-forward and pull back maps of σ_s.

$$[\pounds_{\mathbf{v}}\mathbf{x}]_p = \lim_{[s\rightarrow 0]} s^{-1}[\ (\sigma^{-1}{}_s)_*[\mathbf{x}(\sigma_s(p))] - \mathbf{x}(p)]$$

$$[\pounds_{\mathbf{v}}\mathbf{w}]_p = \lim_{[s\rightarrow 0]} s^{-1}[\ (\sigma_s)^*[\mathbf{w}(\sigma_s(p))] - \mathbf{w}(p)]$$

<u>submanifolds</u>

For a detailed treatment, see Choquet-Bruhat et al [1977, p.228]. Let **S** be a subset of **M**, **S** $\subset$ **M**. If every point q $\in$ **S** belongs to some chart $(\mathcal{V},\Phi)$ of **M** such that:

$$\Phi: \mathcal{V}\cap \mathbf{S} \rightarrow R^p \times A^{n-p} \text{ by } \Phi(x) = (x^1, x^2, \ldots, x^p, a^1, \ldots, a^{n-p})$$

[That is, in such charts, (n-m)-coordinates are fixed. The set of charts $\{(\overline{\mathcal{V}}_i, \overline{\Phi}_i)\}$ where $\overline{\mathcal{V}}_i = \mathcal{V}_i \cap \mathbf{S}$ and $\overline{\Phi}_i(x) = (x^1, x^2, \ldots, x^p)$ form an atlas on **N** of the same class as the atlas of **M**.] then **N** is a p-dimensional submanifold of **M**. The <u>co-dimension</u> of **N** is dim**M** - dim**N** = n - p.

Theorem: <u>Submanifolds Defined By A System of Equations</u>

Let a subset **Z** of **M** be defined by a system of r equations (p < n).

$$\mathbf{Z} = \{x \in \mathbf{M} \mid \varphi^{\mathbf{x}}(x) = 0,\ \mathbf{x} = 1, \ldots, r\}$$

such that $\varphi^{\mathbf{x}}(x)$ are differentiable functions and such that the mapping from **M** into R^p defined by $x \to (\varphi^1(x), \varphi^2(x), \ldots, \varphi^r(x))$ is of rank r for all $x \in$ **Z**. Then **Z** is a submanifold of **M** of dimension n - r.

immersions

Let $\mathcal{S}$ be a differentiable manifold of dimension p and let **M** be a differentiable manifold of dimension n $(p \leq n)$. If the differentiable mapping

$$\mathbf{B}:\mathcal{S} \to \mathrm{M}$$

is of rank p for every point $y \in \mathcal{S}$, then **B** is called an immersion. An immersion is not in general injective. The induced mapping on the tangent bundle $\mathbf{B}_*$ is, however, injective. The image of $\mathcal{S}$ under the immersion **B** is denoted $\mathbf{S} = \mathbf{B}[\mathcal{S}]$.

imbeddings

The pair $(\mathbf{B},\mathcal{S})$ where **B** is injective is called an imbedding. A manifold structure on $\mathcal{S}$ can be carried over from $\mathcal{S}$ in the following manner: Let $\{ (\mathcal{U}_i,\psi_i) \}$ be an atlas on $\mathcal{S}$, and let $\mathbf{B}_1$ be the surjective map:

$$\mathbf{B}_1 : \mathcal{S} \to \mathbf{S}$$

Then the collection $\{(\mathbf{B}_1(\mathcal{U}_i),\psi_i \circ \mathbf{B}_1^{-1})\}$ forms an atlas on **S** and one can show that it induces a differentiable manifold structure on $\mathbf{B}[\mathcal{S}]$. This induced manifold structure may not be equivalent to a submanifold structure on **S** given by **M**. When they are equivalent, the imbedding is said to be regular. If **B** is a regular imbedding, then **S** is a submanifold of **M**.

foliations and distributions

[See Reinhart [1983] for a detailed treatment of foliations.] A map which assigns to each $x \in \mathbf{M}$ a p-dimensional subspace $E_x \subset T_x\mathbf{M}$ is called a p-dimensional distribution or plane-field on **M**. The set $E = \cup E_x \subset T\mathbf{M}$ is a sub-bundle of the tangent bundle of **M**. E is a vector bundle over **M** and is a sub-manifold of **TM**. E is involutive if for any two vector fields **X** and **Y** belonging to E defined on an open set $\mathbf{U} \subset \mathbf{M}$, their Lie Bracket $[\mathbf{X},\mathbf{Y}]$ belongs to E.

integrable distributions

A distribution E is integrable if for any $x \in \mathbf{M}$ there is a locally defined submanifold **S**, called an integral manifold of E, at x such that its tangent bundle is E (restricted to points of **S**).

Frobenius' Theorem

A p-distribution E has an integral manifold if and only if it is involutive. In an open set **U** where E is involutive, there passes through each point only one integral manifold of a distribution. No point belongs to more than one integral manifold. Such a system of integral manifolds is called a foliation of **U**.

exterior Systems

Given a p-distribution plane E_x at any point x, there exists at each point a n-p-dimensional set of forms $\omega^{\mathbf{Y}}$ such that $\omega^{\mathbf{Y}}$ annhilates each vector belonging to E_x ($\omega^{\mathbf{Y}}(\mathbf{z}) = 0$ for each $\mathbf{z} \in E_x$). The p-distribution E defines such a set at each point called a Pfaff system of differential forms or exterior system. The set of forms at each point form a co-distribution field D_x of dimension n-p which forms a sub-bundle D of the cotangent bundle $T^*\mathbf{M}$. Conversely, a co-distribution D determines a distribution E. We may ask what conditions on D insures the existence of integral manifolds of E. The answer, called the dual version of Frobenius' Theorem is that the closure of D defines the same distribution as D itself. The closure of D is the set of 2-forms $\mathbf{d}\omega^y$ at each point. This means that

$$\mathbf{d}\omega^y = \tau_x \wedge \omega^x$$

where τ is a one-form.

Definition: A Pfaffian system of rank $r = n - p$ is completely integrable if there exists a set of r differentiable functions $\phi^x(x)$ whose exterior deriviatives define the same co-distribution as D. Then obviously, each point x belongs to a submanifold of $\mathbf{M}$ defined by the system of equations $\phi^1(x) = c^1$, $\phi^2(x) = c^2$, etc.

APPENDIX B

TWO + TWO PROJECTION OF THE RIEMANN AND RICCI TENSORS IN AFFINE AND METRIC SPACES

Gauss' Equation in an Affine Space

The proof here is for a general anholonomic subspace $\mathcal{N}$ imbedded in an affine space **M**. From the definition of the induced covariant derivatives defined in Section 2.2, we have for a vector p^A of $\mathcal{N}$

$$D_{[D}D_{C]}p^A = B^{\mu}{}_D{}^{\nu}{}_C{}^A{}_{\kappa}{}'\nabla_{[\mu}{}'\nabla_{\nu]}p^{\kappa}$$

$$= B^{\mu}{}_D{}^{\nu}{}_C{}^A{}_{\kappa}\nabla_{[\mu}\{B^{\rho}{}_{\nu]}{}^{\kappa}{}_{\lambda}\nabla_{\rho}p^{\lambda}\}$$

$$= B^{\mu}{}_D{}^{\nu}{}_C{}^A{}_{\kappa}\nabla_{[\mu}\{B^{\rho}{}_{\nu]}{}^{\kappa}{}_{\lambda}\}\nabla_{\rho}p^{\lambda} + B^{\mu}{}_D{}^{\nu}{}_C{}^A{}_{\kappa}\nabla_{[\mu}\nabla_{|\rho|}p^{\lambda}B^{\rho}{}_{\nu]}{}^{\kappa}{}_{\lambda}$$

$$= B^{\mu}{}_D{}^{\nu}{}_C{}^A{}_{\lambda}\nabla_{[\mu}\{B^{\rho}{}_{\nu]}\}\nabla_{\rho}p^{\lambda} + B^{\mu}{}_D{}^{\nu}{}_C{}^A{}_{\kappa}\nabla_{[\mu}\{B^{\kappa}{}_{|\lambda|}\}B^{\rho}{}_{\nu]}\nabla_{\rho}p^{\lambda}$$

$$+ B^{\mu}{}_D{}^{\nu}{}_C{}^A{}_{\lambda}\nabla_{[\mu}\nabla_{\nu]}p^{\lambda}$$

$$= - B^{\mu}{}_D{}^{\nu}{}_C{}^A{}_{\lambda}\nabla_{[\mu}\{C^{\rho}{}_{\nu]}\}\nabla_{\rho}p^{\lambda} - B^{\mu}{}_D{}^{\nu}{}_C{}^A{}_{\kappa}\nabla_{[\mu}\{C^{\kappa}{}_{|\lambda|}\}B^{\rho}{}_{\nu]}\nabla_{\rho}p^{\lambda}$$

$$+ 1/2\ B^{\mu}{}_D{}^{\nu}{}_C{}^{\kappa}{}_B{}^A{}_{\lambda}R_{\mu\nu\kappa}{}^{\lambda}p^B$$

$$= -B^{\mu}{}_D{}^{\nu}{}_C{}^A{}_{\lambda}C_{x}{}^{\rho}\nabla_{[\mu}C^{x}{}_{\nu]}\nabla_{\rho}p^{\lambda}$$

$$-B^{\mu}{}_D{}^{\nu}{}_C{}^A{}_{\kappa}C^{x}{}_{\lambda}\nabla_{[\mu}C_{x}{}^{\kappa}B^{\rho}{}_{\nu]}\ \nabla_{\rho}p^{\lambda} + 1/2B^{\mu}{}_D{}^{\nu}{}_C{}^{\kappa}{}_B{}^A{}_{\lambda}R_{\mu\nu\kappa}{}^{\lambda}p^B \qquad [B-1]$$

The first term on the right-hand side contains $H^{x}{}_{[DC]}$ and vanishes for holonomic surfaces although we include it here for generality. Then

$$D_{[D}\ D_{C]}p^A = H^{\mathbf{x}}{}_{[DC]}D_{\mathbf{x}}p^A - B^{\mu}{}_D{}^{\nu}{}_C{}^A{}_{\kappa}C^{\mathbf{x}}{}_{\lambda}\nabla_{[\mu}C_{\mathbf{x}}{}^{\kappa}B^{\rho}{}_{\nu}\nabla_{\rho}p^{\lambda}$$

$$+ 1/2\ R_{DCB}{}^A p^B$$

$$= H^{\mathbf{x}}{}_{[DC]}D_{\mathbf{x}}p^A + B^{\mu}{}_D{}^{\nu}{}_C{}^A{}_{\kappa}p^{\lambda}\nabla_{[\mu}C_{\mathbf{x}}{}^{\kappa}B^{\rho}{}_{\nu]}\nabla_{\rho}C^{\mathbf{x}}{}_{\lambda}$$

$$+ 1/2\ R_{DCB}{}^A p^B$$

$$= H^{\mathbf{x}}{}_{[DC]}D_{\mathbf{x}}p^A + L_{[D}{}^A{}_{|\mathbf{x}|}H^{\mathbf{x}}{}_{C]B}p^B$$

$$+ 1/2\ R_{DCB}{}^A p^B \qquad [B-2]$$

where is the projection of the Riemann tensor of **M** into the surface.

The Riemann tensor of an anholonomic two-surface element has a more complicated expression in terms of the commutator of the covariant derivatives than it does in a holonomic space. According to Schouten [1954], the Riemann tensor $'R_{DCB}{}^A$ of an anholonomic surface element is defined by

$$2D_{[D}\ D_{C]}p^A = 2H^{\mathbf{x}}{}_{[DC]}\mathfrak{D}_{\mathbf{x}}p^A + {}'R_{DCB}{}^A p^B \qquad [B-3]$$

where

$$'R_{DCB}{}^A \equiv 2\partial_{[D}\Gamma^A{}_{C]B} + 2\Gamma^A{}_{[D|E|}\Gamma^E{}_{C]B} - 2\Gamma^E{}_{[DC]}\Gamma^A{}_{EB} + 2\Omega^{\mathbf{x}}{}_{DC}\Omega^A{}_{\mathbf{x}B}$$

$$[B-4]$$

Combining equations B-2 and B-3 gives

$$H^{\mathbf{x}}{}_{[DC]}\mathfrak{D}_{\mathbf{x}}p^A + 1/2\ {}'R_{DCB}{}^A p^B = H^{\mathbf{x}}{}_{[DC]}D_{\mathbf{x}}p^A + L_{[D}{}^A{}_{|\mathbf{x}|}H^{\mathbf{x}}{}_{C]B}p^B$$

$$+ 1/2\ R_{DCB}{}^A p^B \qquad [B-5]$$

Recalling the relation between $\mathfrak{D}_{\mathbf{x}}p^A$ and $D_{\mathbf{x}}p^A$, defined in Section 2.3, we get

$$1/2\ {}'R_{DCB}{}^A p^B = -H^{\mathbf{x}}{}_{[DC]}L_C{}^A{}_{\mathbf{x}}p^C + L_{[D}{}^A{}_{|\mathbf{x}|}H^{\mathbf{x}}{}_{C]B}p^B$$

$$+ 1/2\ R_{DCB}{}^{A}p^{B}$$

yielding Gauss' equation

$$'R_{DCB}{}^{A} = R_{DCB}{}^{A}p^{B} - 2H^{\mathbf{x}}{}_{[DC]}L_{C}{}^{A}{}_{\mathbf{x}} + 2L_{[D}{}^{A}{}_{|\mathbf{x}|}H^{\mathbf{x}}{}_{C]B} \qquad [B-6]$$

Rearranging terms, we have the total-surface projection of the Riemann tensor of **M**

$$R_{DCB}{}^{A} = {}'R_{DCB}{}^{A} + 2H^{\mathbf{x}}{}_{[DC]}L_{C}{}^{A}{}_{\mathbf{x}} - 2L_{[D}{}^{A}{}_{|\mathbf{x}|}H^{\mathbf{x}}{}_{C]B} \qquad [B-7]$$

When $\mathcal{N}$ is a holonomic surface, the second term on the right-hand side vanishes giving the well-known form of Gauss's equation

$$R_{DCB}{}^{A} = {}'R_{DCB}{}^{A} - 2L_{[D}{}^{A}{}_{|\mathbf{x}|}H^{\mathbf{x}}{}_{C]B} \qquad [B-8]$$

Contracting indices gives an expression for the Ricci tensor of $\mathcal{N}$ in the holonomic case

$$R_{ACB}{}^{A} = {}'R_{CB} - L_{A}{}^{A}{}_{\mathbf{x}}H^{\mathbf{x}}{}_{CB} + L_{C}{}^{A}{}_{\mathbf{x}}H^{\mathbf{x}}{}_{AB} \qquad [B-9]$$

Codazzi's Equation in an Affine Space

Codazzi's equation for an anholonomic two-surface $\mathcal{N}$ can be derived by expanding $D_{[C}H_{A]B}{}^{\mathbf{x}}$

$$D_{[C}H_{A]B}{}^{\mathbf{x}} = B^{\mu}{}_{C}{}^{\nu}{}_{A}{}^{\kappa}{}_{B}C^{\mathbf{x}}{}_{\rho}\nabla_{[\mu}H_{\nu]\kappa}{}^{\rho}$$

$$= - B^{\mu}{}_{C}{}^{\nu}{}_{A}{}^{\kappa}{}_{B}C^{\mathbf{x}}{}_{\rho}\nabla_{[\mu}(B^{\alpha}{}_{\nu]}{}^{\beta}{}_{\kappa}\nabla_{\alpha}C^{\rho}{}_{\beta})$$

$$= - B^{\mu}{}_{C}{}^{\nu}{}_{A}{}^{\kappa}{}_{B}C^{\mathbf{x}}{}_{\rho}\nabla_{[\mu}(B^{\alpha}{}_{\nu]}{}^{\beta}{}_{\kappa})\ \nabla_{\alpha}C^{\rho}{}_{\beta}$$

$$+ B^{\mu}{}_{C}{}^{\nu}{}_{A}{}^{\beta}{}_{B}C^{x}{}_{\rho}B^{\alpha}{}_{[\nu}(\nabla_{\mu]}\nabla_{\alpha}C^{\rho}{}_{\beta})$$

$$= - B^{\mu}{}_{C}{}^{\nu}{}_{A}{}^{\kappa}{}_{B}C^{x}{}_{\rho}\nabla_{[\mu}(B^{\alpha}{}_{\nu]})B^{\beta}{}_{\kappa}\nabla_{\alpha}C^{\rho}{}_{\beta}$$

$$+ B^{\mu}{}_{C}{}^{\nu}{}_{A}{}^{\kappa}{}_{B}C^{x}{}_{\rho}B^{\alpha}{}_{[\nu}\nabla_{\mu]}B^{\beta}{}_{\kappa}(\nabla_{\alpha}C^{\rho}{}_{\beta})$$

$$+ B^{\mu}{}_{C}{}^{\nu}{}_{A}{}^{\beta}{}_{B}C^{x}{}_{\rho}B^{\alpha}{}_{[\nu}(\nabla_{\mu]}\nabla_{\alpha}C^{\rho}{}_{\beta}) \qquad [B-10]$$

Since the second term on the right hand side of equation B-10 vanishes, we arrive at

$$D_{[C}H_{A]B}{}^{x} = -H_{[CA]}{}^{y}{}^{*}L_{y}{}^{x}{}_{B} + B^{\mu}{}_{C}{}^{\nu}{}_{A}{}^{\beta}{}_{B}C^{x}{}_{\rho}B^{\alpha}{}_{[\nu}(\nabla_{\mu]}\nabla_{\alpha}C^{\rho}{}_{\beta})$$

$$= -H_{[CA]}{}^{y}{}^{*}L_{y}{}^{x}{}_{B} - B^{\mu}{}_{C}{}^{\nu}{}_{A}{}^{\beta}{}_{B}C^{x}{}_{\rho}\nabla_{[\mu}\nabla_{\nu]}C^{\rho}{}_{\beta}$$

$$= -H_{[CA]}{}^{y}{}^{*}L_{y}{}^{x}{}_{B} + 1/2\, R_{CAB}{}^{x} \qquad [B-11]$$

or

$$R_{CAB}{}^{x} = 2H_{[CA]}{}^{y}{}^{*}L_{y}{}^{x}{}_{B} + 2D_{[C}H_{A]B}{}^{x} \qquad [B-12]$$

This is <u>Codazzi's equation</u> for $\mathcal{N}$. The first term on the right-hand side vanishes when $\mathcal{N}$ is holonomic and then we get

$$R_{CAB}{}^{x} = 2D_{[C}H_{A]B}{}^{x} \qquad [B-13]$$

<u>$R_{xAB}{}^{y}$ Projection of the Riemann Tensor in an Affine Space</u>

Another set of projections of the Riemann tensor that play an important role in the analysis of the field equations is $R_{\mathbf{x}AB}{}^{\mathbf{y}}$. This set will be obtained by writing out $D_{\mathbf{x}}H_{AB}{}^{\mathbf{y}}$ in four-dimensional form

$$D_{\mathbf{x}}H_{AB}{}^{\mathbf{y}} = C^{\rho}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\beta}B^{\mu}{}_{A}{}^{\lambda}{}_{B}\nabla_{\rho}H_{\mu\lambda}{}^{\beta} = -\ C^{\rho}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\beta}B^{\mu}{}_{A}{}^{\lambda}{}_{B}\nabla_{\rho}(B^{\alpha}{}_{\mu}{}^{\gamma}{}_{\lambda}\nabla_{\alpha}C^{\beta}{}_{\gamma})$$

$$= -\ C^{\rho}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\beta}B^{\mu}{}_{A}{}^{\gamma}{}_{B}\nabla_{\rho}(B^{\alpha}{}_{\mu})\nabla_{\alpha}C^{\beta}{}_{\gamma} - C^{\rho}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\beta}B^{\alpha}{}_{A}{}^{\lambda}{}_{B}\nabla_{\rho}(B^{\gamma}{}_{\lambda})\nabla_{\alpha}C^{\beta}{}_{\gamma}$$

$$-\ C^{\rho}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\beta}B^{\alpha}{}_{A}{}^{\gamma}{}_{B}\nabla_{\rho}\nabla_{\alpha}C^{\beta}{}_{\gamma} \qquad [B-14]$$

Using the fact that the triple **B**- or **C**-projection of $\nabla_{\rho}B^{\alpha}{}_{\mu}$ is identically zero, one can show that the second term on right-hand side of the last line of B-14 vanishes giving

$$D_{\mathbf{x}}H_{AB}{}^{\mathbf{y}} = {}^{*}L_{\mathbf{x}}{}^{\mathbf{z}}{}_{A}{}^{*}L_{\mathbf{z}}{}^{\mathbf{y}}{}_{B} - C^{\rho}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\beta}B^{\alpha}{}_{A}{}^{\gamma}{}_{B}\nabla_{\alpha}\nabla_{\rho}C^{\beta}{}_{\gamma}$$

$$-\ C^{\rho}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\beta}B^{\alpha}{}_{A}{}^{\gamma}{}_{B}\nabla_{[\rho}\nabla_{\alpha]}C^{\beta}{}_{\gamma}$$

$$= {}^{*}L_{\mathbf{x}}{}^{\mathbf{z}}{}_{A}{}^{*}L_{\mathbf{z}}{}^{\mathbf{y}}{}_{B} - C^{\rho}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\beta}B^{\alpha}{}_{A}{}^{\gamma}{}_{B}\nabla_{\alpha}\nabla_{\rho}C^{\beta}{}_{\gamma} - C^{\rho}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\beta}B^{\alpha}{}_{A}{}^{\gamma}{}_{B}R_{\rho\alpha\kappa}{}^{\beta}C^{\kappa}{}_{\gamma}$$

$$+\ C^{\rho}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\beta}B^{\alpha}{}_{A}{}^{\gamma}{}_{B}R_{\rho\alpha\gamma}{}^{\sigma}C^{\beta}{}_{\sigma} \qquad [B-15]$$

The third term on the right hand side of B-15 vanishes because of the contraction of $B^{\gamma}{}_{B}$ and $C^{\kappa}{}_{\gamma}$ so that

$$D_{\mathbf{x}}H_{AB}{}^{\mathbf{y}} = {}^*L_{\mathbf{x}}{}^{\mathbf{z}}{}_A{}^*L_{\mathbf{z}}{}^{\mathbf{y}}{}_B - C^{\rho}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\beta}B^{\alpha}{}_A{}^{\gamma}{}_B\nabla_{\alpha}\nabla_{\rho}C^{\beta}{}_{\gamma} + R_{\mathbf{x}AB}{}^{\mathbf{y}}$$

$$= {}^*L_{\mathbf{x}}{}^{\mathbf{z}}{}_A{}^*L_{\mathbf{z}}{}^{\mathbf{y}}{}_B - C^{\nu}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\beta}B^{\alpha}{}_A{}^{\sigma}{}_B\nabla_{\alpha}(C^{\rho}{}_{\nu}B^{\gamma}{}_{\sigma}\nabla_{\rho}C^{\beta}{}_{\gamma})$$

$$+ C^{\nu}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\beta}B^{\alpha}{}_A{}^{\sigma}{}_B\nabla_{\alpha}(C^{\rho}{}_{\nu}B^{\gamma}{}_{\sigma})\nabla_{\rho}C^{\beta}{}_{\gamma} + R_{\mathbf{x}AB}{}^{\mathbf{y}}$$

$$= {}^*L_{\mathbf{x}}{}^{\mathbf{z}}{}_A{}^*L_{\mathbf{z}}{}^{\mathbf{y}}{}_B - D_AL_{\mathbf{x}}{}^{\mathbf{y}}{}_B + C^{\nu}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\beta}B^{\alpha}{}_A{}^{\sigma}{}_B\nabla_{\alpha}(C^{\rho}{}_{\nu}B^{\gamma}{}_{\sigma})\nabla_{\rho}C^{\beta}{}_{\gamma}$$

$$+ R_{\mathbf{x}AB}{}^{\mathbf{y}}$$

$$= {}^*L_{\mathbf{x}}{}^{\mathbf{z}}{}_A{}^*L_{\mathbf{z}}{}^{\mathbf{y}}{}_B - D_AL_{\mathbf{x}}{}^{\mathbf{y}}{}_B + C^{\rho}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\beta}B^{\alpha}{}_A{}^{\sigma}{}_B\nabla_{\alpha}(B^{\gamma}{}_{\sigma})\nabla_{\rho}C^{\beta}{}_{\gamma}$$

$$+ C^{\nu}{}_{\mathbf{x}}{}^{\mathbf{y}}{}_{\beta}B^{\alpha}{}_A{}^{\gamma}{}_B\nabla_{\alpha}(C^{\rho}{}_{\nu})\nabla_{\rho}C^{\beta}{}_{\gamma} + R_{\mathbf{x}AB}{}^{\mathbf{y}} \qquad [B-16]$$

Finally, one can show that the third term on right hand side of the last line of B-16 vanishes. Thus

$$D_{\mathbf{x}}H_{AB}{}^{\mathbf{y}} = {}^*L_{\mathbf{x}}{}^{\mathbf{z}}{}_A{}^*L_{\mathbf{z}}{}^{\mathbf{y}}{}_B - D_A{}^*L_{\mathbf{x}}{}^{\mathbf{y}}{}_B + L_A{}^C{}_{\mathbf{x}}H_{CB}{}^{\mathbf{y}} + R_{\mathbf{x}AB}{}^{\mathbf{y}} \qquad [B-17]$$

We can express the left-hand side of equation B-17 in terms of the $\mathfrak{D}$-derivatives if we recall the relation between the D- and $\mathfrak{D}$-derivatives developed in Section 2.3. We have

$$\mathfrak{D}_A{}^*L_{\mathbf{x}}{}^{\mathbf{y}}{}_B = D_A{}^*L_{\mathbf{x}}{}^{\mathbf{y}}{}_B - {}^*L_{\mathbf{z}}{}^{\mathbf{y}}{}_B{}^*L_{\mathbf{x}}{}^{\mathbf{z}}{}_A + {}^*L_{\mathbf{x}}{}^{\mathbf{z}}{}_B{}^*L_{\mathbf{z}}{}^{\mathbf{y}}{}_A \qquad [B-18]$$

$$\mathfrak{D}_{\mathbf{x}}H_{AB}{}^{\mathbf{y}} = D_{\mathbf{x}}H_{AB}{}^{\mathbf{y}} - H_{CB}{}^{\mathbf{y}}L_A{}^C{}_{\mathbf{x}} - H_{AC}{}^{\mathbf{y}}L_B{}^C{}_{\mathbf{x}} \qquad [B-19]$$

Hence

$$\mathcal{D}_{\mathbf{x}} H_{AB}{}^{\mathbf{y}} = -\mathcal{D}_A {}^*L_{\mathbf{x}}{}^{\mathbf{y}}{}_B + {}^*L_{\mathbf{x}}{}^{\mathbf{z}}{}_B {}^*L_{\mathbf{z}}{}^{\mathbf{y}}{}_A - H_{AC}{}^{\mathbf{y}} L_B{}^C{}_{\mathbf{x}} + R_{\mathbf{x}AB}{}^{\mathbf{y}} \qquad [B-20]$$

and the desired expression for the Riemann tensor is

$$R_{\mathbf{x}AB}{}^{\mathbf{y}} = \mathcal{D}_{\mathbf{x}} H_{AB}{}^{\mathbf{y}} + \mathcal{D}_A {}^*L_{\mathbf{x}}{}^{\mathbf{y}}{}_B - {}^*L_{\mathbf{x}}{}^{\mathbf{z}}{}_B {}^*L_{\mathbf{z}}{}^{\mathbf{y}}{}_A + H_{AC}{}^{\mathbf{y}} L_B{}^C{}_{\mathbf{x}} \qquad [B-21]$$

Equations B-7, B-12 and B-21 have their counterparts for the rigging space $\mathcal{T}$. The results are stated without proof but are most simply obtained by swapping $\mathcal{N}$-indices for $\mathcal{T}$-indices and vice versa.

Gauss' Equation

$$R_{\mathbf{zxy}}{}^{\mathbf{w}} = {}''R_{\mathbf{zxy}}{}^{\mathbf{w}} + 2\,{}^*H^A{}_{[\mathbf{zx}]} L_{\mathbf{y}}{}^{\mathbf{w}}{}_A - 2\,{}^*L_{[\mathbf{z}}{}^{\mathbf{w}}{}_{|A|} H^A{}_{\mathbf{x}]\mathbf{y}}$$

[B-22]

Codazzi's Equation

$$R_{\mathbf{xyz}}{}^A = = 2\,{}^*H_{[\mathbf{xy}]}{}^B L_B{}^A{}_{\mathbf{z}} + 2D_{[\mathbf{x}} {}^*H_{\mathbf{y}]\mathbf{z}}{}^A \qquad [B-23]$$

$R_{C\mathbf{xy}}{}^D$ Projection Equation

$$R_{C\mathbf{xy}}{}^D = \mathcal{D}_C {}^*H_{\mathbf{xy}}{}^D + \mathcal{D}_{\mathbf{x}} L_C{}^D{}_{\mathbf{y}} - L_C{}^B{}_{\mathbf{y}} L_B{}^D{}_{\mathbf{x}} + {}^*H_{\mathbf{xw}}{}^D {}^*L_{\mathbf{y}}{}^{\mathbf{w}}{}_C \qquad [B-24]$$

Results in Metric Space

When a metric exists on the manifold and the rigging vectors are chosen orthogonal to the two-surfaces, the tensors **H** and **L** are related. The same holds for ***H** and ***L**. This was proved in Section 2.4. We have

$$L_A{}^B{}_{\mathbf{x}} = g_{\mathbf{xy}} g^{BC} H^{\mathbf{y}}{}_{AC} \qquad [B-25]$$

$${}^*L_{\mathbf{y}}{}^{\mathbf{z}}{}_A = g_{AB} g^{\mathbf{wz}} {}^*H^B{}_{\mathbf{yw}} \qquad [B-26]$$

We regard them as being related by the process of raising and lowering indices using the induced metrics. For this reason, we drop the **L**-symbol and always use **H** in a metric space.

All the equations we have developed so far remain the same in a metric space but we replace the **L**'s with **H**'s. We can, however, develop equation B-21 somewhat further.

Since $\mathfrak{D}_{\mathbf{x}} g^{\mathbf{zy}} \equiv D_{\mathbf{x}} g^{\mathbf{zy}} = 0$, we can rewrite equation B-21 as

$$R_{\mathbf{xABy}} = \mathfrak{D}_{\mathbf{x}} H_{\mathbf{ABy}} + g_{\mathbf{yz}} \mathfrak{D}_A {}^{*}H_{\mathbf{x}}{}^{\mathbf{z}}{}_B - {}^{*}H_{\mathbf{x}}{}^{\mathbf{z}}{}_B {}^{*}H_{\mathbf{zyA}} + H_{AC\mathbf{y}} H_B{}^C{}_{\mathbf{x}} \qquad \text{[B-27]}$$

In Section 2.3 we derived

$$H_{AB\mathbf{y}} = -1/2\ \mathfrak{D}_{\mathbf{y}} g_{AB} \qquad \text{[B-28]}$$

and if we substite this into equation B-27, we get

$$R_{\mathbf{x}AB\mathbf{y}} = -1/2 \mathfrak{D}_{\mathbf{x}} \mathfrak{D}_{\mathbf{y}} g_{AB} + g_{\mathbf{yz}} \mathfrak{D}_A {}^{*}H_{\mathbf{x}}{}^{\mathbf{z}}{}_B - {}^{*}H_{\mathbf{x}}{}^{\mathbf{z}}{}_B {}^{*}H_{\mathbf{zy}A} + H_{AC\mathbf{y}} H_B{}^C{}_{\mathbf{x}} \qquad \text{[B-29]}$$

We can derive an equation similar to B-29, but involving the conformal two-metric. Operate on both sides of B-29 with the trace-removing operator $\Lambda^A{}_C{}^B{}_D$. The first term on the right-hand side becomes

$$-1/2 \Lambda^A{}_C{}^B{}_D \mathfrak{D}_{\mathbf{x}} \mathfrak{D}_{\mathbf{y}} g_{AB} = -1/2 \mathfrak{D}_{\mathbf{x}} (\Lambda^A{}_C{}^B{}_D \mathfrak{D}_{\mathbf{y}} g_{AB}) + 1/2 \mathfrak{D}_{\mathbf{x}} (\Lambda^A{}_C{}^B{}_D) \mathfrak{D}_{\mathbf{y}} g_{AB} \qquad \text{[B-30]}$$

We can use equation 2.3-69 to write this as

$$-1/2\Lambda^{A}{}_{C}{}^{B}{}_{D}\ \mathfrak{D}_{\mathbf{x}}\mathfrak{D}_{\mathbf{y}}g_{AB} = -1/2\mathfrak{D}_{\mathbf{x}}(\gamma\mathfrak{D}_{\mathbf{y}}\tilde{g}_{CD}) + 1/2\mathfrak{D}_{\mathbf{x}}(\Lambda^{A}{}_{C}{}^{B}{}_{D})\mathfrak{D}_{\mathbf{y}}g_{AB}$$

$$= -1/2\mathfrak{D}_{\mathbf{x}}(\gamma\mathfrak{D}_{\mathbf{y}}\tilde{g}_{CD}) - 1/4\ \mathfrak{D}_{\mathbf{x}}(g^{AB}g_{CD})\mathfrak{D}_{\mathbf{y}}g_{AB} \qquad [B-31]$$

Then the $\mathcal{N}$-traceless form of equation B-29 is

$$\Lambda^{A}{}_{C}{}^{B}{}_{D}R_{\mathbf{x}AB\mathbf{y}} = -1/2\mathfrak{D}_{\mathbf{x}}(\gamma\mathfrak{D}_{\mathbf{y}}\tilde{g}_{CD}) + \Lambda^{A}{}_{C}{}^{B}{}_{D}\ [g_{\mathbf{yz}}\mathfrak{D}_{A}{}^{*}H_{\mathbf{x}}{}^{\mathbf{z}}{}_{B}$$

$$- {}^{*}H_{\mathbf{x}}{}^{\mathbf{z}}{}_{B}{}^{*}H_{\mathbf{zy}A} + H_{AE\mathbf{y}}H_{B}{}^{E}{}_{\mathbf{x}}] - 1/4\ \mathfrak{D}_{\mathbf{x}}(g^{AB}g_{CD})\mathfrak{D}_{\mathbf{y}}g_{AB}$$

$$= -1/2\mathfrak{D}_{\mathbf{x}}(\gamma\mathfrak{D}_{\mathbf{y}}\tilde{g}_{CD}) + \Lambda^{A}{}_{C}{}^{B}{}_{D}[g_{\mathbf{yz}}\mathfrak{D}_{A}{}^{*}H_{\mathbf{x}}{}^{\mathbf{z}}{}_{B} - {}^{*}H_{\mathbf{x}}{}^{\mathbf{z}}{}_{B}{}^{*}H_{\mathbf{zy}A}]$$

$$+ H_{CE\mathbf{y}}H_{D}{}^{E}{}_{\mathbf{x}} + 1/2g_{CD}H_{AB\mathbf{y}}H^{AB}{}_{\mathbf{x}} - H_{\mathbf{y}}H_{CD\mathbf{x}} \qquad [B-32]$$

This equation can be further reduced to

$$\Lambda^{A}{}_{C}{}^{B}{}_{D}\ R_{\mathbf{x}AB\mathbf{y}} = -1/2\ \gamma\mathfrak{D}_{\mathbf{x}}\mathfrak{D}_{\mathbf{y}}\tilde{g}_{CD}$$

$$+ \Lambda^{A}{}_{C}{}^{B}{}_{D}[g_{\mathbf{yz}}D_{A}{}^{*}H_{\mathbf{x}}{}^{\mathbf{z}}{}_{B} - {}^{*}H_{\mathbf{x}}{}^{\mathbf{z}}{}_{B}{}^{*}H_{\mathbf{zy}A}]$$

$$+ H_{CE\mathbf{y}}H_{D}{}^{E}{}_{\mathbf{x}} + 1/2g_{CD}H_{AB\mathbf{y}}H^{AB}{}_{\mathbf{x}} - H_{\mathbf{y}}H_{CD\mathbf{x}} - 1/2\mathfrak{D}_{\mathbf{x}}\gamma\mathfrak{D}_{\mathbf{y}}\tilde{g}_{CD}$$

[B-33]

and

$$\Lambda^{A}{}_{C}{}^{B}{}_{D}R_{\mathbf{x}AB\mathbf{y}} = -1/2\ \gamma\mathfrak{D}_{\mathbf{x}}\mathfrak{D}_{\mathbf{y}}\tilde{g}_{CD} + \Lambda^{A}{}_{C}{}^{B}{}_{D}[g_{\mathbf{yz}}\mathfrak{D}_{A}{}^{*}H_{\mathbf{x}}{}^{\mathbf{z}}{}_{B} - {}^{*}H_{\mathbf{x}}{}^{\mathbf{z}}{}_{B}{}^{*}H_{\mathbf{zy}A}]$$

$$+ H_{CE\mathbf{y}}H_{D}{}^{E}{}_{\mathbf{x}} + 1/2g_{CD}H_{AB\mathbf{y}}H^{AB}{}_{\mathbf{x}} - H_{\mathbf{y}}H_{CD\mathbf{x}}$$

$$- H_{\mathbf{x}}\Lambda^{A}{}_{C}{}^{B}{}_{D}H_{AB\mathbf{y}}$$

$$= -1/2\ \gamma \mathcal{D}_{\mathbf{x}} \mathcal{D}_{\mathbf{y}} \tilde{g}_{CD} + \Lambda^A{}_C{}^B{}_D [g_{\mathbf{yz}} \mathcal{D}_A {}^*H_{\mathbf{x}}{}^{\mathbf{z}}{}_B - {}^*H_{\mathbf{x}}{}^{\mathbf{z}}{}_B {}^*H_{\mathbf{zy}A}]$$
$$+ H_{CE\mathbf{y}} H_D{}^E{}_{\mathbf{x}} + 1/2 g_{CD} H_{AB\mathbf{y}} H^{AB}{}_{\mathbf{x}} - H_{\mathbf{y}} H_{CD\mathbf{x}}$$
$$- H_{\mathbf{x}} \overline{H}_{CD\mathbf{y}} \qquad [B-34]$$

where $\overline{H}_{CD\mathbf{y}}$ is the traceless extrinsic curvature tensor introduced in Section 2.4.

The right-hand side can be written entirely in terms of $\overline{H}_{CD\mathbf{y}}$

$$\Lambda^A{}_C{}^B{}_D R_{\mathbf{x}AB\mathbf{y}} = -1/2\ \gamma \mathcal{D}_{\mathbf{x}} \mathcal{D}_{\mathbf{y}} \tilde{g}_{CD}$$
$$+ \Lambda^A{}_C{}^B{}_D\ [g_{\mathbf{yz}} \mathcal{D}_A {}^*H_{\mathbf{x}}{}^{\mathbf{z}}{}_B - {}^*H_{\mathbf{x}}{}^{\mathbf{z}}{}_B {}^*H_{\mathbf{zy}A}]$$
$$- 1/2\ [H_{\mathbf{y}} \overline{H}_{CD\mathbf{x}} + H_{\mathbf{x}} \overline{H}_{CD\mathbf{y}}] + \overline{H}_{CE\mathbf{y}} \overline{H}_D{}^E{}_{\mathbf{x}}$$
$$+ 1/2 g_{CD} \overline{H}_{AB\mathbf{y}} \overline{H}^{AB}{}_{\mathbf{x}}$$

[B-35]

or in terms of the traceless extrinsic curvature density $\tilde{H}_{CD\mathbf{y}}$

$$\gamma^{-1} \Lambda^A{}_C{}^B{}_D R_{\mathbf{x}AB\mathbf{y}} = -1/2 \mathcal{D}_{\mathbf{x}} \mathcal{D}_{\mathbf{y}} \tilde{g}_{CD}$$
$$+ \gamma^{-1} \Lambda^A{}_C{}^B{}_D [g_{\mathbf{yz}} \mathcal{D}_A {}^*H_{\mathbf{x}}{}^{\mathbf{z}}{}_B - {}^*H_{\mathbf{x}}{}^{\mathbf{z}}{}_B {}^*H_{\mathbf{zy}A}]$$
$$- 1/2 [H_{\mathbf{y}} \tilde{H}_{CD\mathbf{x}} + H_{\mathbf{x}} \tilde{H}_{CD\mathbf{y}}] + \tilde{H}_{CE\mathbf{y}} \tilde{H}_D{}^E{}_{\mathbf{x}}$$
$$+ 1/2 \tilde{g}_{CD} \tilde{H}_{AB\mathbf{y}} \tilde{H}^{AB}{}_{\mathbf{x}} \qquad [B-36]$$

When a metric exists, the symmetry properties of the Riemann tensor imply that

$$R_{A\mathbf{xy}B} = R_{\mathbf{x}AB\mathbf{y}}$$

so there is no need to develop a separate equation for the $R_{A\mathbf{xy}B}$ projection.

Bianchi Identities

We shall now develop the two+two formulation of the Bianchi identities. Using the conservation law of the stress-energy tensor, we shall write them in the form

$$\nabla_\mu (G^{\mu\nu} - T^{\mu\nu}) = 0$$

or

$$\nabla_\mu S^{\mu\nu} = 0 \qquad [B-37]$$

where

$$S^{\mu\nu} \equiv G^{\mu\nu} - T^{\mu\nu}$$

We can write equation B-37 as

$$B^\nu{}_\mu \nabla_\nu S^\mu{}_\rho + C^\nu{}_\mu \nabla_\nu S^\mu{}_\rho =$$

$$= B^\nu{}_\mu \nabla_\nu S^{\mu'}{}_\rho + B^\nu{}_\mu \nabla_\nu S^{\mu''}{}_\rho + C^\nu{}_\mu \nabla_\nu S^{\mu'}{}_\rho + C^\nu{}_\mu \nabla_\nu S^{\mu''}{}_\rho$$

$$= B^\nu{}_\mu \nabla_\nu S^{\mu'}{}_\rho - S^{\mu''}{}_\rho \nabla_\nu B^\nu{}_\mu - S^{\mu'}{}_\rho \nabla_\nu C^\nu{}_\mu + C^\nu{}_\mu \nabla_\nu S^{\mu''}{}_\rho$$

$$= B^\nu{}_\mu \nabla_\nu S^{\mu'}{}_\rho - S^{\mu''}{}_\rho B^\nu{}_B \nabla_\nu B^B{}_\mu - S^{\mu'}{}_\rho C^\nu{}_{\mathbf{z}} \nabla_\nu C^{\mathbf{z}}{}_\mu + C^\nu{}_\mu \nabla_\nu S^{\mu''}{}_\rho$$

$$= B^\nu{}_\mu \nabla_\nu S^{\mu'}{}_\rho - S^{\mathbf{z}}{}_\rho H_{\mathbf{z}} - S^B{}_\rho {}^*H_B + C^\nu{}_\mu \nabla_\nu S^{\mu''}{}_\rho = 0$$

[B-38]

We first project equation B-38 into the rigging space to get

$$C_{\mathbf{x}}{}^\rho B^\nu{}_\mu \nabla_\nu S^{\mu'}{}_\rho - S^{\mathbf{z}}{}_{\mathbf{x}} H_{\mathbf{z}} - S^B{}_{\mathbf{x}} {}^*H_B + C_{\mathbf{x}}{}^\rho C^\nu{}_\mu \nabla_\nu S^{\mu''}{}_\rho$$

$$= C_{\mathbf{x}}{}^\rho B^\nu{}_\mu \nabla_\nu S^{\mu'}{}_{\rho'} + C_{\mathbf{x}}{}^\rho B^\nu{}_\mu \nabla_\nu S^{\mu'}{}_{\rho''} - S^{\mathbf{z}}{}_{\mathbf{x}} H_{\mathbf{z}} - S^B{}_{\mathbf{x}} {}^*H_B$$

$$+ C_{\mathbf{x}}{}^\rho C^\nu{}_\mu \nabla_\nu S^{\mu''}{}_{\rho'} + C_{\mathbf{x}}{}^\rho C^\nu{}_\mu \nabla_\nu S^{\mu''}{}_{\rho''}$$

$$= D_A S^A{}_{\mathbf{x}} + S^A{}_B H_A{}^B{}_{\mathbf{x}} - S^{\mathbf{z}}{}_{\mathbf{x}} H_{\mathbf{z}} - S^B{}_{\mathbf{x}}{}^* H_B$$

$$+ D_{\mathbf{z}} S^{\mathbf{z}}{}_{\mathbf{x}} - S^{\mathbf{z}}{}_B{}^* H_{\mathbf{x}\mathbf{z}}{}^B$$

$$= \mathfrak{D}_A S^A{}_{\mathbf{x}} + S^A{}_B H_A{}^B{}_{\mathbf{x}} - S^{\mathbf{z}}{}_{\mathbf{x}} H_{\mathbf{z}} - S^B{}_{\mathbf{x}}{}^* H_B + \mathfrak{D}_{\mathbf{z}} S^{\mathbf{z}}{}_{\mathbf{x}}$$

$$= \rho^{-1}\mathfrak{D}_A(\rho S^A{}_{\mathbf{x}}) + S^A{}_B H_A{}^B{}_{\mathbf{x}} + \gamma^{-1}\mathfrak{D}_{\mathbf{z}}(\gamma S^{\mathbf{z}}{}_{\mathbf{x}}) = 0 \qquad \text{[B-39]}$$

We next project equation B-38 into the surface to get

$$B_A{}^\rho B^\nu{}_\mu \nabla_\nu S^{\mu'}{}_\rho - S^{\mathbf{z}}{}_A H_{\mathbf{z}} - S^B{}_A{}^* H_B + B_A{}^\rho C^\nu{}_\mu \nabla_\nu S^{\mu''}{}_\rho$$

$$= B_A{}^\rho B^\nu{}_\mu \nabla_\nu S^{\mu'}{}_{\rho'} + B_A{}^\rho B^\nu{}_\mu \nabla_\nu S^{\mu'}{}_{\rho''} - S^{\mathbf{z}}{}_A H_{\mathbf{z}} - S^B{}_A{}^* H_B$$

$$+ B_A{}^\rho C^\nu{}_\mu \nabla_\nu S^{\mu''}{}_{\rho'} + B_A{}^\rho C^\nu{}_\mu \nabla_\nu S^{\mu''}{}_{\rho''}$$

$$= D_B S^B{}_A - S^A{}_{\mathbf{x}} H_{BA}{}^{\mathbf{x}} - S^{\mathbf{z}}{}_A H_{\mathbf{z}} - S^B{}_A{}^* H_B$$

$$+ D_{\mathbf{x}} S^{\mathbf{x}}{}_A - S^{\mathbf{z}}{}_{\mathbf{x}}{}^* H_{\mathbf{z}}{}^{\mathbf{x}}{}_A$$

$$= \mathfrak{D}_B S^B{}_A + \mathfrak{D}_{\mathbf{x}} S^{\mathbf{x}}{}_A - S^{\mathbf{z}}{}_A H_{\mathbf{z}} - S^B{}_A{}^* H_B$$

$$- S^{\mathbf{z}}{}_{\mathbf{x}}{}^* H_{\mathbf{z}}{}^{\mathbf{x}}{}_A$$

$$= \rho^{-1}\mathfrak{D}_B(\rho S^B{}_A) + \gamma^{-1}\mathfrak{D}_{\mathbf{x}}(\gamma S^{\mathbf{x}}{}_A) - S^{\mathbf{z}}{}_{\mathbf{x}}{}^* H_{\mathbf{z}}{}^{\mathbf{x}}{}_A = 0 \qquad \text{[B-40]}$$

APPENDIX C

TOTALLY SYMMETRIC TRACELESS TENSORS IN TWO-DIMENSIONS

We show how to construct arbitrary totally symmetric traceless tensors in two-dimension from arbitrary totally symmetric tensors. It is not hard to guess at what the traceless operator would look like for order three:

$$\Lambda^{y_1}{}_{x_1}{}^{y_2}{}_{x_2}{}^{y_3}{}_{x_3} = \delta^{y_1}{}_{x_1}{}^{y_2}{}_{x_2}{}^{y_3}{}_{x_3} - 3/4 g^{(y_1 y_2}\delta^{y_3)}{}_{x_3} g_{x_1 x_2} \qquad [C-1]$$

For the case when there are four indices, the operator must have terms which have the single trace of the symmetric tensor and terms which have the double trace. For m indices, the general traceless symmetric tensor would be

$$\begin{aligned} T_{x_1 x_2 \ldots x_m} = {} & \tilde{T}_{x_1 x_2 \ldots x_m} \\ & - a_2 g_{(x_1 x_2} \tilde{T}_{x_3 \ldots x_m) y_1 y_2} g^{y_1 y_2} \\ & + a_4 g_{(x_1 x_2 x_3 x_4} \tilde{T}_{x_5 \ldots x_m) y_1 y_2 y_3 y_4} g^{y_1 y_2} g^{y_3 y_4} + \ldots . \end{aligned} \qquad [C-2]$$

$$T_{x_1 x_2 \ldots x_m}$$

$$= \sum_{k=0}^{J} (-1)^k a_{2k} g_{(x_1 x_2 \ldots} g_{x_{2k-1} x_{2k}} \tilde{T}_{x_{2k+1} \ldots x_m) y_1 y_2 \ldots y_{k-1} y_k} \otimes g^{y_1 y_2} \ldots g^{y_{k-1} y_k} \qquad [C-3]$$

where J is the largest integer less than m/2.

If we take the trace of T with respect to, say $g^{x_1 x_2}$, some of the contractions in the k^{th}-order term will involve taking an additional trace of $\tilde{T}$ which can only be cancelled by terms of $(k+1)^{th}$-order. This gives rise to a recursion relation between a_{2k} and $a_{2(k+1)}$. We need only consider

$$(-1)^k \left[a_{2k} g_{(\mathbf{x}_1} \ldots g_{\mathbf{x}_{2k-1}\mathbf{x}_{2k}} \tilde{T}_{\mathbf{x}_{2k+1} \ldots \mathbf{x}_m)\mathbf{y}_1 \ldots \mathbf{y}_{2k-1}\mathbf{y}_{2k}} g^{\mathbf{y}_1\mathbf{y}_2} \ldots g^{\mathbf{y}_{2k-1}\mathbf{y}_{2k}} \right.$$

$$- a_{2(k+1)} g_{(\mathbf{x}_1} \ldots g_{\mathbf{x}_{2k+1}\mathbf{x}_{2(k+1)}} \tilde{T}_{\mathbf{x}_{2k+3} \ldots \mathbf{x}_m)\mathbf{y}_1 \ldots \mathbf{y}_{2k+1}\mathbf{y}_{2k+2}}$$

$$\left. \otimes\, g^{\mathbf{y}_1\mathbf{y}_2} \ldots g^{\mathbf{y}_{2k+1}\mathbf{y}_{2k+1}} \right] \qquad \text{[C-4]}$$

The first term in the brackets is comprised of m! terms. After contraction with $g^{\mathbf{x}_1\mathbf{x}_2}$, there are $(m-2)!(m-2k)(m-2k-1)$ of these terms which involve an additional contraction of $\tilde{T}$. Of the terms which make up the second symmetrized term in the brackets, some will involve a contraction where $\mathbf{x}_1$ and $\mathbf{x}_2$ belong to the same g. There are $(m-2)!(2k+2)$ of these. But the contraction of g involves an additional factor of 2. Some terms will involve a contraction of indices on different g's. There are $(m-2)!(2k+2)(2k)$ of these. Finally, there will be contractions involving one index on g and one on $\tilde{T}$. There are $2(m-2)!(m-2k-2)(2k+2)$ of these. We thus require that

$$a_{2k}(m-2)!(m-2k)(m-2k-1) =$$
$$a_{2(k+1)}(m-2)![2(2k+2)+(2k+2)(2k)+2(m-2k-2)(2k+2)] \qquad \text{[C-5]}$$

and

$$a_{2k}(m-2k)(m-2k-1)$$
$$= a_{2(k+1)}\, 2(2k+2)\; [1 + k + (m-2k-2)]$$
$$= a_{2(k+1)}\, 2(2k+2)\; (m - k - 1)$$

and

$$a_{2(k+1)} = [(m-2k)(m-2k-1)]/[4(k+1)(m - k - 1)]a_{2k} \qquad \text{[C-6]}$$

Taking $a_0 = 1$, then

$$a_2 = m(m-1)/4(m-1) = m/4$$

$$a_4 = (m-2)(m-3)/4(2)(m-2)\ a_2 = (m-3)/8\ a_2 = (m-3)m/8\cdot 4$$

$$a_6 = (m-4)(m-5)/4(3)(m-3)\ a_4 = (m-4)(m-5)m/12\cdot 8\cdot 4$$

and so forth. [C-7]

For various values of m we have:

m=2 we have $a_2 = 1/2$.

m=3 we have $a_2 = 3/4$.

m=4 we have $a_2 = 1$ and $a_4 = 1/8$.

m=5 we have $a_2 = 5/4$ and $a_4 = 5/16$.

m=6 we have $a_2 = 3/2$, $a_4 = 9/16$ and $a_6 = 1/32$.