5 GW generation by post-Newtonian sources

In Chapter 3 we discussed the generation of GWs assuming that the background space-time can be taken as flat, i.e. that the sources that produce GWs in their far-field region contribute negligibly to the space-time curvature in their near-field region. We then computed the GW production as an expansion in v/c, where v is some typical internal speed of the source. We saw that the leading term is given by the Einstein quadrupole formula, and that higher-order corrections in v/c can be organized in a multipole expansion. This procedure assumes that the background space-time curvature and the velocity of the source can be treated as independent parameters, so that we can keep the space-time flat, while taking into account the v/c corrections. This is indeed the case when the dynamics of the system is governed by non-gravitational forces. For example, a beam of charged particles accelerated by an external electric field could reach highly relativistic speeds, but still it contributes negligibly to the background space-time curvature, and for such a source the formalism developed in Chapter 3 is adequate. For $v/c \ll 1$ we can compute the corrections in powers of v/c to the leading quadrupole result using the multipole expansion, since in this case the lowest multipoles dominate. Even in the extreme relativistic case, where the multipole expansion becomes useless, we can still compute GW production using the exact formula (3.14). An example of the latter type of computation was given in Section 4.4.

However, the astrophysical systems which are more interesting for GW detection are held together by gravitational forces. In this case the assumption that the velocity of the source and the space-time curvature are independent is no longer valid, and the above formalism cannot be applied. In fact, for a self-gravitating system with total mass m we have $(v/c)^2 \sim R_S/d$, where $R_S = 2Gm/c^2$ (so R_S has the meaning of the Schwarzschild radius associated to the mass m) and d is the typical size of the system (e.g. its radius, for an isolated source such as a rotating neutron star, or the orbital distance for a binary system). For a binary system we saw this explicitly in eq. (3.2).[1] More generally, the relation $(v/c)^2 \sim R_S/d$ holds for self-gravitating systems as a consequence of the virial theorem. Since R_S/d is a measure of the strength of the gravitational field near the source, as soon as we switch on the v/c corrections we must also, for consistency, consider the deviation of the background from flat space-time.

[1] In this case, the precise numerical factor is $(v/c)^2 = R_S/(2d)$.

In this chapter we discuss how to go beyond the limit of sources moving in flat space-time. For a self-gravitating system such as a binary star, assuming that space-time is flat means that we describe its dynamics using Newtonian gravity, rather than general relativity. We will see that, when dealing with a (moderately) relativistic system, held together by gravitational forces, the source must rather be described by a post-Newtonian (PN) formalism. In Section 5.1 we recall the PN expansion in general relativity, and we discuss how to obtain the lowest-order correction to the Newtonian equations of motion. GW generation by post-Newtonian sources is described in great detail in Sections 5.2–5.4, and the application to sources with strong gravitational fields, such as neutron stars and black holes, is discussed in Section 5.5.

The results of this chapter have first of all an intrinsic conceptual interest, since we see here at work the full non-linear structure of general relativity. Furthermore, this formalism is of paramount importance in the computation of the waveform from an inspiraling binary system. In fact, as we will see in Section 5.6.1 (and we will further discuss in Chapter 7), a very accurate prediction of the waveform is necessary to extract the GW signal of an inspiraling binary from the experimental data. This waveform has by now been computed to very high order in v/c, as we will review in Section 5.6. It is quite remarkable that non-linear effects in general relativity of apparently very high order, in fact corrections in v/c even up to order $(v/c)^7$, are crucial for the extraction of a coalescing binary signal from the experimental data. Conversely, compact binary systems might turn out to be a unique laboratory for testing the non-linear aspects of general relativity.

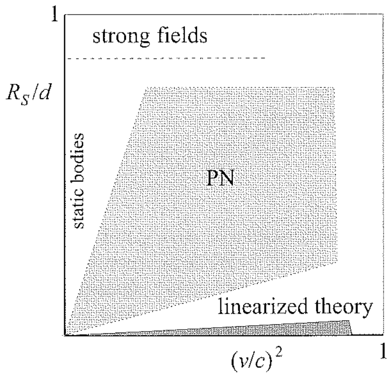

Fig. 5.1 The different regimes in the plane $(v^2/c^2, R_S/d)$.

5.1 The post-Newtonian expansion

5.1.1 Slowly moving, weakly self-gravitating sources

The relation between different possible regimes for the sources, depending on the strength of their self-gravity and on their velocity, is schematically illustrated in Fig. 5.1. In the plane $(v^2/c^2, R_S/d)$, the region close to the horizontal axis, where R_S/d is negligible, corresponds to sources whose dynamics is governed by non-gravitational forces, and which can be described using the linearized theory developed in Chapter 3. The region close to the vertical axis corresponds to essentially static bodies, which are not interesting sources of GWs. Slowly moving, weakly self-gravitating sources correspond to the region of the plane where $(v/c)^2$ and R_S/d are comparable, and none of them is too close to one. As we will see in this chapter, they must be described by a post-Newtonian formalism, so they are marked as "PN" in the figure. When R_S/d gets close to one we are dealing with strong gravity, typically black holes or neutron stars, and we have to resort to strong-field methods.

We now consider a slowly moving and weakly self-gravitating source, which means that v/c and R_S/d are sufficiently small,[2] so we can use

[2] The term "slowly moving sources" can be misleading. For instance, we will be interested in applying the formalism to inspiraling compact binaries made of neutron stars or black holes which, in the last stage of their coalescence, can reach values of v/c as high as 1/2 (in correspondence with the innermost circular orbit, defined by the minimum energy for circular orbits), and in this sense are very relativistic objects. This means that we might need the result to a very high order in v/c. Observe also that the condition $v/c \ll 1$ must be imposed both on the bulk velocities of the objects, such as the orbital velocities of each neutron star in a NS-NS binary system, and also on the internal velocities inside each extended body. This means that we are also requiring that the sources are at most weakly stressed.

them as expansion parameters, and that they are related by $v/c \sim (R_S/d)^{1/2}$. We also demand that the matter energy–momentum tensor $T^{\mu\nu}$ of the source has a spatially compact support, i.e. that it can be enclosed in a time-like world tube $r \leqslant d$ (more precisely, the statement $r \leqslant d$ is assumed to hold in the harmonic coordinate system defined below), and that the matter distribution inside the source is smooth, i.e. that $T^{\mu\nu}(t, \mathbf{x})$ is infinitely differentiable over the whole space-time. We will discuss in Section 5.5 the applicability of the formalism to systems containing black holes.

Our aim is to understand how to compute systematically the corrections to the results of linearized theory, in an expansion in powers of v/c. Just as in electrodynamics, for a non-relativistic source it is convenient to distinguish between the *near zone* and the *far zone*. We found in eq. (3.24) that the typical reduced wavelength of the radiation emitted, $\lambdabar$, is larger than the typical size of the source, d, by a factor of order c/v so, for non-relativistic sources, $d \ll \lambdabar$. The near zone is the region $r \ll \lambdabar$, and the exterior near zone is the region

$$d < r \ll \lambdabar \,. \tag{5.1}$$

In the near zone retardation effects are negligible, and we basically have static potentials. We will see that in this region the post-Newtonian expansion is the correct tool.[3] The far zone (or wave zone) is defined as the region $r \gg \lambdabar$.[4] In the far zone we have waves, retardation effects are crucial and a different treatment is required. The near and the far region are separated by an intermediate region at $r \sim \lambdabar$ (which, in electromagnetism, is called the induction zone).

In a first approximation, we might think that the problem of computing GW generation from a weakly self-gravitating source, in an expansion in v/c, has two aspects:

- We must determine the general-relativistic correction to the equations of motion of the sources up to the desired order in v/c, using a post-Newtonian expansion.
- Given their motion to the desired order, we must compute the GWs emitted by these sources. We have seen in Chapter 3 that GW production can be organized in a multipole expansion, which is an expansion in v/c. Thus, we cannot limit ourselves to the quadrupole formula, but we must include a number of higher multipoles, consistent with the order in v/c to which we wish to work.

The real story is however more complex, and these two aspects cannot really be separated. In particular, the emission of GWs costs energy which is drained from the source so, beyond a certain order, GWs will back-react on the matter sources, influencing their equations of motion. Furthermore, because of the non-linearity of general relativity, the gravitational field is itself a source for GW generation, and the GWs which have been computed to a given order in v/c, at higher orders become themselves a source of further GW production. So, a full-fledged formalism for computing systematically the production of GWs of a self-gravitating source in powers of v/c is necessarily quite complicated.

[3] In the presence of strong-field sources, such as black holes or neutron stars, the near zone can be further separated into a strong-field near zone and a weak-field near zone. The strong-field near zone is the region contained inside balls centered on the sources (e.g. around the two stars in a binary system), and with a radius equal to a few times their Schwarzschild radius. The weak-field near zone is the rest of the near zone, i.e. is the near zone with these strong-field regions excised. We will discuss strong-field sources in Section 5.5.

[4] When studying the propagation of GWs across cosmological distances, it can be convenient to further distinguish among a local wave zone and a distant wave zone. The boundary between the two is where become important effects on the propagation of GWs such as deflection or redshift due to the background curvature of the universe, or the gravitational lensing induced by galaxies, etc. These effects have already been studied in Section 1.5, and in the following we will only consider the local wave zone. Technically, this implies that we will consider background space-times that are asymptotically flat.

5.1.2 PN expansion of Einstein equations

The post-Newtonian approximation is a basic tool of general relativity, developed already in 1916 by Einstein himself, by Droste, de Sitter, and Lorentz, and it has produced a number of classical results. Still, when one tries to extend the lowest-order computations to a systematic all-order expansion, or when one wants to use it for computing the generation of GWs, it raises important conceptual (as well as technical) difficulties, and a fully satisfactory formulation emerged only in relatively recent years.

We begin by analyzing the lowest-order post-Newtonian corrections to the motion of the source, neglecting for the moment the back-reaction due to GWs (as we will see, the back-reaction of GWs on the motion of the source does not enter into play at the level of the first and even the second PN corrections). As discussed above, we assume that the source is non-relativistic, $v/c \ll 1$, and self-gravitating, so that $(R_S/d)^{1/2} \sim v/c$. We introduce the small parameter[5]

$$\epsilon \sim (R_S/d)^{1/2} \sim v/c \,, \tag{5.2}$$

and we also demand that $|T^{ij}|/T^{00} = O(\epsilon^2)$, i.e. that the source be weakly stressed. For instance, for a fluid with pressure p and energy density ρ, this means that $p/\rho = O(\epsilon^2)$. We then expand the metric and the energy–momentum tensor in powers of ϵ. As long as we neglect the emission of radiation, a classical system subject to conservative forces is invariant under time reversal.[6] Under time reversal g_{00} and g_{ij} are even, while g_{0i} is odd. On the other hand, the velocity v changes sign under time reversal so, as long as the invariance under time-reversal is preserved, g_{00} and g_{ij} can contain only even powers of v (and therefore of ϵ), while g_{0i} can contain only odd powers of v. By inspection of Einstein equations one finds that, to work consistently to a given order in ϵ, if we expand g_{00} up to order ϵ^n we must also expand g_{0i} up to order ϵ^{n-1} and g_{ij} up to ϵ^{n-2}. Furthermore, the expansion of g_{0i} starts from $O(\epsilon^3)$. Thus the metric is expanded as follows

$$\begin{aligned} g_{00} &= -1 + {}^{(2)}g_{00} + {}^{(4)}g_{00} + {}^{(6)}g_{00} + \dots \,, \\ g_{0i} &= {}^{(3)}g_{0i} + {}^{(5)}g_{0i} + \dots \,, \\ g_{ij} &= \delta_{ij} + {}^{(2)}g_{ij} + {}^{(4)}g_{ij} + \dots \,, \end{aligned} \tag{5.3}$$

where ${}^{(n)}g_{\mu\nu}$ denotes the terms of order ϵ^n in the expansion of $g_{\mu\nu}$.[7] Similarly, we expand the energy–momentum tensor of matter,

$$\begin{aligned} T^{00} &= {}^{(0)}T^{00} + {}^{(2)}T^{00} + \dots \,, \\ T^{0i} &= {}^{(1)}T^{0i} + {}^{(3)}T^{0i} + \dots \,, \\ T^{ij} &= {}^{(2)}T^{ij} + {}^{(4)}T^{ij} + \dots \,. \end{aligned} \tag{5.4}$$

We can now plug these expansions into the Einstein equations, and equate terms of the same order in ϵ. To determine the order of the various terms we must also take into account that, since we are considering a

[5] When comparing with results in the literature, observe that some authors define $\epsilon \sim v/c$, as we do, while others define $\epsilon \sim (v/c)^2$.

[6] The emission of radiation breaks time-reversal invariance through the boundary conditions, since the no-incoming-radiation boundary conditions (defined in Note 1 on page 102) are transformed into no-outgoing-radiation boundary conditions or, in other words, the retarded Green's function under time-reversal becomes an advanced Green's function. We will come back to this point below.

[7] Actually, one could always generate terms with the wrong parity by performing a gauge transformation. So a more accurate statement is that, as long as radiation-reaction effects are neglected, odd terms in g_{00} such as ${}^{(5)}g_{00}$ (as well as even terms in g_{0i} and odd terms in g_{ij}) satisfy *homogeneous* equations, and can be set to zero with a gauge transformation. In contrast, even terms in g_{00} (as well as odd terms in g_{0i} and even terms in g_{ij}) satisfy inhomogeneous equations, with the appropriate terms from the expansion of the matter energy–momentum tensor on the right-hand side, so we cannot find a gauge transformation that sets them to zero. See Chandrasekhar and Esposito (1970).

source moving with non-relativistic velocity v, the time derivatives of the metric generated by this source are smaller than the spatial derivatives by a factor $O(v)$,

$$\frac{\partial}{\partial t} = O(v)\frac{\partial}{\partial x^i}, \tag{5.5}$$

or $\partial_0 = O(\epsilon)\partial_i$. In particular, the d'Alembertian operator, applied to the metric, to lowest order becomes a Laplacian,

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 = [1 + O(\epsilon^2)]\nabla^2 . \tag{5.6}$$

This means that retardation effects are small corrections, and the lowest-order solution is given in terms of instantaneous potentials. In the PN expansion we are therefore trying to compute some quantity $F(t - r/c)$, such as a given component of the metric, which is intrinsically a function of retarded time $t - r/c$, from its expansion for small retardation,

$$F(t - r/c) = F(t) - \frac{r}{c}\dot{F}(t) + \frac{r^2}{2c^2}\ddot{F}(t) + \dots . \tag{5.7}$$

Each derivative of F carries a factor of ω, the typical frequency of the radiation emitted. Since $\omega/c = 1/\lambda\!\!\!^{-}$, we see that eq. (5.7) is in fact an expansion in powers of $r/\lambda\!\!\!^{-}$. Therefore *the PN expansion is valid only in the near zone, $r \ll \lambda\!\!\!^{-}$, and breaks down in the radiation zone $r \gg \lambda\!\!\!^{-}$.* We will examine in detail this breakdown in the far region in the following sections, where we will see explicitly how a naive extrapolation of the PN iterative scheme up to $r = \infty$ leads to divergences. So, the PN expansion is a formalism that can be used to compute the gravitational field in the near region, but must be supplemented by a different treatment of the far-field region, to compute the fields in the radiation zone.

5.1.3 Newtonian limit

Let us first recall, from elementary general relativity, that the Newtonian limit corresponds to keeping $g_{00} = -1 + {}^{(2)}g_{00}$, $g_{0i} = 0$ and $g_{ij} = \delta_{ij}$ in eq. (5.3). In fact, the equation of motion of a test particle with velocity v, in a gravitational field, is obtained from the geodesic equation

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} . \tag{5.8}$$

In a weak gravitational field we write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$ and, in the limit of low velocities, the proper time τ is the same, to lowest order, as the coordinate time t. Furthermore, $dx^0/dt = c$ while $dx^i/dt = O(v)$. Then, the leading term in v/c is obtained setting $\mu = \nu = 0$ in eq. (5.8),

$$\frac{d^2x^i}{dt^2} \simeq -c^2\Gamma^i_{00} = c^2\left(\frac{1}{2}\partial^i h_{00} - \partial_0 h_0{}^i\right) . \tag{5.9}$$

Since we are considering a source moving with non-relativistic velocity, the time derivative of the metric generated by this source is of higher order with respect to the spatial derivatives, so to leading order eq. (5.9) becomes

$$\frac{d^2x^i}{dt^2} = \frac{c^2}{2}\partial^i h_{00} . \tag{5.10}$$

Writing $h_{00} = -2\phi$ and defining U by $U = -c^2\phi$, we recover the Newtonian equation of motion $\mathbf{a} = \nabla U$, and we see that U is the (sign-reversed) gravitational potential.[8] In a potential U, the virial theorem tells us that a massive particle moves with a velocity $v^2 = O(U)$, so h_{00} is of order v^2/c^2. Comparing with eq. (5.3), we see that the Newtonian limit corresponds to ${}^{(2)}g_{00} = 2U/c^2$, while all other corrections to the flat metric do not affect the Newtonian equation of motion. Observe in particular that ${}^{(2)}g_{ij}$ does not contribute to the Newtonian limit, despite the fact that it is a correction $O(v^2/c^2)$ to the leading term ${}^{(0)}g_{ij} = \delta_{ij}$, just as ${}^{(2)}g_{00}$ is a correction $O(v^2/c^2)$ to the leading term ${}^{(0)}g_{00} = -1$. This is due to the fact that, in the geodesic equation (5.8), $\partial^i g_{00}$ enters through Γ^i_{00}, which is multiplied by $(dx^0/dt)^2 = c^2$, while the gradient of the spatial metric, $\partial^i g_{jk}$, enters through Γ^i_{jk}, which is multiplied by $(dx^j/dt)(dx^k/dt) = O(v^2)$.

It is worth remarking that here it was crucial that we considered the propagation of a massive particle with $v/c \ll 1$. If we rather consider the propagation of a photon in the metric generated by a non-relativistic source, there is no v/c suppression since v is approximately equal to c,[9] and the deviation from flat space in g_{00} and in g_{ij} both contribute to leading order. For instance, the metric generated by a weak and nearly static Newtonian source, in the de Donder gauge, is given by

$$ds^2 \simeq -(1 + 2\phi)dt^2 + (1 - 2\phi)\delta_{ij}dx^i dx^j , \tag{5.11}$$

where $\phi = -U/c^2$ and

$$U(t, \mathbf{x}) = \frac{G}{c^2}\int d^3x' \frac{T^{00}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} . \tag{5.12}$$

(This is easily proved using eq. (3.8), and observing that for a non-relativistic source, to leading order in the source velocity, only T_{00} contributes, so only $\bar{h}_{00}$ is non-vanishing. We neglect retardation effect since we are interested in the near-zone field, and we finally express the result in terms of $h_{\mu\nu} = \bar{h}_{\mu\nu} - (1/2)\eta_{\mu\nu}\bar{h}$.) If we study the propagation of a photon in such a background, of course both the correction -2ϕ to $\eta_{00} = -1$ and the correction $-2\phi\delta_{ij}$ to $\eta_{ij} = \delta_{ij}$ must be taken into account, and give contributions of the same order.[10]

Having established that $g_{00} = -1 + {}^{(2)}g_{00}$, $g_{0i} = 0$ and $g_{ij} = \delta_{ij}$ gives the Newtonian approximation to the dynamics of a massive particle, it follows that the terms ${}^{(4)}g_{00}$, ${}^{(3)}g_{0i}$ and ${}^{(2)}g_{ij}$ give the first post-Newtonian order, denoted as 1PN, the terms ${}^{(6)}g_{00}$, ${}^{(5)}g_{0i}$ and ${}^{(4)}g_{ij}$ gives the 2PN approximation, etc.

[8] It is a nearly universal convention in research papers in general relativity that U denotes the *sign-reversed* gravitational potential, so that $U > 0$. We will refer to U simply as the potential.

[9] More precisely, v differs from c only by terms of order U/c^2, where U is given in eq. (5.12) below.

[10] When studying the deflection of light from the Sun, Einstein at first (in 1911) used the metric

$$ds^2 = -(1 + 2\phi)dt^2 + d\mathbf{x}^2 ,$$

suggested by the Newtonian limit of a massive particle, and obtained a deflection angle of only one half of the correct value, which is the one obtained from (5.11). Einstein himself obtained the correct deflection angle in 1915, when he had the final form of his equations.

5.1.4 The 1PN order

We now discuss the first post-Newtonian correction. First of all, it is useful to choose from the beginning a gauge condition, since this simplifies drastically the equations. A convenient choice is the De Donder gauge condition,

$$\partial_\mu(\sqrt{-g}\, g^{\mu\nu}) = 0\,. \tag{5.13}$$

This is also called the harmonic gauge condition, and the corresponding coordinates are referred to as harmonic coordinates.[11]

It is now in principle straightforward, even if somewhat long, to insert the expansions (5.3) and (5.4) into the Einstein equations, using eq. (5.5) to establish the order of the various term, and the gauge condition (5.13), expanded to the desired order, to simplify the equations (for the explicit computation, see Weinberg 1972, Section 9.1). For ${}^{(2)}g_{00}$ we get the Newtonian equation

$$\nabla^2[{}^{(2)}g_{00}] = -\frac{8\pi G}{c^4}\, {}^{(0)}T^{00}\,, \tag{5.17}$$

while, for the 1PN corrections to the metric, we get

$$\nabla^2[{}^{(2)}g_{ij}] = -\frac{8\pi G}{c^4}\,\delta_{ij}\, {}^{(0)}T^{00}\,, \tag{5.18}$$

$$\nabla^2[{}^{(3)}g_{0i}] = \frac{16\pi G}{c^4}\, {}^{(1)}T^{0i}\,, \tag{5.19}$$

$$\nabla^2[{}^{(4)}g_{00}] = \partial_0^2[{}^{(2)}g_{00}] + {}^{(2)}g_{ij}\partial_i\partial_j[{}^{(2)}g_{00}] - \partial_i[{}^{(2)}g_{00}]\,\partial_i[{}^{(2)}g_{00}] - \frac{8\pi G}{c^4}\left\{{}^{(2)}T^{00} + {}^{(2)}T^{ii} - 2\,{}^{(2)}g_{00}\,{}^{(0)}T^{00}\right\}, \tag{5.20}$$

where $\nabla^2 = \delta^{ij}\partial_i\partial_j$ is the flat-space Laplacian, and the sum over repeated lower (or upper) spatial indices is performed with δ_{ij}. The solution of eq. (5.17), with the boundary condition that the metric vanishes at spatial infinity, is

$${}^{(2)}g_{00} = -2\phi\,, \tag{5.21}$$

where

$$\phi(t,\mathbf{x}) = -\frac{G}{c^4}\int d^3x'\, \frac{{}^{(0)}T^{00}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,, \tag{5.22}$$

so $U = -c^2\phi$ is the (positive) Newtonian potential. Similarly, the 1PN eqs. (5.18) and (5.19) are immediately solved,

$${}^{(2)}g_{ij} = -2\phi\,\delta_{ij}\,, \tag{5.23}$$

$${}^{(3)}g_{0i} = \zeta_i\,, \tag{5.24}$$

where

$$\zeta_i(t,\mathbf{x}) = -\frac{4G}{c^4}\int \frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\, {}^{(1)}T^{0i}(t,\mathbf{x}')\,. \tag{5.25}$$

To solve eq. (5.20) we replace on the right-hand side ${}^{(2)}g_{00}$ by -2ϕ and ${}^{(2)}g_{ij}$ by $-2\phi\,\delta_{ij}$, we use the identity

$$\partial_i\phi\partial_i\phi = \frac{1}{2}\nabla^2(\phi^2) - \phi\nabla^2\phi\,, \tag{5.26}$$

[11] This name originates from the fact that, in this gauge, the coordinates x^ρ satisfy

$$\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\partial_\nu)x^\rho = 0\,. \tag{5.14}$$

On a scalar function ϕ, in curved space, we have

$$\Box\phi \equiv D^\mu D_\mu\phi = \frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\partial_\nu)\phi\,, \tag{5.15}$$

where D_μ is the covariant derivative, and a scalar function ϕ that satisfies $\Box\phi = 0$, i.e.

$$\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\partial_\nu)\phi = 0\,, \tag{5.16}$$

is called a harmonic function. By (a slightly improper) extension, also the coordinates x^ρ that satisfy (5.14) are called harmonic coordinates (even if x^ρ are not four scalar functions indexed by ρ, so the operator $D^\mu D_\mu$ on them is not the same as on scalars).
We will use the denominations De Donder gauge condition and harmonic gauge condition as synonymous. Sometimes in the literature the name "De Donder gauge condition" is reserved to the linearized form given in eq. (1.18), while "harmonic gauge condition" is reserved to eq. (5.13).

and we introduce a new potential ψ defined from

$${}^{(4)}g_{00} = -2(\phi^2 + \psi)\,. \tag{5.27}$$

Then eq. (5.20) becomes

$$\nabla^2\psi = \partial_0^2\phi + \frac{4\pi G}{c^4}\left[{}^{(2)}T^{00} + {}^{(2)}T^{ii}\right], \tag{5.28}$$

which, again with the boundary condition that ψ vanishes at infinity, has the solution

$$\psi(t,\mathbf{x}) = -\int \frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\left\{\frac{1}{4\pi}\partial_0^2\phi + \frac{G}{c^4}\left[{}^{(2)}T^{00}(t,\mathbf{x}') + {}^{(2)}T^{ii}(t,\mathbf{x}')\right]\right\}. \tag{5.29}$$

Observe that ϕ and ζ^i are not independent, since the gauge condition eq. (5.13) imposes the constraint

$$4\partial_0\phi + \nabla\cdot\zeta = 0\,. \tag{5.30}$$

From the explicit expressions (5.22) and (5.25) we see that these are indeed satisfied, because of the conservation of the energy–momentum tensor, expanded to 1PN order.

In agreement with the discussion below eq. (5.6), ϕ, ψ and ζ_i are *instantaneous* potentials: their value at time t depends on the value of the energy–momentum tensor at the same time t, rather than at retarded time. However, we can re-express the solution in terms of *retarded* potentials. This is useful both to understand better the structure of the above solution, and as a starting point for the computation of higher post-Newtonian orders. We begin by observing that, putting together eqs. (5.21) and (5.27), we have

$$\begin{aligned} g_{00} &= -1 - 2\phi - 2(\phi^2 + \psi) + O(\epsilon^6) \\ &= -1 - 2(\phi + \psi) - 2\phi^2 + O(\epsilon^6)\,. \end{aligned} \tag{5.31}$$

Since ψ is of higher order compared to ϕ, in the last term we are free to replace ϕ^2 by $(\phi+\psi)^2$, because the additional terms are beyond the 1PN order anyway. We introduce the quantity

$$V = -c^2(\phi + \psi)\,, \tag{5.32}$$

which has the dimension of a velocity squared, so the solution for g_{00}, to 1PN order, can be written as

$$g_{00} = -1 + \frac{2V}{c^2} - \frac{2V^2}{c^4} + O\left(\frac{1}{c^6}\right). \tag{5.33}$$

(We will often follow the convention, common in the literature on the PN expansion, of writing the remainder as $O(1/c^n)$ rather than $O(\epsilon^n)$.) To this order, this can be written more compactly as

$$g_{00} = -e^{-2V/c^2} + O\left(\frac{1}{c^6}\right). \tag{5.34}$$

The potential ϕ satisfies

$$\nabla^2\phi = \frac{4\pi G}{c^4}\,{}^{(0)}T^{00}\,, \tag{5.35}$$

see eqs. (5.17) and (5.21), while ψ satisfies eq. (5.28). Thus,

$$\nabla^2(\phi+\psi) = \partial_0^2\phi + \frac{4\pi G}{c^4}\left[{}^{(0)}T^{00} + {}^{(2)}T^{00} + {}^{(2)}T^{ii}\right]. \tag{5.36}$$

To this order, $\partial_0^2\phi = \partial_0^2(\phi+\psi)$, so the above equation can be written in terms of the flat-space d'Alembertian $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$, as

$$\begin{aligned}\Box V &= -\frac{4\pi G}{c^2}\left[{}^{(0)}T^{00} + {}^{(2)}T^{00} + {}^{(2)}T^{ii}\right]\\ &= -\frac{4\pi G}{c^2}\left[T^{00} + T^{ii}\right],\end{aligned} \tag{5.37}$$

where, to 1PN order, we could replace ${}^{(0)}T^{00} + {}^{(2)}T^{00}$ with the total value of the 00 component of the energy–momentum tensor, T^{00}, and similarly ${}^{(2)}T^{ii}$ with T^{ii}. We use the active gravitational-mass density defined in eq. (3.205). Then the 1PN equation for g_{00} can be written as

$$\Box V = -4\pi G\,\sigma\,, \tag{5.38}$$

and therefore $V(t,\mathbf{x})$ can be written as a *retarded* integral, as[12]

$$\boxed{V(t,\mathbf{x}) = G\int d^3x'\,\frac{1}{|\mathbf{x}-\mathbf{x}'|}\,\sigma(t-|\mathbf{x}-\mathbf{x}'|/c,\mathbf{x}')\,.} \tag{5.39}$$

[12] Actually, the PN solution could be rewritten equivalently in terms of the advanced integral or of any combination of retarded and advanced Green's functions. What really selects the appropriate Green's function are the boundary conditions. In particular, the retarded Green's function is selected by the no-incoming radiation boundary condition at null infinity. However, we have already seen in eq. (5.7) that the PN expansion only holds in the near region $r \ll \lambdabar$ and therefore, within the PN expansion, it is not possible to impose boundary conditions at infinity. As we will see below, a different approximation scheme, the post-Minkowskian expansion, will be employed in the external source region $d < r < \infty$, and the boundary condition will be consistently imposed on the post-Minkowskian solution, and will select the retarded Green's function. The PN solution and the post-Minkowskian solution will then be matched in the overlap region $d < r \ll \lambdabar$. Even if the PN expansion at a given order could be rewritten in many different forms, e.g. in terms of advanced potentials, or of half-advanced and half-retarded potentials, writing it in terms of retarded Green's function makes it possible the matching, since the post-Minkowskian solutions in $d < r < \infty$ will be unambiguously written in terms of retarded potentials, once the no-incoming radiation boundary condition is imposed on it.

This retarded potential can be written in terms of instantaneous potentials expanding $\sigma(t-|\mathbf{x}-\mathbf{x}'|/c,\mathbf{x}')$ for small retardation effects,

$$\sigma(t-|\mathbf{x}-\mathbf{x}'|/c,\mathbf{x}') = \sigma(t,\mathbf{x}') - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\partial_t\sigma + \frac{|\mathbf{x}-\mathbf{x}'|^2}{2c^2}\partial_t^2\sigma + \ldots\,, \tag{5.40}$$

and of course, given that we are working to 1PN order, for the moment we can only retain the result of this expansion, truncated to 1PN order. We can proceed similarly for g_{0i} and g_{ij}. Using the "active mass-current density" defined in eq. (3.206), and observing that in ζ_i retardation effects are anyway of higher order, we are allowed to rewrite eqs. (5.24) and (5.25), to 1PN order, replacing ζ_i with V_i defined by

$$\boxed{V_i(t,\mathbf{x}) = G\int d^3x'\,\frac{1}{|\mathbf{x}-\mathbf{x}'|}\,\sigma_i(t-|\mathbf{x}-\mathbf{x}'|/c,\mathbf{x}')\,,} \tag{5.41}$$

and similarly we can replace $-c^2\phi$ with V in the solution for g_{ij}, since again the difference is of higher order.

To summarize, in harmonic coordinates the 1PN solution can be written in terms of two functions V and V_i as

$$g_{00} = -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + O\left(\frac{1}{c^6}\right), \tag{5.42}$$

$$g_{0i} = -\frac{4}{c^3}V_i + O\left(\frac{1}{c^5}\right), \tag{5.43}$$

$$g_{ij} = \delta_{ij}\left(1+\frac{2}{c^2}V\right) + O\left(\frac{1}{c^4}\right), \tag{5.44}$$

and V, V_i are given by retarded integrals over the energy–momentum tensor of the source, eqs. (5.39) and (5.41). Observe also that, to this order, the energy–momentum tensor of the source enters only through the two combinations σ and σ_i.

At large distance from the source, i.e. at $r \gg d$, we can expand the potentials V and V_i using

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r} + \frac{\mathbf{x}\cdot\mathbf{x}'}{r^3} + \ldots\,, \tag{5.45}$$

where $r = |\mathbf{x}|$, and we find that the gravitational field at $d \ll r$ (but still within the near region, $r \ll \lambdabar$) is expressed in terms of the multipoles of the energy–momentum tensor of the source. We will examine this multipole expansion in more detail in Section 5.3.2.

5.1.5 Motion of test particles in the PN metric

Once we have the metric in the near zone, we can obtain the equations of motion of a particle of mass m which moves in the near zone from the geodesic equation or, equivalently, writing the action in the given curved background,

$$\begin{aligned}S &= -mc\int dt\left(-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right)^{1/2}\\ &= -mc^2\int dt\left(-g_{00} - 2g_{0i}\frac{v^i}{c} - g_{ij}\frac{v^iv^j}{c^2}\right)^{1/2},\end{aligned} \tag{5.46}$$

and extremizing it. We will be particularly interested in the equations of motion for a binary system. If we limit ourselves to the lowest PN corrections, it is possible to treat the two masses as point-like.[13] In curved space, the energy–momentum tensor of a set of point-like particles with masses m_a and coordinates x_a^μ $(a = 1, 2)$ is

[13] In higher order, some regularization of the Dirac delta becomes necessary. See Section 8 of Blanchet (2006) and the Further Reading for a discussion of the various regularizations which have been used.

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}}\sum_a \gamma_a m_a\,\frac{dx_a^\mu}{dt}\frac{dx_a^\nu}{dt}\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t))\,, \tag{5.47}$$

which generalizes the flat-space expression (3.121). In a N-body system, the metric felt by a particle, labeled as b, is obtained taking as a source the energy–momentum tensor of all the other particles, i.e. replacing $\sum_a$ with $\sum_{a\neq b}$ in eq. (5.47).[14] Expanding the determinant of the metric to second order and using eqs. (5.21) and (5.23) we get

[14] For radiation reaction, a self-force must also be included. However, we will see below that radiation reaction effects enter only in higher orders, and will be discussed in Section 5.3.5.

$$\begin{aligned}-g &= 1 - {}^{(2)}g_{00} + \sum_i {}^{(2)}g_{ij}\\ &= 1 - 4\phi\,.\end{aligned} \tag{5.48}$$

Then the expansion of eq. (5.47) (with $\sum_a \to \sum_{a\neq b}$) gives

$$^{(0)}T^{00}(t,\mathbf{x}) = \sum_{a\neq b} m_a c^2 \delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t))\,, \tag{5.49}$$

$$^{(2)}T^{00}(t,\mathbf{x}) = \sum_{a\neq b} m_a \left(\frac{1}{2}v_a^2 + 2\phi\, c^2\right) \delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t))\,, \tag{5.50}$$

$$^{(1)}T^{0i}(t,\mathbf{x}) = c\sum_{a\neq b} m_a v_a^i \delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t))\,, \tag{5.51}$$

$$^{(2)}T^{ij}(t,\mathbf{x}) = \sum_{a\neq b} m_a v_a^i v_a^j \delta^{(3)}(\mathbf{x}-\mathbf{x}_a(t))\,. \tag{5.52}$$

Plugging these expressions into eqs. (5.22)–(5.29) we obtain the metric in which the particle b propagates and, inserting this metric into eq. (5.46), we get its action, S_b. The total action of the system is the sum over all particles, $S = \sum_b S_b$. Expanding the square root in the action and keeping for consistency only terms up to $O(v^4/c^4)$ gives the first post-Newtonian corrections. In terms of the Lagrangian, the result for a two-body system is $L = L_0 + (1/c^2)L_2$, with

$$L_0 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{Gm_1m_2}{r}\,, \tag{5.53}$$

and

$$L_2 = \frac{1}{8}m_1 v_1^4 + \frac{1}{8}m_2 v_2^4 \tag{5.54}$$
$$+\frac{Gm_1m_2}{2r}\left[3(v_1^2+v_2^2) - 7\mathbf{v}_1\cdot\mathbf{v}_2 - (\hat{\mathbf{r}}\cdot\mathbf{v}_1)(\hat{\mathbf{r}}\cdot\mathbf{v}_2) - \frac{G(m_1+m_2)}{r}\right]\,,$$

where $\mathbf{r}$ is the separation vector between the two particles, $r = |\mathbf{r}|$ and $\hat{\mathbf{r}} = \mathbf{r}/r$. The same computation can be repeated for a system of N particles, and the result is the famous Einstein–Infeld–Hoffmann Lagrangian, $L = L_0 + (1/c^2)L_2$ with

$$L_0 = \sum_a \frac{1}{2}m_a v_a^2 + \sum_{a\neq b}\frac{Gm_am_b}{2r_{ab}}\,, \tag{5.55}$$

and

$$L_2 = \sum_a \frac{1}{8}m_a v_a^4 - \sum_{a\neq b}\frac{Gm_am_b}{4r_{ab}}[7\mathbf{v}_a\cdot\mathbf{v}_b + (\hat{\mathbf{r}}_{ab}\cdot\mathbf{v}_a)(\hat{\mathbf{r}}_{ab}\cdot\mathbf{v}_b)]$$
$$+\frac{3G}{2}\sum_a\sum_{b\neq a}\frac{m_am_bv_a^2}{r_{ab}} - \frac{G^2}{2}\sum_a\sum_{b\neq a}\sum_{c\neq a}\frac{m_am_bm_c}{r_{ab}r_{ac}}\,, \tag{5.56}$$

where $a = 1,\dots,N$ labels the particle, r_{ab} is the distance between particles a and b, and $\hat{\mathbf{r}}_{ab}$ the unit vector from a to b. From this Lagrangian we can derive the equations of motion of a N-particle system, including corrections of order v^2/c^2, i.e. to 1PN order. If one rather performs the expansion up to 2PN order, the equation of motion of a binary system takes the schematic form

$$\frac{d^2x^i}{dt^2} = -\frac{Gm}{r^2}\left\{\hat{x}^i\left[1+O(\epsilon^2)+O(\epsilon^4)\right] + \hat{v}^i\left[O(\epsilon^2)+O(\epsilon^4)\right]\right\}\,, \tag{5.57}$$

where m is the total mass, $\mathbf{x}$ is the relative separation, $\hat{x}^i = x^i/r$, and $\hat{v}^i$ is the unit vector in the direction of the relative velocity. The leading term is of course just Newtonian gravity. The terms $O(\epsilon^2)$ are the 1PN correction to the equations of motion which gives rise, for instance, to the periastron advance of the orbit. The terms $O(\epsilon^4)$ comes from the 2PN correction. The explicit integration of the 1PN equations of motion for a binary system will be discussed in Chapter 6, on pages 317–320, when we need it for the timing formula of binary pulsars.

5.1.6 Difficulties of the PN expansion

The straightforward PN expansion that we have presented, and which was used until, say, the early 1980s, suffers from two serious problems. The first is that, beyond some order, divergences appear. We will see this explicitly in eq. (5.199), and in the discussion below it. However, it is useful to understand first qualitatively the essence of the problem, which is rooted in the fact that general relativity is a non-linear theory. We are trying to solve iteratively the Einstein equations, that have schematically the form

$$\Box h_{\mu\nu} = S_{\mu\nu}[h]\,, \tag{5.58}$$

where $S_{\mu\nu}$ is a source term, that depends both on the matter energy–momentum tensor and, non-linearly, on $h_{\mu\nu}$ (we will see in eq. (5.72) below how to write *exactly* the Einstein equations in this form). One could envisage a systematic weak-field, low-velocity expansion as follows. We write

$$h_{\mu\nu} = {}^{(0)}h_{\mu\nu} + {}^{(1)}h_{\mu\nu} + {}^{(2)}h_{\mu\nu} + \dots\,. \tag{5.59}$$

To zeroth order, we simply set ${}^{(0)}h_{\mu\nu} = 0$. The first-order solution, ${}^{(1)}h_{\mu\nu}$, is obtained setting $h_{\mu\nu} = 0$ on the right-hand side of eq. (5.58), while, according to eq. (5.5), on the left-hand side we neglect the time-derivative. Then we get an equation of the form

$$\nabla^2[{}^{(1)}h_{\mu\nu}] = (\text{matter sources})\,. \tag{5.60}$$

This is integrated by making use of the instantaneous Green's function of the Laplacian, i.e. of the Poisson integral defined on a generic function $f(\mathbf{x})$ by

$$[\Delta^{-1}f](\mathbf{x}) \equiv -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}f(\mathbf{x}')\,, \tag{5.61}$$

(where $\Delta \equiv \nabla^2$) and leads to results such as eq. (5.22). At next order, we insert ${}^{(1)}h_{\mu\nu}$ into $S[h]$ while, in $\Box h_{\mu\nu}$, we replace $\partial_0^2 h_{\mu\nu}$ by $\partial_0^2[{}^{(1)}h_{\mu\nu}]$, so the gravitational field at the next iteration, ${}^{(2)}h_{\mu\nu}$, is determined by an equation of the form

$$\nabla^2[{}^{(2)}h_{\mu\nu}] = (\text{matter sources}) + (\text{terms that depend on } {}^{(1)}h_{\mu\nu})\,, \tag{5.62}$$

which again one would attempt to integrate by using the Poisson integral. The problem is that, beyond some order, the resulting Poisson integrals are necessarily divergent. In fact, first of all, even if the source has a compact support, the second term on the right-hand side of eq. (5.62) extends all over the space, raising an issue of convergence at infinity of the Poisson integral. Furthermore, the higher is the PN order, the higher is also the order of the multipoles that contribute. The gravitational field corresponding to a multipole of order l has a factor $(\mathbf{x}\cdot\mathbf{x}')^l$ which comes from the expansion of $1/|\mathbf{x}-\mathbf{x}'|$ in eq. (5.45). When we use such a field as a source for the next iteration, for l sufficiently large we necessarily get a divergence at large $\mathbf{x}'$ in the retarded integral.[15]

[15] In the earlier works this problem was somehow swept under the rug. The reason is that divergences start to appear only from 2PN order. Furthermore, up to 2.5PN order, the result can be made finite by using some not well justified trick, consisting in bringing some derivative inside the integrals, to make them finite. In this way, early papers managed to get the lowest-order results that nowadays we know to be correct. However, inexorably divergent integrals appear at 3PN order. Therefore this approach is not consistent, and even the validity of the lowest-order results becomes highly questionable.

This problems turns out to be purely technical. Simply, the correct solution to the Poisson equation is not necessarily given by the Poisson integral (5.61). The correct solution is fixed by the boundary conditions, and we will see in Section 5.3.2 that in our problem it is given by a procedure of analytic continuation, that reduces to a Poisson integral only when the latter is convergent, and otherwise is different, and is always finite.

The second problem of the "standard" PN expansion is conceptual, and is that it cannot take into account the boundary conditions at infinity. This can be understood by observing that, as already discussed below eq. (5.7), in the PN expansion we are trying to reconstruct a retarded field, say of the form

$$h_{\mu\nu} = \frac{1}{r}\,F_{\mu\nu}(t-r/c)\,, \tag{5.63}$$

from its expansions for small retardation, $r/c \ll t$,

$$\frac{1}{r}\,F_{\mu\nu}(t-r/c) = \frac{1}{r}\,F_{\mu\nu}(t) - \frac{1}{c}\dot{F}_{\mu\nu}(t) + \frac{r}{2c^2}\ddot{F}_{\mu\nu}(t) - \frac{r^2}{6c^3}\dddot{F}_{\mu\nu}(t) + \ldots \tag{5.64}$$

The coefficients of the higher-order terms therefore blow up as $r \to \infty$. This has nothing to do with the real behavior of the gravitational field at infinity, which should be asymptotically flat, and simply reflects the inadequacy of the PN expansion to study the large r region.

From a mathematical point of view, the PN expansion is an example of singular perturbation theory, or asymptotic expansion, i.e. an expansion of a function $F(r,\epsilon)$ around $\epsilon = 0$,

$$F(r,\epsilon) = \sum_n c_n(r)\epsilon^n\,, \tag{5.65}$$

where the coefficients c_n depend on a second parameter, here r, and they blow up as $r \to \infty$. So, this expansion is not uniformly valid in r, and cannot be used at $r \to \infty$. In particular, as we already observed in Note 12, it is impossible to include in the PN expansion the boundary conditions at infinity, such as the no-incoming radiation boundary condition, appropriate for a radiation problem.[16]

[16] Actually, it can even by shown that the PN expansion cannot be asymptotically flat beyond 2PN or 3PN order (depending on the gauge condition that is used), see Rendall (1992).

The solution to this difficulty, as we will discuss in details in the following sections, is to make use of the PN expansion only in the near region, and to use a different expansion in the far region. Then, the two expansions are matched in an intermediate region, where they are both valid. This procedure is known as "matched asymptotic expansion". The appropriate boundary conditions at infinity will then be imposed on the solution valid in the far zone.

5.1.7 The effect of back-reaction

Once we will have developed a systematic and consistent formalism for computing the gravitational field both in the near and in the far region, we will also be able to compute the modification of the equations of motion of the sources, due to the back-reaction of GWs. Before entering into the technical aspects, however, we can understand with physical arguments what sort of result we should expect.

When we include gravitational radiation the structure of the expansion changes, because invariance under time-reversal is broken by the boundary conditions. To study GWs we impose that there is no incoming radiation at $t = -\infty$ (compare with Note 1 on page 102). Time reversal exchanges outgoing waves with incoming waves, so the argument used above to prove that g_{00} and g_{ij} are even and that g_{0i} is odd in v breaks down. Radiation reaction can generate terms in g_{00} which are odd in v (and cannot be gauged away) and, correspondingly, even terms in g_{0i} and odd terms in g_{ij}.[17]

[17] In higher orders, because of non-linearities, radiation reaction will also contribute to terms in g_{00} which are even. We will see in eq. (5.186) that an even contribution to g_{00} due to back-reaction indeed appears at 4PN order. Thus, beyond 4PN order, all terms (even and odd) contain pieces associated to radiation reaction.

It is not difficult to understand to which order in v/c radiation reaction effects should come into play. We saw in Chapter 3 that the power radiated in GWs by a system with typical velocity v is $P \sim Gm^2v^6/(c^5r^2)$, where m is a mass scale of the system and r its size, see e.g. eq. (3.339). On the other hand, writing the total energy of the system as the sum of its kinetic and potential energy, $E_{\rm tot} = E_{\rm kin} + V$, and using the virial theorem $E_{\rm kin} = -(1/2)V$, we have $E_{\rm tot} = -E_{\rm kin} = -(1/2)mv^2$. If we equate the time derivative of $E_{\rm tot}$ to minus the power radiated in GWs we therefore get, neglecting numerical factors,

$$-mv\frac{dv}{dt} \sim -\frac{Gm^2v^6}{c^5r^2}\,, \tag{5.66}$$

i.e.

$$\frac{dv}{dt} \sim \frac{Gm}{r^2}\left(\frac{v}{c}\right)^5. \tag{5.67}$$

Thus, we expect that radiation-reaction effects enter eq. (5.57) starting from $O(v^5/c^5) = O(\epsilon^5)$, so the equation of motion of a binary system should be of the generic form

$$\begin{aligned}\frac{d^2x^i}{dt^2} = -\frac{Gm}{r^2}\,&\{\hat{x}^i\,[1 + O(\epsilon^2) + O(\epsilon^4) + O(\epsilon^5) + O(\epsilon^6)\ldots]\\ &+\hat{v}^i\,[O(\epsilon^2) + O(\epsilon^4) + O(\epsilon^5) + O(\epsilon^6) + \ldots]\}\,.\end{aligned} \tag{5.68}$$

Given that one traditionally uses the power of $(v/c)^2$ to label the PN order, the term $O(\epsilon^5)$ is called the correction to the equations of motion of order 2.5PN, the term $O(\epsilon^6)$ is the 3PN order, etc. We will see in the next sections how to derive these results.

5.2 The relaxed Einstein equations

First of all, we recast Einstein equations in a form which will be particularly convenient. From the metric $g^{\alpha\beta}(x)$, we define the field $\mathsf{h}^{\alpha\beta}(x)$ by

$$\boxed{\mathsf{h}^{\alpha\beta} \equiv (-g)^{1/2} g^{\alpha\beta} - \eta^{\alpha\beta}\,,} \tag{5.69}$$

where, as usual, g is the determinant of $g_{\alpha\beta}$. This is an *exact* definition, and we are not assuming that $\mathsf{h}_{\alpha\beta}$ is small. Observe that we use the typographical symbol $\mathsf{h}_{\alpha\beta}$ to distinguish it from $h_{\alpha\beta}$, which is rather defined by $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} + O(h^2)$.[18] In the limit of small $h_{\alpha\beta}$ we have $-g = (1+h)$, where $h = \eta^{\mu\nu} h_{\mu\nu}$, and $g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$, so

$$\begin{aligned} -\mathsf{h}^{\alpha\beta} &\simeq \eta^{\alpha\beta} - (1+h)^{1/2}(\eta^{\alpha\beta} - h^{\alpha\beta}) \\ &= h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta} h\,. \end{aligned} \tag{5.70}$$

[18] The quantity $(-g)^{1/2} g^{\alpha\beta}$ is also called the "gothic metric", and denoted by a gothic g, see Landau and Lifshitz, Vol. II (1979), Section 96.

Thus, $\mathsf{h}^{\alpha\beta}$ reduces to the quantity $\bar{h}^{\alpha\beta}$ used in linearized theory, see eq. (1.15), except for an overall sign.[19] We now impose the de Donder, or harmonic, gauge condition (5.13), which in terms of $\mathsf{h}^{\alpha\beta}$ reads

$$\boxed{\partial_\beta \mathsf{h}^{\alpha\beta} = 0\,.} \tag{5.71}$$

[19] For this reason, in the literature $\mathsf{h}^{\alpha\beta}$ is sometimes defined with the opposite sign, i.e. $\mathsf{h}^{\alpha\beta} \equiv \eta^{\alpha\beta} - (-g)^{1/2} g^{\alpha\beta}$. We use the definition (5.69), following the notation of the review Blanchet (2006). This means that, when we compare the results of this chapter with the corresponding linearized limit studied in the previous chapters, we must take into account this overall sign in the GW amplitude. For the same reason, the sign on the right-hand side of eq. (5.72) below is the opposite of that in eq. (1.24).

In this gauge the exact Einstein equations (1.3) take the Landau–Lifshitz form

$$\boxed{\Box \mathsf{h}^{\alpha\beta} = +\frac{16\pi G}{c^4}\tau^{\alpha\beta}\,,} \tag{5.72}$$

where $\Box \equiv -\partial^2/\partial t^2 + \nabla^2$ is the d'Alembertian in *flat* space-time. The quantity on the right-hand side is defined by

$$\boxed{\tau^{\alpha\beta} \equiv (-g)T^{\alpha\beta} + \frac{c^4}{16\pi G}\Lambda^{\alpha\beta}\,,} \tag{5.73}$$

where $T^{\alpha\beta}$ is the matter energy–momentum tensor. The tensor $\Lambda^{\alpha\beta}$ does not depend on the matter variables, and is defined by

$$\Lambda^{\alpha\beta} = \frac{16\pi G}{c^4}(-g)t^{\alpha\beta}_{LL} + (\partial_\nu \mathsf{h}^{\alpha\mu}\partial_\mu \mathsf{h}^{\beta\nu} - \mathsf{h}^{\mu\nu}\partial_\mu\partial_\nu \mathsf{h}^{\alpha\beta})\,, \tag{5.74}$$

where $t^{\alpha\beta}_{LL}$ is called the Landau–Lifshitz energy–momentum pseudotensor,

$$\begin{aligned} \frac{16\pi G}{c^4}(-g)t^{\alpha\beta}_{LL} &= g_{\lambda\mu} g^{\nu\rho}\partial_\nu \mathsf{h}^{\alpha\lambda}\partial_\rho \mathsf{h}^{\beta\mu} + \frac{1}{2} g_{\lambda\mu} g^{\alpha\beta}\partial_\rho \mathsf{h}^{\lambda\nu}\partial_\nu \mathsf{h}^{\rho\mu} \\ &\quad - g_{\mu\nu}(g^{\lambda\alpha}\partial_\rho \mathsf{h}^{\beta\nu} + g^{\lambda\beta}\partial_\rho \mathsf{h}^{\alpha\nu})\partial_\lambda \mathsf{h}^{\rho\mu} \\ &\quad + \frac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau})\partial_\lambda \mathsf{h}^{\nu\tau}\partial_\mu \mathsf{h}^{\rho\sigma}\,. \end{aligned} \tag{5.75}$$

Since $t^{\alpha\beta}_{LL}$ depends explicitly on the metric $g_{\mu\nu}$, it is a highly non-linear function of $\mathsf{h}_{\mu\nu}$. Using the De Donder gauge condition, we see that the last term in eq. (5.74) is a divergence,

$$\partial_\nu \mathsf{h}^{\alpha\mu}\partial_\mu \mathsf{h}^{\beta\nu} - \mathsf{h}^{\mu\nu}\partial_\mu\partial_\nu \mathsf{h}^{\alpha\beta} = \partial_\mu\partial_\nu(\mathsf{h}^{\alpha\mu}\mathsf{h}^{\beta\nu} - \mathsf{h}^{\mu\nu}\mathsf{h}^{\alpha\beta})\,. \tag{5.76}$$

Thus, we can also rewrite eq. (5.72) as

$$\Box \mathsf{h}^{\alpha\beta} = +\frac{16\pi G}{c^4}\left[(-g)(T^{\alpha\beta} + t^{\alpha\beta}_{LL}) + \partial_\mu\partial_\nu\chi^{\alpha\beta\mu\nu}\right]\,, \tag{5.77}$$

where

$$\chi^{\alpha\beta\mu\nu} = \frac{c^4}{16\pi G}(\mathsf{h}^{\alpha\mu}\mathsf{h}^{\beta\nu} - \mathsf{h}^{\mu\nu}\mathsf{h}^{\alpha\beta})\,. \tag{5.78}$$

The important point is that eqs. (5.71) and (5.72) are an *exact* way of recasting the Einstein equations (subject to the assumption that all of space-time can be covered by a harmonic coordinate system), and no approximation has been made yet.[20]

[20] At first sight eq. (5.72) is surprising, since it seems to suggest that $\mathsf{h}_{\mu\nu}$ propagates along the light-cone of *flat* space-time, because on the left-hand side we have the flat-space d'Alembertian. Actually, this is not true because in $\Lambda^{\alpha\beta}$ we have the term $\mathsf{h}^{\mu\nu}\partial_\mu\partial_\nu \mathsf{h}^{\alpha\beta}$, which has two derivatives acting on a field $\mathsf{h}^{\alpha\beta}$. If we wanted to write the equation so that all terms with two derivatives acting on $\mathsf{h}^{\alpha\beta}$ are on the left-hand side, the term $\mathsf{h}^{\mu\nu}\partial_\mu\partial_\nu \mathsf{h}^{\alpha\beta}$ should also go on the left-hand side, so the total differential operator acting on $\mathsf{h}^{\alpha\beta}$ is not a simple flat-space d'Alembertian. Still, eq. (5.72) is a legitimate way of writing the Einstein equations, which is particularly convenient because the flat space d'Alembertian is easily inverted.

Compare this with the standard form of Einstein equations,

$$G_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta}\,, \tag{5.79}$$

where

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R \tag{5.80}$$

is the Einstein tensor. Because of the Bianchi identity $D_\beta G^{\alpha\beta} = 0$, eq. (5.79) implies automatically the covariant conservation of the matter energy–momentum tensor,

$$D_\beta T^{\alpha\beta} = 0\,. \tag{5.81}$$

In turn, eq. (5.81) is an equation of motion for the matter variables. Thus, Einstein equations automatically fix the motion of matter. Einstein equations are completely equivalent to eq. (5.72) *together with* eq. (5.71). However, from a mathematical point of view it makes perfectly sense to first solve eq. (5.72) without requiring, for the moment, that eq. (5.71) be satisfied. Then eq. (5.72), alone, does not constraint the dynamics of the matter variables. In principle, we could assign ourselves an arbitrary time dependence to $T^{\alpha\beta}$, and the equation would still be well defined. For this reason, the 10 tensor components of eq. (5.72) are called the *relaxed Einstein equations*; we have relaxed the condition that the matter variables obey their equations of motion. Of course, this condition must be recovered when, on the solutions of eq. (5.72), we impose eq. (5.71), since the two equations, together, are equivalent to the Einstein equations. Indeed, the gauge condition (5.71) implies that $\tau^{\alpha\beta}$ satisfies the conservation law

$$\partial_\beta \tau^{\alpha\beta} = 0\,, \tag{5.82}$$

with an ordinary, rather than covariant derivative, and this turns out to be fully equivalent to eq. (5.81). Thus, if we first solve eq. (5.72), then eq. (5.71) can be seen as the condition that imposes the equations of motion on the matter variables. Imposing the no-incoming-radiation boundary conditions (defined in Note 1 on page 102), eq. (5.72) can be formally integrated in terms of the retarded Green's function (3.6), just as we did in linearized theory (see eq. (3.8)), and we get

$$\begin{aligned} \mathsf{h}^{\alpha\beta}(t,\mathbf{x}) &= -\frac{4G}{c^4}\int d^4x'\,\frac{\tau^{\alpha\beta}(t',\mathbf{x}')\delta(t'-t+|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|} \\ &= -\frac{4G}{c^4}\int \frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\,\tau^{\alpha\beta}(t-|\mathbf{x}-\mathbf{x}'|/c,\mathbf{x}')\,. \end{aligned} \tag{5.83}$$

On this solution, we can then impose the gauge condition (5.82), which is equivalent to requiring that the matter sources satisfy the equations of motion in the metric $g_{\mu\nu}$.

Contrary to the result (3.5), (3.6) of linearized theory, in eq. (5.83) $\tau^{\alpha\beta}$ is itself a functional of $\mathsf{h}^{\alpha\beta}$ and of its derivatives, so for the moment we have simply converted the differential equation (5.72) into an integro-differential equation for $\mathsf{h}^{\alpha\beta}$. Finding an exact solution of such an equation is hopeless for all realistic astrophysical sources, and we must resort to approximation methods. The crucial observation is that different approximations must be employed, depending on whether we are in the near or in the far zone. In the near zone, the solution for $\mathsf{h}_{\mu\nu}$ will be given in terms of instantaneous potential, and retardation effects can be treated as small corrections. In the far region the post-Newtonian approximation breaks down, and we will rather have gravitational waves, so retardation effects will of course be crucial.

The fact that different expansions must be used in the near and in the far region, in itself is not different from what happens in electrodynamics. The great difference is that electrodynamics is a linear theory, governed by a wave equation of the form $\Box A_\mu = -(4\pi/c)J_\mu$, where the source J_μ depends only on the matter fields, and not on A_μ itself. However, in eq. (5.72) or in eq. (5.83), the field $\mathsf{h}_{\mu\nu}$ appears even on the right-hand side; thus, the gravitational field itself generates gravitational waves and, if we compute iteratively to a sufficiently high order, we will find that the GWs compute at a given order generate themselves more GWs at higher orders. This is an unavoidable consequence of the non-linear structure of general relativity. At the technical level this is reflected in the fact that, even if the matter energy–momentum tensor $T^{\alpha\beta}$ is localized in space, the total source $\tau^{\alpha\beta}$ is not confined to a compact region, but it extends over all of space-time. As a result, a correct treatment is quite complicated (and a naive treatment of the integral in eq. (5.83) typically results in the divergences which plagued early attempts, see Note 15). Nowadays these problems have been solved, and the generation of GWs from post-Newtonian sources has been computed to very high PN order, thanks to the quite remarkable work of two groups, one composed of Blanchet, Damour and coworkers, and one of Will, Wiseman and Pati.[21] Below we discuss these two approaches.

[21] Of course, these works have built on a large body of literature, which extended over decades, see the Further Reading section.

5.3 The Blanchet–Damour approach

In the problem of computing GWs from a non-relativistic, self-gravitating source with typical velocity v there are two length-scales: the size d of the source (which, for a binary system, is the orbital radius), and the length $\mathcal{R}$ that determines the boundary of the near zone, see eq. (5.1). According to eq. (3.24), $\lambda\!\!\!^{-} = (c/v)d$ so, for non-relativistic sources, $\lambda\!\!\!^{-} \gg d$, and therefore the near zone extends up to a radius $\mathcal{R} \gg d$. In the region $r < \mathcal{R}$ the gravitational field can be computed using the post-Newtonian formalism. However, as we have seen in the previous section, the post-Newtonian approach breaks down at $r > \mathcal{R}$.

On the other hand, outside the matter source ($r > d$) the energy–momentum tensor of matter vanishes, and the only contribution to $\tau^{\alpha\beta}$ in eq. (5.73) comes from the gravitational field itself. If the gravitational field inside the matter source is weak, which (for the moment) is an assumptions of the method, already at $r = d$ space-time will not deviate much from flat and, as r increases, it will approach Minkowski space-time more and more. Thus, over the whole region $d < r < \infty$ we can solve the vacuum Einstein equations using a *post-Minkowskian* expansion, that takes into account iteratively the deviation from flat space-time. Since the post-Minkowskian expansion is valid for $d < r < \infty$ and the post-Newtonian for $0 < r < \mathcal{R}$, the two expansions have an overlapping region of validity, $d < r < \mathcal{R}$. The strategy of the Blanchet–Damour formalism is therefore to use the post-Newtonian expansion in the near region, the post-Minkowskian expansion outside the source, and to match them in the intermediate region. In the following subsections we discuss these steps.

5.3.1 Post-Minkowskian expansion outside the source

We first consider the external domain $d < r < \infty$. Since we are outside the source, the energy–momentum tensor of matter vanishes, and we must solve the *vacuum* Einstein equations. By assumption, we are considering sources whose self-gravity is weak. Thus, in a first approximation the metric in the external domain is just $\eta_{\mu\nu}$, i.e. we have Minkowski space-time. At a distance r, the corrections to the Minkowski metric will be given as an expansion in R_S/r where, as in Section 5.1.1, $R_S = 2Gm/c^2$ and m is a characteristic mass of the system. Since R_S is proportional to G, the post-Minkowskian expansion can be written as an expansion in powers of G. We use as basic variable $\mathsf{h}^{\alpha\beta} = \sqrt{-g}\,g^{\alpha\beta} - \eta^{\alpha\beta}$, we choose the De Donder gauge, and we write

$$\sqrt{-g}\,g^{\alpha\beta} = \eta^{\alpha\beta} + G\mathsf{h}_1^{\alpha\beta} + G^2\mathsf{h}_2^{\alpha\beta} + \ldots\,, \tag{5.84}$$

i.e.

$$\mathsf{h}^{\alpha\beta} = \sum_{n=1}^{\infty} G^n \mathsf{h}_n^{\alpha\beta}\,. \tag{5.85}$$

We now plug this expansion into the relaxed Einstein equations (5.72) with $T^{\alpha\beta}=0$

$$\Box \mathsf{h}^{\alpha\beta} = \Lambda^{\alpha\beta}\,, \tag{5.86}$$

and we equate terms of the same order in G. The tensor $\Lambda^{\alpha\beta}$ depends on $g_{\mu\nu}$, which is a highly non-linear functional of $\mathsf{h}_{\mu\nu}$, so it contains all possible powers of $\mathsf{h}_{\mu\nu}$, starting from terms quadratic in h. Thus, we can write

$$\Lambda^{\alpha\beta} = N^{\alpha\beta}[\mathsf{h},\mathsf{h}] + M^{\alpha\beta}[\mathsf{h},\mathsf{h},\mathsf{h}] + L^{\alpha\beta}[\mathsf{h},\mathsf{h},\mathsf{h},\mathsf{h}] + O(\mathsf{h}^5)\,, \tag{5.87}$$

and $N^{\alpha\beta}, M^{\alpha\beta}$, etc. can be found from the explicit expression of $\Lambda^{\alpha\beta}$, with long but straightforward computations. For instance

$$\begin{aligned} N^{\alpha\beta}[\mathsf{h},\mathsf{h}] = &-\mathsf{h}^{\mu\nu}\partial_\mu\partial_\nu\mathsf{h}^{\alpha\beta} + \frac{1}{2}\partial^\alpha\mathsf{h}_{\mu\nu}\partial^\beta\mathsf{h}^{\mu\nu} - \frac{1}{4}\partial^\alpha\mathsf{h}\partial^\beta\mathsf{h} \\ &-\partial^\alpha\mathsf{h}_{\mu\nu}\partial^\mu\mathsf{h}^{\beta\nu} - \partial^\beta\mathsf{h}_{\mu\nu}\partial^\mu\mathsf{h}^{\alpha\nu} + \partial_\nu\mathsf{h}^{\alpha\mu}(\partial^\nu\mathsf{h}^\beta_\mu + \partial_\mu\mathsf{h}^{\beta\nu}) \\ &+\eta^{\alpha\beta}\left[-\frac{1}{4}\partial_\rho\mathsf{h}_{\mu\nu}\partial^\rho\mathsf{h}^{\mu\nu} + \frac{1}{8}\partial_\mu\mathsf{h}\partial^\mu\mathsf{h} + \frac{1}{2}\partial_\mu\mathsf{h}_{\nu\rho}\partial^\nu\mathsf{h}^{\mu\rho}\right]\,, \end{aligned} \tag{5.88}$$

where, on the right-hand side, $\mathsf{h} = \eta_{\alpha\beta}\mathsf{h}^{\alpha\beta}$, and all indices are raised and lower with the Minkowski metric $\eta_{\mu\nu}$.[22] Since $\Lambda^{\alpha\beta}$ starts from a term quadratic in $\mathsf{h}^{\alpha\beta}$, and therefore proportional to G^2, to order G we simply have

[22]For the explicit expression of $M^{\alpha\beta}[\mathsf{h},\mathsf{h},\mathsf{h}]$ see eq. (1.6) of Blanchet (1995).

$$\Box \mathsf{h}_1^{\alpha\beta} = 0\,, \tag{5.89}$$

and, to higher orders, we get

$$\Box \mathsf{h}_2^{\alpha\beta} = N^{\alpha\beta}[\mathsf{h}_1,\mathsf{h}_1]\,, \tag{5.90}$$

$$\Box \mathsf{h}_3^{\alpha\beta} = M^{\alpha\beta}[\mathsf{h}_1,\mathsf{h}_1,\mathsf{h}_1] + N^{\alpha\beta}[\mathsf{h}_1,\mathsf{h}_2] + N^{\alpha\beta}[\mathsf{h}_2,\mathsf{h}_1]\,, \tag{5.91}$$

and so on, together with the gauge conditions

$$\partial_\beta \mathsf{h}_n^{\alpha\beta} = 0\,. \tag{5.92}$$

We write generically the n-th equation in the form

$$\Box \mathsf{h}_n^{\alpha\beta} = \Lambda_n^{\alpha\beta}[\mathsf{h}_1,\mathsf{h}_2,\ldots,\mathsf{h}_{n-1}]\,, \qquad (r>d)\,, \tag{5.93}$$

where we have recalled that the above equations are valid only in the exterior region $r>d$.

General solution of the linearized vacuum equation

We consider first the linearized equation (5.89). We want to find the most general solution, in order to be able to perform later the matching with the near-region post-Newtonian solution. The most general solution of eq. (5.89) in the region $r>d$ (with d any strictly positive constant), can be written in terms of retarded multipolar waves,

$$\mathsf{h}_1^{\alpha\beta} = \sum_{l=0}^{\infty}\partial_L\left[\frac{1}{r}K_L^{\alpha\beta}(t-r/c)\right]\,, \tag{5.94}$$

where we have used the multi-index notation introduced in Section 3.5.1, and the tensors $K_L^{\alpha\beta}$ are traceless and symmetric with respect to the indices $i_1,\ldots,i_l$. From the fact that $K_L^{\alpha\beta}$ is a function of $u=t-r/c$, it follows that $\Box(K_L^{\alpha\beta}(u)/r)=0$ and, since the flat-space d'Alembertian commutes with ∂_L, eq. (5.94) is a solution of eq. (5.89). Since the set of STF tensors $K_L^{\alpha\beta}$, with all possible rank l, provide a complete set of representation of the rotation group, this is the most general solution. Observe that this solution is acceptable since we are in the domain $r>d$, so we have excluded $r=0$ from the domain where it is required to hold. Otherwise all multipoles in eq. (5.94) would become singular.

Equation (5.94) is the most general solution of eq. (5.89), but in general it does not fulfill the De Donder gauge condition. The tensor $K_L^{\alpha\beta}$ is symmetric in the Lorentz indices α,β so, for each L, it has 10 tensor components. Imposing the gauge condition $\partial_\beta\mathsf{h}_1^{\alpha\beta}=0$ reduces the number of independent tensor components to six, and one finds that the most general solution of the equation of motion and gauge condition, in the external region, has the form

$$\mathsf{h}_1^{\alpha\beta} = k_1^{\alpha\beta} + \partial^\alpha\varphi_1^\beta + \partial^\beta\varphi_1^\alpha - \eta^{\alpha\beta}\partial_\mu\varphi_1^\mu\,, \tag{5.95}$$

where the components of $k_1^{\alpha\beta}$ are given by

$$k_1^{00} = -\frac{4}{c^2}\sum_{l\geqslant 0}\frac{(-1)^l}{l!}\partial_L\left[\frac{1}{r}\mathrm{I}_L(u)\right]\,, \tag{5.96}$$

$$k_1^{0i} = \frac{4}{c^3}\sum_{l\geqslant 1}\frac{(-1)^l}{l!}\partial_{L-1}\left[\frac{1}{r}\mathrm{I}^{(1)}_{iL-1}(u) + \frac{l}{l+1}\epsilon_{iab}\partial_a\left(\frac{1}{r}\mathrm{J}_{bL-1}(u)\right)\right]\,,$$

$$k_1^{ij} = -\frac{4}{c^4}\sum_{l\geqslant 2}\frac{(-1)^l}{l!}\partial_{L-2}\left[\frac{1}{r}\mathrm{I}^{(2)}_{ijL-2}(u) + \frac{2l}{l+1}\partial_a\left(\frac{1}{r}\epsilon_{ab(i}\mathrm{J}^{(1)}_{j)bL-2}(u)\right)\right].$$

We have used the notation

$$f^{(n)}(u) \equiv \frac{d^n f}{du^n}\,, \tag{5.97}$$

to denote the n-th derivative with respect to retarded time.[23] The tensor $k_1^{\alpha\beta}$ depends on two families of symmetric and traceless multipole moments,

[23]We also made use of the notation introduced on page 134, so in particular $\partial_{L-2}\equiv\partial_{i_1}\ldots\partial_{i_{l-2}}$, $I_{ijL-2}\equiv I_{iji_1\ldots i_{l-2}}$, and round brackets around indices denote the symmetrization, $a_{(ij)}\equiv(1/2)(a_{ij}+a_{ji})$. On the right-hand side, we freely raised or lowered the spatial indices with δ_{ij}.

$$\mathrm{I}_L(u) = \{\mathrm{I},\mathrm{I}_i,\mathrm{I}_{ij},\mathrm{I}_{ijk},\ldots\}\,, \tag{5.98}$$

and

$$\mathrm{J}_L(u) = \{\mathrm{J}_i,\mathrm{J}_{ij},\mathrm{J}_{ijk},\ldots\}\,, \tag{5.99}$$

which are arbitrary functions of retarded time, except that the gauge condition requires that I, $\mathrm{I}_i^{(1)}$ and J_i are time-independent. This expresses the conservation of the total mass $\mathrm{M}\equiv\mathrm{I}$ of the system, of the total linear momentum $\mathrm{P}_i\equiv\mathrm{I}_i^{(1)}$, and of the total angular momentum $\mathrm{S}_i\equiv\mathrm{J}_i$.[24] The moments I_L and J_L are mass-type and current-type moments, respectively, just as in the multipole expansion of linearized theory, discussed in Section 3.5. The explicit powers of c in

[24]We further impose the condition that the metric is stationary in the far past, i.e. that all I_L and J_L are constants for $t\leqslant -T$, with $T\to\infty$. This is expected to be basically equivalent to the no-incoming radiation boundary condition, but offers some technical advantages. With this boundary condition, the requirement that $\mathrm{I}_i^{(1)}$ be constant implies that also the center-of-mass variable $\mathrm{X}_i=\mathrm{I}_i/\mathrm{I}$ is constant, rather than a priori linearly varying in time.

eqs. (5.96) follow from the choice of dimensions $[I_L] = [\text{mass}] \times [\text{length}]^l$ and $[J_L] = [\text{mass}] \times [\text{velocity}] \times [\text{length}]^l$. These are chosen in anticipation of the fact that I_L and J_L will be related to the mass and current multipoles of the source. The "mass dipole" I_i can be set to zero shifting the origin of the coordinate system.

The function φ_1^μ can be written in terms of four STF moments W_L, X_L, Y_L and Z_L,

$$\varphi_1^0 = \frac{4}{c^3}\sum_{l\geqslant 0}\frac{(-1)^l}{l!}\partial_L\left[\frac{1}{r}W_L(u)\right], \tag{5.100}$$

$$\varphi_1^i = -\frac{4}{c^4}\sum_{l\geqslant 0}\frac{(-1)^l}{l!}\partial_{iL}\left[\frac{1}{r}X_L(u)\right] \tag{5.101}$$

$$-\frac{4}{c^4}\sum_{l\geqslant 1}\frac{(-1)^l}{l!}\partial_{L-1}\left[\frac{1}{r}Y_{iL-1}(u) + \frac{l}{l+1}\epsilon_{iab}\partial_a\left(\frac{1}{r}Z_{bL-1}(u)\right)\right].$$

The appearance of the function φ_1^α in eq. (5.95) reflects the fact that eq. (5.89) is invariant under linearized gauge transformations, $x^\alpha \to x^\alpha + \varphi_1^\alpha(x)$, compare with eq. (1.19). One might be tempted to discard φ_1^α as pure gauge modes (which would give back the result that we found for linearized theory, see eq. (3.204)),[25] but this would not be correct. Our aim is to use the solution (5.95) for $h_1^{\alpha\beta}$ as a starting point for the iterative process that gives $h_2^{\alpha\beta}$, $h_3^{\alpha\beta}$, etc., and therefore to construct a solution of the full, rather than linearized, Einstein equations. Taking as starting point two different solution for h_1 which differ by a *linearized* gauge transformation φ_1 will produce, through the iterative procedure, two solutions of the full Einstein equations that are not related by the full non-linear invariance under diffeomorfisms of general relativity, and which therefore are not physically equivalent. So, beyond linear level the two sets of multipole moments $(I_L, J_L, W_L, X_L, Y_L, Z_L)$ and $(I_L, J_L, 0, 0, 0, 0)$ are not gauge-equivalent. Rather, the six multipole moments $(I_L, J_L, W_L, X_L, Y_L, Z_L)$ are gauge-equivalent to a reduced set $(M_L, S_L, 0, 0, 0, 0)$, in which $M_L = I_L$ and $S_L = J_L$ only to lowest order, and more generally M_L and S_L depend on all the six moments $(I_L, J_L, W_L, X_L, Y_L, Z_L)$. For example, for the quadrupole moment one finds that M_{ij} starts to differ from I_{ij} at $O(1/c^5)$, i.e. at 2.5PN order,[26]

$$M_{ij} = I_{ij} + \frac{4G}{c^5}\left[W^{(2)}I_{ij} - W^{(1)}I_{ij}^{(1)}\right] + O\left(\frac{1}{c^7}\right), \tag{5.102}$$

where W is W_L for $l = 0$. The six sets of moments $(I_L, \ldots, Z_L)$ are referred to as "algorithmic multipole moments".[27]

[25] When comparing with eq. (3.204), recall that in the linearized limit $h_{\mu\nu}$ reduces to $-\bar{h}_{\mu\nu}$, see Note 19, and also that a factor G has been explicitly extracted from $h_1^{\mu\nu}$, see eq. (5.84).

[26] We include in eq. (5.102) a sign correction pointed out in Arun, Blanchet, Iyer and Qusailah (2004), see their eq. (3.8).

[27] They are also called the "source multipole moments", see the review Blanchet (2006). The term "algorithmic multipole moments" used in the original papers stresses that they are intermediate quantities that allow us to connect, via a well defined algorithm, properties of the source to the "multipole moments at infinity", to be defined later. The term "source moments" stresses that they have explicit closed-form expressions as integrals over the source, see below.

Iteration of the solution. Multipolar post-Minkowskian expansion

We have found above the most general solution of the linearized equation (5.89), together with $\partial_\beta h_1^{\alpha\beta} = 0$, in the domain $r > d$. Next we want to plug this solution into the right-hand side of eq. (5.90), and solve the resulting equation for h_2, and so on. The general problem is therefore how to integrate a wave equation such as eq. (5.93), when the source term Λ_n has been determined by the previous recursive level.

Given the function Λ_n, the problem amounts to inverting the $\Box$ operator. Physicists are well accustomed to some possible solutions for the inversion of the d'Alembert operator: the retarded and the advanced Green's functions, familiar from classical electrodynamics, or the Feynman propagator, which is a basic object in quantum field theory. However, from a mathematical point of view, the inversion of the $\Box$ operator has many other possible solutions, which depend on the boundary conditions of the problem.

In the problem at hand the retarded Green's function is simply not the correct solution (and even less any other combination of retarded and advanced Green's functions). The point is that the use of the retarded (or of the advanced) integral requires the knowledge of Λ_n over all of space, while eq. (5.93) is valid only for $r > d$. Observe that, since we are outside the source, we can write each h_n in a multipole expansion, which is an expansion valid for $d/r < 1$, so Λ_n in eq. (5.93) is composed of the product of many multipole expansions. If we naively extended eq. (5.93) down to $r = 0$, we would find that the right-hand side of eq. (5.93) is highly singular at $r = 0$ and, if we make the convolution with the retarded Green's function, the retarded integral diverges.

The appropriate solution has been found by Blanchet and Damour, with a clever mathematical procedure. First, we observe that we are finally interested in computing to some *finite* order in the PN expansion, and to each given order only a finite number of multipoles contribute. This means that, outside the source, we do not really need the exact expression of $h_2^{\alpha\beta}, h_3^{\alpha\beta}$, etc. but only their multipole expansion, truncated to some *finite* order, that depends on the order of the PN expansion that we wish to compute. It is therefore very convenient (in fact, technically inevitable) to perform a multipole expansion of the post-Minkowskian solution, up to a given finite order, and to iterate not $h_1^{\alpha\beta}$ but rather its (truncated) multipole expansion. This method is therefore called the multipolar post-Minkowskian (MPM) expansion.

Since, in the MPM computation of $h_n^{\alpha\beta}$ with n given, only a maximum number of multipoles are relevant, we can find a positive real number B, sufficiently large, so that $r^B\Lambda_n^{\alpha\beta}$ is regular at the origin. Thus, the retarded integral

$$I_n^{\alpha\beta}(B) \equiv \Box_{\rm ret}^{-1}\left(r^B\Lambda_n^{\alpha\beta}\right) \tag{5.103}$$

is well defined, where we denoted by $\Box_{\rm ret}^{-1}$ the convolution with the retarded Green's function,[28]

$$(\Box_{\rm ret}^{-1}f)(t,\mathbf{x}) \equiv -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}f(t-|\mathbf{x}-\mathbf{x}'|/c, \mathbf{x}'). \tag{5.104}$$

Now, it can be proved that $I_n^{\alpha\beta}(B)$ admits a unique analytic continuation in the complex B-plane, except at some integer values of B and, when $B \to 0$, develops some multipole poles. Thus, near $B = 0$ we can write

[28] We also impose the boundary condition in the form given in Note 24. In the retarded integral, the integration is actually over d^4x, along the past null light cone. As $r \to \infty$ along the past null light cones, t goes toward $-\infty$, so this boundary condition forces $\Lambda_n^{\alpha\beta}$ to become strictly zero beyond some value of r. Therefore there is no problem of convergence of the integral at $r \to \infty$.

$I_n^{\alpha\beta}(B)$ in a Laurent expansion,

$$I_n^{\alpha\beta}(B) = \sum_{p=p_0}^{\infty} B^p \iota_{n,p}^{\alpha\beta}\,, \tag{5.105}$$

where $p_0 \in \mathbb{Z}$. If $p_0 < 0$ there are poles. Applying the flat-space d'Alembertian to both sides of this equation and using eq. (5.103) we get

$$r^B \Lambda_n^{\alpha\beta} = \sum_{p=p_0}^{\infty} B^p \Box \iota_{n,p}^{\alpha\beta}\,. \tag{5.106}$$

Writing $r^B = e^{B\log r}$, expanding the exponential, and equating terms with the same powers of B we find that, for $p_0 \leqslant p \leqslant -1$, $\Box\iota_{n,p}^{\alpha\beta} = 0$, while for $p \geqslant 0$

$$\Box\iota_{n,p}^{\alpha\beta} = \frac{(\log r)^p}{p!}\Lambda_n^{\alpha\beta}\,. \tag{5.107}$$

In particular, the term with $p = 0$, i.e. $u_n^{\alpha\beta} \equiv \iota_{n,p=0}^{\alpha\beta}$, satisfies $\Box u_n^{\alpha\beta} = \Lambda_n^{\alpha\beta}$, so we succeeded in finding a particular solution of eq. (5.93). In other words, a solution of eq. (5.93) is given by the coefficient of B^0 in the Laurent expansion (5.105). This is called the *finite part at* $B = 0$ of the retarded integral, and denoted as $\mathrm{FP}_{B=0}$, so[29]

$$\boxed{u_n^{\alpha\beta} = \mathrm{FP}_{B=0}\left\{\Box_{\rm ret}^{-1}\left[r^B\Lambda_n^{\alpha\beta}\right]\right\}\,.} \tag{5.108}$$

We can write this even more compactly introducing the symbol $\mathcal{FP}$, defined on any function $f(x)$ by

$$\mathcal{FP}\,\Box_{\rm ret}^{-1} f \equiv \mathrm{FP}_{B=0}\left\{\Box_{\rm ret}^{-1}\left[r^B f\right]\right\}\,. \tag{5.109}$$

This finite part operation is a prescription which makes well-defined the otherwise divergent retarded integral. The important point is that it is not just a prescription superimposed by hand on a would-be divergent quantity. Rather, we have seen explicitly that it is a correct way to find a solution of eq. (5.93), valid in the region $r > d$. Observe that, when the retarded integral of a function f is well-defined, $\mathcal{FP}\,\Box_{\rm ret}^{-1} f$ reduces simply to $\Box_{\rm ret}^{-1} f$.

Actually, eq. (5.108) is just one particular solution of the inhomogeneous equation (5.93). The most general solution is obtained adding the general solution of the homogeneous equation $\Box \mathsf{h}_n^{\alpha\beta} = 0$. Indeed, the solution (5.108) in general will not satisfy automatically the harmonic gauge condition. So, the solution that we are looking for is really of the form

$$\mathsf{h}_n^{\alpha\beta} = u_n^{\alpha\beta} + v_n^{\alpha\beta}\,, \tag{5.110}$$

where $v_n^{\alpha\beta}$ is a solution of the homogeneous equation, chosen so that $\partial_\alpha v_n^{\alpha\beta} = -\partial_\alpha u_n^{\alpha\beta}$. Since we have the explicit form of $u_n^{\alpha\beta}$, the function $v_n^{\alpha\beta}$ can be determined exactly.[30] The conclusion is that the MPM expansion provides a well-defined algorithm for computing the post-Minkowskian corrections, in principle to arbitrary order.

[29] More precisely in the definition of the $\mathrm{FP}_{B=0}$ operation, eq. (5.108), for dimensional reasons we use $(r/r_0)^B$ rather than r^B. The constant r_0 is arbitrary, and will cancel from physical quantities, as we will see in Section 5.3.4. For the moment, we set $r_0 = 1$, to simplify the notation.

[30] The explicit expression is somewhat involved, and can be found in Blanchet (2006), eqs. (41) and (42).

At this stage, the multipole moments $(\mathrm{I}_L, \mathrm{J}_L, \mathrm{W}_L, \mathrm{X}_L, \mathrm{Y}_L, \mathrm{Z}_L)$ (or, equivalently, M_L and S_L) know nothing about the properties of the source, since they simply parametrize the most general solution of the vacuum Einstein equation. We will determine them in terms of properties of the source, matching the MPM result to the multipole expansion of the post-Newtonian result, in the region $d < r < \mathcal{R}$, where both the post-Minkowskian and the post-Newtonian formalism are applicable.

5.3.2 PN expansion in the near region

We now consider the near region. The 1PN solution, in harmonic coordinates, has already been given in the Section 5.1.4. First of all, it is useful to re-express it in terms of the variable $\mathsf{h}^{\mu\nu}$, defined in eq. (5.69), rather than in terms of $g^{\mu\nu}$.[31] In terms of $\mathsf{h}^{\mu\nu}$, the solution at the Newtonian level is particularly simple, $\mathsf{h}^{00} = -4V/c^2 + O(1/c^4)$, $\mathsf{h}^{0i} = O(1/c^3)$ and $\mathsf{h}^{ij} = O(1/c^4)$. We can now plug this solution into the right-hand side of eq. (5.72). This gives

$$\Box\mathsf{h}^{00} = \frac{16\pi G}{c^4}\left(1 + \frac{4V}{c^2}\right)T^{00} - \frac{14}{c^4}\partial_k V\partial_k V + O\left(\frac{1}{c^6}\right)\,, \tag{5.111}$$

$$\Box\mathsf{h}^{0i} = \frac{16\pi G}{c^4}T^{0i} + O\left(\frac{1}{c^5}\right)\,, \tag{5.112}$$

$$\Box\mathsf{h}^{ij} = \frac{16\pi G}{c^4}T^{ij} + \frac{4}{c^4}\left\{\partial_i V\partial_j V - \frac{1}{2}\delta_{ij}\partial_k V\partial_k V\right\} + O\left(\frac{1}{c^6}\right). \tag{5.113}$$

The solution of these equations is

$$\mathsf{h}^{00} = -\frac{4}{c^2}V + \frac{4}{c^4}(W - 2V^2) + O\left(\frac{1}{c^6}\right)\,, \tag{5.114}$$

$$\mathsf{h}^{0i} = -\frac{4}{c^3}V_i + O\left(\frac{1}{c^5}\right)\,, \tag{5.115}$$

$$\mathsf{h}^{ij} = -\frac{4}{c^4}W_{ij} + O\left(\frac{1}{c^6}\right)\,, \tag{5.116}$$

where V, V_i are given in eqs. (5.39) and (5.41). W_{ij} is a new retarded potential, defined by

$$W_{ij}(t,\mathbf{x}) = G\int d^3x\,\frac{1}{|\mathbf{x}-\mathbf{x}'|}\left[\sigma_{ij} + \frac{1}{4\pi G}(\partial_i V\partial_j V - \frac{1}{2}\delta_{ij}\partial_k V\partial_k V)\right](\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)\,, \tag{5.117}$$

where $\sigma_{ij} = T^{ij}$. This is the same as the 1PN solution given in eqs. (5.42)–(5.44), written in terms of $\mathsf{h}^{\mu\nu}$ rather than $g^{\mu\nu}$, except that this iterative procedure automatically gives h_{ij} up to $O(1/c^6)$ (which is needed to iterate consistently the solution to higher orders), rather than just up to $O(1/c^4)$ as in eq. (5.44). Observe that the integrals in the definition of V and V_i are convergent since the source, and therefore σ and σ_i, have a compact support. The integrand in the definition of W_{ij} rather depends

[31] Recall that $\mathsf{h}^{\mu\nu}$ denotes the combination (5.69), and is not simply the deviation from the flat metric $g^{\mu\nu} - \eta^{\mu\nu}$. The relation between $\mathsf{h}^{\mu\nu}$ and $g^{\mu\nu}$ is therefore non-linear.

on the function V and does not have a compact support. However, from eq. (5.39), we find that, when $|\mathbf{x}'| \to \infty$,

$$V\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}'\right) \to \frac{GM}{|\mathbf{x}'|} \tag{5.118}$$

where

$$M = \int d^3\mathbf{y}\,\sigma(\mathbf{y}, -\infty) + O(1/c^2) \tag{5.119}$$

is the initial mass of the source. From this, we can check that the integral defining the potential W_{ij} is convergent.

Multipolar PN expansion

We can now introduce another important ingredient of the method, the "multipolar post-Newtonian expansion", which combine the PN expansion with the multipole expansion.

The post-Newtonian expansion is valid both inside the source ($r < d$), and in the external near zone $d < r < \mathcal{R}$. In the external near zone we can then expand each post-Newtonian order in a multipole expansion, since the expansion parameter of the multipole expansion is d/r. This gives rise to the "multipolar post-Newtonian expansion", and provides crucial simplifications when performing the matching with the solution in the far region. To 1PN order, we just need the multipole expansion of the potentials V and V_i. This can be written in full generality as

$$V(t, \mathbf{x}) = G\sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\,\partial_L\left[\frac{1}{r}F_L(t - r/c)\right], \tag{5.120}$$

$$V_i(t, \mathbf{x}) = G\sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\,\partial_L\left[\frac{1}{r}G_{iL}(t - r/c)\right]. \tag{5.121}$$

Using eqs. (3.184), (3.185) and (3.188), together with the fact that V and V_i satisfy $\Box V = -4\pi G\sigma$ and $\Box V_i = -4\pi G\sigma_i$ (see eqs. (5.39) and (5.41)), we get

$$F_L(u) = \int d^3y\,\hat{y}_L \int_{-1}^{1} dz\,\delta_l(z)\sigma(u + z|\mathbf{y}|/c, \mathbf{y}), \tag{5.122}$$

$$G_{iL}(u) = \int d^3y\,\hat{y}_L \int_{-1}^{1} dz\,\delta_l(z)\sigma_i(u + z|\mathbf{y}|/c, \mathbf{y}), \tag{5.123}$$

where $u = t - r/c$, and the function $\delta_l(z)$ is defined in eq. (3.189).

The PN expansion to arbitrary order

We now tackle the problem of finding the PN solution to all orders. We write the PN expansion of $\mathsf{h}_{\mu\nu}$ in the form

$$\mathsf{h}^{\mu\nu} = \sum_{n=2}^{\infty} \frac{1}{c^n}\,{}^{(n)}\mathsf{h}^{\mu\nu}, \tag{5.124}$$

where we have extracted explicitly the powers of $1/c$, to help the book-keeping (just as we did with G in the post-Minkowskian expansion).[32] Similarly, we expand the effective energy–momentum tensor as[33]

$$\tau^{\mu\nu} = \sum_{n=-2}^{\infty} \frac{1}{c^n}\,{}^{(n)}\tau^{\mu\nu}. \tag{5.125}$$

Inserting this into the relaxed Einstein equations, and equating terms with the same powers of c, we get a recursive set of Poisson-type equations,

$$\boldsymbol{\nabla}^2[{}^{(n)}\mathsf{h}^{\mu\nu}] = 16\pi G\,[{}^{(n-4)}\tau^{\mu\nu}] + \partial_t^2[{}^{(n-2)}\mathsf{h}^{\mu\nu}]. \tag{5.126}$$

We could now try to solve these equations using the Poisson integral (5.61). However, as we already discussed in Section 5.1.6, beyond some value of n the resulting Poisson integrals diverge. This does not mean that eq. (5.126) admits no solution, but simply that the Poisson integral is not the correct one. The problem here is purely technical, and consists in finding the correct inversion of the Laplacian. The Poisson integral is the right solution only when the boundary condition is that the field vanishes at spatial infinity; otherwise, the solution is different. As an obvious example, consider the equation

$$\boldsymbol{\nabla}^2 U = -\rho\,, \tag{5.127}$$

where ρ is constant all over space (physically, this equation gives a model of Newtonian cosmology). If we attempt to solve for U using the Poisson integral (5.61), we find a divergent result. However, on a function $U(r)$,

$$\boldsymbol{\nabla}^2 U = \frac{1}{r}\frac{\partial^2}{\partial r^2}[rU(r)]\,, \tag{5.128}$$

so we see immediately that $U(r) = (-1/6)\rho r^2$ is a solution. In this case, it was simply not appropriate to impose the boundary condition that $U(r)$ vanishes at infinity, since the source ρ does not vanish either.

In our case the problem is similar, but more subtle. The point is that we cannot enforce the boundary conditions at infinity within the PN expansion, because this expansion becomes singular as $r \to \infty$, as we saw in Section 5.1.6. The correct way to incorporate the boundary conditions is to match the PN solution in the near zone to the post-Minkowskian solution in the external source region, and to impose the no-incoming radiation boundary conditions on the post-Minkowskian solution.

A possible strategy is therefore to find one particular solution of the set of equations (5.126). This is the same as a particular solution of the relaxed Einstein equation (5.72), which is an inhomogeneous equation, so the most general solution is obtained adding an arbitrary solution of the homogeneous equation $\Box\mathsf{h}^{\mu\nu} = 0$ (subject to a regularity condition at the origin, see below). This homogeneous solution will then be fixed matching the PN solution to the post-Minkowskian solution. Once we

[32] However in this case one finds that, starting from 4PN order, ${}^{(n)}\mathsf{h}^{\mu\nu}$ has also a logarithmic dependence on c, and the expansion contains arbitrary powers of $\log c$.

[33] Observe that this expansion starts from $n = -2$, since $\tau^{\mu\nu}$ has dimension of ρc^2.

have found a particular solution of eq. (5.126), the addition of an arbitrary solution of the homogeneous equation provides the most general solution in the near region, while in the previous section we found, with the post-Minkowskian expansion, the most general solution in the external source region. Thus, the matching condition will admit a solution.

A particular solution of the set of equations (5.126) has been found by Poujade and Blanchet using a variant of the analytic continuation technique discussed in Section 5.3.1. Given a function $f(\mathbf{x})$, we consider the integral

$$[\Delta^{-1}(r^B f)](\mathbf{x}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}\,|\mathbf{x}'|^B f(\mathbf{x}')\,. \tag{5.129}$$

If we take B sufficiently large and negative, the factor $|\mathbf{x}'|^B$ regularizes any potential divergence of the integral at $|\mathbf{x}'| \to \infty$.[34] Equation (5.129) then defines a function of B, for B sufficiently large and negative. One can then prove that this function admits a unique analytic continuation to the complex B-plane, except for $B = 0$, where it can develop multipole poles and can be written in a Laurent expansion. The coefficient u of B^0 is again denoted by $\mathrm{FP}_{B=0}$,

$$u = \mathrm{FP}_{B=0}\{\Delta^{-1}[r^B f]\}\,. \tag{5.130}$$

With the same argument used on page 258 for the inversion of the d'Alembertian operator, we can now show that u satisfies $\boldsymbol{\nabla}^2 u = f$, so u provides a well-defined inversion of the Laplacian. When the Poisson integral converges $\mathrm{FP}_{B=0}\{\Delta^{-1}[r^B f]\}$ is the same as $\Delta^{-1} f$. Therefore, we recover the lowest-order results obtained in the early works on the PN expansion. However, now all higher-order terms are manifestly finite and calculable.

[34]Observe that here the factor $|\mathbf{x}'|^B$ regularizes the divergence at infinity, while in the analytic continuation technique discussed in Section 5.3.1 it regularizes the divergence at the origin. For this reason, here we start from B large and negative, while in Section 5.3.1 we started from B large and positive. More precisely, one assumes that the source is extended and made of some perfectly regular (i.e. C^∞) distribution of fluid. We then separate the original Poisson integral into a part from $r = 0$ up to a finite value, say to the boundary $\mathcal{R}$ of the near zone, and a part from $r = \mathcal{R}$ to $r = \infty$. The inner integral converges (assuming that f is a smooth function of the source), and no factor $|\mathbf{x}'|^B$ is inserted there, while the outer integral is regularized by the insertion of the factor $|\mathbf{x}'|^B$. By the uniqueness of the analytic continuation, the result is the sum of the near zone and far zone integrals.

We denote by an overbar the expansion of a quantity up to n-th order in the PN expansion, e.g.

$$\bar{\mathsf{h}}^{\mu\nu} = \sum_{m=2}^{n}\frac{1}{c^m}\,{}^{(m)}\mathsf{h}^{\mu\nu}\,, \tag{5.131}$$

Taking the sum over n of both sides of eq. (5.126), we see that the particular solution that we have found can also be written compactly as

$$\boxed{\bar{\mathsf{h}}^{\mu\nu}_{\rm part} = \frac{16\pi G}{c^4}\,\mathcal{FP}\Box^{-1}_{\rm ret}\bar{\tau}^{\mu\nu}\,.} \tag{5.132}$$

To this solution we must add the most general solution of the homogeneous equation, subject to the condition of regularity at the origin. This has the form

$$\mathsf{h}^{\alpha\beta}_{\rm hom} = \frac{16\pi G}{c^4}\sum_{l=0}^{+\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{\mathcal{R}^{\alpha\beta}_L(t-r/c)-\mathcal{R}^{\alpha\beta}_L(t+r/c)}{2\,r}\right]\,, \tag{5.133}$$

where $\mathcal{R}^{\alpha\beta}_L(u)$ are arbitrary functions of u, and are STF tensors in the indices $i_i \ldots i_l$. The fact that this is a solution follows from the fact that,

for any function $f(u)$, where $u = t - r/c$, we have $\Box[f(u)/r] = 0$, and similarly $\Box[f(v)/r] = 0$, where $v = t + r/c$. The inclusion of all STF tensor provides a full set of representations of the rotation group, so for instance the first term in eq. (5.133) gives the most general retarded solution. The condition of regularity at $r = 0$ fixes the antisymmetric combination of retarded and advanced waves. Observe that, under time reversal, $\mathsf{h}^{\alpha\beta}_{\rm hom}$ is odd. According to the discussion above eq. (5.3), it therefore describes radiation reaction.[35] We will indeed see that this term gives a correction to the leading term of the radiation reaction force.

[35]Furthermore, the fact that it is a solution of the homogeneous equation means that a source is not needed to sustain this field, which again leads to an interpretation in terms of a pure radiation field. It is well known already in classical electrodynamics that the antisymmetric combination of advanced and retarded waves is associated with radiation reaction. See, e.g. Poisson (1999) for a review.

5.3.3 Matching of the solutions

In the external source region, $d < r < \infty$, we have found the solution in the form of a post-Minkowskian expansion, eq. (5.85). For $d/r < 1$ the multipole expansion is applicable, so we could write the solution for h_1 in terms of the multipole moments $(\mathrm{I}_L, \mathrm{J}_L, \mathrm{W}_L, \mathrm{X}_L, \mathrm{Y}_L, \mathrm{Z}_L)$ or, equivalently, in terms of $(\mathrm{M}_L, \mathrm{S}_L)$. Through the iterative procedure that we have discussed, all higher-order terms $\mathsf{h}_2, \mathsf{h}_3, \ldots$ are then determined, in the form of a multipole expansion.

On the other hand, in the region $0 < r < \mathcal{R}$, with $\mathcal{R}$ is the boundary of the near region, we have found the solution in terms of a post-Newtonian expansion. Since we are considering a source with $v \ll c$, we have $\mathcal{R} \gg d$, and the region of validity of the PN expansion overlaps with the region of validity of the post-Minkowskian expansion. In the post-Minkowskian scheme, the moments $(\mathrm{I}_L, \ldots, \mathrm{Z}_L)$ are quantities that parametrize the most general vacuum solution, but for the moment know nothing about the specific source under consideration. In PN solution, on the contrary, the energy–momentum tensor of the source enter explicitly, see eq. (5.132). Comparing these two solutions in the overlapping region, we can therefore fix the multipole moments $(\mathrm{I}_L, \ldots, \mathrm{Z}_L)$ in terms of the energy–momentum tensor of the source.

To perform this matching we observe that, in the overlap region $d < r < \mathcal{R}$, we have $d/r < 1$ so each term of the post-Newtonian expansion can be in turn re-expanded in powers of d/r, i.e. in a multipole expansion. This is the multipolar post-Newtonian expansion discussed in Section 5.3.2. On the other hand, again in the overlap region, each term of the multipolar post-Minkowskian expansion can be expanded in a post-Newtonian way, i.e. in powers of v/c. A crucial point is that the n-th term of the post-Minkowskian expansion, i.e. the term $\mathsf{h}^{\alpha\beta}_n$ in eq. (5.85), when expanded in a PN fashion, is such that[36]

[36]For the proof see Blanchet and Damour (1986), eq. (5.5).

$$\mathsf{h}^{00}_n = O\left(\frac{1}{c^{2n}}\right)\,,\quad \mathsf{h}^{0i}_n = O\left(\frac{1}{c^{2n+1}}\right)\,,\quad \mathsf{h}^{ij}_n = O\left(\frac{1}{c^{2n}}\right)\,. \tag{5.134}$$

This means that, to work to a given order in the PN expansion, we need to take into account only a finite number of iterations of the post-Minkowskian expansion. For example, suppose that we want to perform a computation to 2PN order, i.e. that we want to compute the correction

$O(1/c^4)$ to the Newtonian metric. This means that we need g_{00} up to $O(1/c^6)$, g_{0i} to $O(1/c^5)$, and g_{ij} to $O(1/c^4)$, included. Equation (5.134) shows that we need to compute h_n up to $n = 3$, i.e. that we must perform two iterations of the linearized solution h_1.

Comparing the multipolar post-Newtonian expansion with the PN re-expansion of the post-Minkowskian solution, allows us to fix the multipole moments $(\mathrm{I}_L, \ldots, \mathrm{Z}_L)$ in terms of the energy–momentum tensor of the source. Remarkably, it is possible to compute them analytically, for any l, and formally to arbitrary order in the PN expansion. For I_L and J_L one finds[37]

[37] For the proof, see Section 5 of the review Blanchet (2006). For the explicit expression of the moments $(\mathrm{W}_L, \mathrm{X}_L, \mathrm{Y}_L, \mathrm{Z}_L)$, which are needed only when performing computations to relatively high order, see eqs. (87)–(90) of Blanchet (2006).

$$\mathrm{I}_L(u) = \mathcal{FP}\int d^3x \int_{-1}^{1} dz \left\{ \delta_l(z)\hat{x}_L\Sigma - \frac{4(2l+1)\delta_{l+1}(z)}{c^2(l+1)(2l+3)}\hat{x}_{iL}\Sigma_i^{(1)} \right.$$
$$\left. + \frac{2(2l+1)\delta_{l+2}(z)}{c^4(l+1)(l+2)(2l+5)}\hat{x}_{ijL}\Sigma_{ij}^{(2)} \right\}(u + z|\mathbf{x}|/c, \mathbf{x}), \quad (5.135)$$

$$\mathrm{J}_L(u) = \mathcal{FP}\int d^3x \int_{-1}^{1} dz\, \epsilon_{ab\langle i_l} \left\{ \delta_l(z)\hat{x}_{L-1\rangle a}\Sigma_b \right.$$
$$\left. - \frac{(2l+1)\delta_{l+1}(z)}{c^2(l+2)(2l+3)}\hat{x}_{L-1\rangle ac}\Sigma_{bc}^{(1)} \right\}(u + z|\mathbf{x}|/c, \mathbf{x}). \quad (5.136)$$

We have defined

$$\Sigma = \frac{\bar{\tau}^{00} + \bar{\tau}^{ii}}{c^2}, \quad (5.137)$$

$$\Sigma_i = \frac{\bar{\tau}^{0i}}{c}, \quad (5.138)$$

$$\Sigma_{ij} = \bar{\tau}^{ij}, \quad (5.139)$$

where $\bar{\tau}^{ii} \equiv \delta_{ij}\bar{\tau}^{ij}$, $\tau^{\mu\nu}$ is given in eq. (5.73), the bar over a quantity denotes its PN expansion up to the desired order, and the integration in d^3x is over the whole space $\mathbb{R}^3$. The function $\delta_l(z)$ has been defined in eq. (3.189), and the remaining notation is explained on page 134 and in eq. (5.97).

Comparing with eqs. (3.207) and (3.208), we see a truly remarkable fact: despite all complications of the non-linear theory, *the full non-linear result for* $\mathsf{h}_1^{\mu\nu}$, *to all orders in the PN expansion, is obtained from the result of linearized theory simply replacing* $T^{\mu\nu}$ *with* $\tau^{\mu\nu}$, and inserting the $\mathcal{FP}$ prescription.[38]

[38] Observe also that the factor of G have been reabsorbed in the definition of $\mathsf{h}_1^{\mu\nu}$, see eq. (5.85).

The integration over z is computed, in an expansion in powers of $1/c$, using eq. (3.209). In particular, to 1PN order one finds from the above equations that the mass quadrupole I_{ij} (i.e. I_L with $l = 2$) is given by

$$I_{ij}(u) = \int d^3x\, \hat{x}_{ij}\, \bar{\sigma}(u, \mathbf{x}) + \frac{1}{14c^2}\frac{\partial^2}{\partial u^2}\int d^3x\, \hat{x}_{ij}\, |\mathbf{x}|^2\, \bar{\sigma}(u, \mathbf{x})$$
$$- \frac{20}{21c^2}\frac{\partial}{\partial u}\int d^3x\, \hat{x}_{ijk}\, \bar{\sigma}^{0k}(u, \mathbf{x}) + O\left(\frac{1}{c^4}\right), \quad (5.140)$$

with σ and σ^k defined in eqs. (3.205) and (3.206). According to the notation introduced on page 134, $\hat{x}_{ij} = x_i x_j - (1/3)\delta_{ij}|\mathbf{x}|^2$, and similarly

for $\hat{x}_{ijk}$. Observe that, for the mass quadrupole, to 1PN order the integral is over a function with compact support, so the $\mathcal{FP}$ prescription is not necessary. It is remarkable that, to 1PN order, this general-relativistic result is actually identical to the linearized gravity result obtained from eq. (3.207). That is, for I_{ij} at 1PN order, we do not even need to make the replacement $T_{\mu\nu} \to \tau_{\mu\nu}$, since the contribution due to the gravitational field in $\tau_{\mu\nu}$ actually cancels out.

The second aspect of the matching problem is that it allows us to fix the functions $\mathcal{R}_L^{\alpha\beta}$ that appear in the homogeneous solution (5.133). The result is

$$\mathcal{R}_L^{\alpha\beta}(u) = \frac{1}{2\pi}\mathcal{FP}\int d^3x'\, \hat{x}'_L \int_1^{+\infty} dz\, \delta_l(z)\, \mathcal{M}(\tau^{\alpha\beta})(u - z|\mathbf{x}'|/c, \mathbf{x}'), \quad (5.141)$$

where $\mathcal{M}(\tau^{\alpha\beta})$ denotes the multipole expansion of $\tau^{\alpha\beta}$. This homogeneous term is associated with radiation reaction effects at 4PN order, due to the so-called tail effects, that will be examined in detail in Section 5.3.4. We will also see that the leading radiation reaction term appears at 2.5PN order (indeed, we already understood this from eq. (5.67)), so this homogeneous term describes a 1.5PN correction to the leading term of the back-reaction force.

There is one more comment to be made on the validity of the whole formalism that we have discussed. A crucial point of the whole procedure is the existence of a region where the domain of validity of the post-Minkowskian expansion, $d < r < \infty$, overlaps with the domain of validity of the PN expansion, $0 < r < \mathcal{R}$. If this were not the case, the general form of the post-Minkowskian solution would still be valid, since it is the most general solution of the vacuum Einstein equations. However, we would not be able to connect the multipole moments $\mathrm{I}_L, \mathrm{J}_L, \ldots, \mathrm{Z}_L$ that parametrize it, to the properties of the source. As we saw, the PN expansion breaks down at distances $r \sim \lambdabar \sim (c/v)d$. Since for a material source $v/c < 1$, one might hope that there is always at least a small overlap between the near zone and the external region. However we have seen, already at the level of linearized theory, that a source oscillating at a frequency ω_s emits quadrupole radiation at frequency $\omega = 2\omega_s$, while its mass octupole and current quadrupole radiation is at $\omega = \omega_s$ and at $\omega = 3\omega_s$, and a multipole of order n distributes its radiation among a set of lines in frequency, up to a maximum frequency $n\omega_s$. In a computation to n-th PN order, we must include multipoles up to order $\sim n$, which therefore generate GWs with frequencies up to $\omega_n = O(n)\omega_s$, and, correspondingly, reduced wavelength $\lambdabar_n = O(1/n)\lambdabar_0$, where $\lambdabar_0 \sim (c/v)d$. If n is larger than $O(c/v)$, it is no longer true that $\lambdabar_n \gg d$, and for GWs of such wavelengths the near zone no longer overlaps with the exterior region, so we cannot compute them with this formalism. At the same time, while for $v/c \ll 1$ the lowest multipoles dominate, this is no longer true when v/c approaches one, so the contributions that we are unable to compute are also no longer negligible.

In other words, for a system with typical velocity v, we can compute only up to a post-Newtonian order $O(c/v)$. Thus, the formalism that we have discussed becomes asymptotically exact for $v/c \to 0$, while it gradually breaks down in the opposite limit $v/c \to 1$.

5.3.4 Radiative fields at infinity

Having computed the moments $\mathrm{I}_L, \dots, \mathrm{Z}_L$, the solution outside the source is now determined, and we can study it at future null infinity, i.e. at $r \to \infty$ with $u = t - r/c$ fixed, where we expect to find gravitational waves. One finds that, in this limit, the n-th term $\mathsf{h}_n^{\alpha\beta}$ of the post-Minkowskian expansion (5.85) has the formal structure

$$\mathsf{h}_n^{\mu\nu} = \sum_{k=1}^{\infty}\sum_{p=0}^{n-1} G^{\mu\nu}_{L,(k,p,n)}(u)\,\frac{\hat{n}_L(\log r)^p}{r^k}\,, \tag{5.142}$$

where, as usual, the summation over the multi-index L is understood. The appearance of terms involving $\log r$ at future null infinity is a coordinate effect, due to our use of harmonic coordinates. It is known, since the classical works of Bondi *et al.*, Sachs, and Penrose in the 1960s on the asymptotic structure of space-time at future null infinity, that it is possible to find other coordinate systems, called *radiative coordinates* or Bondi-type coordinates, where the logarithmic terms are absent.[39] We denote one such coordinate system by capital letters, $X^\mu = (T, \mathbf{X})$, and we introduce $R = |\mathbf{X}|$ and $U = T - R/c$. The unit radial vector in these coordinates is $\mathbf{N} = \mathbf{X}/R$, and as usual N_L is the multi-index notation for $N_{i_1} \dots N_{i_l}$. We write the metric in this coordinate system as $G_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}$. Then, the post-Minkowskian expansion at future null infinity has the general structure

$$H_n^{\mu\nu} = \sum_{k=1}^{\infty} K^{\mu\nu}_{L,(k,n)}(U)\frac{\hat{N}_L}{R^k}\,, \tag{5.143}$$

without logarithmic terms. The coordinate transformation from harmonic to radiative coordinates can be obtained, order by order in G, computing explicitly the behavior of $\mathsf{h}_n^{\alpha\beta}$ at future null infinity.

We now introduce two sets of STF multipole moments, called the *radiative multipole moments*, or the multipole moments at infinity, and denoted as $\mathrm{U}_L(U)$ and $\mathrm{V}_L(U)$ (where the argument U is retarded time $T - R/c$ in radiative coordinates, and should not be confused with U_L with $l = 0$), defined as follows. At future null infinity, we select the $1/R$ part of H_{ij}, and we project it onto the TT gauge making use of the projection operator $\Lambda_{ij,kl}$ as in eq. (1.40). We denote the resulting expression by H_{ij}^{TT}. Then, $\mathrm{U}_L(U)$ and $\mathrm{V}_L(U)$ are defined by

$$H_{ij}^{\mathrm{TT}}(U,\mathbf{N}) = \frac{4G}{c^2R}\Lambda_{ijab}(\mathbf{N})\sum_{l=2}^{+\infty}\frac{1}{c^l l!}\Big\{N_{L-2}\mathrm{U}_{abL-2}(U) - \frac{2l}{c(l+1)}N_{cL-2}\epsilon_{cd(a}\mathrm{V}_{b)dL-2}(U)\Big\}\,. \tag{5.144}$$

[39] It should be observed however that the Bondi–Penrose expansion does not exist if the source has been active in the infinite past, as can be seen with physical arguments, see Damour (1986), and with rigorous mathematical results, see Christodoulou and Klainerman (1993). However, in the Blanchet–Damour formalism one always consider a source that was quiet at time t smaller than some value $-T$, see Note 24 on page 255. Physically, this restriction is not an important limitation, since the binary system was obviously not there before the epoch of formation of its stars.

Once we have the explicit expressions of U_L and V_L in terms of the algorithmic moments $\mathrm{I}_L, \mathrm{J}_L, \dots \mathrm{Z}_L$, including all $1/c$ corrections consistent with the PN order to which we want to work, we have completed the solution to the problem of GW generation, since we then have the waveform at infinity, in terms of the energy–momentum tensor of the source.

Lowest-order determination of $\mathrm{U}_L, \mathrm{V}_L$

We first compute the relation between the radiative moments $\mathrm{U}_L, \mathrm{V}_L$ and the algorithmic moments $\mathrm{I}_L, \dots, \mathrm{Z}_L$, to lowest order in $1/c$. So, we limit ourselves to the lowest order in the post-Minkowskian expansion, $\mathsf{h}^{ij} = G\mathsf{h}_1^{ij}$, with h_1^{ij} given in eq. (5.95). We neglect the function φ_1^μ, which at the linearized level is a gauge mode, so it contributes only to higher orders in $1/c$. Then h_1^{ij} is the same as the tensor k_1^{ij} given in eq. (5.96). To this order, there is also no difference between harmonic and radiative coordinates, since no logarithmic factor appears at infinity. To get the leading term for $r \to \infty$ we simply extract the factor $1/r$ from the derivatives in eq. (5.96), and we use the fact that, on a function $f(u)$ of retarded time $u = t - r/c$, we have $\partial_i f(u) = (\partial_i r) df/dr$. Since $\partial_i r = x_i/r = n_i$ and $df/dr = (-1/c)df/du$, we get

$$\partial_L f(u) = \frac{(-1)^l}{c^l}\, n_L f^{(l)}(u)\,. \tag{5.145}$$

We insert this into eq. (5.96) and we write $h_{ij}^{\mathrm{TT}} = \Lambda_{ijab}h_{ab}$. Comparing with the definition (5.144), and observing that, to lowest order $\mathrm{I}_L = \mathrm{M}_L$ and $\mathrm{J}_L = \mathrm{S}_L$, we find that[40]

$$\mathrm{U}_L(U) = \mathrm{M}_L^{(l)}(U)\,, \qquad \mathrm{V}_L(U) = \mathrm{S}_L^{(l)}(U)\,. \tag{5.146}$$

Thus, to lowest order in $1/c$, the radiative multipole moments U_L and V_L are simply equal to the l-th time derivative of M_L and S_L, respectively. At this level, we have simply reproduced the result of linearized theory. Indeed, we saw in Sections 3.3 that the coefficient of $1/r$ in the amplitude (what we have now called a radiative moment) is given by the second derivative of the mass quadrupole moment, see eq. (3.59), by the third derivative of the mass octupole, see eq. (3.141), etc. (see eq. (3.204) or eqs. (3.291) and (3.293) for the general result).

Of course, we cannot limit ourselves to this lowest-order result, but we must include all $1/c$ corrections to the relations (5.146), consistent with the PN order to which we wish to work. So, we next consider the corrections in $1/c$ coming from the first iteration of the post-Minkowskian algorithm, i.e. from the inclusion of h_2^{ij}. We will see that this study also reveals a very interesting conceptual feature, the presence of so-called "hereditary terms".

Higher-order corrections

To illustrate in a simpler setting the computation of the first post-Minkowskian iteration, we take as starting point for the iterative pro-

[40] We must also take into account the minus sign discussed in Note 19 on page 250.

cedure a linearized metric $\mathsf{h}_1^{\alpha\beta}$ of the form given in eq. (5.96), retaining only the two lowest-order mass multipoles, $\mathrm{I} \equiv \mathrm{M}$ and I_{ij}. The latter, up to 2.5PN corrections, is the same as M_{ij}, see eq. (5.102). Then we can write $\mathsf{h}_1^{\alpha\beta}$ as

$$\mathsf{h}_1^{\alpha\beta} = \mathsf{h}_{(\mathrm{M})}^{\alpha\beta} + \mathsf{h}_{(\mathrm{M}_{ij})}^{\alpha\beta}\,, \tag{5.147}$$

where, from eq. (5.96), the monopole part is given by

$$\mathsf{h}_{(\mathrm{M})}^{00} = -\frac{4\mathrm{M}}{c^2 r}\,, \tag{5.148}$$

together with $\mathsf{h}_{(\mathrm{M})}^{0i} = \mathsf{h}_{(\mathrm{M})}^{ij} = 0$ (recall also that M is time-independent), and the quadrupole part is

$$\mathsf{h}_{(\mathrm{M}_{ij})}^{00} = -\frac{2}{c^2}\partial_k\partial_l\left[\frac{1}{r}\mathrm{M}_{kl}(u)\right]\,, \tag{5.149}$$

$$\mathsf{h}_{(\mathrm{M}_{ij})}^{0i} = \frac{2}{c^3}\partial_k\left[\frac{1}{r}\mathrm{M}_{ki}^{(1)}(u)\right]\,, \tag{5.150}$$

$$\mathsf{h}_{(\mathrm{M}_{ij})}^{ij} = -\frac{2}{c^4}\frac{1}{r}\mathrm{M}_{ij}^{(2)}(u)\,. \tag{5.151}$$

We can now determine the next post-Minkowskian iteration, $\mathsf{h}_2^{\alpha\beta}$, solving eq. (5.90) with $N^{\alpha\beta}[\mathsf{h}_1,\mathsf{h}_1]$ given in eq. (5.89). Since $N^{\alpha\beta}$ is quadratic in h_1, when we insert eq. (5.147) we get three terms, one quadratic in the monopole part $\mathsf{h}_{(\mathrm{M})}^{\alpha\beta}$, one quadratic in the quadrupole part $\mathsf{h}_{(\mathrm{M}_{ij})}^{\alpha\beta}$, and a mixed monopole–quadrupole term,

$$\mathsf{h}_2^{\alpha\beta} = \mathsf{h}_{(\mathrm{M}^2)}^{\alpha\beta} + \mathsf{h}_{(\mathrm{M}\times\mathrm{M}_{ij})}^{\alpha\beta} + \mathsf{h}_{(\mathrm{M}_{ij}\times\mathrm{M}_{kl})}^{\alpha\beta}. \tag{5.152}$$

The monopole–monopole part $\mathsf{h}_{(\mathrm{M}^2)}^{\alpha\beta}$ is easily computed. Since $\mathsf{h}_{(\mathrm{M})}^{\alpha\beta}$ is non-vanishing only for $\alpha = \beta = 0$, $N^{\alpha\beta}[\mathsf{h}_{(\mathrm{M})},\mathsf{h}_{(\mathrm{M})}]$ collapses to a very simple expression. For instance,

$$N^{00}[\mathsf{h}_{(\mathrm{M})},\mathsf{h}_{(\mathrm{M})}] = -\frac{14\mathrm{M}^2}{c^4 r^4}\,. \tag{5.153}$$

Thus, $\mathsf{h}_{(\mathrm{M}^2)}^{00}$ is obtained solving the equation $\Box \mathsf{h}_{(\mathrm{M}^2)}^{00} = -14\mathrm{M}^2/(c^4 r^4)$. Since the right-hand side is time-independent, in this case there is no need to go through the procedure of taking the retarded integral with the $\mathcal{FP}$ prescription. Simply, the solution will be time-independent (so the d'Alembertian becomes a Laplacian), and will be a function only of r. From the expression of the Laplacian in spherical coordinates we get

$$\frac{1}{r}\frac{d^2}{dr^2}[r\mathsf{h}_{(\mathrm{M}^2)}^{00}] = -\frac{14\mathrm{M}^2}{c^4 r^4}\,, \tag{5.154}$$

which (together with the boundary condition that the metric vanishes at spatial infinity) gives

$$\mathsf{h}_{(\mathrm{M}^2)}^{00} = -\frac{7\mathrm{M}^2}{c^4 r^2}\,. \tag{5.155}$$

Similarly one finds that $\mathsf{h}_{(\mathrm{M}^2)}^{0i} = 0$ and $\mathsf{h}_{(\mathrm{M}^2)}^{ij} = -n_i n_j \mathrm{M}^2/(c^4 r^2)$.[41] Since it proportional to $1/r^2$, $\mathsf{h}_{(\mathrm{M}^2)}^{\alpha\beta}$ does not contribute to the $1/r$ part of the field, and hence to the radiative moments at infinity.

Consider next the mixed monopole–quadrupole term. When we plug $\mathsf{h}_1^{\alpha\beta} = \mathsf{h}_{(\mathrm{M})}^{\alpha\beta} + \mathsf{h}_{(\mathrm{M}_{ij})}^{\alpha\beta}$ into $N^{\alpha\beta}[\mathsf{h}_1,\mathsf{h}_1]$ and we retain the terms bilinear in $\mathsf{h}_{(\mathrm{M})}^{\alpha\beta}$ and $\mathsf{h}_{(\mathrm{M}_{ij})}^{\alpha\beta}$, we find that $N^{\alpha\beta}[\mathsf{h},\mathsf{h}]$ takes the general form

$$N^{\alpha\beta}[\mathsf{h},\mathsf{h}]_{(\mathrm{M}\times\mathrm{M}_{ij})} = \sum_{k=2}^{\infty}\frac{n_L}{r^k}\,H_L^{(k)}(u)\,, \tag{5.156}$$

for some functions $H_L^{(k)}(u)$. This follows simply from the fact that $\mathsf{h}_{(\mathrm{M})}^{\alpha\beta}$ is proportional to $1/r$, times a constant, while $\mathsf{h}_{(\mathrm{M}_{ij})}^{\alpha\beta}$ is the sum of terms with negative powers of r, such as $1/r, 1/r^2$, etc. multiplied by functions of $u = t - r/c$. The factors n_L come out when taking the spatial derivatives, using $\partial_i r = n_i$.

Thus, to find the contribution of this mixed monopole–quadrupole term to $\mathsf{h}_2^{\alpha\beta}$, we must compute the finite part at $B = 0$ of retarded integrals of the form

$$\Box_{\mathrm{ret}}^{-1}[n_L r^{B-k} H(u)]\,, \tag{5.157}$$

for some function $H(u)$ (we omit the index L in $H(u)$, in order not to imply a summation over l). The computation of this retarded integral gives two completely different results when $k=2$ and when $k \geqslant 3$. For $k \geqslant 3$ (but still $k \leqslant l+2$, which is the case that we will need) one finds[42]

$$\mathrm{FP}_{B=0}\,\Box_{\mathrm{ret}}^{-1}[n_L r^{B-k} H(u)] = n_L \sum_{j=0}^{k-3}\frac{c_{jkl}}{r^{j+1}}\frac{d^{k-3-j}}{du^{k-3-j}}H(u)\,, \tag{5.158}$$

where c_{jkl} are some numerical coefficients,

$$c_{jkl} = -\frac{2^{k-3-j}(k-3)!\,(l+2-k)!(l+j)!}{(l+k-2)!j!(l-j)!}\,. \tag{5.159}$$

Observe that, since j takes the values $0,\ldots,k-3$, the order of the derivative of $H(u)$, $k-3-j$, is between $k-3$ and zero. Since $k \geqslant 3$, this is never negative. The important point about eq. (5.158) is that the result is local in time: its value at a given (retarded) time u depends on the function $H(u)$ and on a finite number of its derivatives, evaluated at the same retarded time u. We will refer to terms with this property as "instantaneous".

It is clear, however, that the above result cannot hold for $k=2$, since in this case the order of the derivative of $H(u)$, $k-3-j$, can become negative. Indeed, for $k=2$ on finds that the result is given by an integral of $H(u)$, rather than by its derivatives,

$$\Box_{\mathrm{ret}}^{-1}\left[\frac{n_L}{r^2}H\left(t-\frac{r}{c}\right)\right] = -\frac{n_L}{r}\int_r^{\infty}dz\,Q_l\left(\frac{z}{r}\right)H\left(t-\frac{z}{c}\right)\,, \tag{5.160}$$

where $Q_l(x)$ is a special function known as the Legendre function of the second kind. (Observe that for $k=2$ the retarded integral converges at

[41] The correctness of the result can be checked observing that

$$\mathsf{h}_{\mathrm{monopole}}^{\alpha\beta} = G\mathsf{h}_{(\mathrm{M})}^{\alpha\beta} + G^2\mathsf{h}_{(\mathrm{M}^2)}^{\alpha\beta}$$

is nothing but the Schwarzschild metric, written in harmonic coordinates, and in terms of $\mathsf{h}^{\alpha\beta} = (-g)^{1/2}g^{\alpha\beta} - \eta^{\alpha\beta}$, expanded to second order in $R_S/r = 2GM/(c^2 r)$.

[42] For the explicit computations see Blanchet and Damour (1986) and (1988), as well as the appendixes of Blanchet (1998a) and (1998b).

$r = 0$, so the $\mathcal{FP}$ prescription is superfluous.) The crucial point is that, since the integration variable z runs from r to ∞ , this result does not depend just on $H(u)$ at the value $u = t - r/c$, but also on all its values at earlier times, from $u = -\infty$ to $u = t - r/c$. In other words, the result depends on the whole past history of the source. A term of this kind is called "hereditary", and we will discuss its physical meaning below.

The asymptotic behavior of eq. (5.160) for $r \to \infty$ and $t - r/c$ fixed is computed using the known behavior of $Q_l(x)$ in the limit $x \to 1^+$,

$$Q_l(x) = -\frac{1}{2}\log\left(\frac{x-1}{2}\right) - a_l + O\left[(x-1)\log(x-1)\right] , \qquad (5.161)$$

where the constant $a_l \equiv \sum_{k=1}^{l} k^{-1}$. From this, changing variable from z to $y = (z - r)/c$, we get

$$\Box_{\rm ret}^{-1}\left[\frac{n_L}{r^2}H(t-r/c)\right] = \frac{cn_L}{2r}\int_0^{\infty} dy\, H\left(t - \frac{r}{c} - y\right)\left[\log\left(\frac{cy}{2r}\right) + 2a_l\right] + O\left(\frac{\log r}{r^2}\right) . \qquad (5.162)$$

Thus, asymptotically this is $O(1/r)$, and contributes to the radiation field. We can now compute the hereditary term in $\mathsf{h}^{\alpha\beta}_{(\mathrm{M}\times\mathrm{M}_{ij})}$, considering for definiteness the component $\alpha = \beta = 0$. Inserting eq. (5.147) into $N^{\alpha\beta}[\mathsf{h}_1, \mathsf{h}_1]$ and keeping only the mixed monopole–quadrupole terms, we find[43]

$$N^{00}[\mathsf{h},\mathsf{h}]_{(\mathrm{M}\times\mathrm{M}_{ij})} = -\frac{8\mathrm{M}}{r^2c^4}\, n_i n_j \mathrm{M}^{(4)}_{ij} + O\left(\frac{1}{r^3}\right) . \qquad (5.163)$$

Using eq. (5.160) we therefore find

$$\mathsf{h}^{00}_{(\mathrm{M}\times\mathrm{M}_{ij})} = \frac{8\mathrm{M}}{rc^4}\, n_i n_j \int_r^{\infty} dz\, Q_2\left(\frac{z}{r}\right) \mathrm{M}^{(4)}_{ij}(t - z/c) + \text{instantaneous terms} . \qquad (5.164)$$

The instantaneous terms are straightforwardly computed using eq. (5.158). Similarly one can compute the other tensor components of $\mathsf{h}^{\alpha\beta}_{(\mathrm{M}\times\mathrm{M}_{ij})}$, as well as the quadrupole–quadrupole term $\mathsf{h}^{\alpha\beta}_{(\mathrm{M}_{ij}\times\mathrm{M}_{kl})}$, and one finds that they all have hereditary contributions, besides the instantaneous terms.[44]

Using the asymptotic expansion (5.162) we can now compute the contribution of $\mathsf{h}_2^{\alpha\beta}$ to the radiation field at infinity, and therefore to the radiative multipole moments. For U_L we get

$$\mathrm{U}_L(u) = \mathrm{M}^{(l)}_L(u) + \frac{2G\mathrm{M}}{c^3}\int_0^{+\infty} d\tau\, \mathrm{M}^{(l+2)}_L(u-\tau)\left[\log\left(\frac{c\tau}{2r}\right) + \kappa_l\right] + O\left(\frac{1}{c^5}\right) , \qquad (5.165)$$

where, as usual, $u = t - r/c$ and

$$\kappa_l = \frac{2l^2 + 5l + 4}{l(l+1)(l+2)} + \sum_{k=1}^{l-2}\frac{1}{k} . \qquad (5.166)$$

[43]It is easy to extract the term $1/r^2$ from $N^{00}[\mathsf{h},\mathsf{h}]_{(\mathrm{M}\times\mathrm{M}_{ij})}$. Just observe that both $\mathsf{h}^{\alpha\beta}_{(\mathrm{M})}$ and $\mathsf{h}^{\alpha\beta}_{(\mathrm{M}_{ij})}$ already carry each one at least one factor $1/r$, and that $\partial_i(1/r) = -n_i/r^2$, so all terms involving spatial derivatives of $1/r$ are at least overall $O(1/r^3)$. Taking further into account that $\mathsf{h}^{\alpha\beta}_{(\mathrm{M})}$ is non-vanishing only for $\alpha = \beta = 0$ and is time-independent we see immediately that, in eq. (5.89), in the mixed term, the only contribution $O(1/r^2)$ comes from the first term, and is $-\mathsf{h}^{\alpha\beta}_{(\mathrm{M})}\partial_0^2\mathsf{h}^{\alpha\beta}_{(\mathrm{M}_{ij})}$.

[44]See Blanchet (2006), eq. (110), for $\mathsf{h}^{\alpha\beta}_{(\mathrm{M}\times\mathrm{M}_{ij})}$, and Blanchet (1998a) for $\mathsf{h}^{\alpha\beta}_{(\mathrm{M}_{ij}\times\mathrm{M}_{kl})}$.

We see that the second term in eq. (5.165) is a $O(1/c^3)$ correction (i.e. a 1.5PN correction) to the leading result given in eq. (5.146). The $\log r$ term in the correction is typical of the harmonic coordinate system that we are using and, as we mentioned, can be eliminated going to radiative coordinates. From the study of the logarithmic terms at infinity one finds that retarded time U in radiative coordinates is related to the harmonic coordinates (t, r) by

$$U = t - \frac{r}{c} - \frac{2G\mathrm{M}}{c^3}\ln\left(\frac{r}{r_0}\right) + O\left(G^2\right) , \qquad (5.167)$$

where r_0 is the arbitrary constant discussed in Note 29 on page 258, and provides a scale for the logarithm. Its arbitrariness corresponds to a freedom in the choice of a system of radiative coordinates, and in particular we see from eq. (5.167) that it can be reabsorbed into a shift in the origin of retarded time U. So it is a a gauge-dependent constant that will not influence any physical result, as it is already clear from the fact that the starting expression (5.165) is independent of r_0. In terms of these radiative coordinates, eq. (5.165) becomes

$$\mathrm{U}_L(U) = \mathrm{M}^{(l)}_L(U) + \frac{2G\mathrm{M}}{c^3}\int_0^{+\infty} d\tau\, \mathrm{M}^{(l+2)}_L(U-\tau)\left[\log\left(\frac{c\tau}{2r_0}\right) + \kappa_l\right] + O\left(\frac{1}{c^5}\right) , \qquad (5.168)$$

i.e. inside the logarithm, r is replaced by r_0. Similarly, for the current-type multipoles one finds

$$\mathrm{V}_L(U) = \mathrm{S}^{(l)}_L(U) + \frac{2G\mathrm{M}}{c^3}\int_0^{+\infty} d\tau\, \mathrm{S}^{(l+2)}_L(U-\tau)\left[\log\left(\frac{c\tau}{2r_0}\right) + \pi_l\right] + O\left(\frac{1}{c^5}\right) , \qquad (5.169)$$

where

$$\pi_l = \frac{l-1}{l(l+1)} + \sum_{k=1}^{l-1}\frac{1}{k} . \qquad (5.170)$$

Of course, the relation between the radiative and algorithmic moments is needed with the highest accuracy for the lowest multipole moments, since the contribution of higher multipoles to the radiation field is suppressed by higher powers of $1/c$. In particular, to compute the GW production in a binary system up to 3PN, we need the relation between the $l = 2$ mass moment U_{ij} and M_{ij} up to 3PN order, which is

$$\begin{aligned}\mathrm{U}_{ij}(U) = {} & \mathrm{M}^{(2)}_{ij}(U) + \frac{2G\mathrm{M}}{c^3}\int_0^{+\infty} d\tau\, \mathrm{M}^{(4)}_{ij}(U-\tau)\left[\log\left(\frac{c\tau}{2r_0}\right) + \frac{11}{12}\right] \\ & -\frac{2G}{7c^5}\int_0^{+\infty} d\tau\, \mathrm{M}^{(3)}_{a\langle i}(U-\tau)\mathrm{M}^{(3)}_{j\rangle a}(U-\tau) \\ & -\frac{G}{c^5}\left[\frac{2}{7}\mathrm{M}^{(3)}_{a\langle i}\mathrm{M}^{(2)}_{j\rangle a} + \frac{5}{7}\mathrm{M}^{(4)}_{a\langle i}\mathrm{M}^{(1)}_{j\rangle a} - \frac{1}{7}\mathrm{M}^{(5)}_{a\langle i}\mathrm{M}_{j\rangle a} - \frac{1}{3}\epsilon_{ab\langle i}\mathrm{M}^{(4)}_{j\rangle a}\mathrm{S}_b\right]\end{aligned}$$

$$+\frac{2G^2M^2}{c^6}\int_0^{+\infty} d\tau\, M_{ij}^{(5)}(U-\tau)$$
$$\times\left[\log^2\left(\frac{c\tau}{2r_0}\right)+\frac{57}{70}\log\left(\frac{c\tau}{2r_0}\right)+\frac{124627}{44100}\right]$$
$$+O\left(\frac{1}{c^7}\right). \quad (5.171)$$

The 1.5PN correction, in the first line, is the hereditary monopole–quadrupole term in $h_2^{\alpha\beta}$ that we have computed above. In the second line we have a hereditary 2.5PN contribution, due to the quadrupole–quadrupole term in $h_2^{\alpha\beta}$, and in the third line we have the instantaneous contribution from this quadrupole–quadrupole term. The term proportional to G^2M^2 is a monopole–monopole–quadrupole term in the second post-Minkowskian iteration $h_3^{\alpha\beta}$, and again it is a hereditary term.

Below we discuss the physics behind these non-local contributions. First, we observe that the above results allow us to compute the waveform and, from it, we can obtain the energy radiated at infinity. This can be obtained inserting eq. (5.144) into the expression for the radiated energy, eq. (1.153). The result for the radiated power P, as a function of retarded time U, is (compare with the linearized result, eq. (3.210))

$$P=\sum_{l=2}^{+\infty}\frac{G}{c^{2l+1}}\left\{\frac{(l+1)(l+2)}{(l-1)l l!(2l+1)!!}\langle U_L^{(1)}(U)U_L^{(1)}(U)\rangle\right.$$
$$\left.+\frac{4l(l+2)}{c^2(l-1)(l+1)!(2l+1)!!}\langle V_L^{(1)}(U)V_L^{(1)}(U)\rangle\right\}. \quad (5.172)$$

For example, up to 2PN order, this equation gives

$$P=\frac{G}{c^5}\left\{\frac{1}{5}U_{ij}^{(1)}U_{ij}^{(1)}+\frac{1}{c^2}\left[\frac{1}{189}U_{ijk}^{(1)}U_{ijk}^{(1)}+\frac{16}{45}V_{ij}^{(1)}V_{ij}^{(1)}\right]\right.$$
$$\left.+\frac{1}{c^4}\left[\frac{1}{9072}U_{ijkm}^{(1)}U_{ijkm}^{(1)}+\frac{1}{84}V_{ijk}^{(1)}V_{ijk}^{(1)}\right]+O\left(\frac{1}{c^6}\right)\right\}, \quad (5.173)$$

(where the average is understood), compare with the 1PN result of linearized theory given in eq. (3.156).

Physical meaning of hereditary terms. "Non-linear memory" and "tails"

At first sight, the appearance of terms that are not instantaneous is quite surprising. For an interaction that propagates in flat space at the speed of light, the signal detected at time t_0, coming from a source at a distance r, depends only on the instantaneous state of the source at retarded time $u_0 = t_0 - r/c$, so it depends on the source multipole moments and on its derivatives, all evaluated at $u = u_0$. However, the sources that generates GWs also curve space-time, so GWs necessarily propagate in a curved space. We see that, as a consequence, besides the instantaneous terms there are also hereditary terms, given by integrals

over retarded time from $-\infty$ to u_0, which therefore depend on the value of the multipole moments at all times $u \leqslant u_0$.

In other words, the intuition stemming from flat space-time suggested that GWs propagate *on* the light cones, while we find that they rather propagate both on and *inside* the light cones.[45] It is as if the gravitational interaction did not propagate just with speed c, but with all possible speeds $0 < v \leqslant c$.

[45] Indeed it was already known since the 1950s, from the study of the initial value problem in general relativity, that, given some initial data on a space-like hypersurface S, to determine the gravitational field at a point P we need not only the values of the initial data on the intersection of S with the past (curved-space) light cones of P, but also the data *inside* this intersection, see the Further Reading.

Physically, this result can be better understood using a field-theoretical language, in terms of back-scattering of gravitons. For instance, the 1.5PN hereditary term, in the first line of eq. (5.171), depends both on derivatives of the quadrupole moment M_{ij}, which in a field-theoretical language is associated to a graviton line, and on the mass M of the source, so it corresponds to scattering of gravitons off the background curvature generated by the mass M of the source. The 2.5PN hereditary term in the second line of eq. (5.171) rather corresponds to a graviton–graviton scattering process (in the language of Feynman graphs, it would be related to a three-graviton vertex), while the terms proportional to G^2M^2 is a higher-order correction to the scattering of a graviton off the external curvature. In this sense, gravitons always propagate locally at the speed of light. However, their arrival time is delayed because they can repeatedly scatter back and forth, either with the background gravitational field or among themselves. The same effect is known to take place in the propagation of the electromagnetic field in curved space-time. However, while conceptually it is legitimate to consider the propagation of electromagnetic waves in flat space-time, where they just propagate on the light cone, for GWs this limiting case strictly speaking does not exist, since the source that generates GWs necessarily produces also a curvature of the background space-time, and also because GWs scatter among themselves (or, in the language of Feynman graphs, because of non-Abelian graviton vertices). So, this propagation inside the light cone unavoidably occurs, when one takes into account the PN corrections. We see here a reflection of the fact that gravity is an intrinsically non-linear theory.

Looking more closely at the hereditary terms in eq. (5.171), we can distinguish between two types of terms, with and without the factor $\log\tau$ inside the integral. We examine first the 2.5PN hereditary term in the second line. Introducing $V = U - \tau$, this integral can be written as

$$F_{ij}(U)\equiv\int_{-\infty}^{U}dV\, M_{a\langle i}^{(3)}(V)M_{j\rangle a}^{(3)}(V) \quad (5.174)$$
$$=\int_{-\infty}^{+\infty}dV\,\mathcal{K}(U,V)\, M_{a\langle i}^{(3)}(V)M_{j\rangle a}^{(3)}(V),$$

where the kernel $\mathcal{K}(U,V)=\theta(U-V)$ is flat and equal to one for $V<U$, and vanishes for $V>U$. Consider a source whose multipole moments were constant in the far past,[46] then it becomes active, and finally is turned off at some value of time which, for a far observer at a given distance r, corresponds to a given value of retarded time, say $U = U_0$. For $U > U_0$ we can split the integral in eq. (5.174) as $\int_{-\infty}^{U} dV =$

[46] This was among the basic assumptions of the method, see Note 24 on page 255 and guarantees the convergence of the integral at $V=-\infty$.

$\int_{-\infty}^{U_0} dV + \int_{U_0}^{U} dV$. However, since the integrand in eq. (5.174) vanishes for $V > U_0$, the integration between U_0 and U gives zero, and for all $U > U_0$ we have $F_{ij}(U) = F_{ij}(U_0)$: the integral remains frozen forever at the value it had at $U = U_0$. Thus, the contribution to the GW amplitude due to this term remains non-zero even after the source has been switched off.

This is due to the fact that, in eq. (5.174), very remote times are weighted as much as more recent times, since the kernel $\mathcal{K}(U,V)$ is a flat function of the integration variable V, for $V < U$. Therefore, the result is really determined by the cumulative history of the source, including its very remote past. For this reason, it is called a "memory effect", and the 2.5PN hereditary term in the second line of eq. (5.171), which is non-linear in M_{ij}, is called the "2.5PN non-linear memory integral".[47] Observe however that, taking the time derivative of $F_{ij}(U)$, we obtain an instantaneous term. Thus in the energy flux, which is determined by $\dot{h}_{ij}^{\mathrm{TT}}$, the 2.5PN non-linear memory term gives an instantaneous contribution.

[47] The existence of such a memory effect was also found by Christodoulou, from a rigorous mathematical study of the asymptotic behavior of the gravitational field at null infinity, and is also known as the "Christodoulou memory". The term "non-linear" distinguishes it from a memory effect that exists already in linearized theory, see the Further Reading section.

Consider next the 1.5PN hereditary term in the first line of eq. (5.171). Introducing again $V = U - \tau$, we get an integral of the form

$$G_{ij}(U) \equiv \int_{-\infty}^{U} dV\, \mathrm{M}_{ij}^{(4)}(V) \log\left(\frac{U-V}{2P_0}\right), \tag{5.175}$$

where $P_0 = r_0/c$ is an arbitrary constant with dimensions of time. We split

$$\int_{-\infty}^{U} dV = \int_{-\infty}^{U-2P} dV + \int_{U-2P}^{U} dV, \tag{5.176}$$

and, in the first integral, we integrate twice by parts. Using the fact that the derivatives of M_{ij} go to zero sufficiently fast for $U \to -\infty$, we get

$$G_{ij}(U) = \frac{1}{2P}\mathrm{M}_{ij}^{(2)}(U-2P) + \int_{U-2P}^{U} dV\, \mathrm{M}_{ij}^{(4)}(V) \log\left(\frac{U-V}{2P_0}\right) - \int_{-\infty}^{U-2P} \frac{dV}{(U-V)^2}\mathrm{M}_{ij}^{(2)}(V). \tag{5.177}$$

The terms in the first line involve only values of V in the "recent past", $U - 2P \leqslant V \leqslant U$, while the contribution from the remote past, $-\infty < V \leqslant U - 2P$ is in the integral in the second line. This way of rewriting $G_{ij}(U)$ allows us to understand that the contribution of very remote times, $V \to -\infty$, is weighted with a quadratically decreasing kernel $(U-V)^{-2}$, contrary to what happens in the memory integral, where the kernel is flat and very remote times contribute as much as recent times. A contribution of the type (5.177) is called a "tail integral". Thus, the first line in eq. (5.171) gives the 1.5PN tail integral, and the last line gives the 3PN tail integral.

In conclusion, the GWs emitted by a source which at some value of time suddenly switches off, can be considered as made by three distinct

pieces: the wavefront, which is due to the instantaneous terms; a tail, which effectively travels at a slower speed and therefore arrives later, and which smoothly fades away with time; and, finally, a "memory", which is a persistent DC (i.e. zero-frequency) contribution.

5.3.5 Radiation reaction

Radiation radiation in electromagnetism

Radiation reaction is a classical problem that was first studied in electromagnetism (the pioneering works were by Lorentz, in 1892 and 1902, and by Planck, in 1897).[48] As a warm-up, let us recall these classical results. Consider a system of electric charges e_a moving under their mutual influence. Being accelerated by their interaction, they radiate electromagnetic waves. This emission costs energy, which must be drained from the mechanical energy of the system. This means that there must be a force which acts on these charges and performs the work necessary to account for the energy loss. This force is called the back-reaction force due to the emission of radiation, or simply the radiation reaction.

[48] Actually, the fact that a finite speed of propagation of the interaction induces a radiation-reaction force was already proposed by Laplace in 1776, in the context of the gravitational interaction of the Earth-Moon system, and for this reason it has also been called the Laplace effect. See Chapter 2 of Kennefick (2007).

In other words we expect that, when we compute the total electromagnetic field due to a system of charges in mutual interaction, to a term describing electromagnetic waves in the far zone should correspond a term in the near zone, that describes a radiation reaction force acting on the charges. This is indeed the case. The dynamics in the near zone can be studied starting from the expression for potential $(\phi, \mathbf{A})$ in terms of the charge density ρ and the current density $\mathbf{j}$,

$$\phi(t,\mathbf{x}) = \int d^3x'\, \frac{1}{|\mathbf{x}-\mathbf{x}'|}\rho(t - |\mathbf{x}-\mathbf{x}'|/c, \mathbf{x}'), \tag{5.178}$$

$$\mathbf{A}(t,\mathbf{x}) = \frac{1}{c}\int d^3x'\, \frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathbf{j}(t - |\mathbf{x}-\mathbf{x}'|/c, \mathbf{x}'). \tag{5.179}$$

We perform an expansion for small retardation effects, just as we have done for the gravitational field when we have discussed the PN expansion, and we insert the result in the Lagrangian for a point-like charge e_a in this external field, which is

$$L_a = -mc^2\sqrt{1-\frac{v_a^2}{c^2}} - e_a\phi(\mathbf{x}_a) + \frac{e_a}{c}\mathbf{v_a}\cdot\mathbf{A}(\mathbf{x}_a). \tag{5.180}$$

The result (see Landau and Lifshitz, Vol. II (1979), Sections 65 and 75) is that, up to second in v/c, the Lagrangian is conservative, i.e. depends only on the positions and velocities of the particles, and reads

$$L = \sum_a \frac{1}{2}m_a v_a^2 + \frac{m_a v_a^4}{8c^2} - \sum_{a>b}\frac{e_a e_b}{r_{ab}}\left\{1 - \frac{1}{2c^2}[\mathbf{v}_a\cdot\mathbf{v}_b + (\mathbf{v}_a\cdot\hat{\mathbf{r}}_{ab})(\mathbf{v}_b\cdot\hat{\mathbf{r}}_{ab})]\right\}. \tag{5.181}$$

(The analogous result for the gravitational field was given in eqs. (5.55) and (5.56).) To this order only even powers of v/c enter, and the term linear in v/c vanishes because of charge conservation. Indeed, were it not for electromagnetic-wave emission, only even powers of v/c could appear, for the same argument based on time-reversal that we discussed on page 239 for the gravitational field. Starting from $O(v^3/c^3)$ however, we have a non-vanishing contribution to the expansion of the potentials $\phi, \mathbf{A}$, which therefore must be the sought-for radiation-reaction term. Indeed, expanding ϕ and $\mathbf{A}$ to this order, one finds that the corresponding electric field is

$$\mathbf{E} = \frac{2}{3c^3}\dddot{\mathbf{d}}, \tag{5.182}$$

where $\mathbf{d} = \sum_a e_a \mathbf{x}_a$ is the electric dipole moment of the system, while the corresponding magnetic field vanishes. This electric field exerts on a charge e_a a force $\mathbf{F}_a = e_a\mathbf{E}$, so the total work performed on all the charges is

$$\sum_a \mathbf{F}_a \cdot \mathbf{v}_a = \frac{2}{3c^3}\dddot{\mathbf{d}} \cdot \sum_a e_a \mathbf{v}_a \,. \tag{5.183}$$

Taking the time average and integrating by part, we get

$$\langle \sum_a \mathbf{F}_a \cdot \mathbf{v}_a \rangle = -\frac{2}{3c^3}\,\langle \ddot{\mathbf{d}}^2 \rangle \,. \tag{5.184}$$

This is just the negative of the energy radiated away in electromagnetic waves in the dipole approximation. We see that the work done by the radiation reaction force, computed from a near-zone expansion, matches exactly the energy carried by the radiation field at future null infinity.

Radiation reaction from GWs

In Section 4.1.3 we computed the effect of GW emission on the orbit of a binary system simply requiring that the energy and angular momentum carried away from the GWs at a given time, were drained from the orbital energy and angular momentum of the source at the corresponding value of retarded time. This is unavoidable in linearized theory, since energy and angular momentum must be conserved (and, for compact bodies, we will see that their internal structure influences the dynamics only starting from 5PN order, see page 288, so to order smaller than 5PN there is no internal degree of freedom that can relax, supplying the required energy). However in the full non-linear theory, given the non-linear phenomena in the propagation of the GWs from the source to infinity that we have discussed, it is no longer obvious that the energy and angular momentum carried away by GWs at a large distance r and time t, are balanced by losses of the system at the corresponding retarded time $t - r/c$.

Anyway, we have by now all the tools necessary to verify explicitly the correctness of this energy-balance argument, since we have in principle determined, in an expansion in v/c, both the radiation field at infinity, and the metric in the near region. The latter determines the equation of motion of the matter source. For a binary system, the equation of motion takes the form already schematically written down in eq. (5.68). The terms $O(\epsilon^2)$ and $O(\epsilon^4)$ in eq. (5.68) are the 1PN and 2PN corrections, respectively, and are non-dissipative (the Lagrangian giving the equations of motion up to $O(\epsilon^2)$ was explicitly written down in eqs. (5.55) and (5.56)). They describe various general relativistic corrections to the orbit, such as the periastron advance, etc.

We now want to find the leading term in the back-reaction, i.e. the first term which is odd in v (and that cannot be set to zero with a gauge transformation). This leading term can obtained by replacing $\tau_{\mu\nu}$ with the energy–momentum tensor of matter $T^{\alpha\beta}$. Then, eq. (5.132) becomes simply

$$\mathsf{h}^{\alpha\beta}(t,\mathbf{x}) = -\frac{4G}{c^4}\int d^3x' \,\frac{1}{|\mathbf{x}-\mathbf{x}'|}\,T^{\alpha\beta}(t - |\mathbf{x}-\mathbf{x}'|/c, \mathbf{x}') \,. \tag{5.185}$$

Since $T^{\alpha\beta}$ has compact support, we can expand its argument $t - |\mathbf{x}-\mathbf{x}'|/c$ in powers of $|\mathbf{x}-\mathbf{x}'|/c$, just as we did for electromagnetism, and the resulting integrals are convergent.[49] One then finds that a number of lowest-order term vanish because of mass and momentum conservation, or they can be set to zero with a gauge transformation. The first non-vanishing terms which are odd under time-reversal and cannot be gauged away are $O(1/c^7)$ in g_{00}, $O(1/c^6)$ in g_{0i}, and $O(1/c^5)$ in g_{ij}, i.e., they are a 2.5PN correction to the metric (5.3). One can then compute the rate of dissipation of energy due to these non-conservative terms, and one finds that the result reproduces the Einstein formula for the emission of radiation in the quadrupole approximation, in linearized theory, eq. (3.75). This computation was first performed by Chandrasekhar and Esposito (1970).

[49] Since the integrand has compact support, the finite part prescription is unnecessary. Observe also that the homogeneous term given by eqs. (5.133) and (5.141) does not contribute to the leading radiation-reaction term (it contributes only starting from order 4PN).

In the 1970s and early 1980s, however, the subtleties in the use of the PN expansion, that we have discussed at length in this chapter, were not yet fully understood, and the PN expansion was used over all of space, including the far region. As we have seen, this unavoidably produces divergences in higher order. So, one had reproduced the correct radiation reaction, but as a term of an expansion in which subsequent terms diverge; not a very satisfying state of affairs. This was at the basis of a controversy over the validity of determining the back-reaction on the sources from the energy balance argument, and over the validity of Einstein quadrupole formula itself, when applied to self-gravitation systems. The issue is particularly important, as we will see in the next chapter, for its application to the change in orbital periods of binary pulsars, which constitutes the first experimental evidence of GW emission.

Nowadays, we know that the correct formalism implies a different treatment of the near- and far-field regions, and the PN result to all order is given by eqs. (5.132), (5.133) and (5.141), and is explicitly finite thanks to the FP prescription, that comes out from a correct use of the formalism. Thus, the 2.5PN radiation reaction is now part of a systematic and well-defined expansion. For a compact binary system this term is responsible for the decrease in the orbital period P_b. The fact that we can compute it directly from the PN expansion in the near region, without invoking any energy balance argument, provides a direct and satisfying way of deriving the theoretical prediction for $\dot{P}_b$. As we will see in Chapter 6, this prediction has been confirmed by the observation in binary pulsars.

A check of the energy balance argument to even higher orders is technically more difficult. We have seen that the quadrupole radiation at infinity corresponds, in the near region, to a 2.5PN correction to the metric. Thus, the 1PN correction to the radiation field corresponds to a 3.5PN correction to the near-region metric. Of course, to check the energy balance argument beyond leading order becomes more and more difficult, since it requires the computation of higher and higher orders in the PN expansion of the near metric. For compact binary, the full near-zone metric has been explicitly computed up to 3.5PN order, and shown to be consistent with the loss of energy and angular momentum at infinity, see the Further Reading. Furthermore, even for the tail integral it has been possible to check explicitly the energy balance argument, and the tail term in the radiation at infinity has been shown to be correctly reproduced, for general PN sources, by a corresponding non-hereditary term in the near-region field. Since the tail integral is a 1.5PN correction to the radiation at infinity, the corresponding hereditary term appears in the 4PN near-region metric.[50] To this order, one finds indeed a hereditary correction to g_{00} given by

[50] Actually, the existence of hereditary terms was first observed in the 4PN near-region metric, see Blanchet and Damour (1988) and later in the radiation field at infinity.

$$\delta g_{00}(t,\mathbf{x}) = -\frac{8}{5c^{10}} x^i x^j \mathrm{M}(t) \int_{-\infty}^{t} dt' \log\left(\frac{t-t'}{2P}\right) \mathrm{M}_{ij}^{(7)}(t'), \quad (5.186)$$

with P a time-scale. We see that it depends on the mass M and on the mass quadrupole M_{ij}, just as the monopole–quadrupole terms $\mathsf{h}^{\alpha\beta}_{(\mathrm{M}\times\mathrm{M}_{ij})}$ computed, in the far region, in eq. (5.164), and it has the same logarithmic singularity at the upper limit of the integral.

The explicit computation shows that the terms in the metric that correspond to back-reaction (i.e. the terms described by antisymmetric waves, see Note 35) can be written as

$$\left(\mathsf{h}_1^{00}\right)_{\text{antisym}} = -\frac{4}{Gc^2} V_{\text{react}}\,, \quad (5.187)$$

$$\left(\mathsf{h}_1^{0i}\right)_{\text{antisym}} = -\frac{4}{Gc^3} V^i_{\text{react}}\,, \quad (5.188)$$

$$\left(\mathsf{h}_1^{ij}\right)_{\text{antisym}} = -\frac{4}{Gc^4} V^{ij}_{\text{react}}\,, \quad (5.189)$$

(plus terms that can be set to zero with a gauge transformation). Up to 4PN order in the near-zone metric (which gives the corrections up to 1.5PN order to the radiation reaction force), the tensor potential V^{ij}_{react} can be neglected, while the scalar potential V_{react} and the vector potential V^i_{react} are given by[51]

[51] See Blanchet (1997), eq. (4.33).

$$V_{\text{reac}}(t,\mathbf{x}) = -\frac{G}{5c^5} x_{ij} \mathrm{I}^{(5)}_{ij}(t) + \frac{G}{c^7}\left[\frac{1}{189} x_{ijk} \mathrm{I}^{(7)}_{ijk}(t) - \frac{1}{70}\mathbf{x}^2 x_{ij} \mathrm{I}^{(7)}_{ij}(t)\right]$$
$$-\frac{4G^2\mathrm{M}}{5c^8} x_{ij} \int_0^{+\infty} d\tau \log\left(\frac{\tau}{2}\right) \mathrm{I}^{(7)}_{ij}(t-\tau) + O\left(\frac{1}{c^9}\right), \quad (5.190)$$

$$V^i_{\text{reac}}(t,\mathbf{x}) = \frac{G}{c^5}\left[\frac{1}{21}\hat{x}_{ijk}\mathrm{I}^{(6)}_{jk}(t) - \frac{4}{45}\epsilon_{ijk} x_{jm} \mathrm{J}^{(5)}_{km}(t)\right] + O\left(\frac{1}{c^7}\right). \quad (5.191)$$

To 2.5PN order in the metric, only the term proportional to $\mathrm{I}^{(5)}_{ij}$ in eq. (5.190) contributes while the terms proportional to $1/c^7$ in eq. (5.190) are a 3.5PN correction, and the tail integral in V_{reac} is a 4PN correction. Similarly, taking into account the factor $1/c^3$ in eq. (5.188), and comparing with eq. (5.3), we see that V^i_{reac} is a 3.5PN correction. Thus, to 2.5PN order, the result reduces to the Burke-Thorne potential that we already discussed in eq. (3.114) (since I^{ij}, to lowest order in v/c, reduces to the quadrupole moment Q^{ij} of linearized theory).

The energy loss can be computed directly in terms of the potentials V_{reac} and V^i_{reac}, and fully agrees with the energy loss computed from the radiative field at infinity.[52] It is interesting to observe that the vector component V_i of the radiation-reaction potential is responsible for the loss of linear momentum, i.e. for the recoil of the center-of-mass of the source due to GW emission, and balances exactly the value that can be found computing the flux of linear momentum at infinity.

[52] See again Blanchet (1997).

5.4 The DIRE approach

We now discuss an approach due to Will, Wiseman and Pati and termed DIRE (Direct Integration of the Relaxed Einstein Equation) by the authors. This formalism is similar in spirit but different in technical details from the Blanchet–Damour approach. It can be proved, however, that the two formalisms are completely equivalent. For reasons of space we will limit ourselves to a brief description, referring the reader to Pati and Will (2000) for more details.

The basic strategy of this method is to start from eq. (5.83), and to iterate the solution in a slow-motion ($v \ll c$), weak-field ($|\mathsf{h}_{\mu\nu}| \ll 1$) approximation, in order to obtain in a systematic way the corrections to linearized theory. To zeroth order we set $\mathsf{h}_{\mu\nu} \equiv {}^{(0)}\mathsf{h}_{\mu\nu} = 0$ over all space-time, which means that ${}^{(0)}g_{\mu\nu} = \eta_{\mu\nu}$. If we denote by ${}^{(N)}\mathsf{h}_{\mu\nu}$ the result of the N-th iteration and by ${}^{(N)}\tau^{\alpha\beta}$ the value of $\tau^{\alpha\beta}$ when $\mathsf{h}_{\mu\nu} = {}^{(N)}\mathsf{h}_{\mu\nu}$, then the iterative rule is

$${}^{(N+1)}\mathsf{h}^{\alpha\beta}(t,\mathbf{x}) = -\frac{4G}{c^4}\int d^4x' \,\frac{{}^{(N)}\tau^{\alpha\beta}(t',\mathbf{x}')\delta(t'-t+|\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}. \quad (5.192)$$

The equation of motion of matter is then obtained imposing the N-th iteration of the De Donder gauge condition (5.71), $\partial_\beta\,{}^{(N)}\mathsf{h}^{\alpha\beta} = 0$.

Setting ${}^{(0)}\mathsf{h}_{\mu\nu} = 0$, we have ${}^{(0)}\tau^{\alpha\beta} = T^{\alpha\beta}$, and the first iteration ${}^{(1)}\mathsf{h}_{\mu\nu}$ gives back the result of linearized theory, eq. (3.8). Since ${}^{(0)}\tau^{\alpha\beta} = T^{\alpha\beta}$ has compact support, the integral is well defined and no divergence appears at this stage. The first PN correction is obtained computing ${}^{(2)}\mathsf{h}^{\alpha\beta}$. This requires plugging ${}^{(1)}\mathsf{h}^{\alpha\beta}$ in the expression for $\tau^{\alpha\beta}$. Since ${}^{(1)}\mathsf{h}^{\alpha\beta}$ already includes GWs propagating to infinity, now the source is no longer restricted to a compact region, and one must be careful in

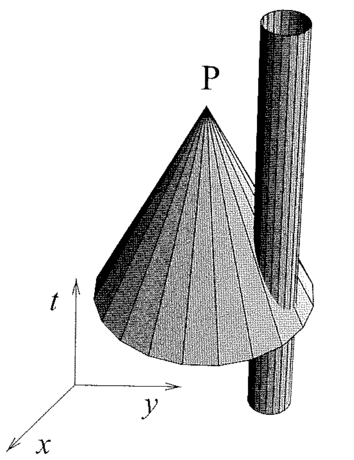

Fig. 5.2 The past light cone of the point P, and the cylinder which bounds the near zone $\mathcal{D}$. Here the point P is in the far region.

handling the integral.

To compute the right-hand side of eq. (5.192) we proceed as follows. We consider a bound system, whose center of mass is taken to be at the origin of the coordinate system, and whose radial extension is always smaller or equal than a value d. We define the *source zone* as the world tube

$$\mathcal{T} = \{x^\alpha | r < d, -\infty < t < \infty\}. \tag{5.193}$$

Outside $\mathcal{T}$ the energy–momentum tensor of matter vanishes, $T^{\alpha\beta} = 0$. We next introduce the length-scale $\mathcal{R} = (c/v)\, d$. The near zone is defined by the world tube

$$\mathcal{D} = \{x^\alpha | r < \mathcal{R}, -\infty < t < \infty\}, \tag{5.194}$$

while the radiation zone is defined as the region at $r > \mathcal{R}$.[53] The Dirac delta in eq. (5.192) tells us that, in order to compute the field ${}^{(N+1)}\mathsf{h}^{\alpha\beta}$ at a point $P = (t, \mathbf{x})$, we must integrate over the past flat-space null cone of P. As we see from Fig. 5.2, this past null cone intersects the world tube $\mathcal{D}$ in a hypersurface $\mathcal{N}$ (which of course is a three-dimensional hypersurface, but in Fig. 5.2 we suppressed one spatial dimension). In the figure we illustrated the situation in which P is in the far zone. In general, we wish to compute ${}^{(N+1)}\mathsf{h}^{\alpha\beta}$ both when P is in the far zone (since this allows us to compute the GWs emitted) and when P is in the near zone, since this determines the equation of motion of matter. A picture similar to Fig. 5.2 can be drawn when the tip of the cone, P, is inside the cylinder that (in this simplified 2+1 dimensional picture) bounds $\mathcal{D}$. For definiteness, in the following we specialize to the case where P is in the far region.

The integration over the past null cone $\mathcal{C}$ can then be split into an integration over $\mathcal{N}$ (defined as the part of the null cone which is in the near region) and an integration over the remainder $\mathcal{C} - \mathcal{N}$. The integration over $\mathcal{N}$ gives, carrying out first the integration over t',

$${}^{(N+1)}\mathsf{h}^{\alpha\beta}_{\mathcal{N}}(t, \mathbf{x}) = -\frac{4G}{c^4} \int_{\mathcal{N}} d^3x' \, \frac{{}^{(N)}\tau^{\alpha\beta}(t - |\mathbf{x} - \mathbf{x}'|/c, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}. \tag{5.195}$$

Within $\mathcal{N}$ the integration variable $\mathbf{x}'$ satisfies $|\mathbf{x}'| < \mathcal{R}$ while, since P is in the far zone, we have $r \equiv |\mathbf{x}| > \mathcal{R}$. Therefore we can expand the $\mathbf{x}'$ dependence, in both occurrences of $|\mathbf{x} - \mathbf{x}'|$ in the integrand, in powers of $|\mathbf{x}'|/r$. This gives

$${}^{(N+1)}\mathsf{h}^{\alpha\beta}_{\mathcal{N}}(t, \mathbf{x}) = -\frac{4G}{c^4} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \, \partial_L \left[\frac{1}{r} M^{\alpha\beta L}(u) \right], \tag{5.196}$$

where

$$M^{\alpha\beta L}(u) = \int_{\mathcal{M}} d^3x' \, \tau^{\alpha\beta}(u, \mathbf{x}') x'^L. \tag{5.197}$$

Here $u = t - r/c$, and $\mathcal{M}$ is the intersection of the near-zone world-tube $\mathcal{D}$ with the constant-time hypersurface $t_{\mathcal{M}} = u$. The integral is therefore expressed in terms of the multipole moments of $\tau^{\alpha\beta}$, and is explicitly

[53]This is slightly different from the definition of $\mathcal{R}$ that we used discussing the Blanchet–Damour formalism, where we preferred to keep $\mathcal{R}$ (much) smaller than $(c/v)d$, to make sure that the expansion parameter in the near zone is much smaller than one. Here however $\mathcal{R}$ is a formal parameter that separates the integral into an inner and an outer part, and whose cancellation, when we resum the two parts, will be checked explicitly, see below, so its precise value is irrelevant. For definiteness we assign it the value $(c/v)d$, following Pati and Will (2000).

convergent because the region $\mathcal{M}$ is bounded. For GWs, we are interested only in the spatial components h^{ij}, and in the term decreasing as $1/r$, so we can bring the factor $1/r$ in eq. (5.196) outside the derivatives. Using

$$\partial_i M(u) = \frac{\partial u}{\partial x^i} \frac{dM}{du} = -n^i \frac{dM}{du}, \tag{5.198}$$

where n^i is the unit vector in the observation direction, and $dM(u)/du = dM/dt$, for GWs eq. (5.196) gives

$${}^{(N+1)}\mathsf{h}^{ij}_{\mathcal{N}}(t, \mathbf{x}) = -\frac{4G}{rc^4} \sum_{l=0}^{\infty} \frac{1}{m!} \frac{\partial^l}{\partial t^l} \int_{\mathcal{M}} d^3x' \, \tau^{ij}(u, \mathbf{x}')(\hat{\mathbf{n}} \cdot \mathbf{x}')^l. \tag{5.199}$$

The next step is the computation of the outer integral, that is, of the integral over the region $\mathcal{C} - \mathcal{N}$. In this outer region the energy–momentum tensor of matter vanishes, and the only contribution to ${}^{(N)}\tau^{\alpha\beta}$ comes from ${}^{(N)}\mathsf{h}^{\alpha\beta}$. We can therefore compute it using the expression of ${}^{(N)}\mathsf{h}^{\alpha\beta}$, as determined at the previous iteration level. The domain of integration is slightly complicated geometrically, as we see from Fig. 5.2, but the integration region can be expressed in a manageable form with an appropriate change of variables (see Pati and Will 2000 for details).

The original integral over the light cone $\mathcal{C}$, eq. (5.192), was of course independent of $\mathcal{R}$, which is an arbitrary constant that we have introduced to split it into two pieces, one at $r < \mathcal{R}$ and one at $r > \mathcal{R}$. In contrast, the inner and outer integrals for $\mathsf{h}^{\alpha\beta}$ depend separately on the radius $\mathcal{R}$, and in particular they are divergent when $\mathcal{R} \to \infty$. Observe in fact that the contribution of $\mathsf{h}^{\alpha\beta}$ to $\tau^{\alpha\beta}$ falls off at large distances as some power of r. Thus, if $\mathcal{R}$ is taken to infinity, for sufficiently high values of l, i.e. for large multipoles, the integral in eq. (5.199) diverges. This is in fact the divergence to be expected if we try to extrapolate the PN expansion to the far region, as we discussed in Section 5.1.6. The correct procedure, then, is to keep $\mathcal{R}$ finite, and show that these $\mathcal{R}$-dependent terms cancel against similar contributions from the remainder of the null cone, $\mathcal{C} - \mathcal{N}$. This cancellation has indeed been proved explicitly to 2PN order (for terms proportional to positive powers of $\mathcal{R}$) in Wiseman and Will (1996) and to all orders, with a proof by induction, in Pati and Will (2000). This cancellation is of course inevitable, but it has a very important practical consequence. When computing for instance the inner integral, one will generally find terms which are independent of $\mathcal{R}$ as well as terms that depend on $\mathcal{R}$ (as a power, or logarithmically). However, we know that all $\mathcal{R}$-dependent terms must cancel against similar terms from the outer integral, so in the computation one can simply drop them, and retain only the $\mathcal{R}$-independent contributions. Similarly, one can drop all $\mathcal{R}$-dependent terms in the outer integral.

Thus, one finally has a manifestly finite and well-defined procedure for computing systematically all higher-order corrections to linearized theory.

5.5 Strong-field sources and the effacement principle

Until now we have assumed that the gravitational field is never strong. Our final aim, however, is to apply this formalism to systems containing compact objects such as neutron stars or black holes, in particular to a compact binary system which is slowly inspiraling, with an orbital velocity $v \ll c$. For a binary system, the quantity d that determines the characteristic source size is the typical orbital separation, and it satisfies $Gm/d \sim (v/c)^2$, where m is the total mass of the system. We denote instead by r_0 the characteristic size of the two stars (assumed for notational simplicity to be comparable). Since we are considering the slow inspiral phase, we are in the regime $d \gg r_0$.

Using for definiteness the Blanchet–Damour formalism, the three basic ingredients are: the post-Minkowskian expansion in the outer source region, the post-Newtonian expansion in the near region, and the fact that there is an intermediate region where we can match the two expansions. Even for a binary made of compact objects, when we are at a distance r (measured from the center-of-mass of the system) of order of a few times the orbital separation d, gravity is already sufficiently weak. Then, from say $r = 1.5d$ up to $r = \infty$ the post-Minkowskian expansion is justified, even for compact binaries containing neutron stars or black holes. Furthermore, when $v \ll c$, the near zone extends up to distances $\mathcal{R} \gg d$, and we have at our disposal a wide region where we can perform the matching to the PN solution. So the issue is whether, with the methods that we have discussed, we can reliably compute the PN solution in the near region $r < \mathcal{R}$. Even in most of the near region the gravitational field is weak; however, within two balls centered on the two stars, of radius of order a few times r_0, see Fig. 5.3, the gravitational field becomes strong. It is therefore unclear whether the PN expansion is applicable.

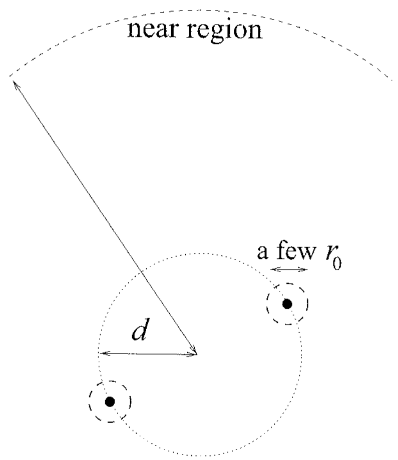

Fig. 5.3 The near region of a compact binary system. Within two balls, centered on the two stars, of radius of order a few times r_0, the gravitational field is strong.

We will see in this section that, in spite of this legitimate concern, the formalism that we have developed is indeed valid for compact objects. We will also estimate the effect of the internal structure of a compact object, and we will see that it shows up in the equations of motion only at the very high 5PN order, well beyond the accuracy of existing computations. This remarkable fact can be traced to a property of general relativity, which has been termed "the effacement of the internal structure". Since the same phenomenon takes place in Newtonian gravity, we begin with a discussion of this simpler case.[54]

[54] We follow closely Damour (1987).

Effacement of the internal structure in Newtonian gravity

Treating the bodies as perfect fluids, a N-body system in Newtonian gravity is described by the velocity field of the fluid, $v^i(t, \mathbf{x})$, and by the mass density $\rho(t, \mathbf{x})$, which is subject to the constraint to have a compact support consisting of N non-overlapping connected regions. We denote by p the pressure, and we assume an equation of state $p = p(\rho)$. The dynamics is governed by the continuity equation,

$$\partial_t \rho + \partial_i(\rho v_i) = 0\,, \tag{5.200}$$

by the Euler equation

$$\rho(\partial_t v^i + v^j \partial_j v^i) = -\partial_i p + \rho \partial_i U\,, \tag{5.201}$$

where U is the sign-reversed gravitational potential (therefore $U > 0$), and by Poisson equation

$$\boldsymbol{\nabla}^2 U = -4\pi G \rho\,. \tag{5.202}$$

We denote by V_a the volume occupied by the a-th body. By definition, on its boundary ∂V_a we have $\rho = p = 0$. The mass of the a-th body is given by

$$m_a = \int_{V_a} d^3x\, \rho(t, \mathbf{x})\,, \tag{5.203}$$

and is a constant, thanks to the continuity equation. The center-of-mass coordinates of the a-th body are defined by

$$z_a^i(t) = \frac{1}{m_a} \int_{V_a} d^3x\, x^i \rho(t, \mathbf{x})\,. \tag{5.204}$$

Differentiating twice with respect to time and using eqs. (5.200) and (5.201) we get

$$m_a \frac{d^2 z_a^i}{dt^2} = \int_{V_a} d^3x\, f_i\,, \tag{5.205}$$

with a force density

$$f_i = -\partial_i p + \rho \partial_i U\,. \tag{5.206}$$

The potential U is obtained solving eq. (5.202) with the boundary condition that it vanishes at infinity,

$$\begin{aligned} U(t, \mathbf{x}) &= G \int d^3x'\, \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \\ &= \sum_{a=1}^{N} G \int_{V_a} d^3x'\, \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\,, \end{aligned} \tag{5.207}$$

where, in the second line, we made use of the fact that ρ is non-vanishing only on the volumes V_a, with $a = 1, \ldots N$. The potential acting on the a-th body can therefore be split into a "self-part",

$$U^{(\mathrm{self}),a}(t, \mathbf{x}) = G \int_{V_a} d^3x'\, \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\,, \tag{5.208}$$

and an "external part",

$$U^{(\mathrm{ext}),a}(t, \mathbf{x}) = G \sum_{b \neq a} \int_{V_b} d^3x'\, \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\,. \tag{5.209}$$

Correspondingly, the force acting on the a-th body is decomposed into a self-force

$$F_i^{(\mathrm{self}),\mathrm{a}} = \int_{V_a} d^3x \, [-\partial_i p + \rho \partial_i U^{(\mathrm{self}),\mathrm{a}}] \,, \tag{5.210}$$

(in which we also included the pressure term) and an external force

$$F_i^{(\mathrm{ext}),\mathrm{a}} = \int_{V_a} d^3x \, \rho \partial_i U^{(\mathrm{ext}),\mathrm{a}} \,. \tag{5.211}$$

If one were to make a naive estimate of these two forces, based solely on dimensional analysis, one would write (assuming for definiteness that all bodies have comparable masses m) $F^{(\mathrm{self}),\mathrm{a}} \sim Gm^2/r_0^2$ and $F^{(\mathrm{ext}),\mathrm{a}} \sim Gm^2/d^2$, where r_0 is the typical body size and d is the distance between the nearest bodies. Since $d \gg r_0$, the self-force would be much larger than the external force. However, in Newtonian gravity the self-force vanishes exactly. The pressure term in eq. (5.210) vanishes because it is the integral of a gradient, and on the boundary $p = 0$. The second term, using eq. (5.208), is

$$\begin{aligned} F_i^{(\mathrm{self}),\mathrm{a}} &= G \int_{V_a} d^3x \, \rho(t,\mathbf{x}) \frac{\partial}{\partial x^i} \int_{V_a} d^3x' \, \frac{\rho(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \\ &= -G \int_{V_a} d^3x \int_{V_a} d^3x' \, (x-x')^i \frac{\rho(t,\mathbf{x})\rho(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} \,. \end{aligned} \tag{5.212}$$

The integrand is odd under the exchange of x with x' while the integration domain is symmetric under this exchange, so the integral vanishes, and there is no self-force. It is however worth observing that the two factors $\rho(t,\mathbf{x})$ and $\rho(t,\mathbf{x}')$ entered the above integral in two conceptually distinct way. One factor, which is the one explicitly written in the second term in eq. (5.206), is really the density of "passive gravitational mass", which measures the response of matter to an external gravitational field. The second factor, which enters through eq. (5.207), is the density of the "active gravitational mass", which is the source for the gravitational field. The fact that these two densities are in fact equal is crucial to the vanishing of the integral in eq. (5.212). Thus, the vanishing of the self-force in Newtonian gravity is a non-trivial result, rooted in the equality between active and passive gravitational mass.

In the equation of motion (5.205), the first term that depends on the internal structure of the bodies, and not just on their masses, is then obtained performing a multipole expansion of the external force. We introduce a coordinate[55]

$$\mathbf{y} = \mathbf{x} - \mathbf{z}_a(t) \,, \tag{5.213}$$

centered on the a-th body. Since we have shown that the self-force vanishes, eq. (5.205) becomes

$$m_a \frac{d^2 z_a^i}{dt^2} = \int_{V_a} d^3y \, \rho(t, \mathbf{z}_a(t) + \mathbf{y}) \partial_i U^{(\mathrm{ext}),\mathrm{a}}(t, \mathbf{z}_a(t) + \mathbf{y}) \,. \tag{5.214}$$

[55] In principle $\mathbf{y}$ should be written as $\mathbf{y}_a$. However, we will use it as an integration variable, so its dependence on a will only appear through the integration domain, and we omit its index a.

The density of the a-th body is localized around $\mathbf{y} = 0$, so it is convenient to introduce the notation $\rho_a(t,\mathbf{y}) \equiv \rho(t, \mathbf{z}_a(t) + \mathbf{y})$, while the external potential at $\mathbf{z}_a(t) + \mathbf{y}$ can be expanded around the value at the point $\mathbf{z}_a(t)$, plus small corrections. Thus, the multipole expansion is obtained writing

$$\begin{aligned} \partial_i U^{(\mathrm{ext}),\mathrm{a}}(t, \mathbf{z}_a + \mathbf{y}) = [&\partial_i U^{(\mathrm{ext}),\mathrm{a}} + y^j \partial_i \partial_j U^{(\mathrm{ext}),\mathrm{a}} \\ &+ \frac{1}{2} y^j y^k \partial_i \partial_j \partial_k U^{(\mathrm{ext}),\mathrm{a}} + \ldots](t, \mathbf{z}_a) \,. \end{aligned} \tag{5.215}$$

Inserting this expansion into eq. (5.214) we get

$$\begin{aligned} m_a \frac{d^2 z_a^i}{dt^2} = [&m_a \partial_i U^{(\mathrm{ext}),\mathrm{a}} + I_a^j \partial_i \partial_j U^{(\mathrm{ext}),\mathrm{a}} \\ &+ \frac{1}{2} I_a^{jk} \partial_i \partial_j \partial_k U^{(\mathrm{ext}),\mathrm{a}} + \ldots](t, \mathbf{z}_a) \,, \end{aligned} \tag{5.216}$$

where

$$I_a^j = \int_{V_a} d^3y \, \rho_a(t,\mathbf{y}) y^j \,, \tag{5.217}$$

$$I_a^{jk} = \int_{V_a} d^3y \, \rho_a(t,\mathbf{y}) y^j y^k \,, \tag{5.218}$$

and so on. However, the dipole in eq. (5.217) vanishes identically because of the definitions (5.204) and (5.213). In eq. (5.216) we can replace I_a^{jk} by the quadrupole moment $Q_a^{jk} = I_a^{jk} - (1/3)\delta^{jk} I_a^{ll}$, since the factor δ^{jk}, when contracted with $\partial_i \partial_j \partial_k U^{(\mathrm{ext}),\mathrm{a}}$ in eq. (5.215), produces $\partial_i \nabla^2 U^{(\mathrm{ext}),\mathrm{a}}$, and the Laplacian of the external potential is proportional to $\sum_{b \neq a} \rho_b$ and therefore vanishes inside the a-th body. Thus,

$$m_a \frac{d^2 z_a^i}{dt^2} = m_a \partial_i U^{(\mathrm{ext}),\mathrm{a}}(t, \mathbf{z}_a) + \frac{1}{2} Q_a^{jk} \partial_i \partial_j \partial_k U^{(\mathrm{ext}),\mathrm{a}}(t, \mathbf{z}_a) + \ldots \,. \tag{5.219}$$

The monopole term gives Newton's law. The quadrupole term is the first term which depends on the inner structure of the body. Each derivative acting on $U^{(\mathrm{ext}),\mathrm{a}}$ brings a contribution $O(1/d)$, while each factor y^i in the definition of the multipole moments, after integration over V_a, brings a factor $O(r_0)$. Thus, the contribution of I_a^{ij} is smaller than the monopole term by a factor $O(r_0^2/d^2)$. Furthermore, in the quadrupole moment Q_a^{ij}, only the non-spherical part of the matter distribution contributes, and this gives another suppression factor, that we denote by ϵ. In general $\epsilon \leqslant 1$ and, in many cases, $\epsilon \ll 1$. In conclusion, defining

$$\alpha \equiv \frac{r_0}{d} \ll 1 \,, \tag{5.220}$$

the structure-dependent terms give a correction to Newton's law of order $\epsilon\alpha^2$, when a naive dimensional analysis that does not take into account that the self-force vanishes exactly, would have rather suggested that they are larger than the external Newtonian force by a factor $1/\alpha^2$. Overall, the terms that depend on the internal structure of the body are therefore suppressed, with respect to naive expectations, by a factor $\epsilon\alpha^4$. As we have seen, the equality of active and passive gravitational mass is at the origin of this large cancellation, which is known as the "effacement principle".

In order to generalize these results to Einstein gravity, it is useful to observe that there is a suggestive way of rewriting the Newtonian equations of motion, in a form that only involves surface integrals, rather than volume integrals. We start from eqs. (5.205) and (5.206). We neglect the pressure term, since this is a gradient and gives a vanishing contribution, and we use eq. (5.202). Then eq. (5.205) becomes

$$m_a \frac{d^2 z_a^i}{dt^2} = -\frac{1}{4\pi G}\int_{V_a} d^3x\, \nabla^2 U \partial_i U\,, \qquad (5.221)$$

where U is the total potential, including the self-potential. Using the identity

$$\partial_j(\partial_i U \partial_j U - \frac{1}{2}\delta_{ij}\partial_k U \partial_k U) = \nabla^2 U \partial_i U\,, \qquad (5.222)$$

we can rewrite this as

$$\begin{aligned} m_a \frac{d^2 z_a^i}{dt^2} &= \int_{V_a} d^3x\, \partial_j t^{ij} \\ &= \int_{S_a} dS_j\, t^{ij}\,, \end{aligned} \qquad (5.223)$$

where

$$t_{ij} = -\frac{1}{4\pi G}\left(\partial_i U \partial_j U - \frac{1}{2}\delta_{ij}\partial_k U \partial_k U\right)\,, \qquad (5.224)$$

and dS_j is the two-dimensional surface element, on a surface S_a bounding the volume V_a, which is arbitrary except that it does not include any other volume V_b with $b \neq a$. This surface-integral representation of the equation of motion is expecially interesting, since it shows that nothing depends on whether the gravitational field inside the bodies is weak or strong. In principle, one could even have a singularity inside the volume V_a, but the equations of motion can be computed evaluating the "stress tensor" t^{ij} on a surface which is far-away from the body (recall that we assumed $d \gg r_0$, so we can go to distances parametrically larger than r_0 before enclosing any other body), where all fields are weak.

Effacement of the internal structure in Einstein gravity

Just as we have done above for Newtonian gravity, we now show how to write the PN equations of motion of general relativity using surface integrals, using a variant of a classical work by Einstein, Infeld and Hoffmann (1938), developed by Itoh and Futamase. We start from the Einstein equations in relaxed form, in the harmonic gauge. First of all, we want to define the analogous of the center-of-mass coordinates. Observe that in the Newtonian case the center-of-mass coordinates $z_a^i(t)$ are such that the mass dipole I_a^j in eq. (5.217) vanishes. In the general

relativistic case, we define the functions $z_a^i(t)$ considering the dipole moment of τ^{00}

$$D_a^i \equiv \int_{V_a} d^3y\, y^i \tau^{00}(t, \mathbf{z}_a(t) + \mathbf{y})\,, \qquad (5.225)$$

(where $\tau^{\mu\nu}$ is the effective energy–momentum tensor defined in eq. (5.73)) and requiring that D_a^i vanishes or, equivalently, that it takes a specified value. The functions $z_a^i(t)$ generalize the Newtonian notion of center-of-mass coordinates, and can be better called "center-of-fields" coordinates, since τ^{00} include also the contribution to the energy density from the gravitational field. We now define the quantity P_a^μ as

$$P_a^\mu(t) = \int_{V_a} d^3y\, \tau^{0\mu}(t, \mathbf{z}_a(t) + \mathbf{y})\,, \qquad (5.226)$$

so this is an effective four-momentum of the a-th body, which includes also the contribution of the gravitational field. Using the conservation of $\tau^{\mu\nu}$, eq. (5.82), and the notation $v_a^i = \dot{z}_a^i$, we find

$$\begin{aligned} \frac{dP_a^\mu}{dt} &= \int_{V_a} d^3y\, [\partial_0 \tau^{0\mu} + v_a^i \partial_i \tau^{0\mu}](t, \mathbf{z}_a(t) + \mathbf{y}) \\ &= \int_{V_a} d^3y\, [-\partial_i \tau^{i\mu} + v_a^i \partial_i \tau^{0\mu}](t, \mathbf{z}_a(t) + \mathbf{y})\,. \end{aligned} \qquad (5.227)$$

Since v_a^i is just a function of time, independent of y, we can carry it outside the integral and we are left with a total derivative. Hence, the variation of P^μ is given by a surface integral,

$$\frac{dP_a^\mu}{dt} = -\int_{S_a} dS_j \tau^{j\mu} + v_a^j \int_{S_a} dS_j \tau^{0\mu}\,. \qquad (5.228)$$

A relation between the "momentum" P_a^i, the "energy" P_a^0, and the velocity v_a^i of the a-th body can be obtained by taking the time derivative of eq. (5.225). On the left-hand side we get zero since D_a^i vanishes (or is anyway a constant), by definition of "center-of-fields" coordinates. On the right-hand side we use the conservation of $\tau^{\mu\nu}$ and we integrate by parts, keeping the boundary terms. Then we obtain

$$P_a^i = P_a^0 v_a^i + Q_a^i\,, \qquad (5.229)$$

where

$$Q_a^i = \int_{S_a} dS_j\, y_a^i \tau^{j0} - v_a^j \int_{S_a} dS_j\, y_a^i \tau^{00}\,. \qquad (5.230)$$

Finally, taking the time derivative of eq. (5.229) and using eq. (5.228) to compute dP_a^i/dt and dP_a^0/dt, we arrive at an equation for dv_a^i/dt,

$$\begin{aligned} P_a^0 \frac{dv_a^i}{dt} = &-\int_{S_a} dS_j \tau^{ji} + v_a^j \int_{S_a} dS_j \tau^{0i} + v_a^i \int_{S_a} dS_j \tau^{j0} \\ &- v_a^i v_a^j \int_{S_a} dS_j \tau^{00} - \frac{dQ_a^i}{dt}\,. \end{aligned} \qquad (5.231)$$

This is an equation of motion for the a-th body. The remarkable point is that it is written entirely in terms of surface integrals. On the right-hand side this is explicit, while the quantity P_a^0 on the left-hand side can be obtained by integrating the $\mu = 0$ component of eq. (5.228), with the initial condition that, when $v/c \to 0$, $P_a^0 \to m_a c^2$, where m_a is the (ADM) mass of the body, so even P_a^0 is determined by surface integrals.

In conclusion, even if somewhere inside the volumes V_a the gravitational field becomes strong, as is the case for neutron stars, or even if there is a horizon, as for black holes, the evaluation of the equation of motion (5.231) can be done on surfaces far from the sources, at a distance smaller, but of the order of the separation d between the bodies, say at $r = d/3$. All these surface integrals therefore only involve weak fields. In other words, we have been able to replace the knowledge of the detailed internal structure of the source with a knowledge of the gravitational field at large distances from it. The equation of motions are the same, independently of whether a given value of the surface integral, computed say at a distance $r = d/3$, was produced by a very relativistic source with strong self-gravity or by a nearly Newtonian source with negligible self-gravity, spread over a larger volume V_a.[56] Then, the PN expansion is applicable even to strong field sources.[57] Since the surface integral formulation is just an equivalent way of recasting the equations of motion derived from Einstein equations, it follows that even the PN expansion in its original formulation, discussed in Section 5.1, is valid for strong-field sources.

The computation of the PN expansion with the surface integral method has been performed explicitly, up to 3PN order, by Itoh and Futamase, see the Further Reading section, and the results are in full agreement with that found with the direct PN expansion, with the important added bonus that this computation shows explicitly that the result is valid for strong fields.

[56] Observe that this effacement principle is valid in general relativity, but not necessarily in some of its extensions. For instance, in the factor $\epsilon\alpha^2$ that we found in the Newtonian limit, the suppression factor ϵ is related to the fact that the gravitational field describes a massless particle with helicities ± 2, which forces gravitons to couple to the quadrupole moment. In extensions of general relativity that include gravitationally interacting scalar fields this suppression factor is absent, since the scalar couples to the trace of I_{ij}. Furthermore, in certain scalar-tensor theories, the local value of Newton's constant G is controlled by the local value of the scalar field. In this case, the inner structure of a body is affected by the presence of a companion, which modifies the value of G inside the first body. Thus, in this strong form, the effacement principle is really a properties specific to general relativity.

[57] After one has found a suitable regularization of the point-particle singularity, which is a non-trivial issue, see the Further Reading.

Structure-dependent corrections in compact binaries

We are now in the position to estimate the post-Newtonian order at which corrections that depend on the internal structure of the body show up in the equations of motion of an inspiraling binary. The physical effect that induces a correction in the equations of motion is the fact that the tidal force exerted by the first body distorts the second body, inducing in it a quadrupole moment. The interaction between this quadrupole moment and the first body produces a force, which modifies the orbit.

The above discussion shows that, using the surface integral method, it is possible to perform the computation staying always in a Newtonian weak-field regime, so we expect that a simple estimate of this tidal effect based on a Newtonian description should give the correct order of magnitude. In Newtonian gravity, the tidal force exerted by a body at a distance d on a body of radius r_0 is of order

$$F_{\rm tidal} \sim \frac{Gmr_0}{d^3}\,. \tag{5.232}$$

The typical ellipticity ϵ induced by such a force is of order of the ratio of this tidal force to the typical self-gravity,

$$F_{\rm self} \sim \frac{Gm}{r_0^2}\,. \tag{5.233}$$

Therefore

$$\epsilon \sim \frac{F_{\rm tidal}}{F_{\rm self}} \sim \alpha^3\,, \tag{5.234}$$

where $\alpha = r_0/d$. The corresponding induced quadrupole moment is $Q_{ij} \sim \epsilon m r_0^2$. According to eq. (5.219), with $m\partial_k U^{({\rm ext}),a} \sim F_k^{(\rm Newton)}$ (where $F_k^{(\rm Newton)}$ is the Newton gravitational force) this produces a structure-dependent interbody force of order

$$F^{(\rm induced)} \sim \frac{Q_{ij}}{m}\partial_i\partial_j F^{(\rm Newton)} \sim \epsilon r_0^2\,\frac{1}{d^2} F^{(\rm Newton)}\,. \tag{5.235}$$

From eq. (5.234), we then find

$$F^{(\rm induced)} \sim \alpha^5 F^{(\rm Newton)}\,. \tag{5.236}$$

On the other hand, for a compact body we have $r_0 \sim Gm/c^2$, and therefore $\alpha \sim Gm/(c^2 d)$. From the virial theorem, $(Gm/d) \sim v^2$, so $\alpha \sim (v^2/c^2)$. In conclusion, for compact bodies,

$$F^{(\rm induced)} \sim \left(\frac{v}{c}\right)^{10} F^{(\rm Newton)}\,. \tag{5.237}$$

A full relativistic analysis indeed confirms this estimate. So, the first structure-dependent term gives a 5PN effect in the equation of motion. This is well beyond the present state-of-the art which, as we discussed, is the 3.5PN order.

In conclusion (after one has found a consistent regularization of the point-particle singularity) the PN formalism can be legitimately applied even in the presence of strong fields, and the corrections dependent on the internal structure can be neglected up to the extremely high 5PN order. As long as the two bodies are far from the merging stage (hence v/c is not too close to one), these 5PN effects can be neglected, and both the orbital motion and the GWs that are produced are determined uniquely by the masses of the bodies, independently of whether the internal structure is highly relativistic or almost Newtonian.[58]

[58] In our discussion we have considered spinless bodies. Otherwise, the orbital motion and the GW generation is determined by two parameters for each body, its mass and its intrinsic angular momentum. Both can be obtained measuring the gravitational field at large distances.

5.6 Radiation from inspiraling compact binaries

The most important application of the above formalism is to the inspiral of compact binaries. We already saw in Section 4.4.1, in the context of linearized theory, that a binary system gradually spirals inward because of the emission of GWs, and the resulting waveform increases in amplitude and in frequency, producing a characteristic "chirp". This long inspiral phase is followed by a phase in which the two object plunge toward each other, and merge. The resulting system, typically a black hole, finally settles down to its ground state, radiating away the energy stored in its excited modes. This is the so-called "ringdown phase". So, the evolution of a compact binary system can be separated into these three phases: inspiral, merging, and ringdown. The merging phase is particularly difficult to model, and here the detailed nature of the source (e.g. whether we have black holes or neutron stars) is also important. The merger and ringdown phases will be analyzed in detail in Vol. 2, which is devoted to the issues in GW physics which depends on the specific nature of the source. Here we will rather discuss the inspiral phase, which is universal, at least up to a very high PN order.

5.6.1 The need for a very high-order computation

The reason why the computation of the waveform to a very high PN order is crucial, is that GW experiments are hunting for signals which are buried in a noise orders of magnitudes larger than the signal itself. To extract such a small signal from the noise there exists a standard technique, called matched filtering, that we will discuss in great detail in Chapter 7, that works if we know well the form of the signal.

For an inspiraling compact binary, we saw in eq. (4.23) that, in a ground-based interferometer, the signal enters into the detector bandwidth, say at $f_{\rm min} \sim 10$ Hz, about 17 minutes before the coalescence, and the signal sweeps up in frequency, performing a vary large number of cycles,[59] before the two stars merge. In order to exploit optimally the signal present in the detector, and therefore to detect sources at farther distances, we need to have an accurate theoretical prediction of the time evolution of the waveform, and especially of the phase, which is rapidly changing. To understand how stringent is this requirement, and also to write compactly the PN corrections, it is useful first of all to introduce some new notation. In place of the source frequency ω_s, we introduce the dimensionless variable

[59] For instance $O(10^4)$ cycles in the case of two neutron stars with $m_1 = m_2 = 1.4M_{\odot}$.

$$x \equiv \left(\frac{Gm\omega_s}{c^3}\right)^{2/3}, \tag{5.238}$$

where $m = m_1 + m_2$ is the total mass of the system and $\omega_s = 2\pi f_s$ is the orbital frequency of the source. Writing $x = [(Gm/r)\,(r\omega_s/c^3)]^{2/3}$, and observing that $Gm/r \sim v^2$ and $r\omega_s \sim v$, we see that

$$x = O\left(\frac{v^2}{c^2}\right). \tag{5.239}$$

Thus, the v/c corrections can be expressed as correction in powers of $x^{1/2}$. We also define the symmetric mass ratio

$$\nu \equiv \frac{\mu}{m} = \frac{m_1 m_2}{(m_1+m_2)^2}, \tag{5.240}$$

and the post-Newtonian parameter

$$\gamma \equiv \frac{Gm}{rc^2}, \tag{5.241}$$

which is $O(v^2/c^2)$. We finally introduce the dimensionless time variable

$$\Theta \equiv \frac{\nu c^3}{5Gm}(t_c - t), \tag{5.242}$$

where t_c is the time at which the coalescence takes place.

We can now rewrite some Newtonian results for a chirping binary, obtained in Section 4.1, in terms of these parameters. In particular, the relation between the frequency and the time to coalescence for circular orbit, eq. (4.19), reads simply (taking into account that in Section 4.1 we computed the radiation emitted by a circular orbit in the quadrupole approximation, so $\omega_s = \omega_{\rm gw}/2 = \pi f_{\rm gw}$)

$$x = \frac{1}{4}\Theta^{-1/4}. \tag{5.243}$$

The accumulated orbital phase, defined by

$$\phi = \int_{t_0}^{t} dt'\,\omega_s(t'), \tag{5.244}$$

can be written, using eq. (4.30) together with $\Phi = 2\phi$ (see eq. (4.28)), as

$$\phi = \phi_0 - \frac{1}{\nu}\Theta^{5/8}, \tag{5.245}$$

or, eliminating Θ in favor of x,

$$\phi = \phi_0 - \frac{x^{-5/2}}{32\nu}. \tag{5.246}$$

Finally, the number of cycles spent in the detector bandwidth can be written, using eq. (4.23), as $\mathcal{N}_{\rm cyc} = \mathcal{N}(f_{\rm min}) - \mathcal{N}(f_{\rm max})$, where

$$\begin{aligned}\mathcal{N}(f) &\equiv \frac{1}{32\pi^{8/3}}\left(\frac{GM_c}{c^3}\right)^{-5/3} f^{-5/3} \\ &= \frac{x^{-5/2}}{32\pi\nu}.\end{aligned} \tag{5.247}$$

All these relations receive corrections from the PN expansion, that can be written as an expansion in powers of $x^{1/2}$, see eq. (5.239), and that will be examined in detail below. In particular, the PN corrections to eq. (5.247) take the form

$$\mathcal{N}(x) = \frac{x^{-5/2}}{32\pi\nu}[1 + O(x) + O(x^{3/2}) + O(x^2) + O(x^{5/2}) + \ldots]. \tag{5.248}$$

If we want to track the evolution of the GW signal, we need a template which reproduces the number of cycles with a precision at least $O(1)$. We see from eq. (5.248) that, since the leading term in $\mathcal{N}(x)$ is proportional to $x^{-5/2}$, we need to include the corrections up to $O(x^{5/2})$ in order to have an error not larger than $O(1)$ on $\mathcal{N}(x)$. This means that we need to compute the PN corrections to the phase at least up to 2.5PN level, i.e. corrections smaller by a factor $(v/c)^5$ with respect to the leading term. Actually, this is not even enough because, once we have accumulated an error of order one on the number of cycles, our template has clearly gone out of phase with the signal, so a more accurate computation is really required in order to exploit optimally the information contained in the output of a ground-based interferometer, at least up to 3PN order, and better yet to 3.5PN.

An equivalent way of understanding the need for a high-order PN computation is to look at the waveform, rather then at the number of cycles $\mathcal{N}(x)$, and to observe that the last term in the expression (4.37) for the phase $\Psi_+(f)$ of the GW amplitude is, in terms of x,

$$\frac{3}{4}\left(\frac{GM_c}{c^3}\,8\pi f\right)^{-5/3} = \frac{3}{128\nu}\,x^{-5/2}\,. \tag{5.249}$$

Thus, for small x the Newtonian phase Ψ_+ is of order $x^{-5/2}$, and it diverges for $x \to 0$. The 1PN corrections gives a contribution to Ψ_+ of order $x^{-3/2}$ which, even if subleading with respect to the Newtonian term, still diverges as $x \to 0$, and similarly all the contributions up to 2.5PN must be kept since they either diverge (up to 2PN) or anyway stay finite (the 2.5PN term) in the small x limit, and only starting from 3PN level we have corrections which vanish as $x \to 0$.

5.6.2 The 3.5PN equations of motion

The general principles for performing such a computation have been discussed in detail in Sections 5.3 and 5.4, and we have also seen in some detail how to obtain the near-field metric to 1PN order, in Section 5.1.4. A new technical problem that arises in higher orders is due to the fact that, if one uses as energy–momentum tensor of the two bodies the expression in terms of the Dirac delta, eq. (5.47), one finds divergencies in the computation of some integrals. Therefore, a modelization of the two bodies as point-like is not possible, and some regularization of the Dirac delta is necessary. Different regularizations have been considered, and (up to 3.5PN) they all give the same final result, as we expect from the effacement principle discussed in Section 5.5.[60] The equation of motion has the general form

$$\frac{dv^i}{dt} = -\frac{Gm}{r^2}\left[(1+\mathcal{A})\,\frac{x^i}{r} + \mathcal{B}\,v^i\right] + \mathcal{O}\left(\frac{1}{c^8}\right)\,, \tag{5.250}$$

so it has a term proportional to the relative separation x^i and a term proportional to the relative velocity in the center-of-mass frame, v^i. For a generic orbit, the expression of $\mathcal{A}$ and $\mathcal{B}$ is extremely long.[61] However we have seen in Section 4.1.3 that, by the time that the signal enters in the bandwidth of a ground-based interferometer, radiation reaction

[60] Except that, with the so-called Hadamard regularization, some ambiguity appears at 3PN order. This ambiguity does not appear with dimensional regularization, see Damour, Jaranowski and Schäfer (2001b) and Blanchet, Damour, Esposito-Farèse and Iyer (2004), nor with the surface integral method, see Itoh and Futamase (2003), nor with the ADM Hamiltonian formalism, see Damour, Jaranowski and Schäfer (2000), and the results agree.

[61] See Blanchet (2006), eq. (182), where the result spills over two pages!

has circularized the orbit to great accuracy. For orbits that are circular except from the inspiral due to radiation reaction, one finds that the radial velocity is $O(1/c^5)$, so most terms in the expression for $\mathcal{A}$ and $\mathcal{B}$ can be dropped, to 3.5PN order, and eq. (5.250) becomes

$$\frac{dv^i}{dt} = -\omega_s^2 x^i - \zeta v^i\,, \tag{5.251}$$

where

$$\omega_s^2 = \frac{Gm}{r^3}\Big\{1 + (-3+\nu)\gamma + \left(6 + \frac{41}{4}\nu + \nu^2\right)\gamma^2 + \left[-10 + \left(22\log(r/r_0') - \frac{75707}{840} + \frac{41}{64}\pi^2\right)\nu + \frac{19}{2}\nu^2 + \nu^3\right]\gamma^3\Big\} + \mathcal{O}\left(\frac{1}{c^8}\right)\,, \tag{5.252}$$

and

$$\zeta = \frac{32}{5}\frac{G^3m^3\nu}{c^5r^4} + \mathcal{O}\left(\frac{1}{c^7}\right)\,. \tag{5.253}$$

Observe that eq. (5.252) is the PN generalization of Kepler's law. The velocity-dependent term in eq. (5.251) describes the radiation reaction. The term $O(1/c^5)$ in ζ is due to the 2.5PN radiation reaction, and we have not written explicitly the more complicated 3.5PN contribution to ζ.

In eq. (5.252) appears a length-scale r_0', which is a gauge-dependent constant. This is not surprising, since the radius r that appears in the above formulas is the relative separation in harmonic coordinates, so it is not an invariant quantity (similarly, γ in eq. (5.241) is not an invariant quantity). However, x defined in eq. (5.238) is a physical quantity, so if we express a physical observable as a power series in x, the constant r_0' must cancel out. For instance, inverting eq. (5.252) one finds

$$\gamma = x\Big\{1 + \left(1 - \frac{\nu}{3}\right)x + \left(1 - \frac{65}{12}\nu\right)x^2 + \left[1 + \left(-\frac{22}{3}\log(r/r'_0) - \frac{2203}{2520} - \frac{41}{192}\pi^2\right)\nu + \frac{229}{36}\nu^2 + \frac{1}{81}\nu^3\right]x^3 + \mathcal{O}\left(\frac{1}{c^8}\right)\Big\}\,, \tag{5.254}$$

which displays explicitly the dependence of γ on r_0'. On the other hand, the PN expansion of the energy of a circular orbit up to 3.5PN turns out to be

$$E = -\frac{\mu c^2\gamma}{2}\Big\{1 + \left(-\frac{7}{4} + \frac{1}{4}\nu\right)\gamma + \left(-\frac{7}{8} + \frac{49}{8}\nu + \frac{1}{8}\nu^2\right)\gamma^2 + \left[-\frac{235}{64} + \left(\frac{22}{3}\log(r/r_0') + \frac{46031}{2240} - \frac{123}{64}\pi^2\right)\nu + \frac{27}{32}\nu^2 + \frac{5}{64}\nu^3\right]\gamma^3\Big\} + \mathcal{O}\left(\frac{1}{c^8}\right)\,, \tag{5.255}$$

which seems to depend on r_0' both explicitly and through γ. However, inserting eq. (5.254) into eq. (5.255), one finds that r_0' cancels out, and

$$E = -\frac{\mu c^2 x}{2}\left\{1+\left(-\frac{3}{4}-\frac{1}{12}\nu\right)x+\left(-\frac{27}{8}+\frac{19}{8}\nu-\frac{1}{24}\nu^2\right)x^2 \right. \\ \left. +\left[-\frac{675}{64}+\left(\frac{34445}{576}-\frac{205}{96}\pi^2\right)\nu-\frac{155}{96}\nu^2-\frac{35}{5184}\nu^3\right]x^3\right\} \\ +\mathcal{O}\left(\frac{1}{c^8}\right). \tag{5.256}$$

5.6.3 Energy flux and orbital phase to 3.5PN order

The computation of the equations of motion in the near region is one of the outputs of the formalisms that we have discussed in the previous sections. The other is the waveform, and therefore the energy flux, at infinity. A computation of the gravitational waveform to a very high PN order is a daunting task. Currently, the computation of the phase is complete up to 3.5 PN order (for the case where the stars have negligible spins).[62] For the power radiated in GWs, $P_{\rm gw}$, one finds, after an extremely long computation,

[62] See Blanchet, Faye, Iyer and Joguet (2002) and Blanchet, Damour, Esposito-Farèse and Iyer (2004).

$$P_{\rm gw} = \frac{32c^5}{5G}\nu^2 x^5\left\{1+\left(-\frac{1247}{336}-\frac{35}{12}\nu\right)x+4\pi x^{3/2} \right. \\ +\left(-\frac{44711}{9072}+\frac{9271}{504}\nu+\frac{65}{18}\nu^2\right)x^2 \\ +\left(-\frac{8191}{672}-\frac{583}{24}\nu\right)\pi x^{5/2} \\ +\left[\frac{6643739519}{69854400}+\frac{16}{3}\pi^2-\frac{1712}{105}C-\frac{856}{105}\log(16\,x)\right. \\ \left.+\left(-\frac{134543}{7776}+\frac{41}{48}\pi^2\right)\nu-\frac{94403}{3024}\nu^2-\frac{775}{324}\nu^3\right]x^3 \\ \left.+\left(-\frac{16285}{504}+\frac{214745}{1728}\nu+\frac{193385}{3024}\nu^2\right)\pi x^{7/2}+\mathcal{O}\left(\frac{1}{c^8}\right)\right\}. \tag{5.257}$$

where $C = 0.577\ldots$ is the Euler–Mascheroni constant. Observe that the limit $\nu \to 0$ corresponds to a test mass moving in the background geometry generated by the other body, which is just a perturbation of the Schwarzschild metric. In this limit, using methods from black-hole perturbation theory, the result has been computed up to the extremely high 5.5PN order, see the Further Reading. Comparing the limit of eq. (5.257) when $\nu \to 0$, with the result of black hole perturbation theory up to 3.5PN, one finds complete agreement, including rational fractions such as 6643739519/69854400. This is a very non-trivial check of the above computation.

The orbital phase evolution up to 3.5PN can now be obtained by integrating the energy balance equation $dE/dt = -P_{\rm gw}$, with E given in eq. (5.256) and $P_{\rm gw}$ in eq. (5.257). This gives x as a function of time.

The result, expressed in terms of Θ defined in eq. (5.242), is

$$x = \frac{1}{4}\Theta^{-1/4}\left\{1+\left(\frac{743}{4032}+\frac{11}{48}\nu\right)\Theta^{-1/4}-\frac{1}{5}\pi\Theta^{-3/8} \right. \\ +\left(\frac{19583}{254016}+\frac{24401}{193536}\nu+\frac{31}{288}\nu^2\right)\Theta^{-1/2} \\ +\left(-\frac{11891}{53760}+\frac{109}{1920}\nu\right)\pi\Theta^{-5/8} \\ +\left[-\frac{10052469856691}{6008596070400}+\frac{1}{6}\pi^2+\frac{107}{420}C-\frac{107}{3360}\log\left(\frac{\Theta}{256}\right)\right. \\ \left.+\left(\frac{3147553127}{780337152}-\frac{451}{3072}\pi^2\right)\nu-\frac{15211}{442368}\nu^2+\frac{25565}{331776}\nu^3\right]\Theta^{-3/4} \\ \left.+\left(-\frac{113868647}{433520640}-\frac{31821}{143360}\nu+\frac{294941}{3870720}\nu^2\right)\pi\Theta^{-7/8}+\mathcal{O}\left(\frac{1}{c^8}\right)\right\}. \tag{5.258}$$

The orbital phase ϕ is now obtained by integrating $d\phi/dt = \omega_s$ which, expressing t in terms of Θ and ω_s in terms of x, reads

$$\frac{d\phi}{d\Theta} = -\frac{5}{\nu}x^{3/2}. \tag{5.259}$$

Inserting x as function of Θ from eq. (5.258), the integration gives

$$\phi(t) = -\frac{1}{\nu}\Theta^{5/8}\left\{1+\left(\frac{3715}{8064}+\frac{55}{96}\nu\right)\Theta^{-1/4}-\frac{3}{4}\pi\Theta^{-3/8} \right. \\ +\left(\frac{9275495}{14450688}+\frac{284875}{258048}\nu+\frac{1855}{2048}\nu^2\right)\Theta^{-1/2} \\ +\left(-\frac{38645}{172032}+\frac{65}{2048}\nu\right)\pi\Theta^{-5/8}\log\left(\frac{\Theta}{\Theta_0}\right) \\ +\left[\frac{831032450749357}{57682522275840}-\frac{53}{40}\pi^2-\frac{107}{56}C+\frac{107}{448}\log\left(\frac{\Theta}{256}\right)\right. \\ +\left(-\frac{126510089885}{4161798144}+\frac{2255}{2048}\pi^2\right)\nu \\ \left.+\frac{154565}{1835008}\nu^2-\frac{1179625}{1769472}\nu^3\right]\Theta^{-3/4} \\ +\left(\frac{188516689}{173408256}+\frac{488825}{516096}\nu-\frac{141769}{516096}\nu^2\right)\pi\Theta^{-7/8} \\ \left.+\mathcal{O}\left(\frac{1}{c^8}\right)\right\}, \tag{5.260}$$

where $\Theta(t)$ is given in eq. (5.242), and Θ_0 is a constant of integration to be fixed by the initial condition (i.e. by the value of Θ when it enters the detector's bandwidth) which replaces ϕ_0 in the Newtonian formula (5.245). Observe that, due to the $\log\Theta$ term at 2.5PN level, as well as due to negative overall powers of Θ in higher orders, it is no longer true that ϕ_0 in eq. (5.246) is the phase at the coalescence time; rather, now ϕ diverges as $\Theta \to 0$. In terms of x, the above result reads

$$\phi = -\frac{x^{-5/2}}{32\nu}\left\{1+\left(\frac{3715}{1008}+\frac{55}{12}\nu\right)x-10\pi x^{3/2}\right.$$

$$
\begin{aligned}
&+\left(\frac{15293365}{1016064}+\frac{27145}{1008}\nu+\frac{3085}{144}\nu^2\right)x^2\\
&+\left(\frac{38645}{1344}-\frac{65}{16}\nu\right)\pi x^{5/2}\log\left(\frac{x}{x_0}\right)\\
&+\left[\frac{12348611926451}{18776862720}-\frac{160}{3}\pi^2-\frac{1712}{21}C-\frac{856}{21}\log(16\,x)\right.\\
&+\left.\left(-\frac{15737765635}{12192768}+\frac{2255}{48}\pi^2\right)\nu+\frac{76055}{6912}\nu^2-\frac{127825}{5184}\nu^3\right]x^3\\
&+\left(\frac{77096675}{2032128}+\frac{378515}{12096}\nu-\frac{74045}{6048}\nu^2\right)\pi x^{7/2}\\
&+\mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\},
\end{aligned}
\tag{5.261}
$$

where x_0 is another constant of integration. When we consider spinning bodies, there is also a spin–orbit coupling arising at 1.5PN and a spin–spin coupling starting at 2PN. They are known up to 2.5PN order, included.

5.6.4 The waveform

The full waveform is presently known up to 2.5PN order. The two polarizations are defined with respect to two axes $\mathbf{p}$ and $\mathbf{q}$, chosen to lie along the major and minor axis, respectively, of the projection onto the plane of the sky of the circular orbit, with $\mathbf{p}$ oriented toward the ascending node. The general structure of the two polarization amplitudes is

$$
h_{+,\times}(t)=\frac{2G\mu x}{c^2 r}\Big\{H^{(0)}_{+,\times}+x^{1/2}H^{(1/2)}_{+,\times}+xH^{(1)}_{+,\times}+x^{3/2}H^{(3/2)}_{+,\times}+x^2H^{(2)}_{+,\times}
$$
$$
x^{5/2}H^{(5/2)}_{+,\times}+O\left(\frac{1}{c^6}\right)\Big\}. \tag{5.262}
$$

The leading term is[63,64]

$$
H^{(0)}_{+}(t)=-(1+\cos^2\iota)\cos 2\psi(t)\,, \tag{5.263}
$$
$$
H^{(0)}_{\times}(t)=-2\cos\iota\sin 2\psi(t)\,, \tag{5.264}
$$

where ψ is a phase, related to ϕ by

$$
\psi(t)=\phi(t)-\frac{2Gm\omega_s}{c^3}\log\left(\frac{\omega_s(t)}{\omega_0}\right), \tag{5.265}
$$

and ω_0 is a constant frequency that can be conveniently chosen as the entry frequency of an interferometric detector.[65] For the crucial phase ϕ one uses the highest available precision, i.e., at present, the 3.5PN result (5.260), independently of the order at which the waveform has been computed. This is necessary since, as discussed in Section 5.6.1, the phase is given as an expansion in x but, in the limit $x\to 0$ the corrections to ϕ up to 2PN order are divergent. Only starting from 2.5PN correction the correction has a finite limit for $x\to 0$, but is anyway of order one in the phase, and we need better than that. On the other hand, we see from eq. (5.262) that the correction to $H^{(0)}_{+,\times}$ i.e. the terms $x^{1/2}H^{(1/2)}$, $xH^{(1)}$, etc., vanish for $x\to 0$ so, in the small x limit, it makes sense to neglect them, even when we include all available corrections up to 3.5PN in ϕ. The approximation in which only $H^{(0)}$ is retained in eq. (5.262), while all available PN corrections to ϕ are included, is called the "restricted" PN approximation.

[63]The minus sign of eqs. (5.263) and (5.264) with respect to the result in eq. (4.29) is due to the sign difference among $\mathrm{h}_{\mu\nu}$ and $\bar{h}_{\mu\nu}$, see Note 19 on page 250.

[64]See Blanchet (2006), eqs. (236)–(242) for the full result including the 2.5 PN corrections $H^{(5/2)}_{+,\times}$.

[65]The use of ψ instead of the actual phase ϕ of the source is convenient because it allows us to collect the logarithmic terms which come out of the computation of the tail effects discussed in Section 5.3.4.

In practice, however, we are not interested in the waveform for parametrically small values of x, but rather for the typical values of v/c, and hence of x, at which the signal of an inspiraling binary enters in the detector bandwidth. Then, for a ground-based detector, the first few corrections to the amplitude are numerically important, and produce an amplitude modulation of the chirp signal.[66] The first correction to the amplitude (which is of 0.5PN order, i.e. $O(v/c)$) is given by

$$
H^{(1/2)}_{+}=-\frac{\sin\iota}{8}\frac{\delta m}{m}\left[(5+\cos^2\iota)\cos\psi-9(1+\cos^2\iota)\cos 3\psi\right], \tag{5.266}
$$
$$
H^{(1/2)}_{\times}=-\frac{3}{4}\sin\iota\cos\iota\frac{\delta m}{m}\left[\sin\psi-3\sin 3\psi\right], \tag{5.267}
$$

where $\delta m=m_1-m_2$ is the mass difference. Observe that, while the phase of the leading terms $H^{(0)}_{+,\times}$ depends on 2ϕ, that is on the integral of $2\omega_s$, the phase in the next-to-leading terms depend on ϕ and 3ϕ, that is on the integral of ω_s and $3\omega_s$. This can be traced back to the fact, discussed in Sections 3.3 and 3.4, that for a purely circular motion with frequency ω_s, the mass quadrupole radiates GWs with $\omega_{\rm gw}=2\omega_s$, while the mass octupole and current quadrupole both radiate at $\omega_{\rm gw}=\omega_s$ and at $\omega_{\rm gw}=3\omega_s$. Thus, together with the quadrupolar component which is chirping according to $\omega_{\rm gw}(t)=2\omega_s(t)$, the term $H^{(1/2)}_{+,\times}$ in eq. (5.262) describes two "sidebands" chirping at $\omega_{\rm gw}(t)=\omega_s(t)$ and at $\omega_{\rm gw}(t)=3\omega_s(t)$, while $H^{(1)}_{+,\times}$ is again chirping at $\omega_{\rm gw}(t)=2\omega_s(t)$, as well as at $\omega_{\rm gw}(t)=4\omega_s(t)$, etc. Observe that these components enter in the detector bandwidth at different times.[67]

It is also useful to express eq. (5.258) as an explicit relation between time t and the GW frequency $f(t)$ (defined here as twice the source frequency $f_s(t)$, so $f(t)$ is really the frequency at which the quadrupole component $H^{(0)}_{+,\times}$ is chirping). We limit ourselves for simplicity to 2PN order, and we neglect spin corrections. First of all, observe that the Newtonian relation between f and t, eq. (4.19), can be rewritten in the form

$$
t-t_*=\tau_0\left[1-(f(t)/f_*)^{-8/3}\right], \tag{5.268}
$$

where t_* is an arbitrary reference time (e.g. the time of entry of the signal in the interferometer bandwidth), $f_*=f(t_*)$, and the parameter τ_0 is given by

$$
\tau_0=\frac{5}{256\pi}\,f_*^{-1}\,(\pi M f_*)^{-5/3}\,\nu^{-1}\,. \tag{5.269}
$$

Here we introduced the shorthand notation $M\equiv Gm/c^3$, where $m=m_1+m_2$ is the total mass of the system (observe that, dimensionally,

[66]An amplitude modulation can also be obtained if the compact stars have an intrinsic spin, which is the physically realistic case. See the Further reading for details.

[67]This time delay can be quite large. For instance, when the quadrupole component has reached the frequency $f_{\rm gw}=2f_s=10$ Hz, the octupole and current quadrupole give a contribution at $f_{\rm gw}=3f_s=15$ Hz. From eq. (4.20) we see that the quadrupole will reach 15 Hz only after about 5 more minutes.

M is actually a time) and, as usual $\nu = \mu/m$. The post-Newtonian corrections, up to 2PN order, modify this relation as follows,

$$t - t_* = \tau_0 \left[1 - \left(\frac{f}{f_*}\right)^{-8/3}\right] + \tau_1 \left[1 - \left(\frac{f}{f_*}\right)^{-2}\right] \qquad (5.270)$$
$$-\tau_{1.5} \left[1 - \left(\frac{f}{f_*}\right)^{-5/3}\right] + \tau_2 \left[1 - \left(\frac{f}{f_*}\right)^{-4/3}\right],$$

with

$$\tau_1 = \frac{5}{192\pi} f_*^{-1} (\pi M f_*)^{-1} \nu^{-1} \left(\frac{743}{336} + \frac{11}{4}\nu\right),$$
$$\tau_{1.5} = \frac{1}{8} f_*^{-1} (\pi M f_*)^{-2/3} \nu^{-1} \qquad (5.271)$$
$$\tau_2 = \frac{5}{128\pi} f_*^{-1} (\pi M f_*)^{-1/3} \nu^{-1} \left(\frac{3058673}{1016064} + \frac{5429}{1008}\nu + \frac{617}{144}\nu^2\right).$$

In terms of these quantities, the chirping of the GW frequency can be written as

$$\frac{df}{dt} = \frac{3f_*}{8\tau_0} \left(\frac{f}{f_*}\right)^{11/3} \left[1 - \frac{3}{4}\frac{\tau_1}{\tau_0}\left(\frac{f}{f_*}\right)^{2/3}\right. \qquad (5.272)$$
$$\left. + \frac{5}{8}\frac{\tau_{1.5}}{\tau_0}\left(\frac{f}{f_*}\right) - \frac{1}{2}\left(\frac{\tau_2}{\tau_0} - \frac{9}{8}\left(\frac{\tau_1}{\tau_0}\right)^2\right)\left(\frac{f}{f_*}\right)^{4/3}\right],$$

and the accumulated phase $\Phi = 2\phi$ which appears in the quadrupole part of the waveform depends on $f(t)$ as

$$\Phi(f) = \frac{16\pi}{5}\tau_0 f_* \left[\left(1 - \left(\frac{f}{f_*}\right)^{-5/3}\right) + \frac{5}{4}\frac{\tau_1}{\tau_0}\left(1 - \left(\frac{f}{f_*}\right)^{-1}\right)\right.$$
$$\left. - \frac{25}{16}\frac{\tau_{1.5}}{\tau_0}\left(1 - \left(\frac{f}{f_*}\right)^{-2/3}\right) + \frac{5}{2}\frac{\tau_2}{\tau_0}\left(1 - \left(\frac{f}{f_*}\right)^{-1/3}\right)\right]. \qquad (5.273)$$

The Fourier transform of $h_+(t)$ and $h_\times(t)$ are computed in saddle point, just as in Problem 4.1. In the restricted PN approximation, the result is

$$\tilde{h}_+(f) = \left(\frac{5}{6}\right)^{1/2} \frac{1}{2\,\pi^{2/3}} \frac{c}{r} \left(\frac{GM_c}{c^3}\right)^{5/6} f^{-7/6} e^{i\Psi_+(f)} \frac{1+\cos^2\iota}{2}, \qquad (5.274)$$

so it is the same as what we found in the Newtonian case in Problem 4.1, except that the phase Ψ_+, which in the Newtonian case is given by eq. (4.37), now receives corrections. To 2PN order, written in terms of the parameters $\tau_0, \ldots, \tau_2$, it becomes[68]

[68] See Poisson and Will (1995).

$$\Psi_+(f) = 2\pi f(t_c + r/c) - \Phi_0 - \frac{\pi}{4} + 2\pi f_* \left[\frac{3\tau_0}{5}\left(\frac{f}{f_*}\right)^{-5/3}\right.$$

$$\left. + \tau_1 \left(\frac{f}{f_*}\right)^{-1} - \frac{3\tau_{1.5}}{2}\left(\frac{f}{f_*}\right)^{-2/3} + 3\tau_2 \left(\frac{f}{f_*}\right)^{-1/3}\right], \qquad (5.275)$$

while $\tilde{h}_\times$ is obtained from $\tilde{h}_+$, by replacing $(1+\cos^2\iota)/2$ by $\cos\iota$ and with $\Psi_\times = \Psi_+ + (\pi/2)$. Observe that τ_0 depends on the masses only through the combination $M^{-5/3}\nu^{-1}$, which gives $M_c^{-5/3}$, and more generally all Newtonian results depend on the masses m_1 and m_2 of the two stars only through the chirp mass M_c. However, this degeneracy is broken by the PN correction, since the parameters τ_1, etc. depend on different combinations of M and ν. Therefore, the masses m_1 and m_2 can now be separately determined by a comparison of the observed phase with the PN prediction.

Further reading

- The lowest-order post-Newtonian corrections to the gravitational field in the near region are discussed in many general relativity textbooks, see e.g. Chapter 9 of Weinberg (1972), Chapter 4 of Will (1993), or Section 5.2 of Straumann (2004). A review of the problem of motion in general relativity is given in Damour (1987).
- The form (5.72) of the Einstein equations was found by Landau and Lifshitz in the 1940s, see Landau and Lifshitz, Vol. II (1979), Section 96. In a iterative procedure, to lowest order the gravitational field $h^{\mu\nu}$ that appears in $\tau^{\alpha\beta}$ is set to zero, so $\tau^{\alpha\beta}$ reduces to the energy–momentum tensor of matter. Thus the Landau and Lifshitz derivation, see their Section 110, was the first which showed that the Einstein quadrupole formula is the correct lowest-order result even for weakly self-gravitating bodies (even if the problem of the finiteness of the higher-order corrections was addressed only later).

  Early attempt toward the construction of a systematic wave-generation formalism for post-Newtonian sources were performed by Epstein and Wagoner (1975), Wagoner and Will (1976) and Thorne (1980). In particular, in the latter paper are given general expressions for the GW fluxes in terms of radiative multipole moments at infinity. The DIRE approach builds on these earlier work, as well as on the works by Wiseman and Will (1991), Wiseman (1992, 1993), and has been developed in particular in Wiseman and Will (1996), Pati and Will (2000, 2002), and Will (2005).
- The method of matched asymptotic expansions was introduced in the radiation-reaction problem by Burke (1971). The back-reaction of GWs and its relation to the PN expansion is discussed in Chandrasekhar and Esposito (1970) (where references to earlier work can be found). Here the correct 2.5PN is obtained but, as we discussed below eq. (5.185), in this scheme higher-order terms were divergent. For the same reason, one could question also the validity of the result in the far region, i.e. the Einstein quadrupole formula, for self-gravitating systems. The unsatisfactory status of the derivations that were available at that time, for the back-reaction and for the Einstein quadrupole formula, was discussed by Ehlers, Rosenblum, Goldberg and Havas (1976). These criticisms stimulated a better understanding of the radiation reaction problem in general relativity and of the derivation of the quadrupole formula, see Walker and Will (1980a, 1980b), Damour and Deruelle (1981), and Damour (1983a, 1983b).

  A review of the "quadrupole formula controversy" (as well as of the various controversies to which GWs have been subject) is Kennefick (1997) and a very detailed and interesting historical account is given in the book Kennefick (2007). Nowadays, with the full development of the systematic and consistent expansion methods discussed in Sections 5.3 and 5.4, the problem of the validity of the quadrupole formula for self-gravitating systems is settled.

- The Blanchet–Damour formalism has been developed in various papers. The general principles are discussed in Damour (1983b) and (1987). The structure of the fields in the post-Minkowskian expansion is studied in Blanchet and Damour (1986). The expansion of the fields at future null infinity and the relation to Thorne's (1980) radiative moments is done in Blanchet (1987). The 1PN generation of GWs is computed in Blanchet and Damour (1989). The multipole expansion of the gravitational field in linearized gravity in terms of STF tensors is presented in Damour and Iyer (1991a). The 1PN expression for the spin moments is computed in Damour and Iyer (1991b), and the 2PN result for mass and current moments is obtained in Blanchet (1995), and applied to coalescing binaries in Blanchet, Damour and Iyer (1995). The 2.5PN result (where the moments $W_L, \ldots, Z_L$ start to mix with I_L, J_L) is computed in Blanchet (1996). The matching of the post-Newtonian and post-Minkowskian solutions is obtained in full generality in Blanchet (1995, 1998c). The determination of the PN expansion to all orders from the matching conditions is discussed in Blanchet (1993), Poujade and Blanchet (2002), and Blanchet, Faye and Nissanke (2005). A detailed review of the formalism, and its application to inspiraling binaries, is Blanchet (2006).
- Early investigations of tails and back-scattering in the gravitational radiation field were performed by Newman and Penrose (1968) and Bardeen and Press (1973). The tail integral was computed in Blanchet and Damour (1988), looking at the 4PN metric in the near-field zone. Its effect on the radiative moments at infinity (where it shows up as a 1.5PN correction) is computed in Blanchet and Damour (1992). In the DIRE approach, the tail integral is computed in Pati and Will (2000). The hereditary terms up to 3PN order are computed in Blanchet (1998a, 1998b). The possibility of detecting the tail contributions from the experimental data is discussed in Blanchet and Sathyaprakash (1995) and, for the memory terms, in Kennefick (1994).
- The initial value problem in general relativity, and the fact that initial data *inside* the light-cone are required, is discussed in Bruhat (1962). The non-linear memory effect has been found in Christodoulou (1991), using a mathematically rigorous study of Einstein equations at null infinity. Its relation to a 2.5PN contribution is clarified in Wiseman and Will (1991), Blanchet and Damour (1992), and Arun, Blanchet, Iyer and Qusailah (2004).

  The denomination "non-linear" memory effect is used to distinguish it from a linear memory effect which arises already in linearized theory, for instance in a scattering process, as a result of an overall change of the linear momentum of the bodies, see Zeldovich and Polnarev (1974), Braginsky and Grishchuk (1986) and Braginsky and Thorne (1987). The non-linear memory term can be understood as the linear memory term due to the linear momentum of the outgoing gravitons, see Thorne (1992).
- Important contributions to the 3PN dynamics have been obtained with a ADM-Hamiltonian formalism by Jaranowski and Schäfer (1998, 1999, 2000) and Damour, Jaranowski and Schäfer (2000, 2001a); and, with a direct PN iteration in harmonic coordinates, in Blanchet and Faye (2001) (equations of motion), de Andrade, Blanchet and Faye (2001) (Lagrangian and conserved quantities) and Blanchet and Iyer (2003) (reduction to center of mass). The complete determination of the dynamics of binary systems to 3PN is done in Damour, Jaranowski and Schäfer (2001b) and Blanchet, Damour and Esposito-Farese (2004), using dimensional regularization. Observe that, at 3PN order, one computes only the metric at the location of the particles. The metric at a generic space-time point admits a closed form only to 2.5PN order, and has been computed in Blanchet, Faye and Ponsot (1998). The 3PN equations have also been obtained with the surface integral method by Itoh, Futamase and Asada (2000, 2001), Itoh and Futamase (2003) and Itoh (2004). The results obtained with these different methods agree with each other.
- Beyond 1PN order a model of the source as point-like, i.e. in terms of Dirac deltas, gives rise to divergences, and need to be regularized. Hadamard and dimensional regularization are reviewed in Section 8 of Blanchet (2006). Dimensional regularization of point-like sources is introduced in Damour, Jaranowski and Schäfer (2001b) and further used in Blanchet, Damour, Esposito-Farèse and Iyer (2004, 2005). This method, based on analytic continuation in $d = 3 + \epsilon$ spatial dimensions, allows us to resolve some ambiguities that appear at 3PN in Hadamard regularization.

  In the language of quantum field theory, a point-like singularity is an example of an ultraviolet divergence, which reflects our ignorance of short-distance physics, and can be dealt with standard method from effective low-energy field theory. An approach of this type is discussed in Goldberger and Rothstein (2006).
- The surface integral method derives from the classic paper of Einstein, Infeld and Hoffmann (1938). Its application to the derivation of the quadrupole formula for strong-field sources is discussed in Damour (1983a, 1983b). A variant of this method has been developed by Itoh, Futamase, and Asada (2000, 2001), and used in Itoh and Futamase (2003) and Itoh (2004) to give a derivation of the 3PN equations of motion, valid for sources with strong internal gravity. The effacement of the internal structure in Newtonian gravity and in general relativity is discussed in detail in Damour (1987). A discussion of the fact that the tidal interaction between compact bodies shows up only to 5PN order even in the full relativistic theory can be found in Damour (1983b).
- Explicit formulas for the phase and waveform of a compact binary system can be found in Blanchet and Schäfer (1993), Wiseman (1993), Poisson and Will (1995), Blanchet, Iyer, Will and Wiseman (1996) and Blanchet (1996). The orbital phase to 3.5PN is computed in Blanchet, Faye, Iyer and Joguet (2002), based on the 3PN computation of the radiative moments in Blanchet, Iyer and Joguet (2002). The waveform to 2.5PN is computed in Arun, Blanchet, Iyer and Qusailah (2004). A summary of explicit formulas is given in the review Blanchet (2006).
- For compact binaries, the radiation reaction terms

up to 3.5PN order are computed (either with the energy balance argument or with explicit PN computations) in Iyer and Will (1993, 1995), Blanchet (1997), Pati and Will (2002), Konigsdorffer, Faye and Schäfer (2003), and Nissanke and Blanchet (2005). For non-spinning bodies, the 4.5PN back-reaction terms have been computed, from the energy balance argument, in Gopakumar, Iyer and Iyer (1997). For general PN sources, the balance equation has been checked explicitly to 1.5PN order (i.e. 4PN order in the near region metric) in Blanchet (1997).
- The inclusion of spin is discussed in Kidder, Will and Wiseman (1993), Apostolatos, Cutler, Sussman and Thorne (1994), Kidder (1995), Królak, Kokkotas and Scäfer (1995), and Tagoshi, Ohashi and Owen (2001), and has been completed to 2.5PN in Faye, Blanchet and Buonanno (2006) and Blanchet, Buonanno and Faye (2006).

  The effect of the eccentricity is computed in Gopakumar and Iyer (2002) and Damour, Gopakumar and Iyer (2004).
- In the limit in which one of the masses in the binary system tends to zero and becomes a test mass, while the other is a black hole, the computation of the motion in the PN expansion can be studied using linear perturbation of a black-hole space-time, see Poisson (1993). With this technique the PN expansion has been pushed up to the extremely high 5.5PN order by Sasaki (1994), Tagoshi and Sasaki (1994) and Tanaka, Tagoshi and Sasaki (1996). Up to 3.5PN order the result can be compared with the limit $\nu \to 0$ of the formulas presented in Section 5.6, and one finds complete agreement.