Further reading

- For the quantum field-theoretical approach to gravitation see the *Feynman Lectures on Gravitation* by Feynman, Morinigo, and Wagner (1995) (which collects lectures given by Feynman in 1962–63), and also DeWitt (1967) and Veltman (1976). For explicit computations of graviton–graviton scattering see Grisaru, van Nieuwenhuizen and Wu (1975).
- The possibility of deriving Einstein equation from an iteration of linearized theory is discussed, among others, by Gupta (1954), Kraichnan (1955), Feynman, Morinigo, and Wagner (1995), and Ogievetsky and Polubarinov (1965). An explicit and elegant iteration leading from the equations of motion of linearized theory to the full Einstein equations was performed by Deser (1970) using a first order Palatini formalism. The ambiguity concerning boundary terms is discussed by Padmanabhan (2004).
- Phenomenological limits on the graviton mass are discussed by Goldhaber and Nieto (1974). The discontinuity as the graviton mass goes to zero was found by Iwasaki (1970), van Dam and Veltman (1970) and Zakharov (1970). Massive gravitons have been further discussed by Boulware and Deser (1972). The fact that linearized theory becomes singular as $m_g \to 0$ was discovered by Vainshtein (1972). The radiation of massive gravitons in linearized theory is discussed by van Niewenhuizen (1973). Discussions of the fate of the discontinuity are given in Deffayet, Dvali, Gabadadze and Vainshtein (2002) and in Arkani-Hamed, Georgi and Schwartz (2003). The difficulty of performing the matching to an asymptotically flat solution, and the possibility of matching to a De Sitter solution, is discussed in Damour, Kogan and Papazoglou (2003). The fact that beyond linearized theory the trace h becomes a ghost is discussed by Boulware and Deser (1972) and, in full generality, by Creminelli, Nicolis, Papucci and Trincherini (2005).
- Lorentz-violating mass terms for $h_{\mu\nu}$ are discussed in Arkani-Hamed, Cheng, Luty and Mukohyama (2004), Rubakov (2004) and Dubovsky, Tinyakov and Tkachev (2005). In this case the mass of the scalar perturbations can be zero while the mass of the graviton $h_{ij}^{\rm TT}$ can be non-zero, and the bounds on the graviton mass derived from the Yukawa fall-off of the gravitational potential only refer to the scalar sector. Furthermore, these models do not suffer of the vDVZ discontinuity and do not have ghosts.

  A bound on the mass that refers directly to $h_{ij}^{\rm TT}$ can be obtained from pulsar timing, as recognized in Damour and Taylor (1991) and discussed quantitatively in Finn and Sutton (2002). The possibility of bounding the mass of $h_{ij}^{\rm TT}$ from the observation of inspiraling compact binaries is discussed in Will (1998) and Larson and Hiscock (2000).

3 Generation of GWs in linearized theory

We now consider the generation of GWs in the context of linearized theory. This means that we assume that the gravitational field generated by the source is sufficiently weak, so that an expansion around *flat* space-time is justified. For a system held together by gravitational forces, this implies that the typical velocities inside the source are small. For instance, in a gravitationally bound two-body system with reduced mass μ and total mass m, we have $E_{\rm kin} = -(1/2)U$, i.e.

$$\frac{1}{2}\mu v^2 = \frac{1}{2}\frac{G\mu m}{r}\,, \tag{3.1}$$

and therefore

$$\frac{v^2}{c^2} = \frac{R_S}{2r}\,, \tag{3.2}$$

where $R_S = 2Gm/c^2$ is the Schwarzschild radius associated to a mass m. A weak gravitational field means $R_S/r \ll 1$ and therefore $v \ll c$. Thus, for a self-gravitating system, weak fields imply small velocities. On the other hand, for a system whose dynamics is determined by non-gravitational forces, the weak-field expansion and the low-velocity expansion are independent, and in this case it makes sense to consider weak-field sources with arbitrary velocities, as we do in this chapter. This will allow us to understand, in the simple setting of a flat background space-time (and therefore Newtonian or at most special-relativistic dynamics for the sources), how GWs are produced. In Section 3.1 we will derive the formulas for GW production valid in flat space-time, but exact in v/c. Then, expanding the exact result in powers of v/c, we will see that for small velocities the GW production can be organized in a multipole expansion (Section 3.2). In Section 3.3 we discuss in detail the lowest order term, which is the quadrupole radiation. In Section 3.4 we discuss the next-to-leading terms, i.e. the mass-octupole and the current quadrupole radiation, and in Section 3.5 we present the systematic multipole expansion to all orders, using first the formalism of symmetric-trace-free (STF) tensors, and then the spherical tensors formalism. Finally, in a Solved Problems section we discuss some applications of this formalism and we collect additional technical material.

The most interesting astrophysical sources of GWs, such as neutron stars, black holes or compact binaries, are self-gravitating systems. In this case, if we want to compute corrections in v/c, we must take into account that, because of eq. (3.2), space-time cannot be considered flat beyond lowest order, and therefore the dynamics of the sources can no longer be described by Newtonian gravity. The corresponding formalism is the GW generation from post-Newtonian sources, which will be the subject of Chapter 5. Still, the results derived in the present chapter will be useful also as a first step toward the understanding of the post-Newtonian results.

3.1 Weak-field sources with arbitrary velocity

In linearized theory the starting point is eq. (1.24), that we recall here,

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}\,, \tag{3.3}$$

where $T_{\mu\nu}$ is the energy–momentum tensor of matter. Recall also that we are in the Lorentz gauge, $\partial^\mu \bar{h}_{\mu\nu} = 0$, and that $T_{\mu\nu}$ satisfies the flat-space conservation law $\partial^\mu T_{\mu\nu} = 0$. Equation (3.3) is linear in $h_{\mu\nu}$ and can be solved by the method of Green's function: if $G(x - x')$ is a solution of the equation

$$\Box_x G(x - x') = \delta^4(x - x')\,, \tag{3.4}$$

(where $\Box_x$ is the d'Alembertian operator with derivatives taken with respect to the variable x), then the corresponding solution of eq. (3.3) is

$$\bar{h}_{\mu\nu}(x) = -\frac{16\pi G}{c^4} \int d^4x'\, G(x - x') T_{\mu\nu}(x')\,. \tag{3.5}$$

The solution of eq. (3.4) depends of course on the boundary conditions that we impose. Just as in electromagnetism, for a radiation problem the appropriate solution is the *retarded Green's function*,[1]

$$G(x - x') = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|}\,\delta(x^0_{\rm ret} - x'^0)\,, \tag{3.6}$$

where $x'^0 = ct'$, $x^0_{\rm ret} = ct_{\rm ret}$, and

$$t_{\rm ret} = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c} \tag{3.7}$$

is called retarded time. Then the solution of eq. (3.3) is

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{\mu\nu}\left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}'\right). \tag{3.8}$$

Outside the source we can put this solution in the TT gauge using eq. (1.40), $h^{\rm TT}_{ij} = \Lambda_{ij,kl} h_{kl} = \Lambda_{ij,kl} \bar{h}_{kl}$ (in the last equality we used the

[1] More precisely, the retarded Green's function is selected by imposing the Kirchoff–Sommerfeld "no-incoming-radiation" boundary conditions, i.e. one imposes

$$\lim_{t \to -\infty} \left[\frac{\partial}{\partial r} + \frac{\partial}{c\partial t}\right] (r\bar{h}_{\mu\nu})(\mathbf{x}, t) = 0\,,$$

where the limit is taken along any surface $ct + r$ = constant, together with the condition that $r\bar{h}_{\mu\nu}$ and $r\partial_\rho \bar{h}_{\mu\nu}$ be bounded in this limit. Physically, this means that there is no incoming radiation at past null infinity.

property (1.38) of the Lambda tensor). Therefore, outside the source,

$$h^{\rm TT}_{ij}(t, \mathbf{x}) = \frac{4G}{c^4} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \int d^3x' \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{kl}\left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}'\right), \tag{3.9}$$

where we use the notation $\hat{\mathbf{x}} = \hat{\mathbf{n}}$, and we will also denote $|\mathbf{x}| = r$. Observe that $h^{\rm TT}_{ij}$ depends only the integrals of the spatial components T_{kl}. The underlying reason that allowed us to eliminate T_{0k} and T_{00} is that they are related to T_{kl} by the conservation of the energy–momentum tensor.[2] If we denote by d the typical radius of the source, at $r \gg d$ we can expand

$$|\mathbf{x} - \mathbf{x}'| = r - \mathbf{x}' \cdot \hat{\mathbf{n}} + O\left(\frac{d^2}{r}\right), \tag{3.10}$$

see Fig. 3.1. We are particularly interested in the value of $h^{\rm TT}_{ij}$ at large distances from the source, where the detector is located, so we take the limit $r \to \infty$ at fixed t,[3] and we retain only the leading term in eq. (3.9). This is a term $O(1/r)$, obtained setting $|\mathbf{x} - \mathbf{x}'| = r$ in the denominator of eq. (3.9), so at large distances

$$h^{\rm TT}_{ij}(t, \mathbf{x}) = \frac{1}{r}\frac{4G}{c^4} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \int d^3x'\, T_{kl}\left(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \hat{\mathbf{n}}}{c}, \mathbf{x}'\right), \tag{3.11}$$

plus terms $O(1/r^2)$ that we neglect. We now write T_{kl} in terms of its Fourier transform,[4]

$$T_{kl}(t, \mathbf{x}) = \int \frac{d^4k}{(2\pi)^4} \tilde{T}_{kl}(\omega, \mathbf{k}) e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}\,. \tag{3.12}$$

Then

$$\begin{aligned} &\int d^3x'\, T_{kl}\,(t - r/c + \mathbf{x}' \cdot \hat{\mathbf{n}}/c, \mathbf{x}') \\ &= \int d^3x' \int \frac{d\omega}{2\pi c} \frac{d^3k}{(2\pi)^3} \tilde{T}_{kl}(\omega, \mathbf{k}) e^{-i\omega(t - r/c)} e^{i(\mathbf{k} - \omega\hat{\mathbf{n}}/c)\cdot\mathbf{x}'} \\ &= \int \frac{d\omega}{2\pi c} \frac{d^3k}{(2\pi)^3} \tilde{T}_{kl}(\omega, \mathbf{k}) e^{-i\omega(t - r/c)} (2\pi)^3 \delta^{(3)}(\mathbf{k} - \omega\hat{\mathbf{n}}/c) \\ &= \int \frac{d\omega}{2\pi c} \tilde{T}_{kl}\,(\omega, \omega\hat{\mathbf{n}}/c)\, e^{-i\omega(t - r/c)}\,, \end{aligned} \tag{3.13}$$

and we get

$$h^{\rm TT}_{ij}(t, \mathbf{x}) = \frac{1}{r}\frac{4G}{c^5} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{T}_{kl}(\omega, \omega\hat{\mathbf{n}}/c) e^{-i\omega(t - r/c)}\,. \tag{3.14}$$

[2] Indeed, when performing the multipole expansion below, the lowest-order result will be re-expressed in terms of T^{00} only, using again energy–momentum conservation.

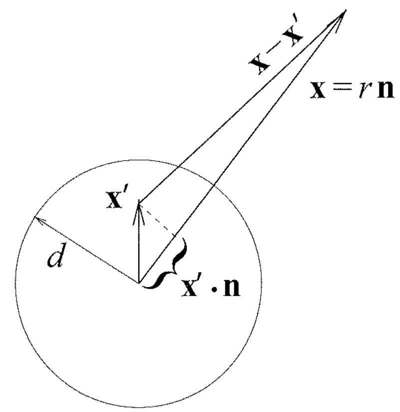

Fig. 3.1 A graphical illustration of the relation given in eq. (3.10).

[3] In linearized theory, GWs are studied at spatial infinity, i.e. $r \to \infty$ at fixed t. We will see in Chapter 5 that, beyond linearized theory, it can be more convenient to work at future null infinity, i.e. $r \to \infty$ with $t - r/c$ fixed.

[4] Our convention on the factors of c is that the four-dimensional wave-vector is $k^\mu = (\omega/c, \mathbf{k})$, and therefore $d^4k = (1/c)d\omega\, d^3k$. Since $x^\mu = (ct, \mathbf{x})$, we then have $k_\mu x^\mu = -\omega t + \mathbf{k}\cdot\mathbf{x}$. Observe that $\mathbf{k}$ has dimensions of the inverse of length. The spatial momentum of a particle with wave-vector $\mathbf{k}$ is $\mathbf{p} = \hbar\mathbf{k}$.

In general, the Fourier components of the energy–momentum tensor of the source will be large around a typical value ω_s, and the characteristic speed at which there is a bulk movement of mass across the source is $v \sim \omega_s d$. For the moment we have made no assumption on the relative values of ω_s and d, and in particular we have not assumed $\omega_s d \ll c$. Therefore eq. (3.14) is valid both for relativistic and for non-relativistic sources, as long as linearized theory applies, and we are at a sufficiently large distance r from the source.

From eq. (1.156), setting $dA = r^2 d\Omega$, we see that the total energy radiated per unit solid angle is

$$\frac{dE}{d\Omega} = \frac{r^2 c^3}{32\pi G} \int_{-\infty}^{\infty} dt\, \dot{h}_{ij}^{\rm TT} \dot{h}_{ij}^{\rm TT} . \tag{3.15}$$

Inserting here the expression (3.14), using $\tilde{T}(-\omega, -\mathbf{k}) = \tilde{T}^*(\omega, \mathbf{k})$ and the property (1.37) of the Lambda tensor, we find[5]

$$\boxed{\frac{dE}{d\Omega} = \frac{G}{2\pi^2 c^7} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \int_0^{\infty} d\omega\, \omega^2 \tilde{T}_{ij}(\omega, \omega\hat{\mathbf{n}}/c) \tilde{T}^*_{kl}(\omega, \omega\hat{\mathbf{n}}/c)\,,} \tag{3.16}$$

and the energy spectrum is therefore

$$\frac{dE}{d\omega} = \frac{G\omega^2}{2\pi^2 c^7} \int d\Omega\, \Lambda_{ij,kl}(\hat{\mathbf{n}}) \tilde{T}_{ij}(\omega, \omega\hat{\mathbf{n}}/c) \tilde{T}^*_{kl}(\omega, \omega\hat{\mathbf{n}}/c)\,. \tag{3.17}$$

A typical source will radiate for a characteristic time Δt. In the idealized case of an exactly monochromatic source, the radiation lasts for $\Delta t = \infty$ and the total radiated energy is formally divergent.[6] Thus, for a monochromatic source the instantaneously radiated power is a more useful quantity. For such a source, radiating at a frequency ω_0, we write $\tilde{T}_{ij}(\omega, \mathbf{k})$ (for positive ω) as

$$\tilde{T}_{ij}(\omega, \mathbf{k}) = \theta_{ij}(\omega, \mathbf{k})\, 2\pi\delta(\omega - \omega_0)\,, \tag{3.18}$$

and eq. (3.16) becomes

$$\frac{dE}{d\Omega} = \frac{G\omega_0^2}{\pi c^7} \Lambda_{ij,kl}(\hat{\mathbf{n}})\, \theta_{ij}(\omega_0, \omega_0\hat{\mathbf{n}}/c) \theta^*_{kl}(\omega_0, \omega_0\hat{\mathbf{n}}/c) T\,. \tag{3.19}$$

We have used $2\pi\delta(\omega = 0) = T$, where T is the total (formally infinite) time. Dividing by T we obtain the power radiated instantaneously,[7]

$$\frac{dP}{d\Omega} = \frac{G\omega_0^2}{\pi c^7} \Lambda_{ij,kl}(\hat{\mathbf{n}})\, \theta_{ij}(\omega_0, \omega_0\hat{\mathbf{n}}/c) \theta^*_{kl}(\omega_0, \omega_0\hat{\mathbf{n}}/c)\,. \tag{3.20}$$

The total power is obtained by integrating over $d\Omega$. To perform the integration one can use the identities

$$\int \frac{d\Omega}{4\pi} n_i n_j = \frac{1}{3}\delta_{ij}\,, \tag{3.21}$$

[5] To compare with Weinberg (1972), Section 10.4, observe that we define the Fourier transform with respect to frequency using $d\omega/(2\pi)$ (see eq. (3.12), or the Notation section) while Weinberg uses $d\omega$; on the other hand, we both use $d^3k/(2\pi)^3$ in the spatial Fourier transform. Therefore our $\tilde{T}(\omega, \mathbf{k})$ is equal to $2\pi\, \tilde{T}^{\rm Weinberg}(\omega, \mathbf{k})$.

[6] Of course, at the latest this divergence is cutoff by the back-reaction due to GW emission. For example, a spinning neutron star with non-vanishing ellipticity emits GWs. The energy of these waves is taken from the rotational energy of the star which therefore gradually slows down, as we will compute in detail in Section 4.2. (Actually, in this case electromagnetic effects dominate and slow down the neutron star even earlier.)

[7] More precisely, we have seen in Chapter 1 that the GW energy is only defined by averaging over a few periods, so this is really the average power radiated over one period of the source motion.

$$\int \frac{d\Omega}{4\pi} n_i n_j n_k n_l = \frac{1}{15}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\,. \tag{3.22}$$

These identities, as well as their generalization to an arbitrary number of n's, can be found as follows. For an odd number of n_i the integral vanishes because the integrand is odd under parity. For an even number of n, we use the fact that the tensor $n_{i_1} n_{i_2} \ldots n_{i_{2l}}$ is totally symmetric and therefore its integral can only depend on the totally symmetrized product of Kronecker deltas. Once the tensor structure is fixed, the overall constant is obtained by contracting all indices. This gives

$$\int \frac{d\Omega}{4\pi} n_{i_1} \ldots n_{i_{2l}} = \frac{1}{(2l+1)!!}\left(\delta_{i_1 i_2}\delta_{i_3 i_4} \ldots \delta_{i_{2l-1} i_{2l}} + \ldots\right)\,, \tag{3.23}$$

where the final dots denote all possible pairing of indices.

3.2 Low-velocity expansion

Just as in electrodynamics, the equations for the generation of radiation are greatly simplified if the typical velocities inside the source are small compared to the speed of light. If ω_s is the typical frequency of the motion inside the source and d is the source size, the typical velocities inside the source are $v \sim \omega_s d$. The frequency ω of the radiation will also be of order[8] ω_s and therefore $\omega \sim \omega_s \sim v/d$. In terms of $\lambdabar = c/\omega$,

$$\lambdabar \sim \frac{c}{v} d\,. \tag{3.24}$$

In a non-relativistic system, $v \ll c$ and the reduced wavelength of the radiation generated is much bigger than the size of the system:

$$\boxed{\text{non-relativistic sources} \quad \Longrightarrow \quad \lambdabar \gg d\,.} \tag{3.25}$$

When the reduced wavelength is much bigger than the size of the system, it is physically clear that we do not need to know the internal motions of the source in all its details, but only the coarse features matter, so the emission of radiation is governed by the lowest multipole moments.[9]

To perform the multipole expansion for gravitational radiation we start from the expression of $h_{ij}^{\rm TT}$ at spatial infinity given in eq. (3.11), that we recall here,

$$h_{ij}^{\rm TT}(t, \mathbf{x}) = \frac{1}{r}\frac{4G}{c^4} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \int d^3x'\, T_{kl}\left(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \hat{\mathbf{n}}}{c}, \mathbf{x}'\right)\,, \tag{3.26}$$

and we write T_{kl} in terms of its Fourier transform,

$$T_{kl}\left(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \hat{\mathbf{n}}}{c}, \mathbf{x}'\right) = \int \frac{d^4k}{(2\pi)^4} \tilde{T}_{kl}(\omega, \mathbf{k}) e^{-i\omega(t - r/c + \mathbf{x}'\cdot\hat{\mathbf{n}}/c) + i\mathbf{k}\cdot\mathbf{x}'}\,. \tag{3.27}$$

For a non-relativistic source, $\tilde{T}_{kl}(\omega, \mathbf{k})$ is peaked around a typical frequency ω_s (or around a range of frequencies, with maximum value ω_s),

[8] Apart from numerical factors which depend on the multipole moment involved and on the details of the motion of the source. We will see below that for a non-relativistic system the dominant contributions come from the lowest multipoles, and for these the numerical factors are $O(1)$; for instance, we will see that a source performing a simple harmonic oscillation at frequency ω_s emits quadrupole radiation $\omega = 2\omega_s$.

[9] A typical example is the electromagnetic radiation from the hydrogen atom. The velocity of the electron inside the hydrogen atom is $v/c \sim \alpha$, where $\alpha \simeq 1/137$ is the fine-structure constant, and the reduced wavelengths of the transitions between the levels of the hydrogen atom are of order $\lambdabar \sim r_B/\alpha$, where r_B is the Bohr radius. Since $\alpha \ll 1$, we have $\lambdabar \gg r_B$, and the multipole expansion is adequate.

with $\omega_s d \ll c$. On the other hand, the energy–momentum tensor is non-vanishing only inside the source, so the integral in eq. (3.26) is restricted to $|\mathbf{x}'| \leqslant d$. Then the dominant contribution to $h_{ij}^{\rm TT}$ comes from frequencies ω that satisfy

$$\frac{\omega}{c}\mathbf{x}'\cdot\hat{\mathbf{n}} \lesssim \frac{\omega_s d}{c} \ll 1\,, \tag{3.28}$$

and therefore we can expand the exponential in eq. (3.27),

$$e^{-i\omega(t-r/c+\mathbf{x}'\cdot\hat{\mathbf{n}}/c)} = e^{-i\omega(t-r/c)} \times\left[1 - i\frac{\omega}{c}x'^i n^i + \frac{1}{2}\left(-i\frac{\omega}{c}\right)^2 x'^i x'^j n^i n^j + \ldots\right]. \tag{3.29}$$

This is equivalent to expanding

$$T_{kl}\left(t-\frac{r}{c}+\frac{\mathbf{x}'\cdot\hat{\mathbf{n}}}{c},\mathbf{x}'\right) \simeq T_{kl}(t-\frac{r}{c},\mathbf{x}') + \frac{x'^i n^i}{c}\,\partial_0 T_{kl} + \frac{1}{2c^2}\,x'^i x'^j n^i n^j\,\partial_0^2 T_{kl} + \ldots \tag{3.30}$$

where all derivatives are evaluated at the point $(t-r/c,\mathbf{x}')$.[10] We now define the momenta of the stress tensor T^{ij},

$$S^{ij}(t) = \int d^3x\, T^{ij}(t,\mathbf{x})\,, \tag{3.31}$$

$$S^{ij,k}(t) = \int d^3x\, T^{ij}(t,\mathbf{x})\, x^k\,, \tag{3.32}$$

$$S^{ij,kl}(t) = \int d^3x\, T^{ij}(t,\mathbf{x})\, x^k x^l\,, \tag{3.33}$$

and similarly for all higher order momenta. In this notation, a comma separates the spatial indices which originates from T^{ij} from the indices coming from $x^{i_1}\ldots x^{i_N}$.[11] The energy–momentum tensor of matter that appears in eq. (3.3) is the one obtained from the variation of the matter action with respect to the metric, so it is in its symmetric form, $T^{ij} = T^{ji}$. Then, its momenta are symmetric separately in the first type of indices and in the second, e.g. $S^{ij,k} = S^{ji,k}$ or $S^{ij,kl} = S^{ij,lk}$, but not necessarily under the exchange of two indices of different type, e.g. in general $S^{ij,k} \neq S^{ik,j}$.

Inserting the expansion (3.30) into eq. (3.26) we get

$$h_{ij}^{\rm TT}(t,\mathbf{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\hat{\mathbf{n}}) \times\left[S^{kl} + \frac{1}{c}n_m\dot{S}^{kl,m} + \frac{1}{2c^2}n_m n_p \ddot{S}^{kl,mp} + \ldots\right]_{\rm ret}\,, \tag{3.34}$$

where the subscript "ret" means that the quantities S^{kl}, $\dot{S}^{kl,m}$, $\ddot{S}^{kl,mp}$, etc. are evaluated at retarded time $t-r/c$. This equation is the basis for

[10] One could have directly written the expansion (3.30), as a formal Taylor expansion in the parameter $\mathbf{x}'\cdot\hat{\mathbf{n}}/c$. However, the above derivation emphasizes that the assumption behind this expansion is the condition $\omega_s d \ll c$, with ω_s the typical source frequency.

[11] Observe that, contrary to most of the literature on general relativity, we never use commas to denote derivatives (nor semicolons to denote covariant derivatives).

the multipole expansion. From the definitions (3.31)–(3.32) we see that, with respect to S^{kl}, $S^{kl,m}$ has an additional factor $x^m \sim O(d)$, while each time derivative brings a factor $O(\omega_s)$. So, with respect to S^{kl}, the tensor $\dot{S}^{kl,m}$ has an additional factor $O(\omega_s d)$, i.e. $O(v)$, where v is a typical velocity inside the source. Then the term $(1/c)n_m\dot{S}^{kl,m}$ is a correction $O(v/c)$ to the term S^{kl}, and similarly the term $(1/c^2)n_m n_p\ddot{S}^{kl,mp}$ is a correction $O(v^2/c^2)$, etc.

The physical meaning of the various terms in this expansion becomes more clear if we eliminate the momenta of T^{ij} in favor of the momenta of the energy density T^{00}, and of the momenta of the linear momentum, T^{0i}/c. We define the momenta of T^{00}/c^2 by[12]

$$M = \frac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x})\,, \tag{3.35}$$

$$M^i = \frac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x})\, x^i\,, \tag{3.36}$$

$$M^{ij} = \frac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x})\, x^i x^j\,, \tag{3.37}$$

$$M^{ijk} = \frac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x})\, x^i x^j x^k\,, \tag{3.38}$$

and so on, while the momenta of the momentum density $(1/c)T^{0i}$ are denoted by

$$P^i = \frac{1}{c}\int d^3x\, T^{0i}(t,\mathbf{x})\,, \tag{3.39}$$

$$P^{i,j} = \frac{1}{c}\int d^3x\, T^{0i}(t,\mathbf{x})\, x^j\,, \tag{3.40}$$

$$P^{i,jk} = \frac{1}{c}\int d^3x\, T^{0i}(t,\mathbf{x})\, x^j x^k\,, \tag{3.41}$$

and similarly for the higher momenta. The time derivatives of these quantities and of the momenta of T^{ij} satisfy relations which follow from energy–momentum conservation. Recall that we are working within linearized theory, which means that the energy–momentum tensor of matter $T^{\mu\nu}$ satisfies the flat-space equation $\partial_\mu T^{\mu\nu} = 0$, as we have seen in eq. (1.25), while non-linearities such as those written schematically in eq. (2.113) are neglected. This means that we are also neglecting the back-action of the GWs on the source.[13]

To obtain these identities, we take a box of volume V larger than the source, and we denote its boundary by ∂V (so $T_{\mu\nu}$ vanishes on ∂V). Using $\partial_\mu T^{\mu 0} = 0$, that is

$$\partial_0 T^{00} = -\partial_i T^{0i}\,, \tag{3.42}$$

(and recalling that $\dot{M} \equiv \partial M/\partial t = c\partial_0 M$) we have

$$c\dot{M} = \int_V d^3x\, \partial_0 T^{00} = -\int_V d^3x\, \partial_i T^{0i}$$

[12] Dimensionally T^{00}/c^2 is a mass density but of course, besides the contribution due to the rest mass of the source, it contains also all contributions to T^{00} coming from the kinetic energy of the particles which make up the source, contributions from the potential energy, etc. For sources that generate a strong gravitational field, such as neutron stars, the gravitational binding energy will also be important. Only for weak-field sources and in the non-relativistic limit, T^{00}/c^2 becomes the mass density. However, since the multipole expansion of the linearized theory assumes weak fields and is a non-relativistic expansion, to lowest order in v/c we can actually replace T^{00}/c^2 with the mass density.

[13] The inclusion of these non-linearities will be discussed in Chapter 5.

$$
\begin{aligned}
&= -\int_{\partial V} dS^i\, T^{0i} \\
&= 0\,.
\end{aligned} \tag{3.43}
$$

The last equality follows from the fact that T^{0i} vanishes on the boundary ∂V, since we have taken the volume V larger than the volume of the source. Of course, a physical system that radiates GWs loses mass. The conservation of the mass M of the radiating body, expressed by eq. (3.43), is due to the fact that in the linearized approximation the back action of the source dynamics due to the energy carried away by the GWs is neglected. Similarly, we obtain the identity

$$
\begin{aligned}
c\dot{M}^i &= \int_V d^3x\, x^i \partial_0 T^{00} \\
&= -\int_V d^3x\, x^i \partial_j T^{0j} \\
&= \int_V d^3x\, (\partial_j x^i) T^{0j} \\
&= \int_V d^3x\, \delta^i_j\, T^{0j} \\
&= cP^i\,.
\end{aligned} \tag{3.44}
$$

In the same way one derives similar identities for the higher momenta of T^{00} and of T^{0i}. For the first few lowest-order momenta of T^{00} we get

$$\dot{M} = 0\,, \tag{3.45}$$

$$\dot{M}^i = P^i\,, \tag{3.46}$$

$$\dot{M}^{ij} = P^{i,j} + P^{j,i}\,, \tag{3.47}$$

$$\dot{M}^{ijk} = P^{i,jk} + P^{j,ki} + P^{k,ij}\,, \tag{3.48}$$

while the lowest-order momenta of T^{0i} satisfy

$$\dot{P}^i = 0\,, \tag{3.49}$$

$$\dot{P}^{i,j} = S^{ij}\,, \tag{3.50}$$

$$\dot{P}^{i,jk} = S^{ij,k} + S^{ik,j}\,. \tag{3.51}$$

The equations $\dot{M} = 0$ and $\dot{P}^i = 0$ are the conservation of the mass and of the total momentum of the source. Another interesting identity is $\dot{P}^{i,j} - \dot{P}^{j,i} = S^{ij} - S^{ji} = 0$, which follows from eq. (3.50) using the fact that S^{ij} is a symmetric tensor, and is the conservation of the angular momentum of the source.

We can now combine these identities to express the momenta S^{ij}, $\dot{S}^{ij,k}$, etc., that appear in the multipole expansion, in terms of the two sets of momenta $\{M, M^i, M^{ij}, \ldots\}$ and $\{P^i, P^{i,j}, \ldots\}$, which have a more immediate physical interpretation.[14] Taking the time derivative of eq. (3.47) and using eq. (3.50), as well as the fact that $S^{ij} = S^{ji}$, we obtain the identity

$$\boxed{S^{ij} = \frac{1}{2}\ddot{M}^{ij}\,.} \tag{3.52}$$

14 In particular, the momenta of T^{ij}, i.e. $\{S^{ij}, S^{ij,k}, \ldots\}$ depend on the distribution of the stresses inside the body, which might be difficult to determine, while the total mass of a body, its mass quadrupole, etc. can be measured more easily.

If we combine eq. (3.48) with eq. (3.51) instead, we get

$$\dddot{M}^{ijk} = 2(\dot{S}^{ij,k} + \dot{S}^{ik,j} + \dot{S}^{jk,i})\,. \tag{3.53}$$

From eq. (3.51) it also follows that $\ddot{P}^{i,jk} = \dot{S}^{ij,k} + \dot{S}^{ik,j}$. Using this relation and eq. (3.53) we can verify that

$$\boxed{\dot{S}^{ij,k} = \frac{1}{6}\dddot{M}^{ijk} + \frac{1}{3}\left(\ddot{P}^{i,jk} + \ddot{P}^{j,ik} - 2\ddot{P}^{k,ij}\right)\,.} \tag{3.54}$$

Equations (3.52) and (3.54) relate S^{ij} and $\dot{S}^{ij,k}$, which are the two lowest-order momenta appearing in the multipole expansion (3.34), to the momenta of T^{00} and of T^{0i}. One can proceed similarly with the higher-order terms. In the next two sections, we examine the leading and the next-to-leading terms, while in Section 3.5 we discuss systematically the expansion to all orders.

3.3 Mass quadrupole radiation

3.3.1 Amplitude and angular distribution

Using eq. (3.52), the leading term of the expansion (3.34) is

$$[h_{ij}^{\rm TT}(t,\mathbf{x})]_{\rm quad} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{\mathbf{n}})\ddot{M}^{kl}(t - r/c)\,. \tag{3.55}$$

From the point of view of the rotation group the tensor M_{kl}, as any symmetric tensor with two indices, decomposes into irreducible representations as

$$M^{kl} = \left(M^{kl} - \frac{1}{3}\delta^{kl}M_{ii}\right) + \frac{1}{3}\delta^{kl}M_{ii}\,, \tag{3.56}$$

where M_{ii} is the trace of M_{ij}. The first term is traceless by construction, and is a pure spin-2 operator, while the trace part is a scalar. Since the Lambda tensor $\Lambda_{ij,kl}$ gives zero when contracted with δ_{kl}, only the traceless term contributes. We use the notation

$$\rho = \frac{1}{c^2}T^{00}\,. \tag{3.57}$$

To lowest order in v/c, ρ becomes the mass density, see Note 12. We also introduce the quadrupole moment

$$
\begin{aligned}
Q^{ij} &\equiv M^{ij} - \frac{1}{3}\delta^{ij}M_{kk} \\
&= \int d^3x\, \rho(t,\mathbf{x})(x^i x^j - \frac{1}{3}r^2\delta^{ij})\,,
\end{aligned} \tag{3.58}
$$

and eq. (3.55) becomes

$$
\boxed{
\begin{aligned}
[h_{ij}^{\rm TT}(t,\mathbf{x})]_{\rm quad} &= \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{\mathbf{n}})\ddot{Q}_{kl}(t - r/c) \\
&\equiv \frac{1}{r}\frac{2G}{c^4}\ddot{Q}_{ij}^{\rm TT}(t - r/c)\,.
\end{aligned}
} \tag{3.59}
$$

Angular distribution of quadrupole radiation

In order to obtain the waveform emitted into an arbitrary direction $\hat{\mathbf{n}}$, we could in principle plug the explicit expression (1.39) of the Lambda tensor into eq. (3.59), and perform the contraction with $\ddot{Q}_{kl}$. It is however more instructive to proceed as follows. First we observe that, when the direction of propagation $\hat{\mathbf{n}}$ of the GW is equal to $\hat{\mathbf{z}}$, P_{ij} is the diagonal matrix diag $(1,1,0)$, i.e. P_{ij} is a projector on the (x,y) plane. Writing $\Lambda_{ij,kl}$ in terms of P_{ij} using eq. (1.36) we have, for an arbitrary 3×3 matrix A_{kl},

$$\begin{aligned}\Lambda_{ij,kl}A_{kl} &= \left[P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\right]A_{kl}\\ &= (PAP)_{ij} - \frac{1}{2}P_{ij}\mathrm{Tr}\,(PA)\,.\end{aligned} \tag{3.60}$$

When P has the form

$$P = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{3.61}$$

we get

$$PAP = \begin{pmatrix} A_{11} & A_{12} & 0\\ A_{21} & A_{22} & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{3.62}$$

while $\mathrm{Tr}\,(PA) = A_{11}+A_{22}$. Therefore

$$\begin{aligned}\Lambda_{ij,kl}A_{kl} &= \begin{pmatrix} A_{11} & A_{12} & 0\\ A_{21} & A_{22} & 0\\ 0 & 0 & 0\end{pmatrix}_{ij} - \frac{A_{11}+A_{22}}{2}\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}_{ij}\\ &= \begin{pmatrix} (A_{11}-A_{22})/2 & A_{12} & 0\\ A_{21} & -(A_{11}-A_{22})/2 & 0\\ 0 & 0 & 0\end{pmatrix}_{ij}.\end{aligned} \tag{3.63}$$

Thus, when $\hat{\mathbf{n}} = \hat{\mathbf{z}}$,[15]

$$\Lambda_{ij,kl}\ddot{M}_{kl} = \begin{pmatrix} (\ddot{M}_{11}-\ddot{M}_{22})/2 & \ddot{M}_{12} & 0\\ \ddot{M}_{21} & -(\ddot{M}_{11}-\ddot{M}_{22})/2 & 0\\ 0 & 0 & 0\end{pmatrix}_{ij}. \tag{3.64}$$

From this we directly read the two polarization amplitudes, for a GW propagating in the z direction,

$$h_+ = \frac{1}{r}\frac{G}{c^4}(\ddot{M}_{11}-\ddot{M}_{22})\,, \tag{3.65}$$

$$h_\times = \frac{2}{r}\frac{G}{c^4}\ddot{M}_{12}\,, \tag{3.66}$$

where it is understood that the right-hand side is computed at the retarded time $t-r/c$. To compute the amplitudes for a wave that, in a frame with axes (x,y,z), propagates in a generic direction $\hat{\mathbf{n}}$, we introduce two unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, orthogonal to $\hat{\mathbf{n}}$ and to each other,

[15] We write the result in terms of the second mass moment M_{ij}, rather than in terms of the quadrupole moment Q_{ij}. Since $\Lambda_{ij,kl}Q_{kl} = \Lambda_{ij,kl}M_{kl}$ (because $\Lambda_{ij,kl}\delta_{kl}=0$, see eq. (1.38)), in the equations below we could use M_{ij} or Q_{ij} equivalently. Typically, it is slightly more practical to use M_{ij} when one makes explicit computations.

chosen so that $\hat{\mathbf{u}}\times\hat{\mathbf{v}} = \hat{\mathbf{n}}$ (so, when $\hat{\mathbf{n}}=\hat{\mathbf{z}}$, we can take $\hat{\mathbf{u}}=\hat{\mathbf{x}}$ and $\hat{\mathbf{v}}=\hat{\mathbf{y}}$), see Fig. 3.2. Then in the (x',y',z') frame, whose axes are in the directions $(\hat{\mathbf{u}},\hat{\mathbf{v}},\hat{\mathbf{n}})$, the wave propagates along the z' axis and we can use the previous result to read h_+ and $h_\times$,

$$h_+(t,\hat{\mathbf{n}}) = \frac{1}{r}\frac{G}{c^4}(\ddot{M}'_{11}-\ddot{M}'_{22})\,, \tag{3.67}$$

$$h_\times(t,\hat{\mathbf{n}}) = \frac{2}{r}\frac{G}{c^4}\ddot{M}'_{12}\,, \tag{3.68}$$

where M'_{ij} are the components of the second mass moment in the frame (x',y',z').[16] These can be related to the components M_{ij} in the (x,y,z) frame observing that in the (x',y',z') frame the vector $\hat{\mathbf{n}}$ has coordinates $n'_i=(0,0,1)$, while in the (x,y,z) frame it has coordinates

$$n_i = (\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)\,, \tag{3.69}$$

Then the components n_i and n'_i are related by a rotation matrix $\mathcal{R}$ such that $n_i = \mathcal{R}_{ij}n'_j$, whose explicit expression is

$$\mathcal{R} = \begin{pmatrix} \cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}. \tag{3.70}$$

Similarly, a tensor $\mathbf{M}$ with two indices has components M_{ij} in the (x,y,z) frame and M'_{ij} in the (x',y',z') frame, related by

$$M_{ij} = \mathcal{R}_{ik}\mathcal{R}_{jl}M'_{kl}\,, \tag{3.71}$$

or, solving for M', $M'_{ij} = (\mathcal{R}^TM\mathcal{R})_{ij}$, where $\mathcal{R}^T$ is the transpose matrix. Inserting $\mathcal{R}$ from eq. (3.70), and plugging the resulting values of M'_{ij} into eqs. (3.67) and (3.68), we get

$$\begin{aligned}h_+(t;\theta,\phi) = \frac{1}{r}\frac{G}{c^4}\big[\ &\ddot{M}_{11}(\cos^2\phi - \sin^2\phi\cos^2\theta)\\ &+\ddot{M}_{22}(\sin^2\phi-\cos^2\phi\cos^2\theta)\\ &-\ddot{M}_{33}\sin^2\theta\\ &-\ddot{M}_{12}\sin 2\phi\,(1+\cos^2\theta)\\ &+\ddot{M}_{13}\sin\phi\,\sin 2\theta\\ &+\ddot{M}_{23}\cos\phi\,\sin 2\theta\big]\,,\\ h_\times(t;\theta,\phi) = \frac{1}{r}\frac{G}{c^4}\big[\ &(\ddot{M}_{11}-\ddot{M}_{22})\sin 2\phi\cos\theta\\ &+2\ddot{M}_{12}\cos 2\phi\cos\theta\\ &-2\ddot{M}_{13}\cos\phi\sin\theta\\ &+2\ddot{M}_{23}\sin\phi\,\sin\theta\big]\,.\end{aligned} \tag{3.72}$$

This equation allows us to compute the angular distribution of the quadrupole radiation, once M_{ij} is given.

[16] Recall that h_+ and $h_\times$ are defined in terms of the components of h_{ij} in the plane transverse to the propagation direction. Therefore these are "the" polarization amplitudes, and are denoted by h_+ and $h_\times$, rather than h'_+ and $h'_\times$.

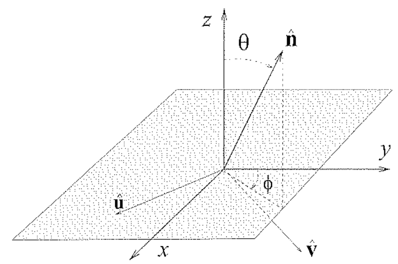

Fig. 3.2 The relation between the $(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$ frame and the $(\hat{\mathbf{u}},\hat{\mathbf{v}},\hat{\mathbf{n}})$ frame. The vector $\hat{\mathbf{u}}$ is in the $(\hat{\mathbf{x}},\hat{\mathbf{y}})$ plane, while $\hat{\mathbf{v}}$ points downward, with respect to the $(\hat{\mathbf{x}},\hat{\mathbf{y}})$ plane.

Absence of monopole and dipole gravitational radiation

We see from eq. (3.59) that the leading term of the multipole expansion is the mass quadrupole. There is neither monopole nor dipole radiation for GWs. This can be understood in two different ways. First of all, observe that a monopole term would depend on M and a dipole term on P^i (the mass dipole moment M^i can be set to zero with a shift of the origin of the coordinate system). Furthermore, $h_{ij}^{\rm TT}$ depends on derivatives of the multipole moments, since a static source does not radiate. However, M and P^i are conserved quantities, so any contribution from M or P^i must vanish.

Actually, M and P^i are conserved only at the level of linearized theory: a radiating system loses mass and, in general, also linear momentum (see page 130). However, the absence of monopole and dipole radiation holds more generally, and is not restricted to linearized theory. One can verify this observing that, even when we include all non-linear terms, as in eq. (2.113), the right-hand side of the wave equation must still be conserved, to be consistent with the Lorentz gauge condition. We will see explicitly in Section 5.2 how to write the equations in such a form. Using that expression, see in particular eqs. (5.69), (5.71) and (5.72), one can verify that the derivation of the absence of monopole and dipole radiation goes through even in the full non-linear theory. The difference with linearized theory is that the lowest-order multipole that contributes, rather than being the quadrupole moment of the energy density of matter, T^{00}, is the quadrupole moment of a more general quantity τ^{00} that includes also the contribution of the gravitational field.

However, there is no need to enter into the details of the non-linear theory: the absence of monopole and dipole radiation is simply the expression of the fact that the graviton is a massless particle with helicity ± 2. We already showed in Problem 1.2 that it is impossible to put a graviton in a state with total angular momentum $j = 0$ or $j = 1$. This emerged as a consequence of the fact that the graviton is a massless particle with helicities ± 2, and it therefore obeys gauge conditions that eliminate the spurious degrees of freedom. Indeed, these conditions allowed us to reduce the five degrees of a traceless symmetric tensor h_{ij}, which would be appropriate for describing a *massive* spin-2 particle, to the two degrees of freedom of a *massless* particle, as we discussed in Section 2.2.2.

Since it is impossible to put a graviton in a state with total angular momentum $j = 0$ or $j = 1$, we can have neither monopole nor dipole radiation, since they correspond to a collection of quanta with $j = 0$ and $j = 1$, respectively. The situation is completely analogous to electrodynamics, where the photon is massless and has helicity ± 1, so it is impossible to put it in a state with total angular momentum $j = 0$ (see Problem 1.2, or Landau and Lifshitz, Vol. IV (1982), Section 6), and therefore monopole radiation is forbidden. In electromagnetism, the leading term of the multipole expansion is therefore dipole radiation.

3.3.2 Radiated energy

Inserting eq. (3.59) in eq. (1.153) and using the property (1.37) of the Lambda tensor, we find the power radiated per unit solid angle, in the quadrupole approximation,

$$\left(\frac{dP}{d\Omega}\right)_{\rm quad} = \frac{r^2c^3}{32\pi G}\langle \dot{h}_{ij}^{\rm TT}\dot{h}_{ij}^{\rm TT}\rangle = \frac{G}{8\pi c^5}\Lambda_{ij,kl}(\hat{\mathbf{n}})\,\langle \dddot{Q}_{ij}\dddot{Q}_{kl}\rangle\,, \tag{3.73}$$

where, as usual, the average is a temporal average over several characteristic periods of the GW, and it is understood that $\dddot{Q}_{ij}$ must be evaluated at the retarded time $t - r/c$. The angular integral can be performed observing that the dependence on $\hat{\mathbf{n}}$ is only in $\Lambda_{ij,kl}$. Using eqs. (3.21) and (3.22) we find

$$\int d\Omega\,\Lambda_{ij,kl} = \frac{2\pi}{15}\left(11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}\right)\,. \tag{3.74}$$

Then, the total radiated power (or, in notation used in astrophysics, the total gravitational luminosity $\mathcal{L}$ of the source) is, in the quadrupole approximation,

$$\boxed{P_{\rm quad} = \frac{G}{5c^5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle\,,} \tag{3.75}$$

where, again, $\dddot{Q}_{ij}$ must be evaluated at the retarded time $t-r/c$. This is the famous quadrupole formula, first derived by Einstein.[17] Sometimes, in explicit computations, it is more practical to use M_{ij} rather than Q_{ij}. Substituting $Q_{ij} = M_{ij} - (1/3)\delta_{ij}M_{kk}$ in eq. (3.75) we have

$$P_{\rm quad} = \frac{G}{5c^5}\langle \dddot{M}_{ij}\dddot{M}_{ij} - \frac{1}{3}(\dddot{M}_{kk})^2\rangle. \tag{3.76}$$

The same result could be obtained by observing that eq. (3.73) is still valid if we replace $\Lambda_{ij,kl}\dddot{Q}_{ij}\dddot{Q}_{kl}$ by $\Lambda_{ij,kl}\dddot{M}_{ij}\dddot{M}_{kl}$, since the contraction of $\Lambda_{ij,kl}$ with δ_{ij} (or with δ_{kl}) gives zero. However, when we use $\Lambda_{ij,kl}\dddot{M}_{ij}\dddot{M}_{kl}$, after integrating in $d\Omega$, on the right-hand side of eq. (3.74) the term $-4\delta_{ij}\delta_{kl}$ (which gave zero when contracted with $\dddot{Q}_{ij}\dddot{Q}_{kl}$) now contributes, since M_{ij} is not traceless, and we find eq. (3.76) again.

The energy radiated per unit solid angle is obtained by integrating the power, eq. (3.73), with respect to time. We write the quadrupole moment in terms of its Fourier transform,

$$Q_{ij}(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\tilde{Q}_{ij}(\omega)e^{-i\omega t}\,, \tag{3.77}$$

and, integrating eq. (3.73) with respect to time, we get

$$\boxed{\left(\frac{dE}{d\Omega}\right)_{\rm quad} = \frac{G}{8\pi^2c^5}\Lambda_{ij,kl}(\hat{\mathbf{n}})\int_0^{\infty}d\omega\,\omega^6\tilde{Q}_{ij}(\omega)\tilde{Q}^*_{kl}(\omega)\,,}$$

[17] Observe that some authors, e.g. Landau and Lifshitz, Vol. II (1979), define the quadrupole moment with a different normalization,

$$(Q_{ij})^{\rm LL} = \int d^3x\,\rho(t,\mathbf{x})(3x^ix^j - r^2\delta^{ij})\,,$$

where the superscript "LL" stands for Landau and Lifshitz. This is larger by a factor of 3 than our definition, eq. (3.58). In term of this quantity, the quadrupole formula therefore reads

$$P_{\rm quad} = \frac{G}{45c^5}\langle(\dddot{Q}_{ij}\dddot{Q}_{ij})^{\rm LL}\rangle\,,$$

and all other equations involving Q_{ij} must be rescaled similarly.

(3.78)

where the integral in $d\omega$ from $-\infty$ to $+\infty$ has been written as twice an integral from zero to ∞ using $\tilde{Q}_{ij}(-\omega) = \tilde{Q}^*_{ij}(\omega)$. Integrating over the solid angle we find the total radiated energy,

$$E_{\rm quad} = \frac{G}{5\pi c^5}\int_0^\infty d\omega\, \omega^6 \tilde{Q}_{ij}(\omega)\tilde{Q}^*_{ij}(\omega)\,, \tag{3.79}$$

and therefore the energy spectrum, integrated over the solid angle, is[18]

$$\left(\frac{dE}{d\omega}\right)_{\rm quad} = \frac{G}{5\pi c^5}\,\omega^6 \tilde{Q}_{ij}(\omega)\tilde{Q}^*_{ij}(\omega)\,. \tag{3.80}$$

[18]To compare this equation with the results in the literature, beside checking the factor of 3 in the normalization of Q_{ij}, one must also check whether or not the Fourier transform is defined using $d\omega/(2\pi)$, as we do, or simply $d\omega$, as, for instance, in Weinberg (1972) or in Straumann (2004).

For a monochromatic source, radiating at a frequency $\omega_0 > 0$, we proceed as in Section 3.1: we write, for positive ω,

$$\tilde{Q}_{ij}(\omega) = q_{ij}\, 2\pi\delta(\omega - \omega_0)\,, \tag{3.81}$$

insert this into eq. (3.78), and again use $2\pi\delta(\omega = 0) = T$, where T is the total (infinite) time interval. The instantaneous power generated by the monochromatic source is obtained by dividing by T, so

$$\left(\frac{dP}{d\Omega d\omega}\right)_{\rm quad} = \frac{G\omega_0^6}{4\pi c^5}\left(\Lambda_{ij,kl}q_{ij}q^*_{kl}\right)\delta(\omega-\omega_0)\,. \tag{3.82}$$

As for the linear momentum, inserting eq. (3.59) into eq. (1.164) we get

$$\frac{dP^i}{dt} = -\frac{G}{8\pi c^5}\int d\Omega\, \dddot{Q}^{\rm TT}_{ab}\, \partial^i \ddot{Q}^{\rm TT}_{ab}\,. \tag{3.83}$$

Under reflection, $\mathbf{x} \to -\mathbf{x}$, the quadrupole moment is invariant while $\partial^i \to -\partial^i$. Therefore the integrand is odd, and the angular integral vanishes. There is no loss of linear momentum in the quadrupole approximation. A non-vanishing result can be obtained by going beyond the quadrupole approximation, from the interference between multipoles of different parity, as we will see in Section 3.4.

3.3.3 Radiated angular momentum

The angular momentum carried away per unit time by GWs can be obtained by plugging the expression for $h_{ij}^{\rm TT}$ in the quadrupole approximation, eq. (3.59), into the general formula for the rate of angular momentum loss, eq. (2.61). Recalling that the first term in eq. (2.61) is the contribution form the orbital angular momentum L^i of the GWs while the second comes from the spin S^i of the field configuration, we write

$$\frac{dJ^i}{dt} = \frac{dL^i}{dt} + \frac{dS^i}{dt}\,. \tag{3.84}$$

For the orbital part we have

$$\left(\frac{dL^i}{dt}\right)_{\rm quad} = -\frac{c^3}{32\pi G}\,\epsilon^{ikl}\int d\Omega\, r^2 \langle \dot{h}^{\rm TT}_{ab}\, x^k \partial^l h^{\rm TT}_{ab}\rangle\,. \tag{3.85}$$

We then substitute $h_{ab}^{\rm TT}(t,\mathbf{x}) = (2G/rc^4)\Lambda_{ab,cd}(\hat{\mathbf{n}})\ddot{Q}_{cd}(t-r/c)$ and we perform the angular integral. The explicit computation is slightly involved, but we find it useful to perform it in detail. The uninterested reader can jump to the result, eq. (3.93).

When we compute $\partial^l h_{ab}^{\rm TT}$, the derivative ∂^l acts on $\Lambda_{ab,cd}(\hat{\mathbf{n}})$ (since $n^i = x^i/r$) as well as on $\ddot{Q}_{cd}(t-r/c)$. However,

$$\begin{aligned}\frac{\partial}{\partial x^l}\ddot{Q}_{cd}(t-r/c) &= \left(\frac{\partial r}{\partial x^l}\right)\frac{d}{dr}\ddot{Q}_{cd}(t-r/c)\\ &= -\frac{x^l}{r}\frac{1}{c}\dddot{Q}_{cd}(t-r/c)\,.\end{aligned} \tag{3.86}$$

In eq. (3.85) this therefore gives a contribution proportional to $\epsilon^{ikl}x^k x^l = 0$. The only non-vanishing term is obtained when ∂^l acts on $\Lambda_{ab,cd}(\hat{\mathbf{n}})$, and

$$\left(\frac{dL^i}{dt}\right)_{\rm quad} = -\frac{G}{2c^5}\,\epsilon^{ikl}\langle \dddot{Q}_{cd}\ddot{Q}_{fg}\rangle\int\frac{d\Omega}{4\pi}\,\Lambda_{ab,cd}\, x^k\partial^l\Lambda_{ab,fg}\,, \tag{3.87}$$

where it is understood that the derivatives of the quadrupole moment are evaluated at the retarded time $t - r/c$. We observe that

$$\begin{aligned}\frac{\partial n^m}{\partial x^l} &= \frac{\partial}{\partial x^l}\left(\frac{x^m}{r}\right)\\ &= \frac{\delta^{lm}}{r} - \frac{x^l x^m}{r^3}\\ &= \frac{1}{r}P^{lm}\,,\end{aligned} \tag{3.88}$$

where $P^{lm} = \delta^{lm} - n^l n^m$ is the projector first introduced in eq. (1.35).[19] Then

[19]As always, we do not need to be careful about raising and lowering spatial indices, since the spatial metric is δ_{ij}. Otherwise, we should write $\partial n^m/\partial x^l = (1/r)P_l^m$.

$$\begin{aligned}\partial^l\Lambda_{ab,fg} &= \frac{\partial n^m}{\partial x^l}\frac{\partial}{\partial n^m}\Lambda_{ab,fg}\\ &= \frac{1}{r}P^{lm}\frac{\partial}{\partial n^m}\Lambda_{ab,fg}\,.\end{aligned} \tag{3.89}$$

Using $\Lambda_{ab,fg}$ in the form (1.36) together with

$$\frac{\partial P^{ij}}{\partial n^m} = -(\delta^{im}n^j + \delta^{jm}n^i)\,, \tag{3.90}$$

which follows from the definition of P^{ij}, we find the identity

$$P^{lm}\frac{\partial}{\partial n^m}\Lambda_{ab,fg} = -(n_f\Lambda_{ab,lg} + n_g\Lambda_{ab,lf} + n_a\Lambda_{lb,fg} + n_b\Lambda_{la,fg})\,. \tag{3.91}$$

The last two terms give zero when contracted with the factor $\Lambda_{ab,cd}$ in eq. (3.87), since the Lambda tensor is transverse on all indices, $n_a\Lambda_{ab,cd} = n_b\Lambda_{ab,cd} = 0$. Then

$$\begin{aligned}\left(\frac{dL^i}{dt}\right)_{\rm quad} &= \frac{G}{2c^5}\,\epsilon^{ikl}\langle \dddot{Q}_{cd}\ddot{Q}_{fg}\rangle\int\frac{d\Omega}{4\pi}\,\Lambda_{ab,cd}\, n_k(n_f\Lambda_{ab,lg} + n_g\Lambda_{ab,lf})\\ &= \frac{G}{2c^5}\,\epsilon^{ikl}\langle \dddot{Q}_{cd}\ddot{Q}_{fg}\rangle\int\frac{d\Omega}{4\pi}\, n_k(n_f\Lambda_{cd,lg} + n_g\Lambda_{cd,lf})\,.\end{aligned} \tag{3.92}$$

The angular integral can now be computed by inserting the explicit form of the Λ tensor (1.39) and using the identities (3.21) and (3.22) (the term with six factors $\hat{\mathbf{n}}$ is proportional to $n^k n^l$ and vanishes when contracted with ϵ^{ikl}, so we only need the integrals with two and with four factors $\hat{\mathbf{n}}$).

The result is

$$\left(\frac{dL^i}{dt}\right)_{\text{quad}} = \frac{2G}{15c^5}\,\epsilon^{ikl}\langle \ddot{Q}_{ka}\dddot{Q}_{la}\rangle\,. \tag{3.93}$$

The calculation of the spin part gives

$$\begin{aligned}\left(\frac{dS^i}{dt}\right)_{\text{quad}} &= \frac{c^3}{16\pi G}\,\epsilon^{ikl}\int r^2 d\Omega\,\langle \dot{h}^{\text{TT}}_{al} h^{\text{TT}}_{ak}\rangle \\ &= \frac{G}{c^5}\,\epsilon^{ikl}\langle \dddot{Q}_{mn}\ddot{Q}_{cd}\rangle \int \frac{d\Omega}{4\pi}\,\Lambda_{al,mn}\Lambda_{ak,cd}\,. \end{aligned} \tag{3.94}$$

Using eq. (1.36) we can prove the identity

$$\Lambda_{al,mn}\Lambda_{ak,cd} = P_{ln}\Lambda_{mk,cd} - \frac{1}{2}P_{mn}\Lambda_{kl,cd}\,. \tag{3.95}$$

The second factor gives zero contracted with ϵ^{ikl} and, again using the identities eqs. (3.21) and (3.22), the remaining angular integration is straightforward,

$$\begin{aligned}\left(\frac{dS^i}{dt}\right)_{\text{quad}} &= \frac{G}{c^5}\,\epsilon^{ikl}\dddot{Q}_{mn}\ddot{Q}_{cd}\int \frac{d\Omega}{4\pi}\,P_{ln}\Lambda_{mk,cd} \\ &= \frac{4G}{15c^5}\,\epsilon^{ikl}\langle \ddot{Q}_{ka}\dddot{Q}_{la}\rangle\,. \end{aligned} \tag{3.96}$$

Summing the spin and orbital contribution, we finally get the angular momentum carried away, per unit time, by the GWs,

$$\boxed{\left(\frac{dJ^i}{dt}\right)_{\text{quad}} = \frac{2G}{5c^5}\,\epsilon^{ikl}\langle \ddot{Q}_{ka}\dddot{Q}_{la}\rangle\,,} \tag{3.97}$$

where we recall again that the derivatives of the quadrupole moment are evaluated at the retarded time $t - r/c$.

3.3.4 Radiation reaction on non-relativistic sources

We have seen that gravitational radiation carries away energy and angular momentum. Ultimately, this energy and angular momentum must come from the source. We therefore expect that the energy and angular momentum carried by the GWs, at a large distance r from the source and at time t, were drained at retarded time $t - r/c$ from the energy and the angular momentum of the source. In the full non-linear theory of gravity (so, in particular, when dealing with self-gravitating sources), one must however take into account non-linear effects in the GW propagation from the source to infinity (such as back-scattering of gravitons on the background space-time, graviton–graviton scattering, etc.). We will see in Section 5.3.4 that, as a result, at higher orders in the post-Newtonian expansion part of the gravitational radiation is delayed, and the total GW consists of a wavefront, which travels at the speed of light, and a "tail", which arrives later. Thus, it is not at all obvious that there is an exact equality, to all orders in the post-Newtonian expansion, between the instantaneous power radiated at large distances at a given time t, and the rate of energy loss of the source at the corresponding retarded time. We will discuss the issue in Section 5.3.5. However, as long as we are in linearized theory, the background space-time is flat, the wave propagates at the speed of light, and this energy balance argument is inescapable.

For $v/c \ll 1$, the leading term is given by quadrupole radiation, so the instantaneous rate of decreases of energy and orbital[20] angular momentum of the source must be given by eqs. (3.75) and (3.97), i.e.

$$\frac{dE_{\text{source}}}{dt} = -\frac{G}{5c^5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle\,, \tag{3.98}$$

$$\frac{dL^i_{\text{source}}}{dt} = -\frac{2G}{5c^5}\,\epsilon^{ikl}\langle \ddot{Q}_{ka}\dddot{Q}_{la}\rangle\,. \tag{3.99}$$

[20] Gravitational waves, as any field, carry away a total angular momentum J^i that, as we saw in Section 2.1.2, is made of a spin contribution and of an orbital angular momentum contribution. This total angular momentum is drained from the total angular momentum of the source which, for a macroscopic source, is a purely orbital angular momentum.

We have required that dE_{source}/dt, computed at retarded time $t - r/c$, be the negative of the power radiated at a large distance r in GWs, at time t. Since the latter is expressed in terms of $\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle$ evaluated at *retarded* time, as in eq. (3.75), we have obtained an equality between dE_{source}/dt and $-(G/5c^5)\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle$, both evaluated at the same value of retarded time or, equivalently, at the same value of time (and similarly for the angular momentum).

The multipole expansion assumes that the sources are non-relativistic so, at least to lowest order, the dynamics of the source is governed by Newtonian mechanics, and it should be possible to describe the back-reaction of the GWs on the source in terms of a force $\mathbf{F}$. Then, we expect that it should be possible to write eq. (3.98) in the form

$$\frac{dE_{\text{source}}}{dt} = \langle F_i v_i\rangle\,, \tag{3.100}$$

or more precisely, for an extended body,

$$\frac{dE_{\text{source}}}{dt} = \langle \int d^3x\,\frac{dF_i}{dV}\dot{x}_i\rangle\,, \tag{3.101}$$

where dF_i/dV is the force per unit volume. Similarly, we expect that eq. (3.99) can be written as

$$\frac{dL^i_{\text{source}}}{dt} = \langle T^i\rangle\,, \tag{3.102}$$

where T^i is the torque associated to the force per unit volume dF_i/dV. The average $\langle \ldots\rangle$ takes into account the fact that the energy and angular momentum of GWs are notions well defined only if we average over several periods. The expression for this force, in the quadrupole approximation, can be found as follows. Since inside $\langle\ldots\rangle$ we can integrate by parts (compare with Note 23 on page 35), we rewrite eq. (3.98) as

$$\frac{dE_{\rm source}}{dt} = -\frac{G}{5c^5}\langle\frac{dQ_{ij}}{dt}\frac{d^5Q_{ij}}{dt^5}\rangle\,. \tag{3.103}$$

From eq. (3.58) we have

$$\frac{dQ_{ij}}{dt} = \int d^3x'\,\partial_t\rho(t,\mathbf{x}')\left(x'_ix'_j - \frac{1}{3}r'^2\delta_{ij}\right)\,. \tag{3.104}$$

The term proportional to δ_{ij} gives zero when contracted with d^5Q_{ij}/dt^5. For a Newtonian source with $T^{00}/c^2 = \rho$ and $T^{0i}/c = \rho v^i$, the conservation of the energy–momentum tensor gives the continuity equation

$$\partial_t\rho + \partial_k(\rho v_k) = 0\,. \tag{3.105}$$

We then replace $\partial_t\rho$ by $-\partial_k(\rho v_k)$ in eq. (3.104) and we integrate by parts. The boundary term at infinity vanishes, because the body has a finite extent, so $\rho = 0$ beyond some value $r > a$. Therefore

$$\int d^3x'\,\partial_t\rho(t,\mathbf{x}')x'_ix'_j = \int d^3x'\,\rho(t,\mathbf{x}')\dot{x}'_k(\delta_{ik}x'_j + \delta_{jk}x'_i)\,, \tag{3.106}$$

and

$$\frac{dE_{\rm source}}{dt} = -\frac{2G}{5c^5}\langle\frac{d^5Q_{ij}}{dt^5}\int d^3x'\,\rho(t,\mathbf{x}')\dot{x}'_ix'_j\rangle\,. \tag{3.107}$$

Of course, d^5Q_{ij}/dt^5 is a function only of time, and does not depend on the dummy integration variable x'. Then it can be carried inside the integral, and eq. (3.107) can be written as

$$\frac{dE_{\rm source}}{dt} = \langle\int d^3x'\,\frac{dF_i}{dV'}\,\dot{x}'_i\rangle\,, \tag{3.108}$$

with a force per unit volume

$$\boxed{\frac{dF_i}{dV'} = -\frac{2G}{5c^5}\rho(t,\mathbf{x}')x'_j\frac{d^5Q_{ij}}{dt^5}(t)\,.} \tag{3.109}$$

Therefore the total force is

$$F_i = -\frac{2G}{5c^5}\frac{d^5Q_{ij}}{dt^5}\int d^3x'\rho(t,\mathbf{x}')x'_j\,. \tag{3.110}$$

Defining the center-of-mass coordinates by

$$x_j(t) \equiv \frac{1}{m}\int d^3x'\rho(t,\mathbf{x}')x'_j\,, \tag{3.111}$$

we find

$$F_i = -\frac{2G}{5c^5}\,mx_j\,\frac{d^5Q_{ij}}{dt^5}\,. \tag{3.112}$$

We have deduced this gravitational force on the source considering the GWs at infinity and imposing the energy balance. However, the motion of a particle under the effect of gravitational forces is completely determined by the value of the metric and its derivatives at the particle location. Thus, it should be possible to deduce the back-reaction force (3.112) also looking directly at the metric in the near-source region, without invoking the energy balance. In other words, if one solves for the gravitational field everywhere in space, in correspondence to a GW solution in the far region, there must be terms in the metric in the near region which, acting directly on the source motion through the geodesic equation, produce exactly the effect that we have inferred from the energy balance argument, i.e. provide the force (3.112). This correspondence between the GWs in the far region and the metric in the near region will be discussed in Chapter 5 in the appropriate context, which is the post-Newtonian formalism. The result is that, indeed, in the near region, among other terms that describe post-Newtonian corrections to the static potential, there is also a correction to the metric coefficient h_{00} of the form

$$\Delta h_{00} = -\frac{2\Phi}{c^2}\,, \tag{3.113}$$

with

$$\Phi(t,\mathbf{x}) = \frac{G}{5c^5}\,x_ix_j\,\frac{d^5Q_{ij}}{dt^5}(t)\,. \tag{3.114}$$

This is known as the Burke–Thorne potential,[21] and generates a Newtonian force

$$F_i = -m\partial_i\Phi\,, \tag{3.115}$$

in agreement with eq. (3.112).[22] As we will discuss in detail in Chapter 5, this term is singled out, and associated to radiation reaction, thanks to the fact that it is odd under time reversal. Terms associated with conservative forces are invariant under time reversal. In contrast, the emission of radiation breaks time-reversal invariance through the boundary conditions, since we impose that there is no incoming radiation (technically, at minus null infinity, see Note 1 on page 102), while at plus null infinity we can have outgoing radiation.

We now check that the force (3.112) also correctly accounts for the angular momentum loss. The torque on a unit volume located at the position x_i is

$$\begin{aligned}\frac{d\mathcal{T}_i}{dV} &= \epsilon_{ijk}x_j\frac{dF_k}{dV}\\ &= -\frac{2G}{5c^5}\,\epsilon_{ijk}\rho(t,\mathbf{x})x_jx_l\frac{d^5Q_{kl}}{dt^5}\,,\end{aligned} \tag{3.116}$$

where we used eq. (3.109). Then

$$\mathcal{T}_i = -\frac{2G}{5c^5}\,\epsilon_{ijk}\frac{d^5Q_{kl}}{dt^5}\int d^3x\,\rho(t,\mathbf{x})x_jx_l\,. \tag{3.117}$$

[21] Higher-order corrections to the back-reaction force are given in eqs. (5.190) and (5.191). When comparing these results with eq. (3.114), observe that eqs. (5.190) and (5.191) are written in terms of the variable $\mathsf{h}_{\mu\nu}$ defined in eq. (5.69) (which in the linearized limit reduces to $-\bar{h}_{\mu\nu}$), rather than in terms of $h_{\mu\nu}$.

[22] Observe that there is here an abuse of notation. In eq. (3.114), x^i is the generic spatial variable. After taking the derivative with respect to x^i in eq. (3.115), we evaluate the force on the actual location of the particle, i.e. on the position $x^i(t)$ defined by eq. (3.111), and this gives eq. (3.112). For instance, for a point-like mass μ the quadrupole moment is $Q_{ij}(t) = \mu[x_i(t)x_j(t) - (1/3)\delta_{ij}|\mathbf{x}(t)|^2]$, where $x_i(t)$ is the actual trajectory of the particle, rather then the generic spatial variable x_i. So Q_{ij} is a given function of time only, and ∂_i in eq. (3.115) does not act on it.

In the last integral we can replace $x_j x_l$ by $x_j x_l - (1/3)\delta_{ij} r^2$, since the term $\sim \delta_{jl}$ gives zero contracted with $\epsilon_{ijk}\, d^5 Q_{kl}/dt^5$, so

$$T_i = -\frac{2G}{5c^5}\,\epsilon_{ijk}\, Q_{jl}\frac{d^5 Q_{kl}}{dt^5}\,. \tag{3.118}$$

In eq. (3.102) actually enters $\langle T_i\rangle$. Inside the average, we can integrate by parts twice, and we get

$$\langle T_i\rangle = -\frac{2G}{5c^5}\,\epsilon_{ikl}\langle \ddot{Q}_{ka}\dddot{Q}_{la}\rangle\,. \tag{3.119}$$

Comparison with eq. (3.99) shows that we have indeed correctly reproduced the expression for the angular momentum loss of the source.

It should be observed that this is also by far the quickest way to derive eq. (3.97), without going through the formalism of Noether's theorem and the more complicated algebra of Section 3.3.3. However, the derivation from the Noether theorem is more general, since it holds independently of the multipole expansion, and of whether or not the back-reaction of GWs on the source can be described by Newtonian mechanics. It is also conceptually more satisfying, since it stresses that the angular momentum carried by GWs is an intrinsic property of the gravitational field, independent of the description of the source, that can be computed by applying the standard methods of classical field theory, as is usually done in all other field theories.

Finally, we expect that the change in linear momentum should be given by $dP^i/dt = \langle F^i\rangle$. This is proportional to $\langle x_j(t) d^5 Q_{ij}/dt^5(t)\rangle$. Since the quadrupole moment is even under parity while x_j is odd, the integrand is odd and, for a periodic motion, its average over one orbital period vanishes and therefore $\langle F_i\rangle = 0$. This is in agreement with the fact that, in the quadrupole approximation, linear momentum is conserved, see the discussion below eq. (3.83).

Gravitational radiation and the equivalence principle

The above results also allow us to clarify an apparent paradox related to the equivalence principle. Consider, for simplifying the setting, a mass μ orbiting a mass M, in the limit $M/\mu \to \infty$. Thus, the light mass μ is accelerated by the gravitational field of the heavy mass M and, according to our computations, it radiates GWs (while M is static and does not radiate.) An observer at large distance from the source, well into the far region, would then in principle be able to detect the waves emitted, and would conclude that the mass μ indeed emits gravitational radiation.

Examine now the same situation from the point of view of an observer freely falling together with the mass μ. According to the equivalence principle, for this observer, in a sufficiently small region around the mass μ, the gravitational field vanishes. We have indeed seen explicitly in Section 1.3.2 how to construct such a freely falling frame all along a geodesic. In this frame the mass μ is not accelerated, and the

corresponding observer should then conclude that the mass μ does not radiate, contrary to the findings of the observer at infinity.

This apparent paradox can be understood recalling that the equivalence principle holds only locally, i.e. in a region around the mass μ much smaller than the typical scales of spatial variation of the gravitational field. One such scale is the length $\lambda\!\!\!^{-}$, over which retardation effects become important (and which determines the wavelength of the GWs detected by the observer at infinity.) Then, conclusions based on the equivalence principle can be valid only up to a distance r from the mass μ, much smaller than $\lambda\!\!\!^{-}$.[23] This means that the equivalence principle at most gives us informations on what happens in the near zone $r \ll \lambda\!\!\!^{-}$; GWs rather appear in the far zone $r \gg \lambda\!\!\!^{-}$, so there is no paradox in the fact that, using arguments valid only for $r \ll \lambda\!\!\!^{-}$, one does not see them. The presence of gravitational radiation at infinity is reflected, in the near zone, in the existence of the force given by eqs. (3.114) and (3.115). However, in the near region retardation effects are negligible, so this term just gives a correction to the static gravitational force, which furthermore is hidden behind other, much larger, corrections. We will see in fact in Chapter 5 that, in an expansion in v/c, the radiation-reaction force is of order $(v/c)^5$ (as it is already clear from the factor $1/c^5$ in eq. (3.114)), while the Newtonian gravitational field receives general-relativistic corrections, corresponding to conservative forces, already at orders $(v/c)^2$ and $(v/c)^4$. All these tidal-like terms, however, in the far region decrease much faster than $1/r$, leaving us with the radiation field only.

[23] In fact, r must also be much smaller than the scale of spatial variation of the quasi-static tidal gravitational fields near the mass μ, which in turn is much smaller than $\lambda\!\!\!^{-}$.

3.3.5 Radiation from a closed system of point masses

For a point-like particle moving on a given trajectory $x_0(t)$, in flat space-time, the energy–momentum tensor is[24]

$$T^{\mu\nu}(t,\mathbf{x}) = \frac{p^\mu p^\nu}{\gamma m}\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_0(t))\,, \tag{3.120}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$, and $p^\mu = \gamma m (dx_0^\mu/dt) = (E/c, \mathbf{p})$ is the four-momentum. If we have a set of point particles labeled by an index A, moving under their mutual influence on trajectories $x_A^\mu(t)$, the total energy–momentum tensor of the system is therefore

$$\begin{aligned} T^{\mu\nu}_{\rm tot}(t,\mathbf{x}) &= \sum_A \frac{p_A^\mu p_A^\nu}{\gamma_A m_A}\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_A(t)) \\ &= \sum_A \gamma_A m_A \frac{dx_A^\mu}{dt}\frac{dx_A^\nu}{dt}\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_A(t))\,, \end{aligned} \tag{3.121}$$

and in particular

$$T^{00}_{\rm tot}(t,\mathbf{x}) = \sum_A \gamma_A m_A c^2 \delta^{(3)}(\mathbf{x}-\mathbf{x}_A(t))\,. \tag{3.122}$$

[24] See, e.g. Landau and Lifshitz, Vol. II (1979), eq. (33.5), or Weinberg (1972), Section 2.8. In curved space-time the expression for $T^{\mu\nu}$ must be further multiplied by $1/\sqrt{-g}$, see, e.g. Straumann (2003), eq. (5.83).

If the system is closed, i.e. no external forces are acting on it, the total energy–momentum tensor of the system is conserved, and we can consistently use eq. (3.121) on the right-hand side of the wave equation (3.3). It is important however to realize that, if we consider a single particle moving on a pre-assigned trajectory $x_0(t)$, we cannot compute the gravitational radiation that it generates by simply plugging $x_0(t)$ into eq. (3.120) and then using eq. (3.9). Such an energy–momentum tensor is in fact not conserved, as a consequence of the fact that, if the particle moves on a path that is not a geodesic of flat space-time, there must be external forces acting on it. To have a conserved energy–momentum tensor, we must also include in our description all objects that generate a force on the particle.

Since the conservation of the energy–momentum tensor in flat space-time is a consequence of the invariance under space-time translations, an equivalent way to pose the problem is to ask what happens to the multipole moments if we perform a shift of the origin of the coordinate system. To understand this point it is sufficient to restrict ourselves to the non-relativistic limit, i.e. to the quadrupole approximation, and to ask what happens to the quadrupole moment if we change the origin of our coordinate system. For a set of non-relativistic particles, labeled by an index A, we see from eq. (3.122) that the second mass moment is[25]

$$\begin{aligned} M^{ij}(t) &= \sum_A m_A \int d^3x\, x^i x^i \delta^{(3)}(\mathbf{x}-\mathbf{x}_A(t)) \\ &= \sum_A m_A x_A^i(t) x_A^j(t)\,. \end{aligned} \tag{3.123}$$

Under a constant translation, $x^i \to x^i + a^i$, M^{ij} acquires an additive contribution

$$M^{ij}(t) \to M^{ij}(t) + a^i \sum_A m_A x_A^j(t) + a^j \sum_A m_A x_A^i(t) + a^i a^j \sum_A m_A\,, \tag{3.124}$$

so its value depends on the choice of the origin. However

$$\begin{aligned} \dot{M}^{ij} &\to \dot{M}^{ij} + a^i \sum_A m_A \dot{x}_A^j + a^j \sum_A m_A \dot{x}_A^i \\ &= \dot{M}^{ij} + a^i P^j_{\rm tot} + a^j P^i_{\rm tot}\,, \end{aligned} \tag{3.125}$$

where $P^i_{\rm tot} = \sum_A m_A \dot{x}_A^i$ is the total momentum of the (non-relativistic) system. For a closed system, $P^i_{\rm tot}$ is constant and therefore $\ddot{M}^{ij}$ is invariant. Since $h_{ij}^{\rm TT}$ depends on the second time derivative of M_{ij}, the gravitational radiation is not affected by a shift of the origin, as it should be, since $h_{ij}^{\rm TT}$ (as any other field, in any field theory) is a scalar under translations. It is important, however, that we have a closed system where all particles have been included, and no external force is present. In the presence of external forces, the energy–momentum tensor of matter is not conserved, or equivalently the multipole moments and all their time derivatives depend on the choice of the origin of the coordinate system, and the whole formalism that we have developed is not consistent.

[25] Recall from eq. (3.37) that M^{ij} is actually the second moment of T^{00}/c^2. However, since we usually compute it in the non-relativistic limit, with an abuse of language we will often call it the second mass moment.

However, the procedure of taking a given trajectory $\mathbf{x}_0(t)$, plugging it into eq. (3.120), and computing the corresponding GW production, becomes correct when $\mathbf{x}_0$ is the *relative* coordinate of an isolated two-body system in the center-of-mass frame, and $\mathbf{x}_0(t)$ is the actual time evolution of $\mathbf{x}_0$, as determined by the mutual interaction between the two bodies. To understand this point, we define as usual the center-of-mass coordinate by

$$\mathbf{x}_{\rm CM} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1+m_2}\,, \tag{3.126}$$

and the relative coordinate $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$, and we denote by $m = m_1 + m_2$ the total mass and by $\mu = m_1 m_2/m$ the reduced mass. Then, for a non-relativistic system, the second mass moment can be written as

$$\begin{aligned} M^{ij} &= m_1 x_1^i x_1^j + m_2 x_2^i x_2^j \\ &= m x_{\rm CM}^i x_{\rm CM}^j + \mu(x_{\rm CM}^i x_0^j + x_{\rm CM}^j x_0^i) + \mu x_0^i x_0^j\,. \end{aligned} \tag{3.127}$$

Therefore, if (and only if) we choose the origin of the coordinate system at $\mathbf{x}_{\rm CM} = 0$, the quadrupole moment becomes the same as that of a particle of mass μ described by the coordinate $x_0(t)$. If we rather opt for an origin with non-vanishing $\mathbf{x}_{\rm CM}$, then the first term in eq. (3.127) is a constant and does not contribute to the production of gravitational radiation, but the second is non-vanishing and time-dependent, since $x = x_0(t)$, and therefore contributes.

Thus, if we describe the system using $\mathbf{x}_1$ and $\mathbf{x}_2$, we can fix the origin of the reference frame at will, since $P^i_{\rm tot}$ is conserved, and therefore the quadrupole moment is invariant under a shift $x^i \to x^i + a^i$. Alternatively, we can adopt the CM point of view, and we are left with a single effective particle of mass μ and coordinate $x_0^i(t)$. This is formally identical to working with $\mathbf{x}_1$ and $\mathbf{x}_2$, so it is a consistent and correct way of computing GW production. However, in this case we no longer have the freedom to shift the origin of the reference frame, since this description is only valid in the CM frame, where $\mathbf{x}_{\rm CM} = 0$. In the CM frame, the mass density is then

$$\rho(t,\mathbf{x}) = \mu\, \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_0(t)\right)\,, \tag{3.128}$$

the second mass moment is

$$M^{ij}(t) = \mu\, x_0^i(t) x_0^j(t)\,, \tag{3.129}$$

and the mass quadrupole is

$$Q^{ij}(t) = \mu \left(x_0^i(t) x_0^j(t) - \frac{1}{3} r_0^2(t) \delta^{ij} \right)\,. \tag{3.130}$$

Having understood this point, it makes sense to ask about the radiation emitted by a two-body system whose relative coordinate performs a given periodic motion, say simple harmonic oscillations. Suppose that the relative coordinate $\mathbf{x}_0(t)$ performs a simple periodic motion with frequency ω_s, say along the z direction,[26]

[26] In a one-dimensional motion this example would be quite unrealistic, since the two bodies would go through each other whenever $\cos\omega_s t = 0$. However, this is just an example to illustrate what happens to a typical oscillatory mode of a system. For instance, one can consider an elliptic motion on a plane, which is the combination of two simple oscillations along the two axes.

$$z_0(t) = a_1 \cos \omega_s t\,. \tag{3.131}$$

Then

$$\begin{aligned} M^{ij}(t) &= \delta^{i3}\delta^{j3}\,\mu z_0^2(t) \\ &= \delta^{i3}\delta^{j3}\,\frac{\mu a_1^2}{2}\,(1+\cos 2\omega_s t)\,. \end{aligned} \tag{3.132}$$

Since the GW amplitude depends on the second derivative of M^{ij}, the constant term does not contribute and the only contribution to $h_{ij}^{\rm TT}$ comes from the term proportional to $\cos 2\omega_s t$. From eq. (3.55), we see that the corresponding waveform $h_{ij}^{\rm TT}$ oscillates as $\cos 2\omega_s t$. This shows that *a non-relativistic source performing simple harmonic oscillations with a frequency ω_s emits monochromatic quadrupole radiation at $\omega = 2\omega_s$.*

However, the fact that the quadrupole radiation is at twice the source frequency is only true if the source performs a simple harmonic motion. For instance, if the motion of the source is a superposition of a periodic motion and of its higher harmonics, e.g. if

$$z_0(t) = a_1 \cos \omega_s t + a_2 \cos 2\omega_s t + \ldots\,, \tag{3.133}$$

then $z_0^2(t)$ contains the term

$$a_1^2 \cos^2 \omega_s t = a_1^2\, \frac{1+\cos 2\omega_s t}{2}\,, \tag{3.134}$$

considered above, plus a term

$$a_2^2 \cos^2 2\omega_s t = a_2^2\, \frac{1+\cos 4\omega_s t}{2}\,, \tag{3.135}$$

which gives radiation at $\omega_{\rm gw} = 4\omega_s$, etc., but also mixed terms such as

$$2a_1 a_2 \cos(\omega_s t)\cos(2\omega_s t) = a_1 a_2 (\cos \omega_s t + \cos 3\omega_s t)\,. \tag{3.136}$$

Therefore in this case quadrupole radiation is emitted at all frequencies $n\omega_s$ for all integers n, both even and odd, including $n = 1$. We will see an example of this type in Section 4.1.2, when we study the radiation emitted from a mass in a Keplerian elliptic orbit.

An even simpler example is given by a system of two masses connected by a spring with rest length L (see Problem 3.1). In this case the relative coordinate satisfies

$$z_0(t) = L + a \cos \omega_s t\,, \tag{3.137}$$

and in $z_0^2(t)$, besides a constant and a term $(a^2/2)\cos 2\omega_s t$, we also have a term $2La \cos \omega_s t$, so the spectrum of gravitational radiation has two lines, one at $\omega = \omega_s$, and one at $\omega = 2\omega_s$. As we have discussed above, the whole procedure of computing GWs from a source on a preassigned trajectory $z_0(t)$ is consistent only if $z_0(t)$ is a relative coordinate in the center-of-mass frame. Therefore we do not have the freedom to shift the origin of the coordinate system, which otherwise would allow us to eliminate L.

3.4 Mass octupole and current quadrupole

We now examine the next-to-leading term of the expansion (3.34),

$$\left(h_{ij}^{\rm TT}\right)_{\rm next-to-leading} = \frac{1}{r}\frac{4G}{c^5}\Lambda_{ij,kl}(\hat{\mathbf{n}})n_m \dot{S}^{kl,m}(t-r/c)\,. \tag{3.138}$$

By definition $\dot{S}^{kl,m}$ is symmetric in kl, and it has no special symmetry with respect to the exchange of k with m. We have seen in eq. (3.54) that

$$\dot{S}^{kl,m} = \frac{1}{6}\dddot{M}^{klm} + \frac{1}{3}\left(\ddot{P}^{k,lm} + \ddot{P}^{l,km} - 2\ddot{P}^{m,kl}\right)\,. \tag{3.139}$$

Therefore $\dot{S}^{kl,m}$ separates into a totally symmetric tensor $\dddot{M}^{klm}$ plus a tensor with mixed symmetry. The meaning of this separation from the point of view of group theory is explained in Problem 3.4, in the Solved Problems section. The totally symmetric term generates mass octupole radiation, while the term with mixed symmetry is called the current quadrupole.

Mass octupole

We consider first the mass octupole radiation. The mass octupole $\mathcal{O}^{klm}$ is defined removing all traces from M^{ijk},

$$\mathcal{O}^{klm} = M^{klm} - \frac{1}{5}\left(\delta^{kl}M^{k'k'm} + \delta^{km}M^{k'lk'} + \delta^{lm}M^{kk'k'}\right)\,. \tag{3.140}$$

Using the fact that $\Lambda_{ij,kl}(\hat{\mathbf{n}})$ is transverse and traceless, we see that the contraction of the trace part with $\Lambda_{ij,kl}(\hat{\mathbf{n}})n_m$ gives zero, and we can use M^{klm} or $\mathcal{O}^{klm}$ interchangeably in the expression for $h_{ij}^{\rm TT}$. Therefore the mass octupole contribution to $h_{ij}^{\rm TT}$ can be written as

$$\left(h_{ij}^{\rm TT}\right)_{\rm oct} = \frac{1}{r}\frac{2G}{3c^5}\Lambda_{ij,kl}(\hat{\mathbf{n}})n_m \dddot{\mathcal{O}}^{klm}\,. \tag{3.141}$$

Similarly to the case of quadrupole radiation, the use of $\mathcal{O}^{klm}$ is nicer from a group-theoretical point of view, since it is a pure spin-3 tensor, see Problem 3.4, while the use of M^{ijk} is simpler in actual computations. We will use $\dddot{M}^{klm}$ and $\dddot{\mathcal{O}}^{klm}$ interchangeably in eq. (3.141).

Observe that, when we consider quantities quadratic in $h_{ij}^{\rm TT}$, as for instance the radiated energy, there is no interference between the mass quadrupole and the mass octupole terms because they have different parity. Under a parity operation, $\mathbf{x} \to -\mathbf{x}$, the mass density is a true scalar, and therefore the quadrupole is invariant while the octupole changes sign (for the same reason, in electrodynamics there is no interference between dipole and quadrupole radiation.) More generally, we will see in Section 3.5 how to systematically organize the multipole expansion so that there are no interference terms to all orders.

Comparing the mass quadrupole and the mass octupole we see that, while the contribution to the GW amplitude from the mass quadrupole is proportional to the second time derivative of M^{ij}, the contribution from the mass octupole is proportional to $(1/c)$ times the third time derivative of M^{ijk}. If d is the typical dimension of the source, the tensor M^{ijk} differ from M^{ij} by a factor $O(d)$. Each time derivative carries a factor $O(\omega_s)$, where ω_s is the typical frequency of the movement of matter inside the source, so $(1/c)\dddot{M}^{ijk} = O(\omega_s d/c)\ddot{M}^{ij}$. Since $\omega_s d \sim v$ is the typical velocity inside the source, the octupole contribution to $h_{ij}^{\rm TT}$ is smaller than the quadrupole contribution by a factor $O(v/c)$, and the contribution of the mass octupole to the radiated power is smaller than the contribution of the mass quadrupole by a factor $O(v^2/c^2)$.

Consider now a two-body non-relativistic system whose relative coordinate in the center-of-mass frame is described by a function $x_0^i(t)$, and has a reduced mass μ. Then, similarly to eq. (3.129), to lowest order in v/c (e.g. replacing T^{00}/c^2 with the mass density), we have

$$M^{ijk}(t) = \mu x_0^i(t) x_0^j(t) x_0^k(t) \, . \tag{3.142}$$

Suppose now that $x_0(t)$ performs simple harmonic oscillations with frequency ω_s. Then, each factor $x_0^i(t)$ is the superposition of a term oscillating as $e^{i\omega_s t}$ and of a term oscillating as $e^{-i\omega_s t}$, so M^{ijk} is the sum of terms oscillating as $e^{\pm i\omega_s t}$ and of terms oscillating as $e^{\pm 3i\omega_s t}$. Therefore *a source performing simple harmonic oscillations at a frequency ω_s emits octupole radiation at $\omega = \omega_s$ and at $\omega = 3\omega_s$.*

If a non-relativistic source performs simple harmonic oscillations, then its energy spectrum is made of a stronger line due to quadrupole radiation at $\omega = 2\omega_s$, plus two smaller "lateral bands" due to the octupole at $\omega = \omega_s$ and at $\omega = 3\omega_s$, as we will see in more detail in Problem 3.3. Recall however from Section 3.3.5 that periodic trajectories that are not simple harmonic motions produce a more complicated spectrum, in which the quadrupole can already contribute to all frequencies $n\omega_s$, for all integer n.

The power emitted per unit solid angle by the octupole moment is computed inserting $\dot{h}_{ij}^{\rm TT}$, obtained from eq. (3.141), into the expression for the power given in eq. (1.153). This gives

$$\begin{aligned} P_{\rm oct} &= \frac{c^3 r^2}{32\pi G} \int d\Omega \, \langle (\dot{h}_{ij}^{\rm TT})_{\rm oct} (\dot{h}_{ij}^{\rm TT})_{\rm oct} \rangle \\ &= \frac{c^3}{32\pi G} \frac{4G^2}{9c^{10}} \langle \frac{d^4\mathcal{O}^{klm}}{dt^4} \frac{d^4\mathcal{O}^{k'l'm'}}{dt^4} \rangle \int d\Omega \, \Lambda_{ij,kl} \Lambda_{ij,k'l'} n_m n_{m'} \, . \end{aligned} \tag{3.143}$$

Using the property (1.37) of the Lambda tensor, together with $\Lambda_{ij,kl} = \Lambda_{kl,ij}$, we have $\Lambda_{ij,kl}\Lambda_{ij,k'l'} = \Lambda_{kl,k'l'}$. Then, renaming the indices, we get

$$P_{\rm oct} = \frac{G}{72\pi c^7} \langle \frac{d^4\mathcal{O}^{ijm}}{dt^4} \frac{d^4\mathcal{O}^{klp}}{dt^4} \rangle \int d\Omega \, \Lambda_{ij,kl}(\hat{\mathbf{n}}) n_m n_p \, . \tag{3.144}$$

To integrate over the solid angle we need the identities (3.21) and (3.22), together with eq. (3.23) with $l = 3$,

$$\int \frac{d\Omega}{4\pi} n_i n_j n_k n_l n_m n_p = \frac{1}{105} (\delta_{ij}\delta_{kl}\delta_{mp} + \ldots) \, , \tag{3.145}$$

where the dots denote the other 14 possible pairings of indices. Using the fact that $\mathcal{O}^{ijk}$ is totally symmetric and traceless, the contractions are straightforward, and we get

$$P_{\rm oct} = \frac{G}{189\, c^7} \langle \frac{d^4\mathcal{O}^{ijk}}{dt^4} \frac{d^4\mathcal{O}^{ijk}}{dt^4} \rangle \, . \tag{3.146}$$

Current quadrupole

The current quadrupole is given by the second term in eq. (3.139). Its physical meaning can be understood observing that, from the definition (3.41),

$$\begin{aligned} & P^{k,lm} + P^{l,km} - 2P^{m,kl} \\ &= \frac{1}{c} \int d^3x \, [T^{0k} x^l x^m + T^{0l} x^k x^m - 2T^{0m} x^k x^l] \\ &= \frac{1}{c} \int d^3x \, [x^l (x^m T^{0k} - x^k T^{0m}) + x^k (x^m T^{0l} - x^l T^{0m})] \\ &= \int d^3x \, \left[x^l j^{mk} + x^k j^{ml} \right] , \end{aligned} \tag{3.147}$$

where

$$j^{jk} = \frac{1}{c} \left(x^j T^{0k} - x^k T^{0j} \right) . \tag{3.148}$$

This is the angular momentum density associated to the (j, k) plane. We write $j^{jk} = \epsilon^{jkl} j^l$, where j^l is the l-th component of the angular momentum density vector, and we define

$$J^{i,j} = \int d^3x \, j^i x^j \, , \tag{3.149}$$

so $J^{i,j}$ is the first moment of the angular momentum density (i.e. the dipole moment of the angular momentum distribution). Then we get

$$P^{k,lm} + P^{l,km} - 2P^{m,kl} = \epsilon^{mkp} J^{p,l} + \epsilon^{mlp} J^{p,k} \, . \tag{3.150}$$

Therefore the current quadrupole is the first moment of the angular momentum density of the source, symmetrized over the indices k, l. Its contribution to the GW amplitude can be written as

$$\left(h_{ij}^{\rm TT} \right)_{\rm curr\ quad} = \frac{1}{r} \frac{4G}{3c^5} \Lambda_{ij,kl}(\hat{\mathbf{n}}) n_m \left(\epsilon^{mkp} \dot{J}^{p,l} + \epsilon^{mlp} \dot{J}^{p,k} \right) . \tag{3.151}$$

We will indeed see in the next section that the full multipole expansion can be organized systematically so that it is determined by two types of objects: the momenta of the energy density of the source (which, to leading order in v/c, are the same as the momenta of the mass density), like the mass quadrupole Q^{ij}, the mass octupole $\mathcal{O}^{ijk}$, etc., and the momenta of the angular momentum density of the source, such as the current quadrupole.

The power associated to the current quadrupole is computed just as we have done for the mass quadrupole and mass octupole: we write

$$\begin{aligned}
P_{\text{curr quad}} &= \frac{c^3 r^2}{32\pi G}\int d\Omega\, \langle (\dot{h}_{ij}^{\rm TT})_{\text{curr quad}} (\dot{h}_{ij}^{\rm TT})_{\text{curr quad}} \rangle \\
&= \frac{c^3}{32\pi G}\,\frac{16G^2}{9c^{10}}\int d\Omega\, \Lambda_{kl,k'l'} n_m n_{m'} \\
&\quad \times \langle \left(\epsilon^{mkp} J^{p,l} + \epsilon^{mlp} J^{p,k}\right) \left(\epsilon^{m'k'p'} J^{p',l'} + \epsilon^{m'l'p'} J^{p',k'}\right)\rangle\,,
\end{aligned} \tag{3.152}$$

where we used again $\Lambda_{ij,kl}\Lambda_{ij,k'l'} = \Lambda_{kl,k'l'}$. The angular integration is performed using the identities (3.21) and (3.22). Observe also that the term in the Lambda tensor with four factors of n, i.e. $n_k n_l n_{k'} n_{l'}$ (see eq. (1.39)) does not contribute since, together with the factors $n_m n_{m'}$ in eq. (3.152), it gives a totally symmetric tensor $n_k n_l n_m n_{k'} n_{l'} n_{m'}$, which vanishes upon contraction with ϵ^{mkp} or ϵ^{mlp}.

When performing contractions, we also make use of the fact that $J^{i,j}$ is traceless, since

$$\begin{aligned}
J^{i,i} &= \int d^3x\, x^i j^i \\
&= \frac{1}{c}\int d^3x\, x^i \epsilon^{ijk} x^j T^{0k} = 0
\end{aligned} \tag{3.153}$$

(the sum over i is understood). Then, we get

$$P_{\text{curr quad}} = \frac{16G}{45c^7}\langle \dddot{\mathcal{J}}^{ij}\, \dddot{\mathcal{J}}^{ij}\rangle\,, \tag{3.154}$$

where we introduced the traceless symmetric matrix

$$\mathcal{J}^{ij} \equiv \frac{J^{i,j} + J^{j,i}}{2}\,, \tag{3.155}$$

that is, the symmetrization of the dipole moment of the angular momentum density. Putting together the power radiated by the mass quadrupole, current quadrupole and mass octupole, we therefore get

$$\boxed{\begin{aligned}
P = \frac{G}{c^5}\Big[&\frac{1}{5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle + \frac{1}{c^2}\frac{16}{45}\langle \dddot{\mathcal{J}}_{ij}\, \dddot{\mathcal{J}}_{ij}\rangle \\
&+\frac{1}{c^2}\frac{1}{189}\langle \frac{d^4\mathcal{O}_{ijk}}{dt^4}\frac{d^4\mathcal{O}_{ijk}}{dt^4}\rangle + O\left(\frac{v^4}{c^4}\right)\Big]\,,
\end{aligned}} \tag{3.156}$$

where $O(v^4/c^4)$ denotes the contributions from higher orders in the multipole expansion. In this formula, one must be careful to include in Q_{ij} also its first corrections $O(v^2/c^2)$, since it gives a contribution to the power of the same order as that due to the mass octupole and current quadrupole. This means that Q_{ij} here is not simply the quadrupole of the mass distribution. Rather, we must go back to its original definition in terms of T^{00}, see eq. (3.37) and include in T^{00} not only the terms of

order ρc^2, where ρ is the mass density, but also the terms of order ρv^2. Furthermore, the time-dependence of the leading term must be computed using the relativistic equations of motion, including corrections up to $O(v^2/c^2)$.

It is quite difficult to imagine a realistic physical system for which the time derivative of the mass quadrupole vanishes, but still the mass octupole is time-varying. For this reason, the mass octupole is always a correction to the leading term. However, there are interesting physical situations where *both* the mass quadrupole and the mass octupole are constant in time and therefore do not contribute to GW production, while the current quadrupole still contributes, so it becomes the leading term. To have M^{ij} and M^{ijk} constant, in fact, it suffices that the energy density T^{00} be constant. Then all the mass momenta are constant, see e.g. eqs. (3.35)–(3.38). Still, the angular momentum density and its momenta, such as its dipole $J^{i,j}$, are not necessarily constant. Consider for example a ring in the (x, y) plane with a uniform mass density ρ, rotating around the z axis. Even if any single volume element of the ring rotates, this rotation does not induce any temporal variation in the density ρ since the ring is spatially uniform, so all mass moments are constant.[27] However, this ring has non-vanishing angular momentum along z. To obtain a system with a dipole moment of angular momentum $J^{i,j}$, we can simply take two rings, both rotating around the z axis, but one counterclockwise and one clockwise, as in Fig. 3.3. The upper ring has a positive J^z while the lower one has a negative J^z, so it is clear that the whole system has a non-vanishing dipole moment $J^{z,z}$.[28]

If the rotational velocities of the rings are not constant, this dipole moment of the angular momentum is time-dependent. A physical example of a system of this type is provided by the torsional oscillations of a neutron star. It is possible that, either because of some external perturbation, or because of a "corequake", the upper hemisphere of a NS suffers a clockwise torsion while the lower hemisphere receives a counterclockwise torsion (with the equator which, by symmetry reasons, stays fixed). Then the two hemispheres will start oscillating back and forth in opposite directions, so that when one rotates clockwise the other rotates counterclockwise, and vice versa. This system will then emit current quadrupole radiation, but not mass quadrupole nor mass octupole radiation. Another important example of this type is the r-mode of neutron stars, which will be discussed in Vol. 2.

It is therefore worthwhile to study the current quadrupole radiation in more detail, and derive its angular distribution. First, we compute the amplitude of the GWs radiated in the z direction. This is quite simple, since we just have to substitute $n = (0, 0, 1)$ into eq. (3.151). Using the explicit expression for the Lambda tensor, eq. (1.39), and recalling that, for a wave traveling along z, $h_{11} = h_+$ and $h_{12} = h_\times$, we obtain

$$h_+ = \frac{1}{r}\frac{4G}{3c^5}\left(\ddot{J}_{1,2} + \ddot{J}_{2,1}\right)\,, \tag{3.157}$$

$$h_\times = \frac{1}{r}\frac{4G}{3c^5}\left(\ddot{J}_{2,2} - \ddot{J}_{1,1}\right)\,. \tag{3.158}$$

[27] One might wonder how it is possible that there is no quadrupole radiation from the whole ring, given that every single mass element is in rotation and therefore radiates GWs. From this "microscopic" point of view, the answer is that the total amplitude is the sum over the contribution of all the mass elements, and these contributions interfere destructively, so that the total GW amplitude h_{ij} vanishes.

[28] It also has a non-zero $J^{x,x}$ and $J^{y,y}$ (consistent with the condition $J^{i,i} = 0$). This can be seen by drawing the direction of the angular momentum vector $\delta\mathbf{J} = (\delta m)\mathbf{r}\times\mathbf{v}$ of various mass elements, where $\mathbf{r}$ is measured from the origin of the coordinate system, see Fig. 3.3. We see for instance that both the infinitesimal mass elements labeled a and c have $J^y < 0$ and are at a coordinate $y > 0$, therefore their contribution to $J^{y,y}$ is negative. Similarly, for the mass elements b and d we have $J^y > 0$, but they have a coordinate $y < 0$, so their contribution to $J^{y,y}$ is again negative.

To obtain the amplitudes h_+ and $h_\times$ for a GW propagating into an arbitrary direction, we proceed as we did for the mass quadrupole, between eqs. (3.67) and (3.72). We first write

$$h_+ = \frac{1}{r}\frac{4G}{3c^5}\left(\ddot{J}'_{1,2} + \ddot{J}'_{2,1}\right), \tag{3.159}$$

$$h_\times = \frac{1}{r}\frac{4G}{3c^5}\left(\ddot{J}'_{2,2} - \ddot{J}'_{1,1}\right), \tag{3.160}$$

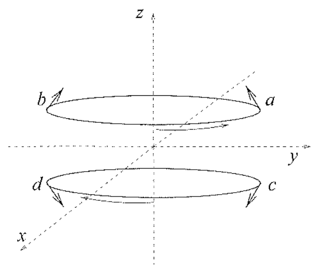

Fig. 3.3 Two rings, both with uniform mass density, one rotating counterclockwise and one clockwise. The arrows show the direction of the angular momentum $\delta\mathbf{J} = (\delta m)\mathbf{r}\times\mathbf{v}$ (with $\mathbf{r}$ measured from the origin of the coordinate system, rather than from the center of each disk) of four infinitesimal mass elements, labeled a, b, c, d.

where $J'_{i,j}$ are the components of the angular momentum dipole in the $(\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{n}})$ frame defined in Fig. 3.2. Since $J_{i,j}$ is a spatial tensor with two indices, it transforms under rotation just as we found for M_{ij} in eq. (3.71), i.e. $J'_{i,j} = \mathcal{R}_{ki}\mathcal{R}_{lj}J_{k,l}$, with $\mathcal{R}$ given by eq. (3.70). Performing the matrix multiplication, we get

$$\begin{aligned} h_+(t;\theta,\phi)|_{\rm curr\ quad} = \frac{1}{r}\frac{4G}{3c^5}\,[\,&(\ddot{J}_{1,1} - \ddot{J}_{2,2})\sin 2\phi\cos\theta \\ &+(\ddot{J}_{1,2} + \ddot{J}_{2,1})\cos 2\phi\cos\theta \\ &-(\ddot{J}_{1,3} + \ddot{J}_{3,1})\cos\phi\sin\theta \\ &+(\ddot{J}_{2,3} + \ddot{J}_{3,2})\sin\phi\sin\theta\,], \end{aligned} \tag{3.161}$$

$$\begin{aligned} h_\times(t;\theta,\phi)|_{\rm curr\ quad} = \frac{1}{r}\frac{4G}{3c^5}\,[\,&-\ddot{J}_{1,1}(\cos^2\phi - \sin^2\phi\cos^2\theta) \\ &-\ddot{J}_{2,2}(\sin^2\phi - \cos^2\phi\cos^2\theta) \\ &+\ddot{J}_{3,3}\sin^2\theta \\ &+(\ddot{J}_{1,2} + \ddot{J}_{2,1})\sin\phi\cos\phi\,(1+\cos^2\theta) \\ &-(\ddot{J}_{1,3} + \ddot{J}_{3,1})\sin\phi\sin\theta\cos\theta \\ &-(\ddot{J}_{2,3} + \ddot{J}_{3,2})\cos\phi\sin\theta\cos\theta\,]\,. \end{aligned} \tag{3.162}$$

This gives the angular distribution of the current quadrupole radiation,[29] for $J^{i,j}$ arbitrary (but satisfying the zero-trace condition $J_{1,1} + J_{2,2} + J_{3,3} = 0$). To check these equations we can plug them into the expression (1.154) for the angular distribution of the power in terms of h_+ and $h_\times$,

$$\frac{dP}{d\Omega} = \frac{c^3 r^2}{16\pi G}\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle\,. \tag{3.163}$$

Performing explicitly the integral over $d\Omega = d\phi d\theta\,|\sin\theta|$, we verify that we get eq. (3.154) back.

Linear momentum losses

Finally, it is interesting to observe that the leading term contributing to the loss of linear momentum comes from the interference between the quadrupole term and the next-to-leading term (i.e. the octupole plus current quadrupole). In fact, recall from eq. (1.164) that dP^k/dt is proportional to

$$\int d\Omega\, \dot{h}_{ij}^{\rm TT}\partial_k h_{ij}^{\rm TT}\,. \tag{3.164}$$

[29] Comparing with eq. (3.72) we see that the angular dependence is the same as for the mass quadrupole, if we replace M_{ij} by $J_{i,j}$, and we exchange the roles of h_+ and $h_\times$ (with an additional minus sign for $(h_\times)_{\rm curr\ quad}$. This was already clear from a comparison of eqs. (3.157) and (3.158) with eqs. (3.65) and (3.66), and from the fact that both M_{ij} and $J_{i,j}$ transform as rank-2 tensors under rotations.

Using eqs. (3.59) and (3.138) we write

$$\begin{aligned} h_{ij}^{\rm TT} &= [h_{ij}^{\rm TT}]_{\rm quad} + [h_{ij}^{\rm TT}]_{\rm next-to-leading} \\ &= \frac{1}{r}\frac{2G}{c^4}\left[\ddot{Q}_{ij}^{\rm TT} + \frac{2}{c}n_m\ddot{S}_{ij,m}^{\rm TT}\right], \end{aligned} \tag{3.165}$$

where $\ddot{S}_{ij,m}^{\rm TT} = \Lambda_{ij,kl}\ddot{S}_{kl,m}$. In the product $\dot{h}_{ij}^{\rm TT}\partial_k h_{ij}^{\rm TT}$ we have diagonal terms and interference terms between the quadrupole and the next-to-leading term. The diagonal terms are proportional to

$$\int d\Omega\, \dddot{Q}_{ij}^{\rm TT}\partial_k \ddot{Q}_{ij}^{\rm TT}\,, \tag{3.166}$$

for the quadrupole, and to

$$\int d\Omega\left(n_l \dddot{S}_{ij,l}^{\rm TT}\right)\partial_k\left(n_m \ddot{S}_{ij,m}^{\rm TT}\right), \tag{3.167}$$

for the next-to-leading term. Because of parity, these angular integrals vanish if the integrand is the product of an odd number of factors n_i. Therefore, we need to count the number of $\hat{\mathbf{n}}$ in these expressions. The TT projection is performed with the Lambda tensor, which has an even number of factors $\hat{\mathbf{n}}$. As for the derivative ∂_k which appears in eqs. (3.166) and (3.167), recall that, on a function of r, $\partial_k f(r) = (\partial_k r)\,df/dr = n^k\,df/dr$, while $\partial_k n^i = \partial_k(x^i/r) = (1/r)(\delta^{ik} - n^i n^k)$. Therefore the effect of ∂_k is always to lower or to raise by one the number of factors n^i, i.e. to transform a term with an even number of $\hat{\mathbf{n}}$ in a term with an odd number, and vice versa.

Then, we see that the diagonal terms in eqs. (3.166) and (3.167) have an odd number of factors $\hat{\mathbf{n}}$, so their angular integral vanish. In the interference terms, such as

$$\int d\Omega\, \dddot{Q}_{ij}^{\rm TT}\partial_k\left(n_m \ddot{S}_{ij,m}^{\rm TT}\right), \tag{3.168}$$

the integrand is even in $\hat{\mathbf{n}}$ and in general is non-vanishing. So, while the radiated energy only gets contributions from the diagonal terms (as we will verify to all orders in the multipole expansion in the next section), the radiated momentum only gets contributions from the interference between multipoles of different parity, in order to compensate for the minus sign acquired by the derivative ∂_k under $\hat{\mathbf{n}} \to -\hat{\mathbf{n}}$.

3.5 Systematic multipole expansion

In eq. (3.34) the multipole expansion has been organized in terms of tensors such as S^{kl}, $\dot{S}^{kl,m}$, $\ddot{S}^{kl,m_1m_2}$, etc. which have two sets of indices (separated by a comma), the first always made by two indices k, l and the second made by a generic number of indices, $m_1, \ldots, m_N$. These tensors are symmetric under the exchange of k and l, and are also symmetric under the exchange of indices in the $m_1, \ldots, m_N$ set. However, they have no special symmetry property under the exchange of indices between the two sets (and they are in general not traceless under contraction of pairs of indices). At next-to-leading order, we separated by hand the term $\dot{S}^{kl,m}$ into two contributions, one corresponding to mass octupole and one to current quadrupole. The reason underlying this separation is group-theoretical. As we discuss in Problem 3.4, this corresponds to a decomposition into irreducible representations of the rotation group. To generalize such a construction to arbitrary orders in the multipole expansion, we must introduce a complete set of representations of the rotation group, for all multipoles. There are two particularly convenient ways of doing that. One is to consider tensors which are symmetric and trace-free (STF) with respect to all pairs of indices. This formalism will be introduced in Section 3.5.1, and we will use it extensively in particular in Chapter 5. A second possibility is to introduce the spherical components of tensors and the tensor spherical harmonics, which is the approach that we discuss in Section 3.5.2

To illustrate these two different approaches, it can be useful to begin our discussion by recalling how the multipole expansion works in the simpler case of a static situation, governed by a Poisson equation of the form

$$\nabla^2\phi = -4\pi\rho\,, \tag{3.169}$$

which would be the case, e.g. in electrostatics or in Newtonian gravity. We consider a stationary source with density $\rho(\mathbf{x})$ localized in space, so $\rho(\mathbf{x}) = 0$ if $r > d$, where d is the source size. The most general solution in the external region $r > d$ can be written as

$$\phi(\mathbf{x}) = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Q_{lm}}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\,. \tag{3.170}$$

In fact, for $r > 0$,

$$\nabla^2\left[\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\right] = 0\,, \tag{3.171}$$

as can be seen immediately from the expression of the Laplacian in spherical coordinates,

$$\nabla^2\psi(\mathbf{x}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) - \frac{\mathbf{L}^2}{r^2}\psi\,, \tag{3.172}$$

together with the property $\mathbf{L}^2 Y_{lm} = l(l+1)Y_{lm}$ of the spherical harmonics. Then, in the external region $r > d$, all terms in the sum in eq. (3.170) are separately solution of the vacuum equation $\nabla^2\phi = 0$. The fact that eq. (3.170) is the most general vacuum solution follows from the fact that the spherical harmonics provide a complete set of representations of the rotation group. On the other hand, the solution of eq. (3.169) (subject to the boundary condition that ϕ approaches zero as $r\to\infty$) can be written in terms of the Green's function of the Laplacian, as

$$\phi(\mathbf{x}) = \int d^3y\,\frac{1}{|\mathbf{x}-\mathbf{y}|}\,\rho(\mathbf{y})\,, \tag{3.173}$$

which is valid both for $\mathbf{x}$ inside and outside the source. Outside the source we have $|\mathbf{x}| > |\mathbf{y}|$, and we can use the addition theorem for spherical harmonics[30] in the form

[30] See, e.g. Jackson (1975), Section 3.6.

$$\frac{1}{|\mathbf{x}-\mathbf{y}|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r'^l}{r^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)\,, \tag{3.174}$$

where we used the notation $|\mathbf{x}| = r$ and $|\mathbf{y}| = r'$; (θ,ϕ) are the polar angles of $\hat{\mathbf{x}}$ and (θ',ϕ') of $\hat{\mathbf{y}}$. Inserting this identity in eq. (3.173) and comparing with eq. (3.170) we find the expression for the multipole coefficients Q_{lm} in terms of the source density $\rho(\mathbf{x})$,

$$Q_{lm} = \int d^3y\,Y^*_{lm}(\theta',\phi')r'^l\rho(\mathbf{y})\,. \tag{3.175}$$

So, eq. (3.170) gives the most general solution of the vacuum equation, and eq. (3.175) fixes the coefficients Q_{lm} in terms of the density of the source.

An alternative way of performing the multipole expansion is to write

$$\begin{aligned}\frac{1}{|\mathbf{x}-\mathbf{y}|} &= \frac{1}{|\mathbf{x}|} - y^i\partial_i\frac{1}{|\mathbf{x}|} + \frac{1}{2}y^iy^j\partial_i\partial_j\frac{1}{|\mathbf{x}|} + \ldots\\ &= \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}y^{i_1}\ldots y^{i_l}\partial_{i_1}\ldots\partial_{i_l}\frac{1}{|\mathbf{x}|}\,.\end{aligned} \tag{3.176}$$

We then make use of the fact that, for $r > 0$, and therefore in particular outside the source, we have $\nabla^2(1/|\mathbf{x}|) = 0$, as can be checked again from eq. (3.172).[31] Then, in eq. (3.176) we can replace y^iy^j with the traceless combination $y^iy^j - (1/3)\delta^{ij}|\mathbf{y}|^2$, and similarly we can remove all traces from the tensors $y^{i_1}\ldots y^{i_l}$. Then, inserting eq. (3.176) into eq. (3.173), we get

[31] More generally,

$$\nabla^2\frac{1}{|\mathbf{x}|} = -4\pi\delta^{(3)}(\mathbf{x})\,, \tag{3.177}$$

i.e. $G(\mathbf{x}-\mathbf{y}) = -1/(4\pi|\mathbf{x}-\mathbf{y}|)$ is the Green's function of the Laplacian.

$$\phi(\mathbf{x}) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}Q_{i_1\ldots i_l}\partial_{i_1}\ldots\partial_{i_l}\frac{1}{|\mathbf{x}|}\,, \tag{3.178}$$

where

$$Q_{i_1\ldots i_l} = \int d^3y\,y^{\langle i_1}\ldots y^{i_l\rangle}\,\rho(\mathbf{y})\,, \tag{3.179}$$

and the brackets in $y^{\langle i_1}\ldots y^{i_l\rangle}$ means that we must take the symmetric-trace-free (STF) part of the tensor $y^{i_1}\ldots y^{i_l}$. We therefore have two equivalent formalisms for the multipole expansion of a scalar field that satisfies the Poisson equation (3.169), either in terms of spherical harmonics, or in terms of STF tensors.

For applications to electrodynamics or to gravitational waves, we need to generalize the above results in two ways. First, we do not have a scalar field but rather a vector field A_μ or a tensor field $\bar{h}_{\mu\nu}$. Second, we do not have static fields governed by a Poisson equation, but rather time-dependent fields whose wave equation involve a d'Alembertian operator, such as $\Box A_\mu = -(4\pi/c)J_\mu$ in electromagnetism, and eq. (3.3) for GWs. For dealing with non-stationary fields, a possible route is to perform a Fourier transform in time (but not in space), thus reducing the d'Alembertian operator $\Box\phi(t,\mathbf{x})$ to $(\nabla^2+k^2)\tilde{\phi}_k(\mathbf{x})$, so that outside the source the field satisfies a Helmholtz equation $(\nabla^2+k^2)\tilde{\phi}_k(\mathbf{x})=0$. For vector or tensor fields this can be combined with the use of vector and tensor spherical harmonics, respectively. In electrodynamics, this is the approach discussed in Chapter 16 of Jackson (1975). However, this approach does not make explicit the time integration linking the multipole moments to the actual evolution of the source, and furthermore yields somewhat complicated final expressions. Thus, in Section 3.5.1 we rather follow an elegant approach, based on STF tensors, which gives a simple and unified treatment of scalar, vector and tensor fields. This formalism will be used extensively in Chapter 5, in the study of post-Newtonian sources.

The generalization of the approach based on spherical harmonics to the vector and tensor case leads to a more complicated formalism, which nevertheless can be useful in various instances (e.g. for classifying the extra polarization states of GWs in alternative theories of gravity), and we study it in Section 3.5.2.

3.5.1 Symmetric-trace-free (STF) form

Multi-index notation

We begin by introducing a useful multi-index notation, due to Blanchet and Damour, where a tensor with l indices $i_1 i_2 \ldots i_l$ is labeled simply using a capital letter L,

$$F_L \equiv F_{i_1 i_2 \ldots i_l}\,. \qquad (3.180)$$

Similarly, G_{iL} denotes a tensor with $l+1$ indices, $G_{iL}\equiv G_{i i_1 i_2\ldots i_l}$, while, e.g. F_{iL-1} is a notation for $F_{i i_1 i_2\ldots i_{l-1}}$. Furthermore, ∂_L is a notation for $\partial_{i_1}\ldots\partial_{i_l}$, and we will also use the notation $x_L\equiv x_{i_1}x_{i_2}\cdots x_{i_l}$ and $n_L\equiv n_{i_1}n_{i_2}\cdots n_{i_l}$, where $n_i=x_i/r$ is the unit vector in the radial direction. In expressions such as $F_L G_L$, with repeated L indices, the summation over all indices $i_1,i_2,\ldots,i_l$ is understood, so

$$F_L G_L = \sum_{i_1\ldots i_l} F_{i_1\ldots i_l} G_{i_1\ldots i_l}\,. \qquad (3.181)$$

We use round brackets around indices to denote the symmetrization, e.g. $a_{(ij)}\equiv(1/2)(a_{ij}+a_{ji})$, and we denote by a hat the symmetric-trace-free (STF) projection. That is, the notation $\hat{K}_L$ means that, starting from the tensor with l indices $K_{i_1\ldots i_l}$, we symmetrize it over all indices, and remove all the traces. The operation of taking the STF projection can also be denote by brackets around the indices, so $\hat{K}_L$ can be equivalently written as $K_{\langle L\rangle}$. The latter notation allows us to write compactly the STF operation between indices belonging to different tensors, e.g. $\epsilon_{ij\langle k}A_{L-1\rangle i_l}$ means that we perform the STF operation among the index k of ϵ_{ijk} and the first $l-1$ indices of $A_{i_1\ldots i_{l-1}i_l}$.

A STF tensor with l indices (i.e. of rank l) $A_{i_1\ldots i_l}$ has $2l+1$ independent components and is therefore a representation of dimension $2l+1$ of the rotation group $SO(3)$. The crucial point is that these representations are irreducible.[32] On the other hand, we know from the theory of representation of the rotation group that the irreducible representations of $SO(3)$ are labeled by an integer value $l=0,1,,\ldots$, and have dimension $2l+1$, so we see that the set of all STF tensor, for all possible ranks l, gives a complete set of representations of $SO(3)$.

[32] In general, it can be shown that the irreducible tensor representations of a group are those that give zero when contracted with the invariant tensors (i.e. with the tensors whose form is unchanged by a group transformation). For $SO(3)$ the invariant tensors are δ_{ij} and ϵ_{ijk}. In fact, if $\mathcal{R}_{ij}$ is rotation matrix, δ_{ij} transforms as $\delta_{ij}\to\mathcal{R}_{ik}\mathcal{R}_{jl}\delta_{kl}$, as for any tensor with two indices. However, $\mathcal{R}_{ik}\mathcal{R}_{jl}\delta_{kl}=\mathcal{R}_{ik}\mathcal{R}_{jk}=\delta_{ij}$ because $\mathcal{R}$ is an orthogonal matrix. Similarly ϵ_{ijk} is invariant because the determinant of $\mathcal{R}$ is equal to one. The condition that the contraction of any two indices of a tensor with an epsilon tensor gives zero singles out totally symmetric tensor, and the condition that the contraction with δ_{ij} gives zero gives the condition of zero trace, for any pair of indices.

A generic tensor does not provide an irreducible representation of the rotation group, and can be decomposed in irreducible representation, i.e. expressed in terms of STF tensors and factors δ_{ij} and ϵ_{ijk}. The simplest example is the decomposition of a generic tensor with two indices T_{ij}. Writing

$$T_{ij}=\frac{1}{2}(T_{ij}+T_{ji})+\frac{1}{2}(T_{ij}-T_{ji}) \equiv S_{ij}+A_{ij}\,, \qquad (3.182)$$

we have decomposed T_{ij} into its symmetric part S_{ij} and its antisymmetric part A_{ij}. Defining $A_k=\epsilon_{ijk}A_{ij}$, we have $A_{ij}=(1/2)\epsilon_{ijk}A_k$. Furthermore, defining S as the trace of S_{ij}, i.e. $S=S_{ii}$, we can rewrite eq. (3.182) as

$$T_{ij}=\frac{1}{3}S\delta_{ij}+\frac{1}{2}\epsilon_{ijk}A_k+(S_{ij}-\frac{1}{3}S\delta_{ij})\,, \qquad (3.183)$$

which shows explicitly the decomposition of a generic rank-2 tensor T_{ij} into a scalar S, a vector A_k and a rank-2 STF tensor $S_{ij}-(1/3)S\delta_{ij}$.[33]

[33] Observe that scalar and vectors are trivially STF tensors of rank zero and one, respectively.

We now examine separately the application of the STF formalism to the multipole expansion of relativistic scalar, vector and tensor fields.

Scalar fields

Consider a scalar field ϕ satisfying the relativistic wave equation

$$\Box\phi=-4\pi\rho\,, \qquad (3.184)$$

where the source $\rho(t,\mathbf{x})$ is in general time-dependent, but is localized in space, so it vanishes if $|\mathbf{x}|>d$. In the region outside the source, the most general solution can be written as

$$\phi(t,\mathbf{x})=\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{F_L(t-r/c)}{r}\right]\,, \qquad (3.185)$$

where we have used the multi-index notation explained above. This result follows from the fact that, for $r>0$ and F_L an arbitrary function of retarded time $u=t-r/c$,

$$\Box\left[\frac{F_L(t-r/c)}{r}\right]=0\,, \qquad (3.186)$$

as can be checked immediately with the help of eq. (3.172). Therefore each term of the sum in eq. (3.185) is separately a solution of the vacuum equation $\Box\phi=0$. The fact that this is the most general solution follows from the fact that the set of tensors F_L, with all possible ranks l, provides a complete set of representations of the rotation group. So, eq. (3.185) is the generalization of eq. (3.178) to fields governed by a relativistic wave equation, with a non-stationary source. On the other hand, eq. (3.184) can be solved exactly using the retarded Green's function (3.6), which is the appropriate Green's function for a radiation problem, so

$$\phi(t,\mathbf{x}) = \int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|}\,\rho\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c},\mathbf{x}'\right). \tag{3.187}$$

Comparing this expression, which holds everywhere and therefore in particular outside the source, with eq. (3.185), we can obtain the relativistic multipoles F_L in terms of the source density ρ. The result is[34]

$$\boxed{F_L(u) = \int d^3y\,\hat{y}_L \int_{-1}^{1} dz\,\delta_l(z)\rho(u+z|\mathbf{y}|/c,\mathbf{y})\,,} \tag{3.188}$$

where, according to the multi-index notation discussed above, $\hat{y}_L$ is the STF projection of y_L. The function $\delta_l(z)$ is defined as

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1}l!}\,(1-z^2)^l\,, \tag{3.189}$$

and satisfies the identities

$$\int_{-1}^{1} dz\,\delta_l(z) = 1\,, \tag{3.190}$$

and

$$\lim_{l\to\infty}\delta_l(z) = \delta(z)\,, \tag{3.191}$$

where $\delta(z)$ is the usual Dirac delta. Physically, the integration over dz in eq. (3.188) performs a weighted time average, different for each multipole moment l, and originates in the different time delay of the radiation emitted from different points inside the source. Equation (3.191) implies that, for sufficiently large l, this time delay becomes negligible.[35]

[34] The computation is performed in detail in Appendix B of Blanchet and Damour (1989).

[35] Observe that the argument $u+z|\mathbf{y}|/c$ in eq. (3.188) can be changed to $u-z|\mathbf{y}|/c$, since $\delta_l(z)=\delta_l(-z)$.

Vector field

We next consider the electromagnetic field A^μ which, in the Lorentz gauge $\partial_\mu A^\mu = 0$ (and unrationalized units for the electric charge), satisfies the wave equation

$$\Box A^\mu = -\frac{4\pi}{c}J^\mu\,, \tag{3.192}$$

and again we consider a source $J^\mu = (c\rho,\mathbf{J})$ which is time-dependent but localized, so it vanishes if $|\mathbf{x}|>d$. Each component of A^μ can be treated just like the scalar field of the previous subsection, and therefore in the external source region we find

$$A^0(t,\mathbf{x}) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{F_L(u)}{r}\right], \tag{3.193}$$

$$A^i(t,\mathbf{x}) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{G_{iL}(u)}{r}\right], \tag{3.194}$$

where $u = t - r/c$. The explicit expression of $F_L(u)$ in terms of the source density is given again by eq. (3.188), while

$$G_{iL}(u) = \int d^3y\,\hat{y}_L \int_{-1}^{1} dz\,\delta_l(z)J_i(u+z|\mathbf{y}|/c,\mathbf{y})\,. \tag{3.195}$$

This is not yet the most convenient final form of the result, because the tensor G_{iL} is symmetric and traceless with respect to the indices $i_1,\ldots i_l$, since it depends on $\hat{y}_L$, but not under exchange of the index i with one of the indices $i_1,\ldots i_l$, so it is not irreducible. It can however be decomposed in irreducible STF tensors as follows,

$$G_{iL} = U_{iL} + \frac{l}{l+1}\epsilon_{ai\langle i_l}C_{L-1\rangle a} + \frac{2l-1}{2l+1}\delta_{i\langle i_l}D_{L-1\rangle}\,, \tag{3.196}$$

where $U_{L+1}\equiv G_{\langle L+1\rangle}$, $C_L \equiv G_{ab\langle L-1}\epsilon_{i_l\rangle ab}$ and $D_{L-1}\equiv G_{aaL-1}$. Then, $A^i(t,\mathbf{x})$ is given by the sum of three STF terms. It is convenient to perform a gauge transformation $A_\mu \to A_\mu - \partial_\mu\theta$ (with $\Box\theta = 0$ in order not to spoil the Lorentz gauge) which removes the last term in eq. (3.196) from A^i, at the price of adding a new contribution to A^0. The final result is[36]

$$A^0(t,\mathbf{x}) = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{Q_L(u)}{r}\right], \tag{3.197}$$

$$A^i(t,\mathbf{x}) = -\frac{1}{c}\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\partial_{L-1}\left[\frac{Q^{(1)}_{iL-1}(u)}{r} + \frac{l}{l+1}\epsilon_{iab}\partial_a\left(\frac{M_{bL-1}(u)}{r}\right)\right],$$

where $Q^{(1)}_{iL-1}$ denotes the first derivative of Q_{iL-1} with respect to retarded time.[37] The explicit expression of the moments Q_L and M_L in terms of the source is

$$Q_L(u) = \int d^3y \int_{-1}^{1} dz\left[\delta_l(z)\hat{y}_L\,\rho(u+z|\mathbf{y}|/c,\mathbf{y}) \right. \tag{3.199}$$
$$\left. -\frac{1}{c^2}\frac{2l+1}{(l+1)(2l+3)}\delta_{l+1}(z)\hat{y}_{iL}J^{(1)}_i(u+z|\mathbf{y}|/c,\mathbf{y})\right],$$

where $l \geqslant 0$, and

$$M_L(u) = \int d^3y \int_{-1}^{1} dz\,\delta_l(z)\hat{y}_{\langle L-1}m_{i_l\rangle}(u+z|\mathbf{y}|/c,\mathbf{y})\,, \tag{3.200}$$

where $l \geqslant 1$ and $m_i = \epsilon_{ijk}y_jJ_k$ is the "magnetization density". These results show that the electromagnetic field outside the source can be expressed in terms of two families of STF time-dependent multipole moments, the "electric moments" $Q_L(u)$ and the "magnetic moments" $M_L(u)$.

[36] The computation is performed in detail in Damour and Iyer (1991a).

[37] For a function $f(u)$ of retarded time, we will use the notation

$$f^{(n)}(u) \equiv \frac{d^nf}{du^n}\,. \tag{3.198}$$

Gravitational field

We now consider the linearized gravitational field $\bar{h}_{\mu\nu}$ that, in the Lorentz gauge (1.18), satisfies eq. (1.24). Again, we assume that the source $T_{\mu\nu}$

has compact support. Then, in the exterior source region the same argument used above for scalar and electromagnetic fields allows us to write the most general solution of the vacuum equations, as

$$\bar{h}^{00}(t,\mathbf{x}) = \frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{F_L(u)}{r}\right], \tag{3.201}$$

$$\bar{h}^{0i}(t,\mathbf{x}) = \frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{G_{iL}(u)}{r}\right], \tag{3.202}$$

$$\bar{h}^{ij}(t,\mathbf{x}) = \frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{H_{ijL}(u)}{r}\right]. \tag{3.203}$$

The computation proceeds similarly to the electromagnetic case, but is technically more involved.[38] We first decompose G_{iL} in STF tensors, just as we have done in eq. (3.196). Similarly, we decompose H_{ijL} in STF tensors. After performing a suitable gauge transformation (that preserves the gauge condition $\partial_\mu\bar{h}^{\mu\nu}=0$), the result can be written as

[38] See Damour and Iyer (1991a), where it is performed in all detail.

$$\bar{h}^{00} = +\frac{4G}{c^2}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{M_L(u)}{r}\right], \tag{3.204}$$

$$\bar{h}^{0i} = -\frac{4G}{c^3}\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\partial_{L-1}\left[\frac{M^{(1)}_{iL-1}(u)}{r} + \frac{l}{l+1}\epsilon_{iab}\partial_a\left(\frac{S_{bL-1}(u)}{r}\right)\right],$$

$$\bar{h}^{ij} = +\frac{4G}{c^4}\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\partial_{L-2}\left[\frac{1}{r}M^{(2)}_{ijL-2}(u) + \frac{2l}{l+1}\partial_a\left(\frac{1}{r}\epsilon_{ab(i}S^{(1)}_{j)bL-2}(u)\right)\right].$$

The result therefore depends again on two families of STF tensors, M_L and S_L. Their explicit expression in terms of the energy–momentum tensor of the source can be written introducing the "active gravitational-mass density" σ,

$$\sigma \equiv \frac{1}{c^2}\left[T^{00}+T^{ii}\right]. \tag{3.205}$$

and the "active mass-current density",

$$\sigma_i \equiv \frac{1}{c}T^{0i}, \tag{3.206}$$

as well as $\sigma_{ij}=T^{ij}$. Then

$$M_L(u) = \int d^3x\int_{-1}^{1}dz\left\{\delta_l(z)\hat{x}_L\sigma - \frac{4(2l+1)\delta_{l+1}(z)}{c^2(l+1)(2l+3)}\hat{x}_{iL}\sigma^{(1)}_i + \frac{2(2l+1)\delta_{l+2}(z)}{c^4(l+1)(l+2)(2l+5)}\hat{x}_{ijL}\sigma^{(2)}_{ij}\right\}(u+z|\mathbf{x}|/c,\mathbf{x}), \tag{3.207}$$

$$S_L(u) = \int d^3x\int_{-1}^{1}dz\,\epsilon_{ab\langle i_l}\left\{\delta_l(z)\hat{x}_{L-1\rangle a}\sigma_b - \frac{(2l+1)\delta_{l+1}(z)}{c^2(l+2)(2l+3)}\hat{x}_{L-1\rangle ac}\sigma^{(1)}_{bc}\right\}(u+z|\mathbf{x}|/c,\mathbf{x}). \tag{3.208}$$

This result holds at the level of linearized theory. In Chapter 5 we will study the full non-linear theory and we will discover that, remarkably, the exact solution of the full general-relativistic problem is constructed using a quantity $\mathsf{h}_1^{\mu\nu}$ that is obtained from eqs. (3.204)–(3.208) by means of a very simple modification, that is, with the replacement of the energy momentum tensor of matter $T_{\mu\nu}$ by an effective energy–momentum tensor $\tau_{\mu\nu}$ that includes also the non-linearities of the gravitational field,[39] see eqs. (5.135) and (5.136). Thus, eqs. (3.207) and (3.208) already contain the blueprint of the solution to the full non-linear problem.

[39] Together with a prescription for rendering finite the integral, since the source $\tau_{\mu\nu}$ no longer has compact support.

The integration over z can be computed, in an expansion in powers of $1/c$, using the formula

$$\int_{-1}^{1}dz\,\delta_l(z)f(u+z|\mathbf{x}|/c,\mathbf{x}) = \sum_{k=0}^{+\infty}\frac{(2l+1)!!}{2^k k!(2l+2k+1)!!}\left(\frac{|\mathbf{x}|}{c}\frac{\partial}{\partial u}\right)^{2k}f(u,\mathbf{x}). \tag{3.209}$$

From this we see that, in eqs. (3.207) and (3.208) (as well as in the analogous formulas for the scalar and vector fields), the integration over z allows us to take into account, in a compact way, an infinite series of derivatives.

Finally, we can use these multipolar expressions for $\bar{h}_{\mu\nu}$ to compute the total power radiated in GWs. We use eq. (1.40), plug it into eq. (1.153) retaining only the terms $O(1/r)$ in $h_{ij}^{\rm TT}$, and we perform the angular integration.[40] The result is

[40] The power is quadratic in $\dot{h}_{ij}^{\rm TT}$ and therefore in the multipole moments. However, mixed terms of the form $M_L S_{L'}$ give vanishing contribution, after the angular integration, because of parity, while terms $M_L M_{L'}$ or $S_L S_{L'}$ contribute only if $L=L'$. In fact, if $l>l'$, the indices in $M_{i_1\ldots i_l}$ cannot be all contracted with the indices of $M_{i_1\ldots i_{l'}}$, and the remaining indices of the $i_1\ldots i_l$ group are necessarily contracted among them, via the Kronecker deltas that come of from the angular integration, see eq. (3.23).

$$\frac{dE}{dt} = \sum_{l=2}^{+\infty}\frac{G}{c^{2l+1}}\left\{\frac{(l+1)(l+2)}{(l-1)l\,l!(2l+1)!!}\langle M_L^{(l+1)}(u)M_L^{(l+1)}(u)\rangle + \frac{4l(l+2)}{c^2(l-1)(l+1)!(2l+1)!!}\langle S_L^{(l+1)}(u)S_L^{(l+1)}(u)\rangle\right\}. \tag{3.210}$$

Similarly, for the linear momentum losses one finds

$$\frac{dP_i}{dt} = \sum_{l=2}^{+\infty}\frac{G}{c^{2l+2}}\left\{\frac{2(l+2)(l+3)}{l(l+1)!(2l+3)!!}\langle M_{iL}^{(l+2)}(u)M_L^{(l+1)}(u)\rangle + \frac{8(l+3)}{c^2(l+1)!(2l+3)!!}\langle S_{iL}^{(l+2)}(u)S_L^{(l+1)}(u)\rangle + \frac{8(l+2)}{c^2(l-1)(l+1)!(2l+1)!!}\langle\epsilon_{ijk}S_{jL-1}^{(l+1)}(u)S_{kL-1}^{(l+1)}(u)\rangle\right\}. \tag{3.211}$$

Observe that the linear momentum losses come from the interference between multipoles of different rank, such as the mass quadrupole/mass octupole mixed term, as we already saw on pages 130–131.[41]

[41] See Thorne (1980), eq. (4.23′) for the corresponding expression for the angular momentum losses.

3.5.2 Spherical tensor form

This section is quite technical, and can be omitted at a first reading.

In this section we discuss an alternative formalism for performing the multipole expansion to all orders, which is based on the generalization of the notion of spherical harmonics to a spin-2 field.

Spherical components of tensors

To introduce spherical tensors we first consider the quadrupole moment or, more generally, any traceless symmetric tensor with two indices, whose Cartesian components we denote by Q_{ij}. As a first step we introduce a basis in the space of traceless symmetric tensors with two indices, which is chosen so to have a simple relation with the $l=2$ spherical harmonics.

We recall that the spherical harmonics $Y^{lm}(\theta,\phi)$ with $l=2$ are[42]

$$Y^{22}(\theta,\phi)=\left(\frac{15}{32\pi}\right)^{1/2}(e^{i\phi}\sin\theta)^2\,, \tag{3.212}$$

$$Y^{21}(\theta,\phi)=-\left(\frac{15}{8\pi}\right)^{1/2}e^{i\phi}\sin\theta\cos\theta\,, \tag{3.213}$$

$$Y^{20}(\theta,\phi)=\left(\frac{5}{16\pi}\right)^{1/2}(3\cos^2\theta-1)\,, \tag{3.214}$$

while the expressions for negative values of m are obtained using

$$Y^{l,-m}=(-1)^m Y^{lm*}\,. \tag{3.215}$$

Consider now the unit radial vector $\hat{\mathbf{n}}$. In polar coordinates we have $n_x=\sin\theta\cos\phi$, $n_y=\sin\theta\sin\phi$, $n_z=\cos\theta$, and therefore

$$e^{i\phi}\sin\theta=n_x+in_y\,,\qquad \cos\theta=n_z\,. \tag{3.216}$$

Plugging this into the explicit expressions for the spherical harmonics Y^{2m}, and using the fact that $n_in_i=1$, we see that we can write

$$\boxed{Y^{2m}(\theta,\phi)=\mathcal{Y}^{2m}_{ij}n_in_j\,,} \tag{3.217}$$

where $\mathcal{Y}^{2m}_{ij}$ is independent of θ,ϕ, and the sum over i,j on the right-hand side is understood. The above equation fixes the part of $\mathcal{Y}^{2m}_{ij}$ which is symmetric in (i,j), and we complete the definition of $\mathcal{Y}^{2m}_{ij}$ requiring that the antisymmetric part vanishes. The explicit form of the tensors $\mathcal{Y}^{2m}_{ij}$ is then

$$\mathcal{Y}^{22}_{ij}=\sqrt{\frac{15}{32\pi}}\begin{pmatrix}1 & i & 0\\ i & -1 & 0\\ 0 & 0 & 0\end{pmatrix}_{ij}\,,$$
$$\mathcal{Y}^{21}_{ij}=-\sqrt{\frac{15}{32\pi}}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & i\\ 1 & i & 0\end{pmatrix}_{ij}\,, \tag{3.218}$$
$$\mathcal{Y}^{20}_{ij}=\sqrt{\frac{5}{16\pi}}\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 2\end{pmatrix}_{ij}\,,$$

together with $\mathcal{Y}^{2,-m}_{ij}=(-1)^m(\mathcal{Y}^{2m}_{ij})^*$. We see from the explicit expressions that the five matrices $\mathcal{Y}^{2,m}_{ij}$ are traceless in the (i,j) indices. This could have also been understood by integrating eq. (3.217) over the solid angle and using $\int d\Omega\, Y^{2m}=0$ and $\int d\Omega\, n_in_j\sim\delta_{ij}$. From the explicit expressions, we also see that the five tensors $\mathcal{Y}^{2m}_{ij}$ are an orthogonal basis, in the sense that

$$\sum_{ij}\mathcal{Y}^{2m}_{ij}(\mathcal{Y}^{2m'}_{ij})^*=\frac{15}{8\pi}\delta^{mm'}\,. \tag{3.219}$$

It is sometimes useful to invert eq. (3.217). The result is

$$n_in_j-\frac{1}{3}\delta_{ij}=\sum_{m=-2}^{2}c^m_{ij}\,Y^{2m}(\theta,\phi)\,, \tag{3.220}$$

where

$$c^m_{ij}=\frac{8\pi}{15}(\mathcal{Y}^{2m}_{ij})^*\,. \tag{3.221}$$

This can be proved multiplying both sides of eq. (3.220) by $\mathcal{Y}^{2m}_{ij}$, summing over i,j and using eqs. (3.221) and (3.219), which gives back eq. (3.217). Since $\mathcal{Y}^{2m}_{ij}$ is traceless, the coefficient of the term proportional to δ_{ij} on the left-hand side of eq. (3.220) is not yet fixed in this way, but we can fix it by observing that the right-hand side of eq. (3.220) is traceless, so also the left-hand side must be traceless.

The five symmetric and traceless matrices $\mathcal{Y}^{2m}_{ij}$, with $m=-2,\ldots,2$, are linearly independent and therefore are a basis for the five-dimensional space of traceless symmetric tensors Q_{ij}. This means that we can expand an arbitrary traceless symmetric tensors Q_{ij} as

$$\boxed{Q_{ij}=\sum_{m=-2}^{2}Q_m\mathcal{Y}^{2m}_{ij}\,.} \tag{3.222}$$

The quantities Q_m are called the *spherical components* of Q_{ij}. Multiplying by n_in_j and using eq. (3.217) we obtain

$$Q_{ij}n_in_j=\sum_{m=-2}^{2}Q_mY^{2m}(\theta,\phi)\,. \tag{3.223}$$

(As always, the summation over the repeated i,j indices is understood; instead, we write explicitly the sum over m.) This equation could also have been obtained directly by observing that $Q_{ij}n_in_j$ is a function of θ,ϕ (with the dependence hidden in $n_i(\theta,\phi)$, while Q_{ij} is a constant tensor) and therefore can be expanded in spherical harmonics as $\sum_{l,m}Q_{lm}Y^{lm}(\theta,\phi)$. However, Q_{ij} is symmetric and traceless, so it is a spin-2 operator and therefore in the expansion in spherical harmonics, only $l=2$ contributes.

The five independent components of the symmetric traceless tensor Q_{ij} are therefore expressed in terms of the five independent quantities Q_m, with $m=-2,\ldots,2$. If Q_{ij} is real, as in the case of the mass

[42] Here, according to the standard definition of spherical harmonics, the angle ϕ is measured from the x axis, so for instance the unit vector in the radial direction $\hat{\mathbf{n}}$ has components $n_x=\sin\theta\cos\phi$, $n_y=\sin\theta\sin\phi$ and $n_z=\cos\theta$. Observe that this definition differs from that used, e.g. in Fig. 3.2 and in equations such as eq. (3.72), where ϕ is measured from the y axis.

quadrupole, the five complex quantities Q_m satisfy $Q_m^* = (-1)^m Q_{-m}$, because of eq. (3.215). Using eq. (3.219) we can invert eq. (3.222),

$$Q_m = \frac{8\pi}{15} Q_{ij} (\mathcal{Y}_{ij}^{2m})^* , \tag{3.224}$$

or, explicitly,

$$\begin{aligned} Q_{\pm 2} &= \left(\frac{2\pi}{15}\right)^{1/2} (Q_{11} - Q_{22} \mp 2iQ_{12}) , \\ Q_{\pm 1} &= \mp \left(\frac{8\pi}{15}\right)^{1/2} (Q_{13} \mp iQ_{23}) , \\ Q_0 &= -\left(\frac{4\pi}{5}\right)^{1/2} (Q_{11} + Q_{22}) . \end{aligned} \tag{3.225}$$

We can now write the power emitted by the quadrupole radiation, given in eq. (3.75), in terms of the spherical components Q_m. Using eq. (3.223) we write

$$\dddot{Q}_{ij} n_i n_j = \sum_{m=-2}^{2} \dddot{Q}_m Y^{2m}(\theta, \phi) , \tag{3.226}$$

and we take the squared modulus,

$$\dddot{Q}_{ij} \dddot{Q}_{kl} n_i n_j n_k n_l = \sum_{m,m'} \dddot{Q}_m^* \dddot{Q}_{m'} Y^{2m^*}(\theta, \phi) Y^{2m'}(\theta, \phi) . \tag{3.227}$$

Integrating over $d\Omega$ with the help of eq. (3.22) and using the orthogonality of the spherical harmonics,

$$\int d\Omega \, Y^{lm^*}(\theta, \phi) Y^{l'm'}(\theta, \phi) = \delta^{ll'} \delta^{mm'} , \tag{3.228}$$

we get

$$\frac{8\pi}{15} \dddot{Q}_{ij} \dddot{Q}_{ij} = \sum_{m=-2}^{2} |\dddot{Q}_m|^2 . \tag{3.229}$$

Therefore, eq. (3.75) becomes

$$\boxed{P_{\rm quad} = \frac{3G}{8\pi c^5} \sum_{m=-2}^{2} \langle |\dddot{Q}_m|^2 \rangle .} \tag{3.230}$$

We can now generalize the above construction to traceless symmetric tensors with an arbitrary number of indices. We consider a (real) STF tensor with l indices, $T_{i_1 \ldots i_l}$.[43] A basis in this tensor space can be obtained by observing that the spherical harmonics $Y^{lm}(\theta, \phi)$ with $m \geqslant 0$ are given explicitly by

$$\begin{aligned} Y^{lm}(\theta, \phi) &= C^{lm} e^{im\phi} P^{lm}(\cos\theta) \\ &= C^{lm} (e^{i\phi} \sin\theta)^m \sum_{k=0}^{[(l-m)/2]} a_k^{lm} (\cos\theta)^{l-m-2k} , \end{aligned} \tag{3.231}$$

[43] In this section we prefer to write explicitly the indices $i_1 \ldots i_l$, rather than using the multi-index notation defined on page 134.

and $Y^{lm} = (-1)^m (Y^{l,-m})^*$ for $m < 0$. The notation $[(l-m)/2]$ denotes the largest integer smaller or equal to $(l-m)/2$, and the coefficients are given by

$$C^{lm} = (-1)^m \left(\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right)^{1/2} , \tag{3.232}$$

$$a_k^{lm} = \frac{(-1)^k}{2^l \, k!(l-k)!} \frac{(2l-2k)!}{(l-m-2k)!} . \tag{3.233}$$

Comparing with eq. (3.216) we see that Y^{lm} is the sum of a term containing l factors n_i, a term containing $l-2$ factors n_i, a term containing $l-4$ factors n_i, etc. Using $n_i n_i = 1$, a term with $l-2$ factors $n_{i_1} \ldots n_{i_{l-2}}$ can be rewritten trivially as a term with l factors n_i, as $\delta_{ij} n_{i_1} \ldots n_{i_{l-2}} n_i n_j$, and similarly for all terms with $l - 2k$ factors. Then we can write

$$Y^{lm}(\theta, \phi) = \mathcal{Y}_{i_1 \ldots i_l}^{lm} n_{i_1} \cdots n_{i_l} , \tag{3.234}$$

where the tensors $\mathcal{Y}_{i_1 \ldots i_l}^{lm}$ are independent of θ, ϕ, and the sum over the l indices $i_1 \ldots i_l$ is understood. We will not need the explicit form of $\mathcal{Y}_{i_1 \ldots i_l}^{lm}$, which anyway can be read from eqs. (3.231) and (3.216). Just as in the case $l = 2$ discussed above, one can show that the tensors $\mathcal{Y}_{i_1 \ldots i_l}^{lm}$ are a basis in the space of traceless symmetric tensors with l indices. This means that we can expand

$$T_{i_1 \ldots i_l} = \sum_{m=-l}^{l} T_{lm} \mathcal{Y}_{i_1 \ldots i_l}^{lm} , \tag{3.235}$$

and this defines the spherical components T_{lm} of the tensor $T_{i_1 \ldots i_l}$. Multiplying by $n_{i_1} \cdots n_{i_l}$ we have the identity

$$T_{i_1 \ldots i_l} n_{i_1} \cdots n_{i_l} = \sum_{m=-l}^{l} T_{lm} Y^{lm}(\theta, \phi) , \tag{3.236}$$

which expresses the fact that in the expansion in spherical harmonics of the left-hand side contribute only the spherical harmonics whose angular momentum l is equal to the number of indices of $T_{i_1 \ldots i_l}$. This is a consequence of the fact that both a STF tensor with l indices, and the spherical harmonics Y^{lm}, provide an irreducible representation of dimension $2l + 1$ of the rotation group.

Using the orthogonality of spherical harmonics we can invert eq. (3.236) and we obtain the spherical components T_{lm} in terms of the Cartesian components $T_{i_1 \ldots i_l}$,

$$\begin{aligned} T_{lm} &= T_{i_1 \ldots i_l} \int d\Omega \, (Y^{lm})^* n_{i_1} \cdots n_{i_l} \\ &= T_{i_1 \ldots i_l} (\mathcal{Y}_{j_1, \ldots j_l}^{lm})^* \int d\Omega \, n_{i_1} \cdots n_{i_l} n_{j_1} \cdots n_{j_l} , \end{aligned} \tag{3.237}$$

where in the second line we used eq. (3.234). The integral can be performed using eq. (3.23). Since $\mathcal{Y}_{j_1, \ldots j_l}^{lm}$ and $T_{i_1 \ldots i_l}$ are traceless, in the sum over permutations in eq. (3.23) the terms with Kronecker deltas of the type $\delta_{i_k i_{k'}}$ or $\delta_{j_k j_{k'}}$ give zero, and the only contributions come from the term $\delta_{i_1 j_1} \ldots \delta_{i_l j_l}$ and from its permutations. Since $\mathcal{Y}^{lm}_{j_1 \ldots j_l}$ is totally symmetric, these $l!$ permutations all give the same result, so

$$T_{lm} = 4\pi \frac{l!}{(2l+1)!!} T_{i_1 \ldots i_l} (\mathcal{Y}^{lm}_{i_1 \ldots i_l})^* . \tag{3.238}$$

For $l = 2$, we recover the result obtained in eq. (3.224). Finally, a useful identity which generalizes eq. (3.229) is obtained taking the modulus squared of eq. (3.236) and integrating over $d\Omega$ with the help of eq. (3.23). This gives

$$4\pi \frac{l!}{(2l+1)!!} T_{i_1 \ldots i_l} T^{i_1 \ldots i_l} = \sum_{m=-l}^{l} |T_{lm}|^2 . \tag{3.239}$$

The transformation properties under rotations of the spherical components of tensors are fixed by the transformation properties of the spherical harmonics. For instance, consider a rotation by an angle φ around the z-axis, $\phi \to \phi + \varphi$. The left-hand side of eq. (3.236) is a scalar so it is invariant, while on the right-hand side $Y_{lm} \to e^{im\varphi} Y_{lm}$. Therefore T_{lm} transforms into itself, as[44]

$$T_{lm} \to e^{-im\varphi} T_{lm} . \tag{3.240}$$

More generally, under arbitrary rotations the $2l+1$ components of T_{lm}, with $m = -l, \ldots, l$ and l given, transform among themselves in the same way as $Y^*_{lm}(\theta, \phi)$.

[44] For this reasons, the spherical components find a typical application in quantum mechanics, for writing selection rules in atomic transitions.

Vector and tensor spherical harmonics

The spherical components of tensors, introduced above, are one of the tools useful for the construction of a systematic multipole expansion. Here we introduce another necessary ingredient of this formalism, the tensor spherical harmonics. In the same way as the usual (scalar) spherical harmonics are useful to describe the angular dependence of a scalar field, tensor spherical harmonics are useful for describing the angular dependence of a field with spin.

We denote by $\mathbf{L}$ the orbital angular momentum operator, by $\mathbf{S}$ the spin operator and by $\mathbf{J} = \mathbf{L} + \mathbf{S}$ the total angular momentum. All these quantities are measured in units of $\hbar$, so for instance, as operator acting on functions, $\mathbf{L} = \mathbf{r} \times (-i\boldsymbol{\nabla})$. Since the operators $\mathbf{J}^2$, J_z^2, $\mathbf{L}^2$ and $\mathbf{S}^2$ commute, we can diagonalize them simultaneously. The eigenfunctions are the tensor spherical harmonics and are denoted by $Y^{ls}_{jj_z}(\theta, \phi)$. Therefore, by definition, the functions $Y^{ls}_{jj_z}(\theta, \phi)$ are the solutions of

$$\mathbf{J}^2 Y^{ls}_{jj_z} = j(j+1) Y^{ls}_{jj_z} , \tag{3.241}$$

$$J_z Y^{ls}_{jj_z} = j_z Y^{ls}_{jj_z} , \tag{3.242}$$

$$\mathbf{L}^2 Y^{ls}_{jj_z} = l(l+1) Y^{ls}_{jj_z} , \tag{3.243}$$

$$\mathbf{S}^2 Y^{ls}_{jj_z} = s(s+1) Y^{ls}_{jj_z} . \tag{3.244}$$

Their explicit form can be obtained coupling the (scalar) spherical harmonics Y_{lm} to the spin function χ_{ss_z}, with the appropriate Clebsch–Gordan coefficients which gives a state with total angular momentum $|j, j_z\rangle$,

$$Y^{ls}_{jj_z}(\theta, \phi) = \sum_{l_z=-l}^{l} \sum_{s_z=-s}^{s} \langle s l s_z l_z | j j_z \rangle \, Y_{ll_z}(\theta, \phi) \chi_{ss_z} . \tag{3.245}$$

It is easy to check that this expression indeed satisfies eqs. (3.241)-(3.244). For instance, the operators L_i act only on the variables θ, ϕ of $Y_{ll_z}(\theta, \phi)$, and then eq. (3.243) follows from $\mathbf{L}^2 Y_{ll_z} = l(l+1) Y_{ll_z}$. Similarly, eq. (3.244) follows from the fact that the spin operator acts only on the spin wavefunction χ, with $\mathbf{S}^2 \chi_{ss_z} = s(s+1) \chi_{ss_z}$. Finally, the Clebsch–Gordan coefficients $\langle s l s_z l_z | j j_z \rangle$ couple a state with orbital angular momentum $|l l_z\rangle$ to a state with spin $|s s_z\rangle$ to give a state with total angular momentum $|j j_z\rangle$, so that eqs. (3.241) and (3.242) follow.

Depending on the value of s, one has spinor spherical harmonics ($s = 1/2$), vector spherical harmonics ($s = 1$), spin-2 tensor spherical harmonics, etc.[45]

Tensor spherical harmonics describe the angular distribution and polarization of particles of spin s, in a state with definite values of the total angular momentum j, of j_z, and of the orbital angular momentum l. For gravitational waves, we are interested in spin-2 tensor spherical harmonics. Observe that, beside the indices l, s, j, j_z written explicitly, $Y^{ls}_{jj_z}$ carries also an index which depends on the nature of the spin wavefunction χ; e.g. a spinor index for $s = 1/2$, a vector index for $s = 1$, a pair of spatial indices (i, i') for $s = 2$ (with $(Y^{l2}_{jj_z})_{ii'}$ symmetric and traceless in i, i'), etc.

[45] One should not be misled by this nomenclature. Of course, the properties of $Y^{ls}_{jj_z}$ under rotations depend on the value of the *total* angular momentum j, not on the spin s, so for instance a vector spherical harmonics $Y^{l1}_{jj_z}$ with $j = 2$ (and therefore $l = 1, 2$ or 3) has the transformation properties of a spin-2 operator, not of a vector, just as the usual scalar spherical harmonics Y_{lm} transform of course as a spin-l operator, not as a scalar.

Vector spherical harmonics

Let us first examine the vector spherical harmonics. The spin wave function χ in this case is a vector, and we denote it by $\boldsymbol{\xi}$. The wavefunctions with a definite value of $s_z = 0, \pm 1$ can be constructed from the unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ as

$$\boldsymbol{\xi}^{(\pm 1)} = \mp \frac{1}{\sqrt{2}} (\mathbf{e}_x \pm i \mathbf{e}_y) , \qquad \boldsymbol{\xi}^{(0)} = \mathbf{e}_z . \tag{3.246}$$

Then the vector spherical harmonics are

$$\mathbf{Y}^{l}_{jj_z}(\theta, \phi) = \sum_{l_z=-l}^{l} \sum_{s_z=0,\pm 1} \langle 1 l s_z l_z | j j_z \rangle \, Y_{ll_z}(\theta, \phi) \, \boldsymbol{\xi}^{(s_z)} . \tag{3.247}$$

Observe that in $\boldsymbol{\xi}^{(s_z)}$ the index $s_z = \pm 1, 0$ tells us which vector we must consider, according to eq. (3.246); the spatial components of the vector are instead denoted by $\xi^{(s_z)}_i$. Correspondingly, $\mathbf{Y}^{l}_{jj_z}$ is a vector with components $(Y^{l}_{jj_z})_i$. Note also that we have written $\mathbf{Y}^{l1}_{jj_z}$ simply as $\mathbf{Y}^{l}_{jj_z}$, since the fact that $s = 1$ is already implicit in the vector notation $\mathbf{Y}$.

By construction, the vector spherical harmonics are eigenfunctions of $\mathbf{L}^2$,

$$\mathbf{L}^2\,(Y^l_{jj_z})_i = l(l+1)\,(Y^l_{jj_z})_i\,, \tag{3.248}$$

and therefore are useful in solving an equation of the form $\Box\mathbf{V}=0$, where $\Box = -\partial_0^2+\boldsymbol{\nabla}^2$ is the flat space d'Alembertian and $\mathbf{V}$ a vector field. In fact, using the expression for the Laplacian in spherical coordinates, eq. (3.172), we can separate the radial and the angular dependence writing

$$\mathbf{V}(r,\theta,\phi) = \sum_{l,j,j_z} f_{ljj_z}(r)\mathbf{Y}^l_{jj_z}(\theta,\phi)\,. \tag{3.249}$$

Observe that the vector spherical harmonics are orthonormal,

$$\int d\Omega\,\mathbf{Y}^l_{jj_z}\cdot(\mathbf{Y}^{l'}_{j'j'_z})^* = \delta_{ll'}\delta_{jj'}\delta_{j_zj'_z}\,. \tag{3.250}$$

The vectors $\mathbf{Y}^l_{jj_z}(\theta,\phi)$ can have $j=l-1,l,l+1$ (if $l\neq 0$), or $j=1$ if $l=0$, (that is, the possible quantum combinations of spin $s=1$ and orbital angular momentum l). For generic values of l and j within this range, they have no special property with respect to the radial unit vector $\hat{\mathbf{n}}$, that is, they are neither purely transverse nor purely longitudinal. We can however observe that the full set of vectors $\mathbf{Y}^l_{jj_z}$, with $j=l-1,l,l+1$ (if $l\neq 0$), or $j=1$ if $l=0$, can be expressed in terms of the following combinations,

$$\mathbf{Y}^E_{jj_z} = (2j+1)^{-1/2}\left[(j+1)^{1/2}\mathbf{Y}^{j-1}_{jj_z}+j^{1/2}\mathbf{Y}^{j+1}_{jj_z}\right]\,, \tag{3.251}$$

$$\mathbf{Y}^B_{jj_z} = i\mathbf{Y}^j_{jj_z}\,, \tag{3.252}$$

$$\mathbf{Y}^R_{jj_z} = (2j+1)^{-1/2}\left[j^{1/2}\mathbf{Y}^{j-1}_{jj_z}-(j+1)^{1/2}\mathbf{Y}^{j+1}_{jj_z}\right]\,, \tag{3.253}$$

with $j\geqslant 1$, together with $\mathbf{Y}^R_{00}=Y_{00}\hat{\mathbf{n}}$. Observe that, since $s=1$, a given value of $j\geqslant 1$ can be obtained with $l=j-1,j$ or $j+1$. In eqs. (3.251) and (3.253) we have combined the vector harmonic with $l=j-1$ and the vector harmonic with $l=j+1$, while $\mathbf{Y}^B_{jj_z}$ is made with $l=j$.[46]

Since $\mathbf{Y}^E_{jj_z}$ and $\mathbf{Y}^R_{jj_z}$ are superposition of vector harmonics with different values of l, they are no longer eigenfunctions of $\mathbf{L}^2$. However, using the properties of spherical harmonics, we can rewrite the above definitions in terms of the scalar spherical harmonics Y_{lm} as[47]

$$\mathbf{Y}^E_{lm} = [l(l+1)]^{-1/2}\,r\boldsymbol{\nabla}Y_{lm} \qquad (l\geqslant 1)\,, \tag{3.254}$$

$$\mathbf{Y}^B_{lm} = [l(l+1)]^{-1/2}\,i\mathbf{L}Y_{lm} \qquad (l\geqslant 1)\,, \tag{3.255}$$

$$\mathbf{Y}^R_{lm} = Y_{lm}\,\hat{\mathbf{n}} \qquad (l\geqslant 0)\,. \tag{3.256}$$

From these expressions we see that $\mathbf{Y}^R_{lm}$ is a longitudinal vector, since it is proportional to $\hat{\mathbf{n}}$, while $\mathbf{Y}^E_{lm}$ and $\mathbf{Y}^B_{lm}$ are transverse. In fact, the operator $\mathbf{L}$ has only components in the $\hat{\theta}$ and $\hat{\phi}$ directions and, since Y_{lm} depends only on θ,ϕ and not on r, also $\boldsymbol{\nabla}Y_{lm}$ has only components in

[46] Using the explicit expression of the spherical harmonics, we can verify immediately that $Y_{10}\boldsymbol{\xi}^{(0)}-Y_{11}\boldsymbol{\xi}^{(-1)}-Y_{1-1}\boldsymbol{\xi}^{(+1)}$ is proportional to $\hat{\mathbf{n}}=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$. On the other hand, apart from a proportionality factor, the combination $Y_{10}\boldsymbol{\xi}^{(0)}-Y_{11}\boldsymbol{\xi}^{(-1)}-Y_{1-1}\boldsymbol{\xi}^{(+1)}$ is just the combination (3.247) with the Clebsch–Gordan coefficients necessary to produce a state with $j=0$ from $l=1$ and $s=1$. Thus, the vector spherical harmonics $\mathbf{Y}^1_{00}$ is proportional to $\hat{\mathbf{n}}$, and is therefore proportional to $\mathbf{Y}^R_{00}$ (observe that $Y_{00}=1/(4\pi)^{1/2}$ is just a constant.) In this way we have taken into account the only vector harmonics that exists for $j=0$ while, for each $j\geqslant 1$, we have three vector harmonics, with $l=j,j-1,j+1$. The one with $l=j$ is rewritten as $\mathbf{Y}^B_{jj_z}$ while the two with $l=j\pm 1$ are combined to form $\mathbf{Y}^E_{jj_z}$ and $\mathbf{Y}^R_{jj_z}$, with $j\geqslant 1$.

[47] In eqs. (3.254)–(3.256), in order to conform to the most common notation used in electrodynamics (for vector spherical harmonics) and in the gravitational-wave literature (for the spin-2 tensor harmonics), we have changed the labeling of the indices of $\mathbf{Y}^E$, $\mathbf{Y}^B$ and $\mathbf{Y}^R$ from j,j_z to l,m. It is however important to understand that these indices refer to the *total* angular momentum, and not to the orbital angular momentum.

the $\hat{\theta}$ and $\hat{\phi}$ directions. Furthermore, since $\mathbf{L}=\mathbf{r}\times(-i\boldsymbol{\nabla})=-ir\hat{\mathbf{n}}\times\boldsymbol{\nabla}$, we have

$$\mathbf{Y}^B_{lm} = \hat{\mathbf{n}}\times\mathbf{Y}^E_{lm}\,. \tag{3.257}$$

Therefore $\mathbf{Y}^E_{lm}$ and $\mathbf{Y}^B_{lm}$ are transverse with respect to $\hat{\mathbf{n}}$ and are orthogonal to each other. Under a parity transformation $\mathbf{Y}^E_{lm}$ and $\mathbf{Y}^R_{lm}$ pick a factor $\pi_l=(-1)^l$. This is the transformation property of the electric field, so this is called an electric-type parity. Instead $\mathbf{Y}^B_{lm}$ picks a factor $\pi_l=(-1)^{l+1}$, so it has a magnetic type parity.[48]

The vector functions $\mathbf{Y}^E_{lm}$, $\mathbf{Y}^B_{lm}$ and $\mathbf{Y}^R_{lm}$ are called "pure-spin vector harmonics" because they are appropriate for describing the polarization states of a vector field, while the vector functions $\mathbf{Y}^{l'}_{lm}$ given in eq. (3.247) are called "pure-orbital vector harmonics" because they are eigenfunctions of the orbital angular momentum.

The pure-spin vector harmonics are orthonormal,

$$\int d\Omega\,\mathbf{Y}^J_{lm}\cdot(\mathbf{Y}^{J'}_{l'm'})^* = \delta_{JJ'}\delta_{ll'}\delta_{mm'}\,, \tag{3.258}$$

where $J=E,B,R$. The angular dependence of an arbitrary vector field can be expanded in pure-spin vector harmonics as

$$\mathbf{V}(t,r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} R_{lm}(t,r)\mathbf{Y}^R_{lm}(\theta,\phi) \tag{3.259}$$
$$+\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left[E_{lm}(t,r)\mathbf{Y}^E_{lm}(\theta,\phi)+B_{lm}(t,r)\mathbf{Y}^B_{lm}(\theta,\phi)\right]\,.$$

Observe that, in the second line, the sum over l runs only over $l\geqslant 1$, since the corresponding pure-spin vector harmonics start from $l=1$.[49]

A massive spin-1 particle has three degrees of freedom, and we see that these degrees of freedom are described by E_{lm}, B_{lm} and R_{lm}, respectively. If we want to describe a *massless* vector particle, however, the situation is different. A massless vector particle has only two physical degrees of freedom (see the discussion of the Poincaré representations in Problem 1.1), with helicities $h=\pm 1$. If we perform a rotation by an angle θ around the $\hat{\mathbf{n}}$ axis the two transverse vectors $\mathbf{Y}^E_{lm}$ and $\mathbf{Y}^B_{lm}$ transform among themselves (see eq. (3.257)) so $\mathbf{Y}^E_{lm}\pm i\mathbf{Y}^B_{lm}$ are multiplied by $\exp\{\pm i\theta\}$. Comparing with eq. (2.197), we understand that they describe the two components of a massless particle with helicities $h=\pm 1$. Instead $\mathbf{Y}^R_{lm}$, being proportional to $\hat{\mathbf{n}}$, is invariant under rotations around the $\hat{\mathbf{n}}$ axis, and therefore, again from eq. (2.197), we see that it has $h=0$ which, for a massless particle, implies $s=0$; so $\mathbf{Y}^R_{lm}$ describes a spin-zero massless particle.

In electrodynamics, the longitudinal degree of freedom described by $\mathbf{Y}^R_{lm}$ is eliminated by gauge invariance, and electromagnetic radiation is purely transverse. Therefore in the expansion of the vector potential $\mathbf{A}$

[48] We define parity reversing the orientation of the axes of the reference frame while keeping the vectors fixed (compare with Note 34 on page 98). With respect to the Cartesian basis vectors $(\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z)$, a vector spherical harmonic $\mathbf{Y}_{lm}$ has components $(\mathbf{Y}_{lm})_i$, that is $\mathbf{Y}_{lm}=(\mathbf{Y}_{lm})_i\mathbf{e}_i$. Under parity, the components $(\mathbf{Y}^E_{lm})_i$ and $(\mathbf{Y}^R_{lm})_i$ pick a factor $(-1)^{l+1}$, with $(-1)^l$ coming from the scalar spherical harmonic Y_{lm} and a further minus sign from ∂_i and from n_i, respectively. However, $\mathbf{Y}^E_{lm}=(\mathbf{Y}^E_{lm})_i\mathbf{e}_i$ transform with a factor $(-1)^l$ because of the further minus sign from the transformation of the base vectors $\mathbf{e}_i$. Instead, $(\mathbf{Y}^B_{lm})_i$ picks only the factor $(-1)^l$ from the scalar spherical harmonics, because angular momentum is a pseudovector and its components L_i are unchanged under parity, and then $\mathbf{Y}^B_{lm}=(\mathbf{Y}^B_{lm})_i\mathbf{e}_i$ is multiplied by $\pi_l=(-1)^{l+1}$.

[49] Recall that we have changed notation between eq. (3.253) and eq. (3.254), see Note 47, and the quantity that we are now labeling by l is the *total* angular momentum, previously denoted by j.

we have $R_{lm}=0$,

$$\mathbf{A}(t,r,\theta,\phi)=\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left[E_{lm}(t,r)\mathbf{Y}^{E}_{lm}(\theta,\phi)+B_{lm}(t,r)\mathbf{Y}^{B}_{lm}(\theta,\phi)\right]. \tag{3.260}$$

For each value of the *total* (recall Note 47) angular momentum $l=1,2,\ldots$ the electromagnetic field is therefore characterized by two polarization states with opposite parity, described by the wavefunctions E_{lm} and B_{lm}, called electric and magnetic photons, respectively. Their linear combinations give rise to the two helicity states of the photon. The state with total angular momentum $l=0$ is instead absent.

The fact that, for a massless particle with helicities ± 1, such as the photon, the state with total angular momentum $l=0$ is absent while for all other values of the total angular momentum we have two states with opposite parity agrees with the analysis that we performed in Problem 1.2.

Spin-2 tensor harmonics

We can now introduce the spin-2 tensor harmonics, which are relevant for the description of gravitational radiation. First of all we need the spin wavefunction for $s=2$ with a definite value of s_z; this wavefunction is a traceless symmetric tensor, that we denote by $t^{(s_z)}_{ik}$, and is obtained taking two spin-1 wavefunctions $\xi^{(m_1)}_i$ and $\xi^{(m_2)}_k$, and combining them with the appropriate Clebsch–Gordan coefficients,

$$t^{(s_z)}_{ik}=\sum_{m_1,m_2=-1}^{1}\langle 11m_1m_2|2s_z\rangle\,\xi^{(m_1)}_i\xi^{(m_2)}_k\,. \tag{3.261}$$

Table 3.1 The coefficients which enter in the definition of pure-spin $s=2$ spherical harmonics (from Thorne 1980).

a_{11}	$+\left(\frac{(j+1)(j+2)}{(2j+1)(2j+3)}\right)^{1/2}$
a_{12}	$-\left(\frac{2j(j+1)}{3(2j-1)(2j+3)}\right)^{1/2}$
a_{13}	$+\left(\frac{(j-1)j}{(2j-1)(2j+1)}\right)^{1/2}$
a_{21}	$-\left(\frac{2j(j+2)}{(2j+1)(2j+3)}\right)^{1/2}$
a_{22}	$-\left(\frac{3}{(2j-1)(2j+3)}\right)^{1/2}$
a_{23}	$+\left(\frac{2(j-1)(j+1)}{(2j-1)(2j+1)}\right)^{1/2}$
a_{31}	$+\left(\frac{(j-1)j}{2(2j+1)(2j+3)}\right)^{1/2}$
a_{32}	$+\left(\frac{3(j-1)(j+2)}{(2j-1)(2j+3)}\right)^{1/2}$
a_{33}	$+\left(\frac{(j+1)(j+2)}{2(2j-1)(2j+1)}\right)^{1/2}$
b_{11}	$+\left(\frac{j+2}{2j+1}\right)^{1/2}$
b_{12}	$-\left(\frac{j-1}{2j+1}\right)^{1/2}$
b_{21}	$-\left(\frac{j-1}{2j+1}\right)^{1/2}$
b_{22}	$-\left(\frac{j+2}{2j+1}\right)^{1/2}$

The five tensors $t^{(s_z)}_{ik}$, with $s_z=0,\pm1,\pm2$, are symmetric and traceless, and play the role that the three vectors $\xi^{(s_z)}_i$ with $s_z=0,\pm1$ played for vector spherical harmonics, that is, we combine them with the scalar spherical harmonics to obtain the spin-2 tensor spherical harmonics,

$$(\mathbf{T}^{l}_{jj_z})_{ik}\equiv(Y^{l2}_{jj_z})_{ik}=\sum_{l_z=-l}^{l}\sum_{s_z=-2}^{2}\langle 2ls_zl_z|jj_z\rangle\,Y_{ll_z}(\theta,\phi)\,t^{(s_z)}_{ik}\,. \tag{3.262}$$

Just as in the vector case, $(\mathbf{T}^{l}_{jj_z})_{ik}$ are by construction eigenfunctions of the $\mathbf{L}^2$ operator (and for this reason are called pure-orbital $s=2$ tensor spherical harmonics), but have no special property with respect to the radial unit vector $\hat{\mathbf{n}}$. Similarly to the case of vector harmonics, we can however observe that the full set of tensors $\mathbf{T}^{l}_{jj_z}$, with $j\geqslant 0$ and $j=l\pm2, l\pm1, l$ (if $l\geqslant 2$) or $j=1,2,3$ if $l=1$, or $j=2$ if $l=0$ (that is, the possible quantum combinations of spin $s=2$ and orbital angular momentum l) can be expressed in terms of combinations, called the pure-spin $s=2$ tensor spherical harmonics, with definite properties under rotations along the radial directions. For $j\geqslant 2$, these are given

by

$$\mathbf{T}^{S0}_{jj_z}=a_{11}\,\mathbf{T}^{j+2}_{jj_z}+a_{12}\,\mathbf{T}^{j}_{jj_z}+a_{13}\mathbf{T}^{j-2}_{jj_z}\,, \tag{3.263}$$
$$\mathbf{T}^{E1}_{jj_z}=a_{21}\,\mathbf{T}^{j+2}_{jj_z}+a_{22}\,\mathbf{T}^{j}_{jj_z}+a_{23}\mathbf{T}^{j-2}_{jj_z}\,, \tag{3.264}$$
$$\mathbf{T}^{E2}_{jj_z}=a_{31}\,\mathbf{T}^{j+2}_{jj_z}+a_{32}\,\mathbf{T}^{j}_{jj_z}+a_{33}\mathbf{T}^{j-2}_{jj_z}\,, \tag{3.265}$$
$$\mathbf{T}^{B1}_{jj_z}=b_{11}\,i\mathbf{T}^{j+1}_{jj_z}+b_{12}\,i\mathbf{T}^{j-1}_{jj_z}\,, \tag{3.266}$$
$$\mathbf{T}^{B2}_{jj_z}=b_{21}\,i\mathbf{T}^{j+1}_{jj_z}+b_{22}\,i\mathbf{T}^{j-1}_{jj_z}\,, \tag{3.267}$$

where the coefficients are given in Table 3.1. These combinations can be expressed in terms of the scalar spherical harmonics as follows[50]

[50] As in the vector case, we now switch notation from j,j_z to l,m, but one should be aware that these indices refer to the *total* angular momentum; we reserve instead the notation (i,j) for the spatial indices of vectors.

$$(\mathbf{T}^{S0}_{lm})_{ij}=[n_in_j-(1/3)\delta_{ij}]Y_{lm}\,, \tag{3.268}$$
$$(\mathbf{T}^{E1}_{lm})_{ij}=c^{(1)}_l\,(r/2)(n_i\partial_j+n_j\partial_i)Y_{lm}\,, \tag{3.269}$$
$$(\mathbf{T}^{B1}_{lm})_{ij}=c^{(1)}_l\,(i/2)(n_iL_j+n_jL_i)Y_{lm}\,, \tag{3.270}$$
$$(\mathbf{T}^{E2}_{lm})_{ij}=c^{(2)}_l\,r^2\Lambda_{ij,i'j'}(\hat{\mathbf{n}})\partial_{i'}\partial_{j'}Y_{lm}\,, \tag{3.271}$$
$$(\mathbf{T}^{B2}_{lm})_{ij}=c^{(2)}_l\,r\Lambda_{ij,i'j'}(\hat{\mathbf{n}})(i/2)(\partial_{i'}L_{j'}+\partial_{j'}L_{i'})Y_{lm}\,, \tag{3.272}$$

where, as usual, $\Lambda_{ij,i'j'}$ is the tensor that implements the TT projection, see eq. (1.36), and

$$c^{(1)}_l=\left(\frac{2}{l(l+1)}\right)^{1/2}, \qquad c^{(2)}_l=\left(2\frac{(l-2)!}{(l+2)!}\right)^{1/2}. \tag{3.273}$$

A complete set of $s=2$ spherical harmonic is given by eqs. (3.268)–(3.272) where $\mathbf{T}^{S0}_{lm}$ has $l\geqslant 0$, $\mathbf{T}^{E1}_{lm}$ and $\mathbf{T}^{B1}_{lm}$ have $l\geqslant 1$, while $\mathbf{T}^{E2}_{lm}$ and $\mathbf{T}^{E2}_{lm}$ have $l\geqslant 2$.[51] The above tensors are all symmetric and traceless by construction. On a traceless-symmetric tensor h_{ij}, the transversality condition $n_ih_{ij}=0$ eliminates three degrees of freedom, and indeed we see from the explicit expressions that only $\mathbf{T}^{E2}_{lm}$ and $\mathbf{T}^{B2}_{lm}$ are transverse,

[51] Various useful ways of rewriting eqs. (3.268)–(3.272) can be found in eq. (2.30) of Thorne (1980). Observe also that, using the explicit form of the spherical harmonics Y_{lm}, $\mathbf{T}^{E1}_{lm}$ and $\mathbf{T}^{B1}_{lm}$, as defined in eqs. (3.269) and (3.270), vanish for $l=0$, while $\mathbf{T}^{E2}_{lm}$ and $\mathbf{T}^{B2}_{lm}$, as defined in eqs. (3.271) and (3.272), vanish for $l=0$ and for $l=1$.

$$\boxed{n_i(\mathbf{T}^{E2}_{lm})_{ij}=0\,,\qquad n_i(\mathbf{T}^{B2}_{lm})_{ij}=0\,.} \tag{3.274}$$

The five pure-spin $s=2$ tensor harmonics are appropriate for describing the five independent components of a *massive* spin-2 particle. However, as we discussed in Problem 1.1 a *massless* particle with quantum number s has only two components rather than $2s+1$, with helicities $h=\pm s$. Under a rotation by an angle θ around the $\hat{\mathbf{n}}$ axis, $(\mathbf{T}^{S0}_{lm})_{ij}$ is invariant since it depends only on n_i,n_j. Therefore in the massless case it describes a spin-0 particle. Instead $(\mathbf{T}^{E1}_{lm})_{ij}$ and $(\mathbf{T}^{B1}_{lm})_{ij}$ have one index (i or j) proportional to n_i or n_j, which is invariant, while the other index is carried by a vector in the transverse plane. Therefore they combine to give the two eigenvectors of helicity with $h=\pm1$ and describe a massless vector particle. Finally, the transverse and traceless tensors $(\mathbf{T}^{E2}_{lm})_{ij}$ and $(\mathbf{T}^{B2}_{lm})_{ij}$ have two transverse indices and combine to give rise to the states with $h=\pm2$ which make up a massless particle with $s=2$. Therefore, even if we use the name "spin-2 tensor harmonics", it is important to understand that they can be used to describe either the five polarization states of a massive spin-2 particle, or to describe a *massless* field; in the latter case these five degrees of freedom decompose into the two degrees of freedom of a massless particle with $h = \pm 2$, the two degrees of freedom of a massless particle with $h = \pm 1$, and one degree of freedom corresponding to a massless scalar particle. Observe that the labels $0, 1$ or 2 in $S0, E1, B1, E2, B2$ refer to the (absolute value of the) helicity carried in the massless case.

In particular, in standard general relativity the graviton is a massless particle with helicity ± 2 or, equivalently, the tensor $h_{ij}^{\rm TT}$ that describes GWs in the TT gauge, beside being symmetric and traceless, is also transverse. Therefore in its expansion enter only $\mathbf{T}_{lm}^{E2}$ and $\mathbf{T}_{lm}^{B2}$, while the other components are eliminated by gauge invariance, as discussed in Sections 1.2 and 2.2.2. In the wave zone, where $h_{ij}^{\rm TT}$ decreases as $1/r$, the most general form of $h_{ij}^{\rm TT}(t, r, \theta, \phi)$ is then

$$h_{ij}^{\rm TT} = \frac{1}{r}\frac{G}{c^4}\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\left[u_{lm}(\mathbf{T}_{lm}^{E2})_{ij}(\theta,\phi) + v_{lm}(\mathbf{T}_{lm}^{B2})_{ij}(\theta,\phi)\right], \tag{3.275}$$

where u_{lm} and v_{lm} are functions of retarded time $t - r/c$, and the factor G/c^4 in front is a useful normalization of u_{lm}, v_{lm}. The other pure-spin $s = 2$ tensor harmonics can enter in extensions of general relativity in which further degrees of freedom are present, and the condition of transversality no longer holds. Furthermore, in scalar–tensor extensions of general relativity, h_{ij} is not even traceless and there is a sixth degree of freedom corresponding to the trace of h_{ij}, which is a scalar field. So, in the most general case we must include all five spin-2 tensor harmonics (3.268)–(3.272), and we must further add $\delta_{ij}Y_{lm}$, which is not traceless and accounts for the scalar field corresponding to the trace part. The function $\delta_{ij}Y_{lm}$ can be combined with $(\mathbf{T}_{lm}^{S0})_{ij}$ to give a purely longitudinal and a purely transverse (but not traceless) tensor harmonic,

$$(\mathbf{T}_{lm}^{L0})_{ij} = n_i n_j Y_{lm}, \qquad (\mathbf{T}_{lm}^{T0})_{ij} = \frac{1}{\sqrt{2}}(\delta_{ij} - n_i n_j)Y_{lm}, \tag{3.276}$$

with $l \geqslant 0$. The coefficients in eqs. (3.268)–(3.272) and in eq. (3.276) are chosen so that the pure-spin harmonics are orthonormal,

$$\int d\Omega\, (\mathbf{T}_{lm}^{J})_{ij}(\mathbf{T}_{l'm'}^{J'})_{ij}^{*} = \delta^{JJ'}\delta_{ll'}\delta_{mm'}, \tag{3.277}$$

where the label J takes the values $L0, T0, E1, B1, E2, B2$. Finally, we observe from the explicit expressions that $\mathbf{T}_{lm}^{L0}$, $\mathbf{T}_{lm}^{T0}$, $\mathbf{T}_{lm}^{E1}$ and $\mathbf{T}_{lm}^{E2}$ have "electric-type" parity $\pi_l = (-1)^l$ while $\mathbf{T}_{lm}^{B1}$ and $\mathbf{T}_{lm}^{B2}$ have "magnetic-type" parity $\pi_l = (-1)^{l+1}$.

Equation (3.275) is the main result of this subsection. It is the generalization to the spin-2 field h_{ij} of the more usual expansion of the solution of the wave equation for a relativistic scalar field in the wave zone, in terms of (scalar) spherical harmonics,

$$\phi(t, r, \theta, \phi) = \frac{1}{r}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm}Y_{lm}(\theta,\phi), \tag{3.278}$$

with c_{lm} functions of $t - r/c$, in the wave zone. Comparing eqs. (3.275) and (3.278) we see the following important differences: (1) a scalar field has only one spin degree of freedom and therefore, at each angular momentum level l, m, it is described by a single function $c_{lm}(t - r/c)$. A gravitational wave, instead, has two helicity states. Therefore there are two sets of tensor spherical harmonics, $\mathbf{T}_{lm}^{E2}$ and $\mathbf{T}_{lm}^{B2}$, and correspondingly two sets of functions $u_{lm}(t-r/c)$ and $v_{lm}(t-r/c)$. These states are transverse, see eq. (3.274), one with electric-type parity and the other with magnetic-type parity. Because of eq. (3.277), the pure-spin harmonics provides an orthonormal basis for these modes. (2) The expansion of $h_{ij}^{\rm TT}$ starts from total angular momentum $l = 2$. It is impossible to construct a GW with total angular momentum $l = 0$ or $l = 1$. This counting of degrees of freedom is in full agreement with the discussion of graviton states in Problem 1.2.

Our next task is to relate the coefficients u_{lm}, v_{lm} in eq. (3.275) to the appropriate multipole moments of the source, as we do in the next subsection.[52]

[52] Recall that all our discussion holds in the context of linearized theory, i.e. in a flat background space-time. The determination of the GW at infinity in terms of the source moments in the full non-linear theory is a much more difficult problem, whose solution will be given in Chapter 5. In the non-linear case, we will find that the STF formalism is much more convenient, and in fact the final result that will be given in eqs. (5.135) and (5.136) is a simple generalization of eqs. (3.207) and (3.208).

Relation with the source moments

In eq. (3.34) we found the solution of the equation of motion for $h_{ij}^{\rm TT}$ in the wave zone $r \gg d$, in the form

$$\begin{aligned} h_{ij}^{\rm TT} &= \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,pq}\left[S^{pq} + \frac{1}{c}n_{i_1}\dot{S}^{pq,i_1} + \frac{1}{2c^2}n_{i_1}n_{i_2}\ddot{S}^{pq,i_1i_2} + \ldots\right] \\ &= \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,pq}\sum_{\alpha=0}^{\infty}\frac{1}{\alpha!}\left(\partial_0^{\alpha}S^{pq,i_1\ldots i_\alpha}\right)n_{i_1}\cdots n_{i_\alpha}, \end{aligned} \tag{3.279}$$

where it is understood that all the $S^{pq,i_1\ldots i_\alpha}$ are functions of $t - r/c$. On the other hand, in the previous section we have seen that the most general expansion for $h_{ij}^{\rm TT}$ in the wave zone is given by

$$h_{ij}^{\rm TT} = \frac{1}{r}\frac{G}{c^4}\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\left[u_{lm}(\mathbf{T}_{lm}^{E2})_{ij} + v_{lm}(\mathbf{T}_{lm}^{B2})_{ij}\right], \tag{3.280}$$

where again it is understood that u_{lm} and v_{lm} are functions of $t - r/c$. Comparing the two expressions, we can determine u_{lm} and v_{lm}. To obtain u_{lm}, we multiply both sides of eq. (3.280) by $(\mathbf{T}_{lm}^{E2})_{ij}^{*}$ and we integrate over $d\Omega$. On the right-hand side, using the orthonormality condition (3.277), we single out u_{lm}, while on the left-hand side we insert the expression (3.279) for $h_{ij}^{\rm TT}$. Similarly, to obtain v_{lm} we multiply by

$(\mathbf{T}^{B2}_{lm})^*_{ij}$. Then we get

$$u_{lm} = \sum_{\alpha=0}^{\infty} \frac{4}{\alpha!} \left(\partial_0^\alpha S^{pq,i_1\ldots i_\alpha}\right) \int d\Omega\, (\mathbf{T}^{E2}_{lm})^*_{ij} \Lambda_{ij,pq} n_{i_1} \cdots n_{i_\alpha} , \qquad (3.281)$$

$$v_{lm} = \sum_{\alpha=0}^{\infty} \frac{4}{\alpha!} \left(\partial_0^\alpha S^{pq,i_1\ldots i_\alpha}\right) \int d\Omega\, (\mathbf{T}^{B2}_{lm})^*_{ij} \Lambda_{ij,pq} n_{i_1} \cdots n_{i_\alpha} .$$

Since the Lambda tensor projects on the transverse and traceless part of a tensor, and $\mathbf{T}^{E2}_{lm}$ and $\mathbf{T}^{B2}_{lm}$ are already transverse and traceless, we have

$$(\mathbf{T}^J_{lm})^*_{ij} \Lambda_{ij,pq} = (\mathbf{T}^J_{lm})^*_{pq} , \qquad (3.282)$$

(with $J = E2, B2$), and we can simplify the above expressions,

$$u_{lm} = \sum_{\alpha=0}^{\infty} \frac{4}{\alpha!} \left(\partial_0^\alpha S^{ij,i_1\ldots i_\alpha}\right) \int d\Omega\, (\mathbf{T}^{E2}_{lm})^*_{ij} n_{i_1} \cdots n_{i_\alpha} , \qquad (3.283)$$

$$v_{lm} = \sum_{\alpha=0}^{\infty} \frac{4}{\alpha!} \left(\partial_0^\alpha S^{ij,i_1\ldots i_\alpha}\right) \int d\Omega\, (\mathbf{T}^{B2}_{lm})^*_{ij} n_{i_1} \cdots n_{i_\alpha} . \qquad (3.284)$$

The computation of the integrals is involved, but can be performed order by order in v/c. To leading order in v/c, we perform it in detail in Problem 3.5.[53] The result is

$$u_{lm} = \frac{16\pi}{(2l+1)!!} \left[\frac{l}{2}(l-1)(l+1)(l+2)\right]^{1/2} \mathcal{Y}^{lm*}_{i_1\ldots i_l}\, \partial_0^{l-2} S^{i_1 i_2, i_3 \ldots i_l} \times \left[1 + O\left(\frac{v^2}{c^2}\right)\right] . \qquad (3.285)$$

[53] In general, if one wishes to compute the total radiated power consistently to a given order in v^2/c^2, one cannot stop to the leading order in v^2/c^2 in the computations of u_{lm}. For instance, the subleading terms in the mass quadrupole u_{2m} are of the same order, in the radiated power, as the leading term of the mass octupole u_{3m}. Furthermore, for self-gravitating sources, we cannot use the expansion over flat space as we have done in this chapter, and the computation must be performed using the post-Newtonian formalism presented in Chapter 5.

We now want to write the time derivatives of $S^{i_1 i_2, i_3 \ldots i_l}$ in terms of the derivatives of the momenta of T^{00}, as follows. From the conservation of energy–momentum tensor, $\partial_\mu T^{\mu\nu} = 0$, we have the relations $\partial_0 T^{00} = -\partial_i T^{0i}$ and $\partial_0 T^{0i} = -\partial_j T^{ji}$. We can combine them to get $\partial_0^2 T^{00} = \partial_i \partial_j T^{ij}$. Then, integrating twice by parts,

$$\begin{aligned} \ddot{M}^{i_1\ldots i_l} &= \int d^3x\, (\partial_0^2 T^{00}) x^{i_1} \cdots x^{i_l} \\ &= \int d^3x\, (\partial_i \partial_j T^{ij}) x^{i_1} \cdots x^{i_l} \\ &= \int d^3x\, T^{ij} \partial_i \partial_j (x^{i_1} \cdots x^{i_l}) . \end{aligned} \qquad (3.286)$$

For $l \geqslant 2$, as in our case,

$$\begin{aligned} \partial_i \partial_j (x^{i_1} \cdots x^{i_l}) &= (\partial_i \partial_j x^{i_1} x^{i_2}) x^{i_3} \cdots x^{i_l} + \ldots \\ &= (\delta_i^{i_1} \delta_j^{i_2} + \delta_i^{i_2} \delta_j^{i_1}) x^{i_3} \cdots x^{i_l} + \ldots , \end{aligned} \qquad (3.287)$$

where the dots denote the other similar terms; in total there are $l(l-1)/2$ terms of this type. Therefore

$$\ddot{M}^{i_1\ldots i_l} = 2(S^{i_1 i_2, i_3, \ldots i_l} + \ldots) , \qquad (3.288)$$

where the dots denote all other $l(l-1)/2$ pairing of indices (the permutation of the first two indices i, j in $S^{ij,kl\ldots}$ is already taken into account by the overall factor of 2). This is the generalization to arbitrary l of the relation $\ddot{M}^{ij} = 2S^{ij}$ found in eq. (3.52). Once we contract the left- and right-hand sides of this equation with $\mathcal{Y}^{lm*}_{i_1\ldots i_l}$, which is totally symmetric, all these permutations give the same result, so

$$\mathcal{Y}^{lm*}_{i_1\ldots i_l} S^{i_1 i_2, i_3, \ldots i_l} = \frac{1}{l(l-1)} \mathcal{Y}^{lm*}_{i_1\ldots i_l} \ddot{M}^{i_1\ldots i_l} . \qquad (3.289)$$

Therefore, in eq. (3.285),

$$\mathcal{Y}^{lm*}_{i_1\ldots i_l} \partial_0^{l-2} S^{i_1 i_2, i_3, \ldots i_l} = \frac{1}{c^{l-2}} \frac{1}{l(l-1)} \mathcal{Y}^{lm*}_{i_1\ldots i_l} \frac{d^l}{dt^l} M^{i_1\ldots i_l} . \qquad (3.290)$$

Then we get

$$\boxed{u_{lm} = \frac{d^l}{dt^l} I_{lm} ,} \qquad (3.291)$$

where, to leading order in $(v/c)^2$,

$$I_{lm} = \frac{1}{c^{l-2}} \frac{16\pi}{(2l+1)!!} \left[\frac{(l+1)(l+2)}{2l(l-1)}\right]^{1/2} \mathcal{Y}^{lm*}_{i_1\ldots i_l} M^{i_1\ldots i_l} . \qquad (3.292)$$

Similarly, repeating the same analysis for v_{lm} (and using results discussed in Problem 3.5), we get

$$\boxed{v_{lm} = \frac{d^l}{dt^l} S_{lm} ,} \qquad (3.293)$$

where, again to leading order in $(v/c)^2$,

$$S_{lm} = \frac{1}{c^{l-1}} \frac{32\pi}{(2l+1)!!} \left[\frac{l(l+2)}{2(l-1)(l+1)}\right]^{1/2} \mathcal{Y}^{lm*}_{i i_1\ldots i_{l-1}} \epsilon_{ijk} P^{j,k i_1 \ldots i_{l-1}} , \qquad (3.294)$$

where $P^{j,k i_1 \ldots i_{l-1}}$ are the momenta of the linear momentum.[54]

Comparing with eq. (3.238) we see that I_{lm} and S_{lm} are just the spherical components of the tensors $M^{i_1\ldots i_l}$ and $\epsilon_{ijk} P^{j,k i_1 \ldots i_{l-1}}$, respectively, apart from an l-dependent normalization. The tensor $M^{i_1\ldots i_l}$ represents the moments of T^{00}/c^2; if the source is non-relativistic and has a negligible self-gravity, T^{00}/c^2 is the same as the mass density. Instead, from eqs. (3.40) and (3.41), we see that $\epsilon_{ijk} P^{j,k}$ is the angular momentum, and that $\epsilon_{ijk} P^{j,k i_1 \ldots i_{l-1}}$ are the momenta of the angular momentum. Writing explicitly

[54] The expressions that we have computed in this section for I_{lm} and S_{lm} are valid only to leading order in v/c, i.e. they are Newtonian expressions. The full expansion in v/c is studied in Thorne (1980), Section V.

$$\begin{aligned} M^{i_1\ldots i_l} &= \frac{1}{c^2} \int d^3x\, T^{00} x^{i_1} \cdots x^{i_l} \\ &= \frac{1}{c^2} \int d^3x\, T^{00} r^l n^{i_1} \cdots n^{i_l} , \end{aligned} \qquad (3.295)$$

and using eq. (3.234), we can rewrite eq. (3.292) as

$$I_{lm} = \frac{1}{c^l}\frac{16\pi}{(2l+1)!!}\left[\frac{(l+1)(l+2)}{2l(l-1)}\right]^{1/2}\int d^3x\, r^l T^{00} Y^*_{lm}\,, \tag{3.296}$$

again to leading order in v^2/c^2. Similarly we find

$$S_{lm} = \frac{1}{c^l}\frac{32\pi}{(2l+1)!!}\left[\frac{l(l+2)}{2(l-1)(l+1)}\right]^{1/2} \times\epsilon_{jpq}\int d^3x\, r^l T^{0p}(\mathcal{Y}^{lm}_{ji_1\ldots i_{l-1}})^* n^q n^{i_1}\cdots n^{i_{l-1}}\,. \tag{3.297}$$

This expression can be written in terms of the vector spherical harmonic $\mathbf{Y}^B_{lm}$ defined in eq. (3.255). In fact, inserting eq. (3.234) in the definition of this vector harmonic, one finds

$$(\mathbf{Y}^B_{lm})_i = \left(\frac{l}{l+1}\right)^{1/2}\epsilon_{ijk}\mathcal{Y}^{lm}_{ki_1\ldots i_{l-1}} n^j n^{i_1}\cdots n^{i_{l-1}}\,, \tag{3.298}$$

and therefore, to leading order in $(v/c)^2$,

$$S_{lm} = \frac{1}{c^l}\frac{32\pi}{(2l+1)!!}\left[\frac{(l+2)}{2(l-1)}\right]^{1/2}\int d^3x\, r^l T^{0i}(\mathbf{Y}^B_{lm})^*_i\,. \tag{3.299}$$

The coefficients of the expansion of $h^{\rm TT}_{ij}$ have therefore been written as integrals over the source of quantities which depend on the energy–momentum tensor.

Radiated power

We can finally write the radiated power to all orders in the multipole expansion. In eq. (3.275) we have expressed $h^{\rm TT}_{ij}$ in the basis of $\mathbf{T}^{E2}_{lm}, \mathbf{T}^{B2}_{lm}$. Inserting eq. (3.275) into eq. (1.153) and using the orthonormality relation (3.277), we find

$$\int d\Omega\, \dot{h}^{\rm TT}_{ij}\dot{h}^{\rm TT}_{ij} = \frac{1}{r^2}\frac{G^2}{c^8}\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\left[|\dot{u}_{lm}|^2 + |\dot{v}_{lm}|^2\right]. \tag{3.300}$$

Therefore

$$\frac{dE}{dt} = \frac{G}{32\pi c^5}\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\langle\left|\frac{d^{l+1}I_{lm}}{dt^{l+1}}\right|^2 + \left|\frac{d^{l+1}S_{lm}}{dt^{l+1}}\right|^2\rangle\,, \tag{3.301}$$

where I_{lm} and S_{lm} are functions of retarded time $t - r/c$. The power is therefore a sum of terms each one associated with a single mass multipole or angular momentum multipole, and there are no mixed terms.

As a check of the above result, we can verify that we reproduce the mass quadrupole and the current quadrupole radiation. We start from the mass quadrupole. Equation (3.292), with $l = 2$, gives

$$I_{2m} = \frac{16\pi}{5\sqrt{3}}\mathcal{Y}^{2m*}_{ij}M_{ij} = \frac{16\pi}{5\sqrt{3}}\mathcal{Y}^{2m*}_{ij}Q_{ij}\,, \tag{3.302}$$

where we could replace M_{ij} by Q_{ij} since $\mathcal{Y}^{2m*}_{ij}$ is traceless in the (i,j) indices. From eq. (3.301), the power radiated by the quadrupole is

$$P_{\rm quad} = \frac{G}{32\pi c^5}\left(\frac{16\pi}{5\sqrt{3}}\right)^2\langle\dddot{Q}_{ij}\dddot{Q}_{kl}\rangle\sum_{m=-2}^{2}\mathcal{Y}^{2m*}_{ij}\mathcal{Y}^{2m}_{kl}\,. \tag{3.303}$$

We now use the identity[55]

$$\sum_{m=-2}^{2}\mathcal{Y}^{2m*}_{ij}\mathcal{Y}^{2m}_{kl} = \frac{15}{16\pi}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right), \tag{3.304}$$

and we correctly recover eq. (3.75). The current quadrupole is checked similarly: from eq. (3.294) we have

$$S_{2m} = \frac{1}{c}\frac{64\pi}{15\sqrt{3}}\mathcal{Y}^{2m*}_{il}\epsilon_{ijk}P^{j,kl}\,. \tag{3.305}$$

Since

$$\epsilon_{ijk}P^{j,kl} = \epsilon_{ijk}\int d^3x\, T^{0j}x^k x^l = -\int d^3x\,(\epsilon_{ikj}x^k T^{0j})x^l = -\int d^3x\, j^i x^l = -J^{i,l}\,, \tag{3.306}$$

we get

$$S_{2m} = -\frac{1}{c}\frac{64\pi}{15\sqrt{3}}\mathcal{Y}^{2m*}_{ij}J_{i,j}\,. \tag{3.307}$$

Then, using again the identity (3.304), and recalling that $J_{i,j}\delta_{ij} = 0$,

$$P_{\rm curr\ quad} = \frac{G}{32\pi c^7}\left(\frac{64\pi}{15\sqrt{3}}\right)^2\langle\dddot{J}_{i,j}\dddot{J}_{k,l}\rangle\sum_{m=-2}^{2}\mathcal{Y}^{2m*}_{ij}\mathcal{Y}^{2m*}_{kl} = \frac{8G}{45c^7}\langle\dddot{J}_{i,j}\dddot{J}_{k,l}\rangle(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) = \frac{8G}{45c^7}\langle\dddot{J}_{i,j}(\dddot{J}_{i,j}+\dddot{J}_{j,i})\rangle = \frac{16G}{45c^7}\langle\dddot{\mathcal{J}}_{ij}\dddot{\mathcal{J}}_{ij}\rangle\,, \tag{3.308}$$

in agreement with eq. (3.154).

[55] This identity could be proved using the explicit expressions (3.218) for $\mathcal{Y}^{2m}_{ij}$. However, the simplest way to derive it is to observe that, after summing over $m = -2,\ldots,2$, there is no longer a dependence on the direction chosen as quantization axis for j_z, so the right-hand side of eq. (3.304) can only depend on combinations of Kronecker deltas, i.e. it must be of the form $a\delta_{ik}\delta_{jl} + b\delta_{il}\delta_{jk} + c\delta_{ij}\delta_{kl}$. Since the left-hand side of eq. (3.304) gives zero when contracted with δ_{ij}, we must have $\delta_{ij}(a\delta_{ik}\delta_{jl}+b\delta_{il}\delta_{jk}+c\delta_{ij}\delta_{kl}) = 0$, which fixes $c = -(1/3)(a+b)$. Then a and b are fixed comparing the left and right-hand side of eq. (3.304) for two different values of the indices, using the explicit expressions (3.218).

3.6 Solved problems

Problem 3.1. Quadrupole radiation from an oscillating mass

As a first simple application, we compute the quadrupole radiation emitted by a non-relativistic system with just one degree of freedom, that performs harmonic oscillations along the z axis,

$$z_0(t) = a \cos \omega_s t \,, \tag{3.309}$$

with $a\omega_s \ll c$ and $\omega_s > 0$. As we explained in Section 3.3.5, the whole formalism that we have developed for computing the emission of gravitational radiation is consistent only if we have a closed system, on which no external forces are acting. So, in this case the actual physical system that we have in mind could be made for instance of two masses connected by a spring, and $z_0(t)$ is the relative coordinate in the center-of mass system. For the moment we consider the case where the rest length of the spring is zero (which is not realistic for a one-dimensional spring, but is representative of a number of situations where some degree of freedom performs a simple harmonic motion).

The mass density is then

$$\rho(t, \mathbf{x}) = \mu\, \delta(x)\delta(y)\delta(z - z_0(t)) \,, \tag{3.310}$$

where μ is the reduced mass of the system, and the second mass moment is

$$\begin{aligned} M^{ij}(t) &= \int d^3x\, \rho(t, \mathbf{x}) x^i x^j \\ &= \mu\, z_0^2(t)\, \delta^{i3}\delta^{j3} \\ &= \mu\, a^2\, \frac{1 + \cos 2\omega_s t}{2}\, \delta^{i3}\delta^{j3} \,. \end{aligned} \tag{3.311}$$

Inserting this into eq. (3.72), we obtain

$$\begin{aligned} h_+(t; \theta, \phi) &= -\frac{1}{r}\frac{G}{c^4}\, \ddot{M}_{33}(t_{\rm ret}) \sin^2\theta \\ &= \frac{2G\mu a^2 \omega_s^2}{rc^4} \sin^2\theta \cos(2\omega_s t_{\rm ret}) \,, \end{aligned} \tag{3.312}$$

$$h_\times(t; \theta, \phi) = 0 \,. \tag{3.313}$$

Therefore we have monochromatic radiation at a frequency $\omega = 2\omega_s$, with a purely plus polarization, see Fig. 3.4.

The angular distribution is independent of ϕ (reflecting the cylindric symmetry of the source), and has a maximum at $\theta = \pi/2$, i.e. in the direction orthogonal to the axis along which the source oscillates. Observe that the radiation vanishes along the z axis. This reflects the fact that only the components of the motion of the source transverse to the line-of-sight contribute to the production of GWs. This is a general result, which follows from the fact that $\Lambda_{ij,kl} n^k = \Lambda_{ij,kl} n^l = 0$. That is, the Lambda tensor projects the motion of the source onto the plane transverse to the propagation direction.

Observe also that the pattern of lines of force of the GW shown in Fig. 3.4 is a physical result, independent of our conventions. The fact that we call it a "plus" polarization, instead, is related to our choice of axes $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$ with respect to which the plus and cross polarizations are defined. With our definition, $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$ are obtained from the $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ axes applying the rotation matrix $\mathcal{R}$ given in eq. (3.70), see also Fig. 3.2. Thus, in Fig. 3.4, for the particular propagation direction for which the lines of force are shown (i.e. for a direction $\hat{\mathbf{n}}$ such that $\theta = \pi/2$), we have $\hat{\mathbf{v}} = -\hat{\mathbf{z}}$, while $\hat{\mathbf{u}}$ is along the intersection of this transverse plane with the (x, y) plane. If, in this transverse plane, we instead used $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$ axes rotated by 45 degrees to define h_+ and $h_\times$, we would rather call the pattern of Fig. 3.4 a purely cross polarization, compare with Figs. 1.2 and 1.3, while for axes rotated by a generic angle ψ we would have a mixture of plus and cross polarizations, according to eq. (2.194).

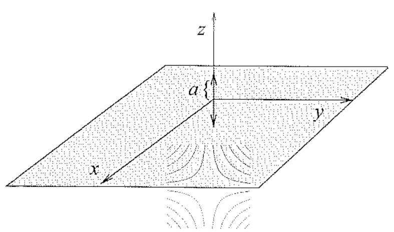

Fig. 3.4 A source oscillating along the z axis (double arrow), and the lines of force of the GW emitted in a direction with $\theta = \pi/2$.

The radiated power is computed from eq. (3.73),

$$\begin{aligned} \left(\frac{dP}{d\Omega}\right)_{\rm quad} &= \frac{r^2 c^3}{16\pi G} \langle \dot{h}_+^2 \rangle \\ &= \frac{G\mu^2 a^4 \omega_s^6}{2\pi c^5} \sin^4\theta \,, \end{aligned} \tag{3.314}$$

where we used $\langle \cos^2(2\omega_s t)\rangle = 1/2$. Alternatively, we can recover the same result using eq. (3.73) in the form

$$\left(\frac{dP}{d\Omega}\right)_{\rm quad} = \frac{G}{8\pi c^5} \Lambda_{33,33}(\hat{\mathbf{n}})\, \langle \dddot{M}_{33}^2 \rangle \,, \tag{3.315}$$

and observing that, from eq. (1.39),

$$\begin{aligned} \Lambda_{33,33} &= \frac{1}{2}(1 - n_3^2)^2 \\ &= \frac{1}{2}\sin^4\theta \,, \end{aligned} \tag{3.316}$$

since $n_3 = \cos\theta$. In Fig. 3.5 we show this angular distribution, in the (x, z) plane. The integration over the solid angle gives

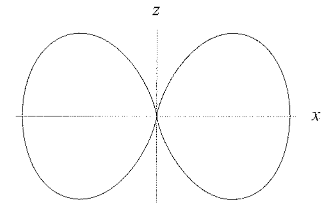

Fig. 3.5 The angular distribution of the quadrupole radiation, for a mass oscillating along the z axis. We represent the angular distribution plotting the function $\rho = \sin^4\theta$, where (ρ, θ) are the polar coordinates in the (x, z) plane and θ is measured from the z axis. The full three-dimensional pattern has cylindrical symmetry around the z axis.

$$P_{\rm quad} = \frac{16}{15}\frac{G\mu^2}{c^5}\, a^4 \omega_s^6 \,. \tag{3.317}$$

The total energy radiated over one period $T = 2\pi/\omega_s$ of the source motion is therefore

$$\langle E_{\rm quad}\rangle_T = \frac{32\pi}{15}\frac{G\mu^2}{c^5}\, a^4 \omega_s^5 \,. \tag{3.318}$$

This result becomes physically more transparent if we rewrite it in terms of $v = a\omega_s$ (which is the maximum speed of the source),

$$\boxed{\langle E_{\rm quad}\rangle_T = \frac{32\pi}{15}\frac{G\mu^2}{a}\left(\frac{v}{c}\right)^5 .} \tag{3.319}$$

Observe that $G\mu^2/a$ is of order of the gravitational self-energy of an object of mass μ and size a. In the quadrupole approximation, the energy radiated over a cycle is suppressed, with respect to this energy scale, by a factor $(v/c)^5$.

Finally, it is instructive to consider the case of two masses connected by a spring with a rest length L, so their relative coordinate obeys

$$z_0(t) = L + a\cos\omega_s t \,. \tag{3.320}$$

Observe that L has an invariant meaning, because $z_0(t)$ is the relative position between the two masses (compare with the discussion in Section 3.3.5), so we cannot set it to zero with a choice of the origin. Now we get

$$z_0^2(t) = \frac{a^2}{2}\cos 2\omega_s t + 2La\cos\omega_s t + \text{const.} \,, \tag{3.321}$$

and eq. (3.312) becomes

$$h_+(t;\theta,\phi) = \frac{2G\mu\omega_s^2}{rc^4}\sin^2\theta\left[a^2\cos(2\omega_s t_{\rm ret}) + La\cos(\omega_s t_{\rm ret})\right], \qquad (3.322)$$

while still $h_\times = 0$. Thus, beside having gravitational radiation at $\omega_{\rm gw} = 2\omega_s$, we also have radiation at $\omega_{\rm gw} = \omega_s$. Observe that, in the power, there is no interference between these two terms because $\langle\cos^2(2\omega_s t)\rangle = 1/2 = \langle\cos^2(\omega_s t)\rangle$, but

$$\langle 2\cos(2\omega_s t)\cos(\omega_s t)\rangle = \langle\cos(3\omega_s t) + \cos(\omega_s t)\rangle = 0\,. \qquad (3.323)$$

Problem 3.2. Quadrupole radiation from a mass in circular orbit

In this problem we consider a binary system with masses m_1 and m_2, and we assume that the relative coordinate is performing a circular motion. We assume for the moment that the orbital motion is given, and we neglect any back-reaction on the motion due to GW emission. In this form, this is just a simple exercise, propedeutic for understanding a real self-gravitating binary system. In Section 4.1 we will include the effect of the GW back-reaction within linearized theory. Furthermore, beyond lowest-order in v/c, we cannot keep the space-time as flat when describing a self-gravitating system, and the correct formalism for computing the v/c corrections will be the subject of Chapter 5.

So, for the moment, we rather assign ourselves the trajectory. We choose the (x, y, z) frame so that the orbit lies in the (x, y) plane, and is given by

$$\begin{aligned} x_0(t) &= R\cos(\omega_s t + \frac{\pi}{2})\,,\\ y_0(t) &= R\sin(\omega_s t + \frac{\pi}{2})\,, \qquad (3.324)\\ z_0(t) &= 0\,. \end{aligned}$$

(The phase $\pi/2$ is a useful choice of the origin of time.) We denote by $\mu = m_1m_2/(m_1+m_2)$ the reduced mass of the system. From eq. (3.129), in the CM frame the second mass moment is $M^{ij} = \mu x_0^i(t)x_0^j(t)$, so

$$M_{11} = \mu R^2\,\frac{1-\cos 2\omega_s t}{2}\,, \qquad (3.325)$$

$$M_{22} = \mu R^2\,\frac{1+\cos 2\omega_s t}{2}\,, \qquad (3.326)$$

$$M_{12} = -\frac{1}{2}\mu R^2\,\sin 2\omega_s t\,, \qquad (3.327)$$

while the other components vanish. Therefore we have

$$\ddot{M}_{11} = 2\mu R^2\omega_s^2\cos 2\omega_s t\,, \qquad (3.328)$$

$$\ddot{M}_{12} = 2\mu R^2\omega_s^2\sin 2\omega_s t\,, \qquad (3.329)$$

and $\ddot{M}_{22} = -\ddot{M}_{11}$. Plugging these expressions into eq. (3.72) we get

$$h_+(t;\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\omega_s t_{\rm ret} + 2\phi)\,, \qquad (3.330)$$

$$h_\times(t;\theta,\phi)) = \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\cos\theta\sin(2\omega_s t_{\rm ret} + 2\phi)\,. \qquad (3.331)$$

Thus, the quadrupole radiation is at twice the frequency ω_s of the source. It is interesting to observe that the dependences $h_+ \sim (1+\cos^2\theta)$ and $h_\times \sim \cos\theta$ are a general consequence of eq. (3.72), whenever $M_{13} = M_{23} = M_{33} = 0$ and $\ddot{M}_{22} = -\ddot{M}_{11}$. These conditions are satisfied also in other problems (e.g. a rigid body rotating around one of its principal axis), so we will meet again this type of angular dependence.

As for the dependence on ϕ, it can be clearly understood physically: contrary to the oscillating mass of the previous problem, now the source is not invariant under rotations around the z axis, since at any given value of t the mass μ is at a specific point along the orbit, which is changed by a rotation around the z axis; therefore h_+ and $h_\times$ have a dependence on ϕ. However, since the orbit is circular, a rotation of the source by an angle $\Delta\phi$ around the z axis is the same as a time translation Δt with $\omega_s\Delta t = \Delta\phi$, and therefore the dependence of h_+ and $h_\times$ on ϕ is only through the combination $\omega_s t_{\rm ret} + \phi$.

From the observational point of view, we have only access to the radiation that a binary star emits in the direction which points from the star toward us. The angle θ is therefore equal to the angle ι between the normal to the orbit and the line-of-sight (see Fig. 3.6). The distance r to an astrophysical source is, for most practical purposes, a constant.[56] As long as, during the observation, we can neglect the proper motion of the source (which however sometimes is not the case, as we will discuss in Section 7.6), also the angle ϕ is fixed, so we have $\omega_s t_{\rm ret} + \phi = \omega_s t + \alpha$, with $\alpha = \phi - \omega_s r/c$ a fixed constant. Then we can shift the origin of time so that $2\omega_s t + 2\alpha \to 2\omega_s t$ plus an integer multiple of 2π, so $\cos(2\omega_s t + 2\alpha) \to \cos(2\omega_s t)$ and $\sin(2\omega_s t + 2\alpha) \to \sin(2\omega_s t)$. Therefore, this observer can write the GWs received by a binary system (as long as the approximation of a fixed circular orbit is valid, see Section 4.1) as

$$\begin{aligned} h_+(t) &= \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\left(\frac{1+\cos^2\iota}{2}\right)\cos(2\omega_s t)\,,\\ h_\times(t) &= \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\cos\iota\sin(2\omega_s t)\,. \end{aligned} \qquad (3.332)$$

If we see the orbit edge-on, $\iota = \pi/2$, then $h_\times$ vanishes and therefore the GW is linearly polarized. Instead, at $\iota = 0$, h_+ and $h_\times$ have the same amplitude; in this case, since the former is proportional to $\sin(2\omega_s t_{\rm ret})$ while the latter to $\cos(2\omega_s t_{\rm ret})$, in the plane $(h_+, h_\times)$ the radiation describes a circle parametrized by t, that is, the radiation is circularly polarized. To understand what this means in more physical terms, we consider the pattern of lines of force corresponding to a circular polarization. If we have a purely plus polarization, $h_+ = A_+\cos 2\omega_s t$, according to eq. (1.96) we have a force field

$$\mathbf{F}_+ = -2\mu\omega_s^2\,A_+\,\mathbf{v}_+\,\cos 2\omega_s t\,, \qquad (3.333)$$

where the vector field $\mathbf{v}_+(x,y)$ has components $(x,-y)$. This force field is shown in Fig. 1.2. If instead we have a purely cross polarization of the form $h_\times = A_\times\sin 2\omega_s t$, according to eq. (1.96) we have a force field

$$\mathbf{F}_\times = -2\mu\omega_s^2\,A_\times\,\mathbf{v}_\times\,\sin 2\omega_s t\,, \qquad (3.334)$$

where the vector field $\mathbf{v}_\times(x,y)$ has components (y,x). This force field is shown in Fig. 1.3. If we have both $h_+ = A_+\cos 2\omega_s t$ and $h_\times = A_\times\sin 2\omega_s t$,

[56] As we will discuss in Section 7.6, there are situations in which we must take into account corrections due to motion of the Earth around the Sun, or more precisely around the Solar System Barycenter (SSB). So, more generally, we can take r to be the distance from the source to the SSB. This however is only important for observations lasting at least few months. We will see in Section 4.1 that, in ground-based interferometers, the gravitational waves emitted by a binary system are observable only for about the last 15 minutes before the system coalesces.

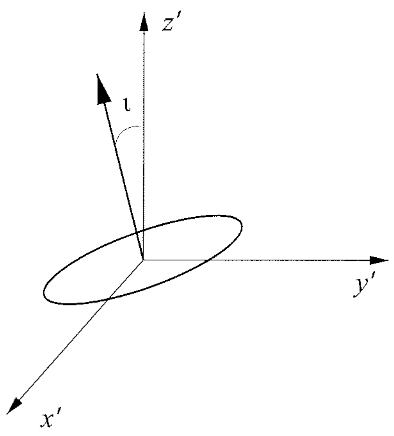

Fig. 3.6 The geometry of the problem in a frame (x', y', z') where a fixed observer is at large distance along the positive z' axis. The normal to the orbit makes an angle ι with the z' axis.

and furthermore $A_+ = A_\times \equiv A$, we have

$$\mathbf{F} = -2\mu\omega_s^2 A \left[\mathbf{v}_+ \cos 2\omega_s t + \mathbf{v}_\times \sin 2\omega_s t\right] . \tag{3.335}$$

Observe that $\mathbf{v}_+$ and $\mathbf{v}_\times$ are orthogonal, $\mathbf{v}_+ \cdot \mathbf{v}_\times = 0$, and therefore the pattern of lines of forces described by eq. (3.335) is the same as the pattern of Fig. 1.2, that rotates uniformly so that at $\omega_s t = 0$ it is the same as Fig. 1.2, at $\omega_s t = \pi/4$ it is the same as Fig. 1.3, and so on.

At intermediate values of ι the amplitudes for h_+ and $h_\times$ are different and therefore we have elliptic polarization, i.e. in the plane $(h_+, h_\times)$ the radiation describes an ellipse parametrized by t. We see that, from a measurement of the degree of polarization, i.e. of the relative amplitude of h_+ and $h_\times$, we can deduce the inclination ι of the orbit.

The angular distribution of the radiated power, in the quadrupole approximation, is obtained as usual using eq. (3.73),

$$\left(\frac{dP}{d\Omega}\right)_{\rm quad} = \frac{r^2 c^3}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle . \tag{3.336}$$

Inserting here eq. (3.332), and using $\langle \cos^2(2\omega_s t)\rangle = \langle \sin^2(2\omega_s t)\rangle = 1/2$, we get

$$\boxed{\left(\frac{dP}{d\Omega}\right)_{\rm quad} = \frac{2G\mu^2 R^4 \omega_s^6}{\pi c^5}\, g(\theta) ,} \tag{3.337}$$

where

$$\boxed{g(\theta) = \left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta .} \tag{3.338}$$

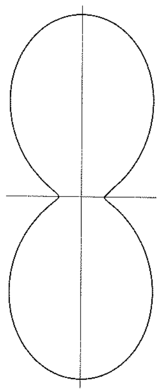

Fig. 3.7 The function $g(\theta)$ in polar coordinates. The angle θ is measured from the vertical axis.

The radiation is maximum at $\theta = 0$, i.e. in the direction normal to the plane of the orbit. A polar plot of $g(\theta)$ is shown in Fig. 3.7. Observe that, contrary to the angular distribution found in Problem 3.1, $g(\theta)$ never vanishes since, whatever the angle θ at which an observer is located, there is always a component of the source motion orthogonal to the observer's line-of-sight.

Integrating eq. (3.337) over the solid angle we get the total power radiated in the quadrupole approximation,

$$P_{\rm quad} = \frac{32}{5}\frac{G\mu^2}{c^5} R^4 \omega_s^6 = \frac{1}{10}\frac{G\mu^2}{c^5} R^4 \omega^6 , \tag{3.339}$$

where $\omega = 2\omega_s$ is the frequency of the GW.[57] The energy radiated in one period $T = 2\pi/\omega_s$ of the source motion is therefore, writing $v = \omega_s R$,

$$\langle E_{\rm quad}\rangle_T = \frac{64\pi}{5}\frac{G\mu^2}{R}\left(\frac{v}{c}\right)^5 . \tag{3.340}$$

Similarly to the result found in eq. (3.319), this is suppressed by a factor $(v/c)^5$ with respect to the energy scale $G\mu^2/R$.

[57] If one is interested only in the total power, rather than in the angular distribution, it is actually simpler to derive it directly from eq. (3.75): we use again the reference frame where the orbit is given by eq. (3.324), so

$$\dddot{M}_{ij} = 4\mu\omega_s^3 R^2 \times \begin{pmatrix} -\sin 2\omega_s t & \cos 2\omega_s t & 0 \\ \cos 2\omega_s t & \sin 2\omega_s t & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} .$$

Since this is traceless, it is also equal to $\dddot{Q}_{ij}$. Plugging this into eq. (3.75), we recover eq. (3.339).

Problem 3.3. Mass octupole and current quadrupole radiation from a mass in circular orbit

In this problem we compute the mass octupole and current quadrupole radiation generated by a binary system of reduced mass μ, whose center-of-mass coordinate describes a circular trajectory.

Before applying these results to a real binary star, it is important to repeat the caveat already made in the previous problem: in binary systems held together by gravitation, when we wish to study the higher-order terms in v/c, we can no longer take the space-time as flat. For instance, the general-relativistic corrections to the $1/r^2$ Newton's law produce modifications to the trajectory, of higher order in v/c. As we will see in Chapter 5 when we study the post-Newtonian formalism, this and other effects produce corrections $O(v^2/c^2)$ to the power radiated in the quadrupole approximation. Since the octupole radiation is itself suppressed by a factor v^2/c^2 compared to quadrupole radiation, if we wish to compute the total radiated power to this order, a consistent treatment requires the inclusion of these relativistic corrections to the trajectory, and will be deferred to Chapter 5.

The purpose of this exercise is therefore only to illustrate, in a simplified but somewhat academic setting of a circular orbit in flat space-time, some interesting features of mass octupole and current quadrupole radiation, which will be useful for understanding physically some aspects of the correct results for self-gravitating systems, that will be discussed in Chapter 5.

With this caveat, we can now assign the orbit and proceed with the computation. If we want to compute the radiation emitted from the star in the direction of the observer, it is simpler to use the geometrical setting of Fig. 3.6 (labeling now the axes of this figure as (x, y, z) rather than (x', y', z')), in which the observer is along the z axis. The equation of the orbit in this frame is

$$\begin{aligned} x_0(t) &= R\cos\omega_s t , \\ y_0(t) &= R\cos\iota \sin\omega_s t , \\ z_0(t) &= R\sin\iota\sin\omega_s t , \end{aligned} \tag{3.341}$$

and is obtained from an orbit lying in the (x, y) plane, performing a rotation by an angle ι around the x axis. We set the observer in the z direction, so we compute the radiation emitted along $\hat{\mathbf{n}} = (0, 0, 1)$. For the octupole radiation, eq. (3.141) gives

$$\left(h_{ij}^{\rm TT}\right)_{\rm oct} = \frac{1}{r}\frac{2G}{3c^5}\Lambda_{ij,kl}(\hat{\mathbf{n}})\dddot{M}_{kl3} . \tag{3.342}$$

(As usual, in actual computations, it is more convenient to use M^{ijk} rather than $\mathcal{O}^{ijk}$.) We found in eq. (3.63) that, when $\hat{\mathbf{n}} = (0, 0, 1)$, in the multiplication of a matrix $\dddot{M}_{kl3}$ by the Lambda tensor the components of $\dddot{M}_{kl3}$ with $k = 3$ or $l = 3$ do not contribute, so we just need to compute M_{ab3} with $a, b = 1, 2$. With the trajectory given in eq. (3.324) we find, for $a, b = 1, 2$

$$\begin{aligned} M_{ab3} &= \mu\, x_a(t) x_b(t) z(t) \\ &= \mu R^3 \sin\iota\sin\omega_s t \begin{pmatrix} \cos^2\omega_s t & \cos\iota\sin\omega_s t\cos\omega_s t \\ \cos\iota\sin\omega_s t\cos\omega_s t & \cos^2\iota\sin^2\omega_s t \end{pmatrix}_{ab} . \end{aligned} \tag{3.343}$$

According to eq. (3.63), when $\hat{\mathbf{n}} = (0, 0, 1)$ the contraction with the Lambda tensor amounts to replacing M_{113} with $(1/2)(M_{113} - M_{223})$ and M_{223} with $(-1/2)(M_{113} - M_{223})$. Therefore

$$\Lambda_{ab,cd}(\hat{\mathbf{n}}) M_{cd3} = \mu R^3 \sin\iota\sin\omega_s t \times \tag{3.344}$$

$$\times \begin{pmatrix} \frac{1}{2}(\cos^2\omega_s t - \cos^2\iota\sin^2\omega_s t) & \cos\iota\sin\omega_s t\cos\omega_s t \\ \cos\iota\sin\omega_s t\cos\omega_s t & -\frac{1}{2}(\cos^2\omega_s t - \cos^2\iota\sin^2\omega_s t) \end{pmatrix}_{ab} .$$

Taking the third time derivative we find

$$(h_+)_{\rm oct} = \frac{1}{r}\frac{G\mu R^3\omega_s^3}{12c^5}\sin\iota\left[(3\cos^2\iota - 1)\cos\omega_s t - 27(1+\cos^2\iota)\cos 3\omega_s t\right] ,$$
$$(h_\times)_{\rm oct} = \frac{1}{r}\frac{G\mu R^3\omega_s^3}{12c^5}\sin 2\iota\left[\sin\omega_s t - 27\sin 3\omega_s t\right] . \quad (3.345)$$

As expected, we have radiation both at $\omega = \omega_s$ and at $\omega = 3\omega_s$. The current quadrupole radiation can be computed similarly; actually, it is even simpler to compute first the sum of the mass octupole and current quadrupole radiation, which is obtained directly from $\dot{S}^{kl,m}$, see eq. (3.54). For a non-relativistic point particle of mass μ (or, equivalently, for a two-body system in the center-of-mass, with reduced mass μ), eqs. (3.32) and (3.120) give

$$S^{kl,m} = \mu\dot{x}^k\dot{x}^l x^m , \quad (3.346)$$

and the radiation along the z axis is

$$\left(h_{ij}^{\rm TT}\right)_{\rm oct+cq} = \frac{1}{r}\frac{4G}{c^5}\Lambda_{ij,kl}(\hat{\mathbf{n}})\ddot{S}^{kl,3} , \quad (3.347)$$

where the subscript "cq" stands for "current quadrupole". The computation is completely analogous to the one that we just performed above, and gives

$$(h_+)_{\rm oct+cq} = \frac{1}{r}\frac{G\mu R^3\omega_s^3}{2c^5}\sin\iota\left[(\cos^2\iota - 3)\cos\omega_s t - 3(1+\cos^2\iota)\cos 3\omega_s t\right] ,$$
$$(h_\times)_{\rm oct+cq} = \frac{1}{r}\frac{G\mu R^3\omega_s^3}{2c^5}\sin 2\iota\left[\sin\omega_s t - 3\sin 3\omega_s t\right] . \quad (3.348)$$

Therefore also the current quadrupole contribution, which is the difference between eqs. (3.348) and eqs. (3.345), is a sum of terms with frequencies ω_s and $3\omega_s$. The contribution to the total radiated power from the mass octupole and the current quadrupole is

$$P_{\rm oct+cq} = \frac{r^2c^3}{16\pi G}2\pi\int_{-1}^{1} d\cos\iota\,\langle\dot{h}_+^2 + \dot{h}_\times^2\rangle$$
$$= \frac{424}{105}\frac{G\mu^2}{c^7}R^6\omega_s^8 . \quad (3.349)$$

It is interesting to compare the power at $\omega = \omega_s, 3\omega_s$ (both generated by the mass octupole plus the current quadrupole), with the power at $\omega = 2\omega_s$, generated by the mass quadrupole (and which also receives corrections $O(v^2/c^2)$ from a post-Newtonian treatment of the orbit). From eq. (3.348) we find

$$P(\omega_s) = \frac{19}{672}\left(\frac{v}{c}\right)^2 P(2\omega_s) , \quad (3.350)$$

and

$$P(3\omega_s) = \frac{135}{224}\left(\frac{v}{c}\right)^2 P(2\omega_s) , \quad (3.351)$$

where $P(2\omega_s)$ is the leading-order quadrupole result, eq. (3.339). In Fig. 3.8 we show the relative intensity of the three spectral lines at $\omega = \omega_s, 2\omega_s$ and $3\omega_s$, for $v/c = 10^{-2}$. Observe that the vertical scale is logarithmic.

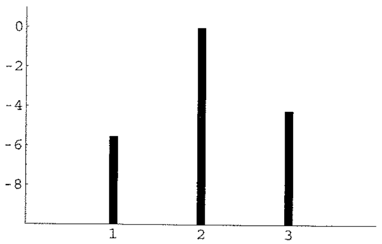

Fig. 3.8 $\log_{10}[P(\omega)/P(2\omega_s)]$, as a function of ω/ω_s, for $v/c = 10^{-2}$, including the contributions of the mass quadrupole, of the mass octupole, and of the current quadrupole. The line at $\omega = 2\omega_s$ is due to the mass quadrupole and each of the two lines at $\omega = \omega_s$ and $\omega = 3\omega_s$ is the sum of the contributions of the mass octupole and current quadrupole.

Problem 3.4. Decomposition of $\dot{S}^{kl,m}$ into irreducible representations of $SO(3)$

We have seen that the next-to-leading term in the multipole expansion is proportional to $\dot{S}^{kl,m}$. In this problem we discuss the decomposition of $\dot{S}^{kl,m}$ into irreducible representations of the rotation group $SO(3)$, and we will understand the group-theoretical origin of the mass octupole and current quadrupole terms.

Let us recall that the irreducible representations of Lie groups, as for instance $O(N)$ or $U(N)$, are conveniently expressed in terms of Young diagrams. A Young diagram is a set of n boxes, organized into r lines of length $n_1, n_2, \ldots, n_r$, with $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r$ and $n_1 + \ldots + n_r = n$. In each box we put an index $i_1, \ldots, i_n$. For $O(N)$ each index takes the values $1, \ldots, N$, which is the dimension of the vector representation. The irreducible tensor representations can be obtained antisymmetrizing first over the indices in the columns and then symmetrizing over the indices in the lines.[58] For $O(N)$ we must also remove the traces on all pairs of symmetric indices. For example, a generic tensor with three indices k, l, m is decomposed into irreducible representations of $O(N)$ as follows. First of all, we remove from the tensor all the traces. Let us call T^{klm} the resulting traceless tensor, so that $T^{kmm} = T^{mlm} = T^{kkm} = 0$ (repeated indices are summed over). Then the decomposition of T^{klm} is shown in Fig. 3.9. The Young diagram (a) represents the tensor obtained symmetrizing T^{klm} over all indices; the diagram (b) represents the tensor obtained antisymmetrizing first over (k,m), which gives $T^{klm} - T^{mlk}$, and then symmetrizing over the pair (k,l). This gives

$$\text{diagram } (b): \quad T^{klm} + T^{lkm} - T^{mlk} - T^{mkl} . \quad (3.352)$$

The tensor corresponding to the diagram (c) is obtained similarly, antisymmetrizing first over (k,l) and then symmetrizing over (k,m), and gives

$$\text{diagram } (c): \quad T^{klm} + T^{mlk} - T^{lkm} - T^{lmk} . \quad (3.353)$$

Finally, the Young diagram (d) represents the tensor obtained antisymmetrizing T^{klm} over all indices.[59] Counting the independent components we see that a traceless, but otherwise generic, tensor with three indices k, l, m, each taking the values $1, 2, 3$, has 18 components, and that the diagram (a) represents a tensor with seven components, (b) and (c) with five components each, and (d) has one component, so we get $18 = 7 + 5 + 5 + 1$. Recall that the representations of the rotation group can also be labeled by the spin s, and the representation labeled by s has dimension $2s + 1$. Then, the representation with dimension seven, i.e. the diagram (a), corresponds to $s = 3$, while the representations with dimension five corresponds to $s = 2$. Denoting by $\mathbf{s}$ the representation with spin s, and by $\oplus$ the direct sum of representations, the decomposition of Fig. 3.9 reads

$$T^{klm} \in \mathbf{3} \oplus \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{1} . \quad (3.354)$$

We can now understand the decomposition given in eq. (3.139), in terms of Fig. 3.9. Our starting point is the tensor $\dot{S}^{kl,m}$. This is symmetric in (k,l), and has no special symmetry with respect to the other indices. Therefore, it is not an irreducible representation of $SO(N)$.

First of all observe that, in eq. (3.138), we can replace $\dot{S}^{kl,m}$ by the tensor in which all the traces have been removed. In fact, subtracting from $\dot{S}^{kl,m}$ a term

[58] It is more common to first symmetrize over the lines and then antisymmetrize over the columns (see, e.g. Hamermesh (1962), Section 10.6). The two procedures are however equivalent and for our purposes the former is more convenient.

[59] If, rather than $O(N)$, we consider $SO(N)$, we must also take into account that there is a globally defined antisymmetric tensor $\epsilon^{i_1\cdots i_N}$. In this case the representations obtained one from the other contracting antisymmetric indices with the ϵ tensor are equivalent. In particular, for $SO(3)$ we have the tensor ϵ^{ijk}. So, for instance, an antisymmetric tensor with two indices A^{kl} is equivalent to a vector $A_i = \epsilon_{ikl}A^{kl}$ while a totally antisymmetric tensor A^{ijk} is equivalent to a scalar A through the relation $A = \epsilon_{ijk}A^{ijk}$. Since the indices take the values $i = 1, \ldots, N$, we cannot antisymmetrize over more than N indices, and there are no diagrams with more than N lines, for $O(N)$. For $SO(N)$ a column with N box is equivalent to a scalar, as we see contracting with the totally antisymmetric tensor, and can be eliminated. Therefore a Young diagram of $SO(N)$ can always be reduced to an equivalent diagram with no more than $N - 1$ rows.

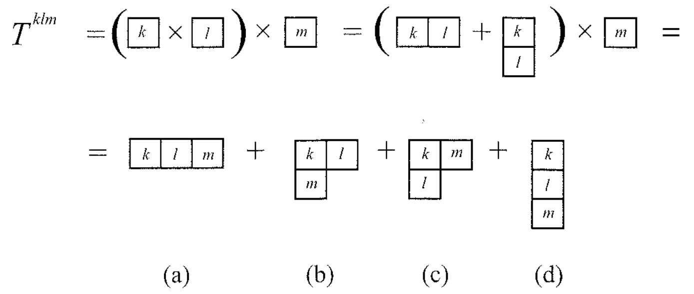

Fig. 3.9 The Young diagrams corresponding to irreducible tensor representations with three indices k, l, m.

proportional to δ_{kl} we get in eq. (3.138) a factor proportional to $\Lambda_{ij,kl} n_m \delta_{kl}$, which vanishes because $\Lambda_{ij,kk} = 0$, while subtracting from $\dot{S}^{kl,m}$ a term proportional to δ_{km} or to δ_{lm} we get a factor proportional to $\Lambda_{ij,kl} n_l = 0$. We can therefore consider $\dot{S}^{kl,m}$ as a tensor from which all traces have already been removed (this is analogous to the fact that, in the quadrupole term, we could substitute M^{ij} with Q^{ij}).

Figure 3.9 gives the decomposition in irreducible representations of a generic traceless tensor T^{klm}. However, $\dot{S}^{kl,m}$ is not generic, but is symmetric in (k, l) (and of course remains symmetric also after we remove all its traces). Then, when we decompose $\dot{S}^{kl,m}$ in irreducible representations as in Fig. 3.9, the diagrams (c) and (d) do not contribute, since the are obtained antisymmetrizing first over (k, l) which, applied to $\dot{S}^{kl,m}$, gives zero. Therefore, in the decomposition of $\dot{S}^{kl,m}$, only two irreducible representations appear:

- diagram (a), which is the totally symmetric and traceless combination, and therefore has the symmetries of the mass octupole;
- diagrams (b), which has the structure given in eq. (3.352). For a tensor T^{klm} symmetric in (k, l), the structure of indices given in eq. (3.352) simplifies to
$$2T^{klm} - T^{mlk} - T^{mkl} . \tag{3.355}$$
Identifying T^{klm} with $\ddot{P}^{m,kl}$, (which is indeed symmetric in (k, l)), we see that this is precisely the structure of indices of the current quadrupole term in eq. (3.139).

We therefore understand that the algebraic identity (3.139) expresses the decomposition of $\dot{S}^{kl,m}$ in irreducible representations of $SO(3)$. The mass octupole corresponds to the Young diagram (a) in Fig. 3.9, and is a spin-3 representation, while the current quadrupole corresponds to the Young diagram (b) in Fig. 3.9, and is a spin-2 representation. The representations corresponding to the Young diagrams (c) and (d) instead do not appear, because $\dot{S}^{kl,m}$ is symmetric in (k, l).

The fact that $\ddot{P}^{k,lm} + \ddot{P}^{l,km} - 2\ddot{P}^{m,kl}$ is a spin-2 tensor is the origin of its name "current quadrupole": the term "quadrupole" refers to the spin-2 nature of the operator, and the term "current" refers to the fact that is is obtained from the momenta of the momentum P^i.

Problem 3.5. Computation of $\int d\Omega\, (\mathbf{T}^{E2,B2}_{lm})^*_{ij} n_{i_1} \cdots n_{i_\alpha}$

In Section 3.5 we have seen that the coefficients of the multipole expansion, u_{lm} and v_{lm}, are given in terms of integrals over the solid angle of the quantity $(\mathbf{T}^{E2}_{lm})^*_{ij} n_{i_1} \cdots n_{i_\alpha}$, for u_{lm}, and $(\mathbf{T}^{B2}_{lm})^*_{ij} n_{i_1} \cdots n_{i_\alpha}$, for v_{lm}, see eqs. (3.283) and (3.284). In this problem we compute explicitly these integrals. As a first step, we consider
$$\int d\Omega\, Y^*_{lm}\, n_{i_1} \cdots n_{i_\alpha} , \tag{3.356}$$
with l, α arbitrary integers. Using eq. (3.234), we can write it as
$$(\mathcal{Y}^{lm}_{j_1,\ldots j_l})^* \int d\Omega\, n_{j_1} \cdots n_{j_l} n_{i_1} \cdots n_{i_\alpha} . \tag{3.357}$$
We can now perform this integral using eq. (3.23). If $\alpha < l$, in the product of Kronecker delta of eq. (3.23) there are not enough indices of the type $i_1, \ldots, i_\alpha$ to be contracted with the indices of the type $j_1, \ldots j_l$, and therefore necessarily we have at least one Kronecker delta involving two indices of the group $j_1, \ldots j_l$, e.g. terms containing $\delta_{j_1 j_2}$. Then, since $\mathcal{Y}^{lm}_{j_1,\ldots j_l}$ is traceless, the result is zero. Therefore the integral in eq. (3.356) can be non-vanishing only if $\alpha \geqslant l$. If $\alpha = l$, we found below eq. (3.237) that
$$\int d\Omega\, Y^*_{lm}\, n_{i_1} \cdots n_{i_l} = 4\pi \frac{l!}{(2l+1)!!} (\mathcal{Y}^{lm}_{i_1 \ldots i_l})^* . \tag{3.358}$$
Now, observe that u_{lm} is expressed as a sum over α of terms containing the α-th time derivatives of $S^{ij,i_1 \ldots i_\alpha}$, see eq. (3.283). In order of magnitude, if d is the size of the source,
$$\begin{aligned} \partial_0^{\alpha+1} S^{ij,i_1 \ldots i_\alpha i_{\alpha+1}} &\sim \frac{\omega_s d}{c}\, \partial_0^\alpha S^{ij,i_1 \ldots i_\alpha} , \\ &= \frac{v}{c}\, \partial_0^\alpha S^{ij,i_1 \ldots i_\alpha} , \end{aligned} \tag{3.359}$$
since $\partial_0 = (1/c)\partial/\partial t$ and each time derivative brings down a factor ω_s, where ω_s is the typical frequency of the source, while the addition of the index $i_{\alpha+1}$ corresponds to the insertion of a factor $x^{i_{\alpha+1}}$ inside the integral in d^3x over the source volume, which therefore gives a contribution $O(d)$. Therefore, in the limit $v/c \ll 1$ in which the multipole expansion is useful, the dominant term in eqs. (3.283) and (3.284) is the one with the smallest value of α for which the integral is non-vanishing. Recall from eqs. (3.265) and (3.267) that
$$\mathbf{T}^{E2}_{lm} = a_{31}\, \mathbf{T}^{l+2}_{lm} + a_{32}\, \mathbf{T}^{l}_{lm} + a_{33} \mathbf{T}^{l-2}_{lm} , \tag{3.360}$$
$$\mathbf{T}^{B2}_{lm} = b_{21}\, i\mathbf{T}^{l+1}_{lm} + b_{22}\, i\mathbf{T}^{l-1}_{lm} , \tag{3.361}$$
where the coefficients are given in Table 3.1. Since $\mathbf{T}^{l'}_{jj_z}$ is proportional to $Y_{l l_z}$, see eq. (3.262), the integral $\int d\Omega\, (\mathbf{T}^{l-2}_{lm})^*_{ij} n_{i_1} \cdots n_{i_\alpha}$ is proportional to $\int d\Omega\, Y^*_{l-2,m} n_{i_1} \cdots n_{i_\alpha}$ and therefore the lowest value of α for which it is non-vanishing is $\alpha = l-2$, while for the integral of $\mathbf{T}^{l}_{lm}$ we need at least $\alpha = l$ and for the integral of $\mathbf{T}^{l+2}_{lm}$ we need at least $\alpha = l+2$. We conclude that in eq. (3.283) the smallest possible value of α which gives a non-vanishing contribution is obtained from $\mathbf{T}^{l-2}_{lm}$ and is $\alpha_{\rm min} = l-2$. Similarly, for v_{lm} the leading contribution is obtained from $\mathbf{T}^{l-1}_{lm}$ and is $\alpha_{\rm min} = l-1$. Therefore
$$\begin{aligned} u_{lm} &\simeq a_{33} \frac{4}{(l-2)!} \left(\partial_0^{l-2} S^{ij,i_1 \ldots i_{l-2}} \right) \int d\Omega\, (\mathbf{T}^{l-2}_{lm})^*_{ij} n_{i_1} \cdots n_{i_{l-2}} , \\ v_{lm} &\simeq -ib_{22} \frac{4}{(l-1)!} \left(\partial_0^{l-1} S^{ij,i_1 \ldots i_{l-1}} \right) \int d\Omega\, (\mathbf{T}^{l-1}_{lm})^*_{ij} n_{i_1} \cdots n_{i_{l-1}} , \end{aligned} \tag{3.362}$$

times a factor $[1+O(v^2/c^2)]$. Now we use the identities

$$(\mathbf{T}^{l-2}_{lm})_{ij} = \left[\frac{l(l-1)}{(2l-1)(2l+1)}\right]^{1/2} \mathcal{Y}^{lm}_{iji_1\ldots i_{l-2}}\, n_{i_1}\cdots n_{i_{l-2}}\,, \tag{3.363}$$

$$(\mathbf{T}^{l-1}_{lm})_{ij} = i\left[\frac{2l(l-1)}{(l+1)(2l+1)}\right]^{1/2} \epsilon_{pq(i}\mathcal{Y}^{lm}_{j)qi_1\ldots i_{l-2}}\, n_p n_{i_1}\cdots n_{i_{l-2}}\,, \tag{3.364}$$

where the parentheses on the indices, in eq. (3.364), denotes the symmetrization over the indices i,j (i.e. $A_{(ij)}\equiv(1/2)(A_{ij}+A_{ji})$). These identities can be obtained (with quite some work) inserting eq. (3.234) into the definition of the spin-2 tensor harmonics. Inserting the explicit values of a_{33}, b_{22} from Table 3.1 (and recalling the change of notation $j\rightarrow l$ that we made in between, see Note 47), we get

$$\begin{aligned} u_{lm} &= \left[\frac{(l+1)(l+2)}{2(2l-1)(2l+1)}\right]^{1/2} \frac{4}{(l-2)!} \left[\frac{l(l-1)}{(2l-1)(2l+1)}\right]^{1/2} \\ &\quad\times\left(\partial_0^{l-2}S^{ij,i_1\ldots i_{l-2}}\right)\mathcal{Y}^{lm*}_{ijj_1\ldots j_{l-2}}\int d\Omega\, n_{i_1}\cdots n_{i_{l-2}}\, n_{j_1}\cdots n_{j_{l-2}} \\ &= \left[\frac{1}{2}\,l(l-1)(l+1)(l+2)\right]^{1/2} \frac{4}{(l-2)!}\,\frac{1}{(2l-1)(2l+1)} \\ &\quad\times\frac{4\pi}{(2l-3)!!}(l-2)!\,\left(\partial_0^{l-2}S^{ij,i_1\ldots i_{l-2}}\right)\mathcal{Y}^{lm*}_{iji_1\ldots i_{l-2}} \\ &= \frac{16\pi}{(2l+1)!!}\,\left[\frac{1}{2}\,l(l-1)(l+1)(l+2)\right]^{1/2}\left(\partial_0^{l-2}S^{ij,i_1\ldots i_{l-2}}\right)\mathcal{Y}^{lm*}_{iji_1\ldots i_{l-2}}\,. \end{aligned} \tag{3.365}$$

where the final integral as been performed using eq. (3.23). The integral for v_{lm} is performed similarly using eqs. (3.363) and (3.364), with the final result given in the text.

Further reading

- The quadrupole radiation is discussed in all general relativity textbooks, see in particular Weinberg (1972), Misner, Thorne and Wheeler (1973), Landau and Lifshitz, Vol. II (1979) and Straumann (2004).
- The radiation from sources with arbitrary velocity is discussed in Weinberg (1972), Section 10.4. Gravitational wave generation is also discussed in detail in the reviews Thorne (1983) and (1987).
- Radiation reaction for slow-motion sources is discussed in Misner, Thorne and Wheeler (1973), Sections 36.8 and 36.11.
- The multipole expansion for time-dependent fields in terms of STF tensors was introduced by Sachs (1961) and Pirani (1964). Thorne (1980) derived the slow-motion expansion of the mass and spin multipole moments, both in STF and in spherical tensor form. The closed-form expression for these moments in STF form is derived in Damour and Iyer (1991a). A detailed review of the multipole expansion for GWs, as well as a historical overview of the relevant literature, is Thorne (1980). A physical discussion of current quadrupole radiation is given in Schutz and Ricci (2001).

Applications

4

In this chapter we apply the formalism that we have developed to various instructive problems. The systems that we examine here are still somewhat idealized, compared to real astrophysical sources. This allows us to understand the essence of the physical mechanisms with a minimum of complications, and forms the basis for a more detailed study of realistic sources, which will be the subject of Vol. 2.

We begin, in Section 4.1, with the study of binary systems, taking the bodies as point-like and moving at first on a Newtonian trajectory. We will compute how the back-reaction of GWs affects the motion of the sources, inducing the inspiral and coalescence of the binary system, and we will see how this, in turn, affects the emission of the GWs themselves.

In Section 4.2 we will compute the radiation emitted by spinning rigid bodies, which are a first idealization of rotating neutron stars.

In Section 4.3 we compute the radiation emitted by a body falling radially into a black hole. A full resolution of this problem requires expansion over the Schwarzschild metric, rather than over flat space-time of linearized theory, and will be deferred to Vol. 2. However, we will see that the low-frequency part of the spectrum can be computed using a flat space-time background, so we can perform here this part of the computation. We will also compare the situation in which the infalling particle is point-like with that of a real star, which can be disrupted by the tidal gravitational force of the black hole. This is particularly instructive because it allows us to compare the coherent and the incoherent emission of GWs.

In Section 4.4 we study the radiation emitted by a mass accelerated by an external force. It will be interesting to compare the results with the electromagnetic radiation from an accelerated charge. We will see that, while the electromagnetic field of a relativistic charge is beamed into a small angle in the forward direction, this does not happen in the gravitational case. Finally, some computational details are collected in a Solved Problems section, at the end of the chapter.

4.1 Inspiral of compact binaries

In this section we consider a binary system made of two compact stars, such as neutron stars or black holes, and we treat them as point-like, with masses m_1, m_2, and positions $\mathbf{r}_1$ and $\mathbf{r}_2$. In the Newtonian approximation, and in the center-of-mass frame (CM), the dynamics reduces to a one-body problem with mass equal to the reduced mass