Part I

Gravitational-wave theory

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In this chapter we discuss how gravitational waves (GWs) emerge from general relativity, and what their properties are. The most straightforward approach, pursued in Sections 1.1–1.3, is "linearized theory", and consists of expanding the Einstein equations around the flat Minkowski metric $\eta_{\mu\nu}$. This allows us to see immediately how a wave equation emerges (Section 1.1) and how the solutions can be put in an especially simple form by an appropriate gauge choice (Section 1.2); then, using standard tools of general relativity such as the geodesic equation and the equation of the geodesic deviation, we can study how these waves interact with a detector, idealized for the moment as a set of test masses (Section 1.3).

We next turn to the issue of what is the energy and momentum carried by GWs. Historically, this is a subject that has been surrounded by much confusion, to the extent that for a long time even the existence of physical effects associated with GWs was considered dubious. The heart of the problem is that general relativity has a huge local gauge invariance, the invariance under arbitrary coordinate transformations, and one can easily fall into the mistake of believing that the effect of GWs can be "gauged away", i.e. set to zero with an appropriate coordinate transformation. We will discuss these issues in details, paying special attention to the conceptual aspects that are hidden behind the derivations. In this chapter we will approach the problem from a geometric point of view, identifying the energy–momentum tensor of GWs from their effect on the background curvature. This approach requires that we depart from linearized theory (i.e. from the expansion over a flat background) and take a broader point of view, where GWs are introduced as perturbations over a slowly varying, but otherwise generic, curved background, as discussed in Section 1.4. In Section 2.1 we will re-examine these conceptual problems from the point of view of field theory. The combination of the geometrical and field-theoretical perspectives gives in general a deeper understanding of the subject. As a byproduct of the study of the interaction between GWs and the background performed in Section 1.4, we will also find the equation governing the propagation of GWs in curves space, which is examined in Section 1.5. Finally, at the end of the chapter we collect in a Solved Problems section some detailed calculations and some more technical issues.

1.1 Expansion around flat space

The gravitational action is $S = S_E + S_M$, where

$$S_E = \frac{c^3}{16\pi G}\int d^4x \sqrt{-g}\, R \tag{1.1}$$

is the Einstein action and S_M is the matter action. The Ricci scalar R, as well as the Ricci tensor $R_{\mu\nu}$ and the Riemann tensor $R_{\mu\nu\rho\sigma}$ are defined in the Notation section. The energy–momentum tensor of matter, $T^{\mu\nu}$, is defined from the variation of the matter action S_M under a change of the metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, according to[1]

$$\delta S_M = \frac{1}{2c}\int d^4x \sqrt{-g}\, T^{\mu\nu}\delta g_{\mu\nu} \,. \tag{1.2}$$

[1] The factor $1/c$ in eq. (1.2) compensates the fact that $d^4x = c\,dtd^3x$, see the Notation.

Taking the variation of the total action with respect to $g_{\mu\nu}$, one finds the Einstein equations,

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \,.} \tag{1.3}$$

General relativity is invariant under a huge symmetry group, the group of all possible coordinate transformations,

$$x^\mu \to x'^\mu(x) \,, \tag{1.4}$$

where x'^μ is an arbitrary smooth function of x^μ. More precisely, we require that $x'^\mu(x)$ be invertible, differentiable, and with a differentiable inverse, i.e. $x'^\mu(x)$ is an arbitrary diffeomorphism. Under eq. (1.4), the metric transforms as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma}(x) \,. \tag{1.5}$$

We will refer to this symmetry as the *gauge symmetry* of general relativity.

As a first step toward the understanding of GWs, we wish to study the expansion of the Einstein equations around the flat-space metric. Therefore we write

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \qquad |h_{\mu\nu}| \ll 1 \,,} \tag{1.6}$$

and we expand the equations of motion to linear order in $h_{\mu\nu}$. The resulting theory is called *linearized theory*. Since the numerical values of the components of a tensor depend on the reference frame, what we really mean is that, in the physical situation in which we are interested, there exists a reference frame where eq. (1.6) holds, on a sufficiently large region of space. Choosing a reference frame breaks the invariance of general relativity under coordinate transformations. Indeed, breaking

a local invariance is in general the best way to get rid of spurious degrees of freedom and exposing the actual physical content of a field theory.

Symmetries of linearized theory

However, after choosing a frame where eq. (1.6) holds, a residual gauge symmetry remains. Consider in fact a transformation of coordinates

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x) \,, \tag{1.7}$$

where the derivatives $|\partial_\mu \xi_\nu|$ are at most of the same order of smallness as $|h_{\mu\nu}|$. Using the transformation law of the metric, eq. (1.5), we find that the transformation of $h_{\mu\nu}$, to lowest order, is

$$\boxed{h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu) \,.} \tag{1.8}$$

If $|\partial_\mu \xi_\nu|$ are at most of the same order of smallness as $|h_{\mu\nu}|$, the condition $|h_{\mu\nu}| \ll 1$ is preserved, and therefore these slowly varying diffeomorphisms are a symmetry of linearized theory.[2]

[2] The first corrections to the right-hand side of eq. (1.8) are $O(\partial\xi\partial\xi)$ and $O(h\partial\xi)$. Note that corrections $O(\xi\partial^2\xi)$, which appear in the intermediate steps of the derivation of eq. (1.8), finally cancel, so it is not necessary to require that $|\xi^\mu|$ themselves are small but only that $|\partial_\mu\xi_\nu|$ are small. The condition $|\partial_\mu\xi_\nu| \ll 1$ is also all we need to invert iteratively the relation $x'^\mu = x^\mu + \xi^\mu(x)$, writing $x^\mu = x'^\mu - \xi^\mu(x) = x'^\mu - \xi^\mu(x'-\xi) \simeq x'^\mu - \xi^\mu(x') + O(\xi\partial\xi)$. When the background metric is not $\eta_{\mu\nu}$ we will also find a condition on $|\xi^\mu|$, see Problem 1.2.

We can also perform finite, global (i.e. x-independent) Lorentz transformations

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu \,. \tag{1.9}$$

By definition of Lorentz transformation, the matrix $\Lambda^\mu{}_\nu$ satisfies

$$\Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \eta_{\rho\sigma} = \eta_{\mu\nu} \,. \tag{1.10}$$

Under a Lorentz transformation,

$$\begin{aligned} g_{\mu\nu}(x) \to g'_{\mu\nu}(x') &= \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma g_{\rho\sigma}(x) \\ &= \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma [\eta_{\rho\sigma} + h_{\rho\sigma}(x)] \\ &= \eta_{\mu\nu} + \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma h_{\rho\sigma}(x) \,, \end{aligned} \tag{1.11}$$

where we used eq. (1.10). Therefore in the Lorentz-transformed frame we have $g'_{\mu\nu}(x') = \eta_{\mu\nu} + h'_{\mu\nu}(x')$, with

$$h'_{\mu\nu}(x') = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma h_{\rho\sigma}(x) \,. \tag{1.12}$$

This shows that $h_{\mu\nu}$ is a tensor under Lorentz transformations. Rotations never spoil the condition $|h_{\mu\nu}| \ll 1$, while for boosts we must limit ourselves to those that do not spoil this condition.

Besides, we see from eq. (1.5) that $h_{\mu\nu}$ is invariant under constant translations, i.e. transformations $x^\mu \to x'^\mu = x^\mu + a^\mu$, where a^μ is not restricted to be infinitesimal, but can be finite. Therefore linearized theory is invariant under finite Poincaré transformations (that is, the group formed by translations and Lorentz transformations), as well as under the infinitesimal local transformation (1.8). In contrast, full general relativity does not have Poincaré symmetry, since the flat space metric plays no special role, but has the full invariance under coordinate transformations, rather than the infinitesimal version (1.8).

To linear order in $h_{\mu\nu}$, the Riemann tensor becomes

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\nu\partial_\rho h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\nu\rho} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho}) \,. \tag{1.13}$$

(We will prove this in Problem 1.1, where we perform explicitly the linearization of the Riemann tensor over an arbitrary curved background). Plugging eq. (1.8) into eq. (1.13) we see that, under the residual gauge transformation (1.8), the linearized Riemann tensor is *invariant* (while, under arbitrary diffeomorfisms in the full non-linearized theory, it is rather covariant).

Equations of motion of linearized theory

The linearized equations of motion are written more compactly defining

$$h = \eta^{\mu\nu} h_{\mu\nu}\,, \tag{1.14}$$

and

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h\,. \tag{1.15}$$

Observe that $\bar{h} \equiv \eta^{\mu\nu}\bar{h}_{\mu\nu} = h - 2h = -h$, so eq. (1.15) can be inverted to give

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}\,. \tag{1.16}$$

In linearized theory we use the convention that indices are raised and lowered with the flat metric $\eta_{\mu\nu}$. Using eq. (1.13) we can compute with straightforward algebra the linearization of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R$, and we find that the linearization of the Einstein equations (1.3) is

$$\boxed{\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - \partial^\rho\partial_\nu\bar{h}_{\mu\rho} - \partial^\rho\partial_\mu\bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4}T_{\mu\nu}\,.} \tag{1.17}$$

We can now use the gauge freedom (1.8) to choose the *Lorentz gauge* (also called the Hilbert gauge, or the harmonic gauge, or the De Donder gauge),[3]

$$\boxed{\partial^\nu\bar{h}_{\mu\nu} = 0\,.} \tag{1.18}$$

To prove that, using the symmetry transformation (1.8), we can impose the condition (1.18), we observe that, in terms of $\bar{h}_{\mu\nu}$, eq. (1.8) becomes

$$\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho)\,, \tag{1.19}$$

and therefore

$$\partial^\nu\bar{h}_{\mu\nu} \to (\partial^\nu\bar{h}_{\mu\nu})' = \partial^\nu\bar{h}_{\mu\nu} - \Box\xi_\mu\,, \tag{1.20}$$

where, in the context of linearized theory, $\Box$ is defined as the flat space d'Alembertian, $\Box = \eta_{\mu\nu}\partial^\mu\partial^\nu = \partial_\mu\partial^\mu$. (Recall also that in linearized theory indices are raised and lowered with the flat metric $\eta_{\mu\nu}$.) Therefore, if the initial field configuration $h_{\mu\nu}$ is such that $\partial^\nu\bar{h}_{\mu\nu} = f_\mu(x)$, with $f_\mu(x)$ some function, to obtain $(\partial^\nu\bar{h}_{\mu\nu})' = 0$ we must choose $\xi_\mu(x)$ so that

$$\Box\xi_\mu = f_\mu(x)\,. \tag{1.21}$$

[3] More generally, the harmonic (or De Donder gauge) is defined, in a curved background, by the condition $\partial_\mu(g^{\mu\nu}\sqrt{-g}) = 0$. In this form, we will use it extensively in Chapter 5. Writing $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and expanding to linear order, the harmonic gauge reduces to the Lorentz gauge (1.18).
The denomination "Lorentz gauge" derives its name by the analogy with the Lorentz gauge of electromagnetism, $\partial_\mu A^\mu = 0$. It is amusing to observe that this is in fact a misnomer. In electromagnetism, this gauge was first used by L. V. Lorenz (without "t", the person who also invented retarded potentials) in 1867, when the better known H. A. Lorentz was just 14 years old! (See J. D. Jackson and L. B. Okun, 2001.) However, this "misprint" has by now entered universally into use, and we will conform to it.
To make things worse, none of the above denominations is historically correct. In general relativity, this gauge choice was in fact first suggested to Einstein by De Sitter, see Chapter 3 of Kennefick (2007).

This equation always admits solutions, because the d'Alembertian operator is invertible. If we denote by $G(x)$ a Green's function of the d'Alembertian operator, so that

$$\Box_x G(x-y) = \delta^4(x-y)\,, \tag{1.22}$$

then the corresponding solution is

$$\xi_\mu(x) = \int d^4x\, G(x-y) f_\mu(y)\,. \tag{1.23}$$

In this gauge the last three terms on the left-hand side of eq. (1.17) vanish, and we get a simple wave equation,

$$\boxed{\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}\,.} \tag{1.24}$$

Observe that eq. (1.18) gives four conditions, that reduce the 10 independent components of the symmetric 4×4 matrix $h_{\mu\nu}$ to six independent components. Equations (1.18) and (1.24) together imply for consistency

$$\partial^\nu T_{\mu\nu} = 0\,, \tag{1.25}$$

which is the conservation of energy–momentum in the linearized theory. This should be contrasted with the conservation in the full theory, $D^\nu T_{\mu\nu} = 0$, where D^ν is the covariant derivative.

Physically, the approximations implicit in the linearized theory can be summarized as follows: the bodies that act as sources of GWs are taken to move in flat space-time, along the trajectories determined by their mutual influence. In particular, for a self-gravitating system such as a binary star, the fact that the background space-time metric is $\eta_{\mu\nu}$ means that we are describing its dynamics using Newtonian gravity, rather than full general relativity. The response of test masses to the GW $h_{\mu\nu}$ generated by these bodies is rather computed using $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and neglecting terms $O(h^2)$ when evaluating the Christoffel symbols or the Riemann tensor.

1.2 The transverse-traceless gauge

Equation (1.24) is the basic result for computing the generation of GWs within linearized theory. To study the propagation of GWs as well as the interaction with test masses (and therefore with a GW detector), we are rather interested in this equation outside the source, i.e. where $T_{\mu\nu} = 0$,

$$\Box\bar{h}_{\mu\nu} = 0 \qquad \text{(outside the source)}\,. \tag{1.26}$$

Since $\Box = -(1/c^2)\partial_0^2 + \boldsymbol{\nabla}^2$, eq. (1.26) implies that GWs travel at the speed of light. Outside the source we can greatly simplify the form of the metric, observing that eq. (1.18) does not fix the gauge completely; in fact, we saw in eq. (1.20) that, under the transformation (1.7), $\partial^\nu\bar{h}_{\mu\nu}$

transforms as in eq. (1.20). Then, the condition $\partial^\nu \bar{h}_{\mu\nu} = 0$ is not spoiled by a further coordinate transformation $x^\mu \to x^\mu + \xi^\mu$ with

$$\Box \xi_\mu = 0 \,. \tag{1.27}$$

If $\Box \xi_\mu$ is zero, then also $\Box \xi_{\mu\nu} = 0$, where

$$\xi_{\mu\nu} \equiv \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho \,, \tag{1.28}$$

since the flat space d'Alembertian $\Box$ commutes with ∂_μ. Then eq. (1.19) tells us that, from the six independent components of $\bar{h}_{\mu\nu}$, which satisfy $\Box \bar{h}_{\mu\nu} = 0$, we can subtract the functions $\xi_{\mu\nu}$, which depend on four independent arbitrary functions ξ_μ, and which satisfy the same equation, $\Box \xi_{\mu\nu} = 0$. This means that we can choose the functions ξ_μ so as to impose four conditions on $h_{\mu\nu}$. In particular, we can choose ξ^0 such that the trace $\bar{h} = 0$. Note that if $\bar{h} = 0$, then $\bar{h}_{\mu\nu} = h_{\mu\nu}$. The three functions $\xi^i(x)$ are now chosen so that $h^{0i}(x) = 0$. Since $\bar{h}_{\mu\nu} = h_{\mu\nu}$, the Lorentz condition (1.18) with $\mu = 0$ reads

$$\partial^0 h_{00} + \partial^i h_{0i} = 0 \,. \tag{1.29}$$

Having fixed $h_{0i} = 0$, this simplifies to

$$\partial^0 h_{00} = 0 \,, \tag{1.30}$$

so h_{00} becomes automatically constant in time. A time-independent term h_{00} corresponds to the static part of the gravitational interaction, i.e. to the Newtonian potential of the source which generated the gravitational wave. The gravitational wave itself is the time-dependent part and therefore, as far as the GW is concerned, $\partial^0 h_{00} = 0$ means that $h_{00} = 0$. So, we have set all four components $h_{0\mu} = 0$ and we are left only with the spatial components h_{ij}, for which the Lorentz gauge condition now reads $\partial^j h_{ij} = 0$, and the condition of vanishing trace becomes $h^i{}_i = 0$. In conclusion, we have set

$$\boxed{h^{0\mu} = 0 \,, \qquad h^i{}_i = 0 \,, \qquad \partial^j h_{ij} = 0 \,.} \tag{1.31}$$

This defines the *transverse-traceless gauge*, or TT gauge. By imposing the Lorentz gauge, we have reduced the 10 degrees of freedom of the symmetric matrix $h_{\mu\nu}$ to six degrees of freedom, and the residual gauge freedom, associated to the four functions ξ^μ that satisfy eq. (1.27), has further reduced these to just two degrees of freedom. We will denote the metric in the TT gauge by $h_{ij}^{\rm TT}$.

Observe that the TT gauge cannot be chosen inside the source, since in this case $\Box \bar{h}_{\mu\nu} \neq 0$. Inside the source, once we have chosen the Lorentz gauge, we still have the freedom to perform a transformation with $\Box \xi_\mu = 0$, and therefore $\Box \xi_{\mu\nu} = 0$. However, now we cannot set to zero any further component of $\bar{h}_{\mu\nu}$, which satisfies $\Box \bar{h}_{\mu\nu} \neq 0$, subtracting from it a function $\xi_{\mu\nu}$ which satisfies $\Box \xi_{\mu\nu} = 0$.[4]

[4] This closely parallels the situation in electrodynamics. The classical equation of motion obtained from the variation of the Maxwell Lagrangian with an external current is $\partial_\mu F^{\mu\nu} = j^\nu$, i.e. $\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = j^\nu$, and it becomes $\Box A^\nu = j^\nu$ when we impose the Lorentz gauge $\partial_\mu A^\mu = 0$. The Lorentz gauge still leaves the residual gauge freedom $A_\mu \to A_\mu - \partial_\mu \theta$ with $\Box \theta = 0$. Outside the source we have $j^\mu = 0$, and therefore $\Box A^\mu = 0$, so the residual gauge freedom, i.e. the function θ which satisfies $\Box \theta = 0$, can be used to set $A^0 = 0$. When $A^0 = 0$, the Lorentz gauge $\partial_\mu A^\mu = 0$ becomes a transversality condition on A^i, $\partial_i A^i = 0$. If instead $j^0 \neq 0$, we have $\Box A^0 \neq 0$ and we cannot remove A^0 using a function θ which satisfies $\Box \theta = 0$.

Equation (1.26) has plane wave solutions, $h_{ij}^{\rm TT}(x) = e_{ij}(\mathbf{k}) e^{ikx}$, with $k^\mu = (\omega/c, \mathbf{k})$ and $\omega/c = |\mathbf{k}|$ (and the usual convention that the real part is taken at the end of the computation). The tensor $e_{ij}(\mathbf{k})$ is called the polarization tensor. For a single plane wave with a given wave-vector $\mathbf{k}$ (or for a superposition of plane waves with different frequencies but all with the same direction of propagation $\hat{\mathbf{n}} = \mathbf{k}/|\mathbf{k}|$), we see from eq. (1.31) that the non-zero components of $h_{ij}^{\rm TT}$ are in the plane transverse to $\hat{\mathbf{n}}$ since, on a plane wave, the condition $\partial^j h_{ij} = 0$ becomes $n^i h_{ij} = 0$. Choosing for definiteness $\hat{\mathbf{n}}$ along the z axis, and, imposing that h_{ij} be symmetric and traceless, we have

$$h_{ij}^{\rm TT}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos[\omega(t - z/c)] \,, \tag{1.32}$$

or, more simply,

$$\boxed{h_{ab}^{\rm TT}(t,z) = \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix}_{ab} \cos[\omega(t - z/c)] \,,} \tag{1.33}$$

where $a, b = 1, 2$ are indices in the transverse (x, y) plane; h_+ and $h_\times$ are called the amplitudes of the "plus" and "cross" polarization of the wave. In terms of the interval ds^2,

$$\begin{aligned} ds^2 = &-c^2 dt^2 + dz^2 + \{1 + h_+ \cos[\omega(t - z/c)]\} dx^2 \\ &+ \{1 - h_+ \cos[\omega(t - z/c)]\} dy^2 + 2h_\times \cos[\omega(t - z/c)] dx dy \,. \end{aligned} \tag{1.34}$$

Projection onto the TT gauge; the Lambda tensor

Given a plane wave solution $h_{\mu\nu}(x)$ propagating in the direction $\hat{\mathbf{n}}$, outside the sources, already in the Lorentz gauge but not yet in the TT gauge, we can find the form of the wave in the TT gauge as follows. First we introduce the tensor

$$P_{ij}(\hat{\mathbf{n}}) = \delta_{ij} - n_i n_j \,. \tag{1.35}$$

This tensor is symmetric, is transverse (i.e. $n^i P_{ij}(\hat{\mathbf{n}}) = 0$), is a projector (i.e. $P_{ik} P_{kj} = P_{ij}$), and its trace is $P_{ii} = 2$. With the help of P_{ij} we construct

$$\boxed{\Lambda_{ij,kl}(\hat{\mathbf{n}}) = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl} \,.} \tag{1.36}$$

This is still a projector, in the sense that

$$\Lambda_{ij,kl} \Lambda_{kl,mn} = \Lambda_{ij,mn} \,. \tag{1.37}$$

Furthermore it is transverse on all indices, $n^i \Lambda_{ij,kl} = 0, n^j \Lambda_{ij,kl} = 0$, etc., it is traceless with respect to the (i,j) and (k,l) indices,

$$\Lambda_{ii,kl} = \Lambda_{ij,kk} = 0 \,, \tag{1.38}$$

and it is symmetric under the simultaneous exchange $(i,j) \leftrightarrow (k,l)$. In terms of $\hat{\mathbf{n}}$, its explicit form is

$$\Lambda_{ij,kl}(\hat{\mathbf{n}}) = \delta_{ik}\delta_{jl} - \frac{1}{2}\delta_{ij}\delta_{kl} - n_j n_l \delta_{ik} - n_i n_k \delta_{jl} + \frac{1}{2}n_k n_l \delta_{ij} + \frac{1}{2}n_i n_j \delta_{kl} + \frac{1}{2}n_i n_j n_k n_l \,. \tag{1.39}$$

We shall meet $\Lambda_{ij,kl}$ often, and we call it the Lambda tensor.[5] We can now show that, given a plane wave $h_{\mu\nu}$ in the Lorentz gauge, but not yet in the TT gauge, the GW in the TT gauge is given in terms of the spatial components h_{ij} of $h_{\mu\nu}$ by

[5] This is the same tensor $\Lambda_{ij,kl}$ defined in Weinberg (1972), eq. (10.4.14). However, Weinberg only uses it in an equation where it is contracted with the tensor $T^{ij}T^{kl}$, which is symmetric under $i \leftrightarrow j$ and $k \leftrightarrow l$ and then he replaces the term $n_j n_l \delta_{ik} + n_i n_k \delta_{jl}$ in eq. (1.39) by $2n_j n_l \delta_{ik}$.

$$\boxed{h_{ij}^{\mathrm{TT}} = \Lambda_{ij,kl} h_{kl} \,.} \tag{1.40}$$

In fact by construction the right-hand side is transverse and traceless in (i,j) while, from the fact that $h_{\mu\nu}$ was a solution of the wave equation in the vacuum and that it was in the Lorentz gauge, it follows that $\Box h_{ij}^{\mathrm{TT}} = 0$. (Observe that it is important that $h_{\mu\nu}$ be in the Lorentz gauge already, otherwise the equation of motion that it satisfies would not simply be $\Box h_{\mu\nu} = 0$.)

In general, given any symmetric tensor S_{ij}, we define its transverse-traceless part as

$$S_{ij}^{\mathrm{TT}} = \Lambda_{ij,kl} S_{kl} \,. \tag{1.41}$$

Observe that S_{ij}^{TT} is still symmetric.

Plane wave expansion

For later calculations, it is useful to spell out clearly our conventions and definitions for the plane wave expansion. In the TT gauge the equation of motion is $\Box h_{ij}^{\mathrm{TT}} = 0$ and therefore h_{ij}^{TT} can be expanded as

$$h_{ij}^{\mathrm{TT}}(x) = \int \frac{d^3k}{(2\pi)^3} \left(\mathcal{A}_{ij}(\mathbf{k}) e^{ikx} + \mathcal{A}_{ij}^*(\mathbf{k}) e^{-ikx} \right) \,. \tag{1.42}$$

The four-vector k^μ, with dimensions of inverse length, is related to the frequency ω and to the wave-vector $\mathbf{k}$ by $k^\mu = (\omega/c, \mathbf{k})$, with $|\mathbf{k}| = \omega/c = (2\pi f)/c$ and $\mathbf{k}/|\mathbf{k}| = \hat{\mathbf{n}}$. Therefore $d^3k = |\mathbf{k}|^2 d|\mathbf{k}| d\Omega = (2\pi/c)^3 f^2 df d\Omega$, with $f > 0$. We denote by $d^2\hat{\mathbf{n}} = d\cos\theta d\phi$ the integration over the solid angle, so the above equation reads

$$h_{ij}^{\mathrm{TT}}(x) = \frac{1}{c^3} \int_0^\infty df\, f^2 \int d^2\hat{\mathbf{n}} \left(\mathcal{A}_{ij}(f, \hat{\mathbf{n}})\, e^{-2\pi i f (t - \hat{\mathbf{n}}\cdot\mathbf{x}/c)} + \text{c.c.} \right) \,. \tag{1.43}$$

Observe that, inside the parentheses, both the contribution written explicitly and its complex conjugate refer to a wave traveling in the direction $+\hat{\mathbf{n}}$, since they depend on t and $\mathbf{x}$ only through the combination $(t - \hat{\mathbf{n}}\cdot\mathbf{x}/c)$. Observe also that, in this form, only "physical" frequencies $f > 0$ enter in the expansion.

The TT gauge conditions (1.31) give $\mathcal{A}^i{}_i(\mathbf{k}) = 0$ and $k^i \mathcal{A}_{ij}(\mathbf{k}) = 0$. Of course, in a superposition of waves with different propagation directions, $h_{ij}(x)$ does not reduce to a 2×2 matrix; for instance, h_{12} gets contributions from the waves with $k^3 \neq 0$, h_{13} gets contributions from the waves with $k^2 \neq 0$, h_{23} from the waves with $k^1 \neq 0$, etc. This will be important when we consider stochastic backgrounds of GWs. However, when we observe on Earth a GW emitted by a single astrophysical source, the direction of propagation of the wave, $\hat{\mathbf{n}}_0$, is very well defined and we can write

$$\mathcal{A}_{ij}(\mathbf{k}) = \mathcal{A}_{ij}(f)\, \delta^{(2)}(\hat{\mathbf{n}} - \hat{\mathbf{n}}_0) \,. \tag{1.44}$$

The transversality condition now states that the only non-vanishing components are those in the plane transverse to the propagation direction $\hat{\mathbf{n}}_0$. We label by $a, b = 1, 2$ the indices in the transverse plane and we omit, for notational simplicity, the superscript TT, since the fact that we are in the TT gauge is already implicit in the use of the indices $a, b = 1, 2$ instead of $i, j = 1, 2, 3$. Then

$$h_{ab}(t, \mathbf{x}) = \int_0^\infty df \left(\tilde{h}_{ab}(f, \mathbf{x}) e^{-2\pi i f t} + \tilde{h}_{ab}^*(f, \mathbf{x}) e^{2\pi i f t} \right) , \tag{1.45}$$

where

$$\tilde{h}_{ab}(f, \mathbf{x}) = \frac{f^2}{c^3} \int d^2\hat{\mathbf{n}}\, \mathcal{A}_{ab}(f, \hat{\mathbf{n}})\, e^{2\pi i f \hat{\mathbf{n}}\cdot\mathbf{x}/c} = \frac{f^2}{c^3} \mathcal{A}_{ab}(f)\, e^{2\pi i f \hat{\mathbf{n}}_0\cdot\mathbf{x}/c} \,. \tag{1.46}$$

As we will see when we discuss the detectors, for resonant bars and ground based interferometers the linear dimensions of the detector are much smaller than the reduced wavelength $\lambda\!\!\!^{-} = \lambda/(2\pi)$ of the GWs to which they are sensitive. In this case, choosing the origin of the coordinate system centered on the detector, we have $\exp\{2\pi i f \hat{\mathbf{n}}\cdot\mathbf{x}/c\} = \exp\{i\hat{\mathbf{n}}\cdot\mathbf{x}/\lambda\!\!\!^{-}\} \simeq 1$ all over the detector. If we are interested in the GW at the detector location, we can therefore neglect all $\mathbf{x}$-dependences and write simply

$$h_{ab}(t) = \int_0^\infty df \left(\tilde{h}_{ab}(f) e^{-2\pi i f t} + \tilde{h}_{ab}^*(f) e^{2\pi i f t} \right) , \tag{1.47}$$

with $\tilde{h}_{ab}(f) = \tilde{h}_{ab}(f, \mathbf{x} = 0)$. Of course, the dependence on $\mathbf{x}$ must be kept when we compare the GW signal at two different detectors (e.g. when we consider the overlap reduction function in a two-detector correlation, see Section 7.8.3) or when we need spatial derivatives of $h_{ab}(t, \mathbf{x})$ (e.g. when we compute spatial components of the energy–momentum tensor).

From eq. (1.33) it follows that

$$\tilde{h}_{ab}(f) = \begin{pmatrix} \tilde{h}_+(f) & \tilde{h}_\times(f) \\ \tilde{h}_\times(f) & -\tilde{h}_+(f) \end{pmatrix}_{ab} \,. \tag{1.48}$$

The $+$ and $\times$ polarizations are defined with respect to a given choice of axes in the transverse plane. If we rotate by an angle ψ the system of axes used for their definition, we show in Problem 1.1 that h_+ and $h_\times$ transform as

$$h_+ \to h_+ \cos 2\psi - h_\times \sin 2\psi\,, \tag{1.49}$$
$$h_\times \to h_+ \sin 2\psi + h_\times \cos 2\psi\,. \tag{1.50}$$

Observe that, until now, only physical frequencies $f > 0$ entered our equations. However, eq. (1.45) can be rewritten in a slightly more compact form, extending the definition of $\tilde{h}_{ab}(f,\mathbf{x})$ to negative frequencies, by defining

$$\tilde{h}_{ab}(-f,\mathbf{x}) = \tilde{h}^*_{ab}(f,\mathbf{x})\,, \tag{1.51}$$

so that eq. (1.45) becomes[6] (we do not explicitly write the $\mathbf{x}$ dependence)

$$h_{ab}(t) = \int_{-\infty}^{\infty} df\, \tilde{h}_{ab}(f)\, e^{-2\pi i f t}\,. \tag{1.52}$$

[6] Of course eq. (1.52) could have been written directly setting $\mathbf{x} = 0$ from the beginning and writing a temporal Fourier transform. However, this derivation makes it clear that negative frequencies are not "unphysical", but are simply a way of rewriting the complex conjugate term indicated by *c.c.* in eq. (1.43), and that this contribution refers to a wave traveling in the $+\hat{\mathbf{n}}$ direction, exactly as the contribution with $f > 0$.

The inversion of eq. (1.52) is

$$\tilde{h}_{ab}(f) = \int_{-\infty}^{\infty} dt\, h_{ab}(t)\, e^{2\pi i f t}\,. \tag{1.53}$$

Another useful form for the plane wave expansion is obtained by introducing the polarization tensors $e^A_{ij}(\hat{\mathbf{n}})$ (with $A = +,\times$ labeling the polarizations) defined as

$$\boxed{e^+_{ij}(\hat{\mathbf{n}}) = \hat{\mathbf{u}}_i\hat{\mathbf{u}}_j - \hat{\mathbf{v}}_i\hat{\mathbf{v}}_j\,, \qquad e^\times_{ij}(\hat{\mathbf{n}}) = \hat{\mathbf{u}}_i\hat{\mathbf{v}}_j + \hat{\mathbf{v}}_i\hat{\mathbf{u}}_j\,,} \tag{1.54}$$

with $\hat{\mathbf{u}}, \hat{\mathbf{v}}$ unit vectors orthogonal to the propagation direction $\hat{\mathbf{n}}$ and to each other. With this definition, the polarization tensors are normalized as

$$e^A_{ij}(\hat{\mathbf{n}}) e^{A',ij}(\hat{\mathbf{n}}) = 2\delta^{AA'}\,. \tag{1.55}$$

In the frame where $\hat{\mathbf{n}}$ is along the $\hat{z}$ direction, we can choose $\hat{\mathbf{u}} = \hat{\mathbf{x}}$ and $\hat{\mathbf{v}} = \hat{\mathbf{y}}$, so

$$e^+_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{ab}\,, \qquad e^\times_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{ab}\,, \tag{1.56}$$

with $a, b = 1, 2$ spanning the (x, y) plane. In a generic frame, we can define the amplitudes $\tilde{h}_A(f,\hat{\mathbf{n}})$ from

$$\frac{f^2}{c^3}\mathcal{A}_{ij}(f,\hat{\mathbf{n}}) = \sum_{A=+,\times} \tilde{h}_A(f,\hat{\mathbf{n}})\, e^A_{ij}(\hat{\mathbf{n}})\,. \tag{1.57}$$

Equation (1.43) then becomes

$$\boxed{h_{ab}(t,\mathbf{x}) = \sum_{A=+,\times} \int_{-\infty}^{\infty} df \int d^2\hat{\mathbf{n}}\, \tilde{h}_A(f,\hat{\mathbf{n}}) e^A_{ab}(\hat{\mathbf{n}})\, e^{-2\pi i f(t - \hat{\mathbf{n}}\cdot\mathbf{x}/c)}\,,} \tag{1.58}$$

where again we have defined $\tilde{h}_A(-f,\hat{\mathbf{n}}) = \tilde{h}^*_A(f,\hat{\mathbf{n}})$.

1.3 Interaction of GWs with test masses

In the previous section we have seen how to describe a GW. In this section we discuss the interaction of GWs with a detector, idealized for the moment as a set of test masses. This is an issue that hides some subtleties because, even if the physics must finally be invariant under coordinate transformations, the language that we use to describe the GWs and the detector, as well as the intermediate steps of the computations, do depend on the reference frame that we choose.

In general relativity, the mathematical procedure of choosing a gauge corresponds, physically, to selecting a specific observer. We have seen that GWs have an especially simple form in the TT gauge, so we want to understand which reference frame corresponds to the TT gauge. We will also see that the description of the detector is more intuitive in another frame, the detector proper frame. It is therefore important, when we discuss the interaction of GWs with the detector, to be aware of which reference frame we are using, and to understand which is the appropriate language in that frame.

Two important tools for understanding the physical meaning of a given gauge choice are the geodesic equation and the equation of the geodesic deviation. We briefly recall these basic concepts of general relativity in the next subsection. We will then explore the interaction of GWs with test masses in different frames, in particular in the TT frame and in the detector proper frame.

1.3.1 Geodesic equation and geodesic deviation

In this subsection and in the next we recall some elementary notions of general relativity, referring the reader, e.g. to Misner, Thorne and Wheeler (1973) or to Hartle (2003) for more details and proofs. Consider, in some reference frame, a curve $x^\mu(\lambda)$, parametrized by a parameter λ. The interval ds between two points separated by $d\lambda$ is given by

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= g_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} d\lambda^2\,. \end{aligned} \tag{1.59}$$

All along a space-like curve we have, by definition, $ds^2 > 0$, and we can use

$$ds = (g_{\mu\nu} dx^\mu dx^\nu)^{1/2} \tag{1.60}$$

to measure proper distances along the curve. A time-like curve is rather defined by the condition that all along it $ds^2 < 0$, and in this case we can define the proper time τ, from

$$c^2 d\tau^2 = -ds^2 = -g_{\mu\nu} dx^\mu dx^\nu\,. \tag{1.61}$$

The proper time τ is the time measured by a clock carried along this trajectory. It is therefore natural to use τ itself as the parameter λ

which parametrizes the trajectory, so that $x^\mu = x^\mu(\tau)$. Observe, from eq. (1.61), that

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -c^2 . \tag{1.62}$$

The four-velocity u^μ is defined as

$$u^\mu = \frac{dx^\mu}{d\tau} , \tag{1.63}$$

so eq. (1.62) reads

$$g_{\mu\nu}u^\mu u^\nu = -c^2 . \tag{1.64}$$

Among all possible time-like curves that satisfy the fixed boundary conditions $x^\mu(\tau_A) = x^\mu_A$ and $x^\mu(\tau_B) = x^\mu_B$, the classical trajectory of a point-like test mass m is obtained by extremizing the action

$$S = -m\int_{\tau_A}^{\tau_B} d\tau . \tag{1.65}$$

This gives the *geodesic equation*,

$$\boxed{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 ,} \tag{1.66}$$

which is the classical equation of motion of a test mass in the curved background described by the metric $g_{\mu\nu}$, in the absence of external non-gravitational forces. In terms of the four-velocity u^μ, the geodesic equation reads

$$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}u^\nu u^\rho = 0 . \tag{1.67}$$

Consider now two nearby geodesics, one parametrized by $x^\mu(\tau)$ and the other by $x^\mu(\tau) + \xi^\mu(\tau)$.[7] Then $x^\mu(\tau)$ satisfies eq. (1.66), while $x^\mu(\tau) + \xi^\mu(\tau)$ satisfies

$$\frac{d^2(x^\mu+\xi^\mu)}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x+\xi)\frac{d(x^\nu+\xi^\nu)}{d\tau}\frac{d(x^\rho+\xi^\rho)}{d\tau} = 0 . \tag{1.68}$$

If $|\xi^\mu|$ is much smaller than the typical scale of variation of the gravitational field, taking the difference between eqs. (1.68) and (1.66), and expanding to first order in ξ, we get

$$\frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{d\xi^\rho}{d\tau} + \xi^\sigma\partial_\sigma\Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 . \tag{1.69}$$

This is the equation of the geodesic deviation. We can rewrite it in a more elegant way by introducing the covariant derivative of a vector field $V^\mu(x)$ along the curve $x^\mu(\tau)$,

$$\frac{DV^\mu}{D\tau} \equiv \frac{dV^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}V^\nu\frac{dx^\rho}{d\tau} . \tag{1.70}$$

Then, eq. (1.69) can be written as

$$\boxed{\frac{D^2\xi^\mu}{D\tau^2} = -R^\mu{}_{\nu\rho\sigma}\,\xi^\rho\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} ,} \tag{1.71}$$

[7] More precisely, each geodesic is parametrized by its own proper time τ, and $\xi^\mu(\tau)$ connects point with the same value of τ on the two geodesics.

or, in terms of the four-velocity u^μ,

$$\frac{D^2\xi^\mu}{D\tau^2} = -R^\mu{}_{\nu\rho\sigma}\,\xi^\rho u^\nu u^\sigma . \tag{1.72}$$

This equation shows that two nearby time-like geodesics experience a tidal gravitational force, which is determined by the Riemann tensor. Writing explicitly the geodesic equation or the equation of the geodesic deviation in the reference frame of interest we can understand how test masses behave for the corresponding observer. We consider the most relevant examples in the next subsections.

1.3.2 Local inertial frames and freely falling frames

Before discussing the TT frame and the detector frame, it may be useful to recall some basic facts about the construction of local inertial frames and of freely falling frames.

It is a standard exercise in general relativity to show that it is always possible to perform a change of coordinates such that, at a given space-time point P, all the components of the Christoffel symbol vanish, $\Gamma^\mu_{\nu\rho}(P) = 0$. In such a frame, at P the geodesic equation (1.66) becomes

$$\left.\frac{d^2x^\mu}{d\tau^2}\right|_P = 0 , \tag{1.73}$$

so in this frame a test mass is free falling, although only at one point in space and at one moment in time. Such a frame is called a *local inertial frame* (sometimes abbreviated as LIF), and gives a realization of the equivalence principle.

An explicit construction of the corresponding system of coordinates can be done as follows (see, e.g. Hartle 2003, Section 8.4). At the point P we choose a basis of four orthonormal four-vectors, e_α, where $\alpha = 0,\ldots 3$ labels the four-vector. We choose them orthogonal to each other with respect to the flat-space metric $\eta_{\mu\nu}$, so $\eta_{\mu\nu}e^\mu_\alpha e^\nu_\beta = \eta_{\alpha\beta}$. Consider now the spatial geodesic that starts at P, in the direction of a space-like unit four-vector n. We parametrize the geodesic using the proper distance. Let Q be the point reached from P, moving along this geodesic, after a proper distance s, and let (n^0, n^1, n^2, n^3) be the components of n in the basis $\{e_\alpha\}$. Then we assign to Q the coordinates $x_Q = (sn^0, sn^1, sn^2, sn^3)$. Thus, for example, if we send out a geodesic along the direction e_3, and we meet a point Q after a proper distance s, the coordinates of Q are $x_Q = (0,0,0,s)$. Similarly, we send out a time-like geodesic in the direction of a time-like unit four-vector n, we parametrize this geodesic using proper time, and we assign the coordinates $(\tau n^0, \tau n^1, \tau n^2, \tau n^3)$ to the point that we reach after a proper time τ.

We fill all of space-time with time-like or space-like geodesics (null geodesics can be approximated to arbitrary accuracy with time-like or space-like geodesics and therefore can be obtained by continuity), so all points are reached by at least one geodesic.

In a sufficiently small region of space, geodesics do not intersect (which of course is no longer true on large regions of space, as vividly shown for instance by the phenomenon of gravitational lensing), so each point in this small region is reached by one and only one geodesic. Thus, the above method allows us to assign coordinates unambiguously to all points of a sufficiently small space-time region around P. This coordinate system is known as *Riemann normal coordinates.* We can now check that it indeed gives an explicit realization of a local inertial frame.

First of all, the fact that $g_{\mu\nu}(P) = \eta_{\mu\nu}$ follows simply from the fact that the coordinates are referred to the basis $\{e_\alpha\}$, built at P, which by definition is orthonormal with respect to the flat space-time metric, $\eta_{\mu\nu}e^\mu_\alpha e^\nu_\beta = \eta_{\alpha\beta}$. To show that we also have $\Gamma^\mu_{\nu\rho}(P) = 0$ in this frame, consider the geodesic equation (1.66). Since, by definition, the coordinates are linear in proper time (if they are reached by a time-like geodesic; or linear in proper distance if they are reached by space-like geodesics, in which case in the geodesic equation (1.66) τ must be replaced by s), the term $d^2x^\mu/d\tau^2$ (or d^2x^μ/ds^2 for space-like geodesics) vanishes, while $dx^\mu/d\tau = n^\mu$. Then the geodesic equation becomes

$$\Gamma^\mu_{\nu\rho}(P)n^\nu n^\rho = 0\,, \tag{1.74}$$

and, since this holds for all n^μ, we conclude that

$$\Gamma^\mu_{\nu\rho}(P) = 0\,. \tag{1.75}$$

Riemann normal coordinates therefore provide an explicit example of a local inertial frame.

In a local inertial frame a test mass moves freely only at one point in space and at one moment in time. We can however do much better than this, building a reference frame where a test mass is in free fall *all along the geodesic.* Such a frame can be built observing that a freely spinning object (like a gyroscope) that moves along a time-like geodesic $x^\mu(\tau)$ obeys the equation

$$\frac{ds^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}s^\nu\frac{dx^\rho}{d\tau} = 0\,, \tag{1.76}$$

where s^μ is the spin four-vector, i.e. the four-vector that in the rest frame reduces to $s^\mu = (0, \mathbf{s})$. This equation is the covariant generalization of the equation $ds^\mu/d\tau = 0$ that expresses the conservation of angular momentum in flat space-time. We start by constructing a local inertial frame at P, as before, but using three gyroscopes to mark the direction of the spatial axes. We then propagate this reference frame along the geodesic, always orienting the spatial axes in the direction pointed out by the gyroscopes (while the time axis is in the direction of the four-velocity along the geodesic). By definition, then, in this frame the gyroscopes do not rotate with respect to the axes since they *define* the orientation of the axes. Then, $ds^\mu/d\tau = 0$ along the entire time-like geodesic and, from eq. (1.76), we see that $\Gamma^\mu_{\nu\rho}$ vanishes along the entire time-like geodesic, and not just at a single point P. Such a reference frame is called a *freely falling frame,* and its coordinates (Riemann normal coordinates with axes marked by gyroscopes) are known as *Fermi normal coordinates.* A freely falling frame is therefore a local inertial frame along an entire geodesic.

Such a frame is practically given by drag-free satellites, in which an experimental apparatus is freely floating inside a satellite, which screens it from external disturbances (e.g. solar wind, micrometeorites, etc.). The satellite senses precisely the position of the experimental apparatus and adjusts its position, using thrusters, to remain centered about it. As we will see, these drag-free techniques are crucial for a space interferometer.

1.3.3 TT frame and proper detector frame

The TT frame

We have seen that there exists a gauge where GWs have an especially simple form, the TT gauge. We denote the corresponding reference frame as the TT frame and we now ask what it means, physically, to be in the TT frame.

Again, the answer can be found by looking at the geodesic equation, eq. (1.66). If a test mass is at rest at $\tau = 0$, we find from eq. (1.66) that

$$\begin{aligned}\left.\frac{d^2x^i}{d\tau^2}\right|_{\tau=0} &= -\left[\Gamma^i_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\right]_{\tau=0}\\ &= -\left[\Gamma^i_{00}\left(\frac{dx^0}{d\tau}\right)^2\right]_{\tau=0},\end{aligned} \tag{1.77}$$

where in the second line we used the fact that, by assumption, at $\tau = 0$ we have $dx^i/d\tau = 0$, since we took the mass initially at rest. Writing $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and expanding to first order in $h_{\mu\nu}$, the Christoffel symbol $\Gamma^\mu_{\nu\rho}$ becomes

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}(\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho})\,, \tag{1.78}$$

and therefore

$$\Gamma^i_{00} = \frac{1}{2}(2\partial_0 h_{0i} - \partial_i h_{00})\,. \tag{1.79}$$

However, in the TT gauge this quantity vanishes, because both h_{00} and h_{0i} are set to zero by the gauge condition. Therefore, if at time $\tau = 0$ $dx^i/d\tau$ is zero, in the TT gauge also its derivative $d^2x^i/d\tau^2$ vanishes, and therefore $dx^i/d\tau$ remains zero at all times. This shows that *in the TT frame, particles which were at rest before the arrival of the wave remain at rest even after the arrival of the wave.*[8]

In other words, the coordinates of the TT frame stretch themselves, in response to the arrival of the wave, in such a way that the position of free test masses initially at rest do not change. A physical implementation of the TT gauge can be obtained using the free test masses themselves to mark the coordinates. We can use a test mass to define the origin

[8]Strictly speaking, this is true only to linear order in $h_{\mu\nu}$ since, if we also include the terms $O(h^2)$ in eq. (1.78), Γ^i_{00} no longer vanishes. However, given that on Earth one typically expects GWs with at most $h = O(10^{-21})$, going beyond the linear order is here of no interest.

of the coordinates, a second one to define, e.g. the point with spatial coordinates ($x = 1, y = 0, z = 0$), and so on; then, we state that, by definition, these masses still mark the origin, the point ($x = 1, y = 0, z = 0$), etc. even when the GW is passing.

If the coordinates of test masses initially at rest remain constant, also their coordinate separation must remain constant, for arbitrary finite separation and therefore, of course, also when the separation is small with respect to the typical length-scale of variation of the GW, which is its reduced wavelength. In this limiting case, the equation of the geodesic deviation applies, and it is instructive to check explicitly, from the equation of the geodesic deviation in the TT frame, that the separation ξ^i between the coordinates of two test masses initially at rest does not change. To this purpose, we use the spatial component ($\mu = i$) of eq. (1.69). Since, at $\tau = 0$, $dx^i/d\tau = 0$ by assumption, while $dx^0/d\tau = c$, we get

$$\left.\frac{d^2\xi^i}{d\tau^2}\right|_{\tau=0} = -\left[2c\Gamma^i_{0\rho}\frac{d\xi^\rho}{d\tau} + c^2\xi^\sigma\,\partial_\sigma\Gamma^i_{00}\right]_{\tau=0} . \tag{1.80}$$

However, we already saw that in the TT gauge Γ^i_{00} vanishes identically (at all values of space and time, since in the TT gauge h_{0i} and h_{00} vanish everywhere), and therefore in the first term in bracket, $\Gamma^i_{0\rho}$ is non-vanishing only if ρ is a spatial index, while in the second term $\partial_\sigma\Gamma^i_{00} = 0$. From eq. (1.78), in the TT gauge $\Gamma^i_{0j} = (1/2)\partial_0 h_{ij}$. Therefore, in the TT gauge, the equation of the geodesic deviation gives

$$\left.\frac{d^2\xi^i}{d\tau^2}\right|_{\tau=0} = -\left[\dot{h}_{ij}\frac{d\xi^i}{d\tau}\right]_{\tau=0} , \tag{1.81}$$

and therefore, if at $\tau = 0$ we have $d\xi^i/d\tau = 0$, then also $d^2\xi^i/d\tau^2 = 0$, and the separation ξ^i remains constant at all times.[9]

Observe also that, since in the TT gauge we have $h_{00} = h_{0i} = 0$, the proper time on a time-like trajectory $x^\mu(\tau) = (x^0(\tau), x^i(\tau))$ is obtained from

$$\begin{aligned} c^2 d\tau^2 &= c^2 dt^2(\tau) - (\delta_{ij} + h^{\rm TT}_{ij})dx^i(\tau)dx^j(\tau) \\ &= c^2 dt^2(\tau) - (\delta_{ij} + h^{\rm TT}_{ij})\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\,d\tau^2 , \end{aligned} \tag{1.82}$$

where we write $x^0(\tau) = ct(\tau)$. However, we have seen for a test mass initially at rest that $dx^i(\tau)/d\tau = 0$ at all times. Then, in the TT gauge the proper time τ measured by a clock sitting on a test mass initially at rest is the same as coordinate time t.[10]

The TT gauge illustrates in a particularly neat way the fact that, in general relativity, the physical effects are not expressed by what happens to the coordinates since the theory is invariant under coordinate transformations. At first sight one might be surprised that in the TT gauge the position of test masses does not change as a GW passes by. Of course, this does not mean that the GW had no physical effect, but only that we used the freedom in choosing the coordinate system to *define* the coordinates in such a way that they do not change. Physical effects can

[9] To avoid misunderstandings, observe that ξ^i is a *coordinate* distance, since it is the difference between the coordinates of two test masses. It is not a *proper* distance. We will see below what happens if we consider proper distances.

[10] Actually, this is true up to irrelevant correction $O(h^4)$ since $dx^i(\tau)/d\tau = O(h^2)$. Compare with Note 8.

instead be found monitoring *proper distances*, or *proper times*. Consider for instance two events at $(t, x_1, 0, 0)$ and at $(t, x_2, 0, 0)$, respectively. In the TT gauge, the *coordinate distance* $x_2 - x_1 = L$ remains constant, even if there is a GW propagating along the z axis. However, we see from eq. (1.34) that the *proper distance* s between these two events is

$$\begin{aligned} s &= (x_2 - x_1)\,[1 + h_+\cos(\omega t)]^{1/2} \\ &\simeq L\,[1 + \frac{1}{2}h_+\cos(\omega t)] , \end{aligned} \tag{1.83}$$

where in the second line we only retained the term linear in h_+. Therefore, the proper distance changes periodically in time because of the GW. More generally, if the spatial separation between the two events is given by a vector $\mathbf{L}$, the proper distance is given by $s^2 = L^2 + h_{ij}(t)L_iL_j$ and, to linear order in h, we have $s \simeq L + h_{ij}(L_iL_j/2L)$, implying

$$\ddot{s} \simeq \frac{1}{2}\ddot{h}_{ij}\frac{L_i}{L}L_j . \tag{1.84}$$

Writing $L_i/L = n_i$ and defining s_i from $s = n_is_i$, we get

$$\begin{aligned} \ddot{s}_i &\simeq \frac{1}{2}\ddot{h}_{ij}L_j \\ &\simeq \frac{1}{2}\ddot{h}_{ij}s_j , \end{aligned} \tag{1.85}$$

where in the second line we used the fact that, to lowest order in h, we have $L_j = s_j$. This is the geodesic equation in terms of proper distances, rather than coordinate distances.

If these two test masses are mirrors between which a light beam travels back and forth, it is the proper distance that determines the time taken by the light to make a round trip, so the fact that GWs affect the proper distance means that they can be detected measuring the round-trip time. We will see in detail in Chapter 9 how to analyze an interferometric GW detector in this way.

The proper detector frame

The TT frame has the advantage that GWs have a very simple form in it. However, it is not the frame normally used by an experimentalist to describe its apparatus. In a laboratory, positions are not marked by freely falling particles; rather, after choosing an origin, one ideally uses a rigid ruler to define the coordinates.[11] In this frame we expect that a test mass which is free to move (at least along some direction) will be displaced by the passage of the GWs, with respect to the position defined by the rigid ruler and by the test mass which defines the origin. This is different from what happens in the TT frame, where the positions of the test masses are, by definition, unchanged by an incoming GW.

Conceptually, the simplest laboratory to analyze is one inside a drag-free satellite, so the apparatus is indeed in free fall in the total gravitational field, both of the Earth and of the GWs which might be present.

[11] A rigid ruler is of course an idealization. When we study resonant bars, in Chapter 8, we will see that a bar (and hence also a ruler) is stretched by an incoming GW. We denote by $\xi_0(t)$ the oscillation amplitude of the fundamental elastic mode of a bar (or of a ruler) of length L, and by ω_0 and γ_0 the frequency and the dissipation coefficient of this elastic mode (with $\gamma_0 \ll \omega_0$), respectively. We will find in eq. (8.32) that, when the bar (or the ruler) is driven by a monochromatic GW with frequency ω and amplitude h_0,

$$\xi_0(t) = (2Lh_0\omega^2/\pi^2) \times \frac{(\omega^2 - \omega_0^2)\cos\omega t - \gamma_0\omega\sin\omega t}{(\omega^2 - \omega_0^2)^2 + \gamma_0^2\omega^2} .$$

For a resonant bar, one chooses the frequency ω_0 of the fundamental elastic mode as close as possible to the frequency ω of the GW that one is searching, so the denominator becomes very small (typically, in a resonant bar $\gamma_0 = \omega_0/Q$ with $Q \sim 10^6$) and $\xi_0(t)$ is enhanced. A ruler is instead rigid, with respect to a GW with a frequency ω, if it has $\omega_0 \gg \omega$; then the above equation becomes $\xi_0(t) \simeq -(\Delta L)\cos\omega t$, with

$$\Delta L/L = (2/\pi^2)\,h_0\,(\omega/\omega_0)^2 .$$

Then $\omega_0 \gg \omega$ implies that $\Delta L/L \ll h$. We will see in eq. (8.14) that $\omega_0 = \pi v_s/L$, where v_s is the speed of sound in the material. Thus, a rigid ruler can be obtained taking L small. Observe also that all experiments do not measure the absolute length of the apparatus, but rather the length variation induced by a GW. Therefore a very small rigid ruler is all that is needed.

This means that, if we restrict our attention to a sufficiently small region of space, we can choose coordinates $(t, \mathbf{x})$ so that *even in the presence of GWs*, the metric is flat,

$$ds^2 \simeq -c^2dt^2 + \delta_{ij}dx^i dx^j \,. \tag{1.86}$$

We have seen in Section 1.3.2 how to explicitly construct such a freely falling frame along an entire geodesic, using Fermi normal coordinates.

To linear order in $|x^i|$ there are no corrections to this metric, since in a freely falling frame the derivatives of $g_{\mu\nu}$ vanish at the point P around which we expand. Pursuing the expansion to second order, and expressing the second derivatives of $g_{\mu\nu}$ in terms of the Riemann tensor (using again the fact that the Christoffel symbol vanishes at the point P around which we are expanding), the result is

$$\begin{aligned} ds^2 \simeq & -c^2dt^2\left[1+R_{0i0j}x^ix^j\right] \\ & -2cdt\,dx^i\left(\frac{2}{3}R_{0jik}x^jx^k\right)+dx^idx^j\left[\delta_{ij}-\frac{1}{3}R_{ikjl}x^kx^l\right], \end{aligned} \tag{1.87}$$

where the Riemann tensor is evaluated at the point P. We see that, if L_B is the typical variation scale of the metric, so that $R_{\mu\nu\rho\sigma} = O(1/L_B^2)$, the corrections to the flat metric are $O(r^2/L_B^2)$, where $r^2 = x^ix^i$.

For an Earthbound detector, the situation seems more complicated since it is not in free fall with respect to the Earth's gravity (that is, it has an acceleration $\mathbf{a} = -\mathbf{g}$ with respect to a local inertial frame), and furthermore it rotates relative to local gyroscopes (as illustrated for instance by a Foucault pendulum). The metric in this laboratory frame can be found by explicitly writing the coordinate transformation from the inertial frame to the frame which is accelerating and rotating, and transforming the metric accordingly. The result, up to $O(r^2)$, is[12]

[12] See Ni and Zimmermann (1978), or eq. (4.1) of Thorne (1983).

$$\begin{aligned} ds^2 \simeq & -c^2dt^2\left[1+\frac{2}{c^2}\mathbf{a}\cdot\mathbf{x}+\frac{1}{c^4}(\mathbf{a}\cdot\mathbf{x})^2-\frac{1}{c^2}(\boldsymbol{\Omega}\times\mathbf{x})^2+R_{0i0j}x^ix^j\right] \\ & +2cdt\,dx^i\left[\frac{1}{c}\epsilon_{ijk}\Omega^jx^k-\frac{2}{3}R_{0jik}x^jx^k\right] \\ & +dx^idx^j\left[\delta_{ij}-\frac{1}{3}R_{ikjl}x^kx^l\right], \end{aligned} \tag{1.88}$$

where a^i is the acceleration of the laboratory with respect to a local free falling frame (i.e. a^i is minus the local "acceleration of gravity") and Ω^i is the angular velocity of the laboratory with respect to local gyroscopes. The term $2\mathbf{a}\cdot\mathbf{x}/c^2$ in eq. (1.88) gives the inertial acceleration. The term $(\mathbf{a}\cdot\mathbf{x}/c^2)^2$ is a gravitational redshift. The term $(\boldsymbol{\Omega}\times\mathbf{x}/c)^2$ gives a Lorentz time dilatation due to the angular velocity of the laboratory. The term $(1/c)\epsilon_{ijk}\Omega^jx^k$ is known as the Sagnac effect. Finally, the terms proportional to the Riemann tensor contain both the effect of slowly varying gravitational backgrounds and the effect of GWs.

The frame where the metric has the form (1.88) is called the *proper detector frame*, and is implicitly used by experimentalists in a laboratory on Earth: first of all, at zeroth order in r/L_B, this metric reduces to eq. (1.86), i.e. as long as we focus on regions of space smaller than the typical variation scale of the background, *we live in the flat space-time of Newtonian physics*. This should be contrasted with the TT gauge, where the GW is always present in the background space-time (and no simplification appears when performing an expansion in r/L_B; the metric in the TT gauge is not an expansion in r/L_B). Next, in the proper frame there are corrections linear in r/L_B. Their effects can be described in terms of Newtonian forces (Newtonian gravity, Coriolis forces, centrifugal forces, etc.). In fact, writing the geodesic equation corresponding to the metric (1.88), and neglecting the terms $O(r^2)$ in the metric, we get

$$\frac{d^2x^i}{d\tau^2} = -a^i - 2(\boldsymbol{\Omega}\times\mathbf{v})^i + \frac{f^i}{m} + O(x^i)\,. \tag{1.89}$$

The term $-a^i$ is the acceleration of gravity while $-2(\boldsymbol{\Omega}\times\mathbf{v})^i$ is the Coriolis acceleration, and we also added an external force f^i which represents, for instance, the suspension mechanism that compensates the acceleration of gravity. Including the terms $O(x^ix^j)$ in the metric, we get corresponding terms $O(x^i)$ on the right-hand side of eq. (1.89), such as a term $-[\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})]^i$ which gives the centrifugal acceleration, etc.[13] Details aside, the point is that in this frame the evolution of the coordinate $x^i(\tau)$ of a test mass is described by the equations of motion of Newtonian physics, i.e. in terms of forces.

[13] See eq. (20) of Ni and Zimmermann (1978), for the full expression including relativistic corrections.

At quadratic order there are also the terms proportional to the Riemann tensor, to which both the slowly varying gravitational field of the Earth, and the GWs contribute. The effect of the GWs is therefore entirely in the term $O(r^2)$. In principle, GWs must therefore compete with a number of other effects, like static gravitational forces, Coriolis forces, etc., that practically are many order of magnitudes larger. What can save the situation is the fact that GWs can have high frequencies, compared to the typical variation time-scales of all other effects. In practice, as we will see when we discuss the various Earthbound detectors, GWs with frequencies lower than a few Hz are hopelessly lost into a sea of much higher Newtonian noises. At higher frequencies, however, it is possible to have a frequency window[14] where sufficient isolation from external noises is possible, and an interesting sensitivity to GWs can be obtained.

[14] Sufficiently high frequency is necessary to overcome the slowly varying Newtonian noises, as well as the seismic noise. However, we will see that above a certain frequency, other types of instrumental noise begin to dominate, and therefore only a frequency window is available for GW detection.

To isolate the effect of GWs, we can therefore focus on the response of the detector in this frequency window. The acceleration a^i is compensated by the suspension mechanism, and all other effects produce slowly varying changes. We can then neglect all terms in eq. (1.88), and we only retain the part proportional to the Riemann tensor. This means that we can use eq. (1.87), i.e. the metric in the freely falling frame, and we deduce from it the geodesic equation. It is understood that we restrict to the components of $x^i(\tau)$ in the direction in which the test masses are left free to move by the suspension mechanism, and that we consider only the Fourier components of the motion in a frequency window where the detector is sensitive to GWs. In this frequency window, we will assume that time-varying Newtonian gravitational forces are sufficiently small, so that only GWs contribute to the Riemann tensor.

Rather than using the geodesic equation, it is actually simpler to use the equation for the geodesic deviation, in the form (1.69). We use the fact that $\Gamma^{\mu}_{\nu\rho}$ vanishes at the expansion point P. Furthermore, since the detector moves non-relativistically, $dx^i/d\tau$ can be neglected with respect to $dx^0/d\tau$, and eq. (1.69) gives

$$\frac{d^2\xi^i}{d\tau^2} + \xi^\sigma \partial_\sigma \Gamma^i_{00} \left(\frac{dx^0}{d\tau}\right)^2 = 0 . \tag{1.90}$$

The metric (1.87) depends quadratically on the distance x^i from the point P around which we are expanding, while it depends on t only through the Riemann tensor. In eq. (1.90), $\partial_\sigma\Gamma^i_{00}$ is evaluated at the point P, i.e. at $x^i = 0$. Since $g_{\mu\nu} = \eta_{\mu\nu} + O(x^i x^j)$, a non-zero contribution comes only from the terms in which the two derivatives on the metric present in $\partial_\sigma\Gamma^i_{00}$ are both spatial derivatives, and act on $x^i x^j$. In particular $\partial_0\Gamma^i_{00}$ evaluated at P gives zero, so $\xi^\sigma \partial_\sigma\Gamma^i_{00} = \xi^j\partial_j\Gamma^i_{00}$. Furthermore, using the fact that, at the point P, both $\Gamma^{\mu}_{\nu\rho} = 0$ and $\partial_0\Gamma^i_{0j} = 0$, we have $R^i{}_{0j0} = \partial_j\Gamma^i_{00} - \partial_0\Gamma^i_{0j} = \partial_j\Gamma^i_{00}$, so eq. (1.90) becomes

$$\frac{d^2\xi^i}{d\tau^2} = -R^i{}_{0j0}\,\xi^j \left(\frac{dx^0}{d\tau}\right)^2 . \tag{1.91}$$

If a test mass is initially at rest, it acquires a velocity $dx^i/d\tau = c\,O(h)$ after the passage of the GW. Therefore

$$\begin{aligned} dt^2 &= d\tau^2 \left[1 + \frac{1}{c^2}\frac{dx^i}{d\tau}\frac{dx^i}{d\tau}\right] \\ &= d\tau^2 \left[1 + O(h^2)\right] . \end{aligned} \tag{1.92}$$

On the other hand, in eq. (1.91) the Riemann tensor $R^i{}_{0j0}$ is already $O(h)$, since we are neglecting all effects of the background and we are considering only the GWs. Therefore, if in eq. (1.91) we limit ourselves to linear order in h, we can write $t = \tau$, so $dx^0/d\tau = c$, and eq. (1.91) becomes

$$\ddot{\xi}^i = -c^2 R^i{}_{0j0}\xi^j , \tag{1.93}$$

where the dot denotes the derivative with respect to the *coordinate time* t of the proper detector frame.

Next, we should compute the Riemann tensor $R^i{}_{0j0}$ due to the GWs in the proper detector frame, where eq. (1.93) holds. However, as we discussed below eq. (1.13), in linearized theory the Riemann tensor is *invariant*, rather than just covariant as in full general relativity, and we can compute it in the frame that we prefer. Clearly, the best choice is to compute it in the TT frame, since in this frame GWs have the simplest form. Then, from eq. (1.13) we immediately find

$$R^i{}_{0j0} = R_{i0j0} = -\frac{1}{2c^2}\,\ddot{h}^{\rm TT}_{ij} . \tag{1.94}$$

In conclusion, the equation of the geodesic deviation in the proper detector frame is

$$\boxed{\ddot{\xi}^i = \frac{1}{2}\,\ddot{h}^{\rm TT}_{ij}\,\xi^j .} \tag{1.95}$$

This equation is remarkable in its simplicity, since it states that, *in the proper detector frame*, the effect of GWs on a point particle of mass m can be described in terms of a *Newtonian force*

$$F_i = \frac{m}{2}\,\ddot{h}^{\rm TT}_{ij}\,\xi^j , \tag{1.96}$$

and therefore the response of the detector to GWs can be analyzed in a purely Newtonian language, without any further reference to general relativity.

This means that, in practice, an experimenter in a laboratory on Earth can describe the situation as follows:

- He/she lives in flat space-time, where Newtonian intuition applies.
- There are a number of static or slowly varying Newtonian forces. The acceleration of gravity is compensated by a suspension mechanism, so an Earthbound detector is not free to move in the z direction. Still, it is free to move in the (x, y) plane, or at least in one direction in this plane, depending on the suspension mechanism and detector geometry, and the effect of GWs can show up in the motion of the detector in this free direction. Other slowly varying Newtonian forces perturb the experiment, and one takes great care to minimize their influence. This can be possible at most in a frequency window $[f_{\rm min}, f_{\rm max}]$, where $f_{\rm min}$ is sufficiently large that slowly varying Newtonian noises are under control (as well as the seismic noise, which from a practical point of view is more important at low frequencies), and $f_{\rm max}$ is not too large, otherwise other noises (e.g., as we will see, the shot noise in an interferometer) begin to dominate.
- Even the effect of GWs on test masses is described in terms of a Newtonian force, given by eq. (1.96).

Before exploring the consequences of eq. (1.95), it is worthwhile to add a few more remarks on its meaning and to stress its limit of validity.

- ⋆ We have defined ξ^i to be a *coordinate* separation (rather than a proper distance), since it was introduced as the difference between the coordinates of two nearby geodesics, see eqs. (1.66) and (1.68). With this definition, we have found that eq. (1.95) holds in the proper detector frame. It does not hold in the TT frame, where the geodesic equation is eq. (1.81), and $d^2\xi^i/d\tau^2$ is proportional to $d\xi^i/d\tau$, rather than to ξ^i (consistent with the fact that, in the TT gauge, if $d\xi^i/d\tau$ initially vanishes, the coordinate separation ξ^i does not change).

Nevertheless, in eq. (1.95) the GW in the TT gauge, h_{ij}^{TT}, enters. This comes out because the Riemann tensor is invariant, and we can compute it in any frame, and in particular in the TT gauge, where the GW has the simplest form. Thus, eq. (1.95) combines two nice features: it holds in a frame where our Newtonian intuition is valid, and therefore the description of the detector is more intuitive; at the same time, the Newtonian force due to GWs is expressed in terms of h_{ij} in the TT gauge, where the form of the GW is simpler.

* ⋆ In the proper detector frame, to a first approximation coordinate distances are the same as proper distances, since the metric is flat, up to $O(r^2/L_B^2)$. Since proper distances are an invariant concept, eq. (1.95) also describes the evolution of proper distances in any other frame (as long as $\tau = t$ to lowest order in h, and the spatial velocities are non-relativistic, otherwise we must use the most general form of the equation of the geodesic deviation, eq. (1.71)). In particular, if we substitute ξ^i with the proper distances s^i, eq. (1.95) holds also in the TT gauge, as we indeed found in eq. (1.85).
* ⋆ In deriving the equation for the geodesic deviation we have expanded the Christoffel symbols to first order in ξ, neglecting all higher orders, see eqs. (1.68) and (1.69). This is valid as long as $|\xi^i|$ is much smaller than the typical scale over which the gravitational field changes substantially. For a GW, this length-scale is the reduced wavelength $\lambdabar$. Thus, if a detector has a characteristic linear size L, we can discuss its interaction with GWs using the equation of the geodesic deviation, if and only if

$$\boxed{L \ll \lambdabar\,.} \tag{1.97}$$

As we will see in the chapters on experiments, this condition is satisfied by resonant bar detectors and (in a first approximation) by ground based interferometers. It is not satisfied by proposed space-borne interferometers such as LISA, nor by the Doppler tracking of spacecraft.
In the former case it is possible to study the detector and its interaction with GWs using a simple and intuitive description in terms of Newtonian forces, supplemented by the "GW force" (1.96). In the latter case, a full general relativistic description, usually performed in the TT frame, is necessary.

* ⋆ One should observe that, in general, analysis in the TT gauge can be more subtle, since in this frame our intuition can be misleading. For instance, in certain GW detectors (e.g. microwave cavities) one can have objects which are natural to treat as having rigid walls. However, this description is correct only in the proper detector frame. In fact, when a GW passes, the relative position of a freely falling mass and of an object which is not free to move (like the endpoint of a rigid ruler) changes; since, in the TT frame, the coordinates are marked by freely falling masses, an object which is described as rigid in the proper detector frame, gets a deformation $\Delta L/L = O(h)$ in the TT frame when a GW passes.

We can now use eq. (1.95) to study the effect of a GW on test masses. We use the language of the proper detector frame, which is more intuitive, so we consider a ring of test masses initially at rest in the proper detector frame and fix the origin in the center of the ring. Then ξ^i describe the distance of a test mass, with respect to this origin (coordinate distance or proper distance, since we have seen that in the proper detector frame, on a sufficiently small region of space, they are the same). We then use eq. (1.95) to see how these positions change under the effect of a GW. (Alternatively, as discussed above, if we wish to take the point of view of an observer in the TT frame, then eq. (1.95) describes the evolution of *proper distances*.)

We consider a GW propagating along the z direction, and a ring of test masses located in the (x, y) plane. First of all, for a wave propagating in the z direction, the components of h_{ij}^{TT} with $i = 3$ or $j = 3$ are zero and therefore we see from eq. (1.95) that, if a test particle is initially at $z = 0$, it will remain at $z = 0$, and the displacement will be confined to the (x, y) plane. Therefore, GWs are transverse not only from a mathematical point of view (i.e. they satisfy $\partial_i h_{ij} = 0$), but also in their physical effect: they displace the test masses transversally, with respect to their direction of propagation.[15]

To study the motion of the test particles in the (x, y) plane, we first consider the + polarization. Then, at $z = 0$ (choosing the origin of time so that $h_{ij}^{\text{TT}} = 0$ at $t = 0$),

$$h_{ab}^{\text{TT}} = h_+ \sin\omega t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.98}$$

and, as usual, $a, b = 1, 2$ are the indices in the transverse plane. We write $\xi_a(t) = (x_0 + \delta x(t), y_0 + \delta y(t))$, where (x_0, y_0) are the unperturbed positions and $\delta x(t), \delta y(t)$ are the displacements induced by the GW. Then eq. (1.95) becomes

$$\delta\ddot{x} = -\frac{h_+}{2}(x_0 + \delta x)\omega^2 \sin\omega t\,, \tag{1.99}$$

$$\delta\ddot{y} = +\frac{h_+}{2}(y_0 + \delta y)\omega^2 \sin\omega t\,. \tag{1.100}$$

Since δx is $O(h_+)$, on the right-hand side, to linear order in h, the terms $\delta x, \delta y$ can be neglected with respect to the constant parts x_0, y_0, and the equations are immediately integrated, to give

$$\delta x(t) = \frac{h_+}{2} x_0 \sin\omega t\,, \tag{1.101}$$

$$\delta y(t) = -\frac{h_+}{2} y_0 \sin\omega t\,. \tag{1.102}$$

Similarly, for the cross polarization, we get

$$\delta x(t) = \frac{h_\times}{2} y_0 \sin\omega t\,, \tag{1.103}$$

[15]These two properties are however logically distinct. The condition $\partial_i h_{ij} = 0$ can be imposed *exactly*, as a gauge condition, and (as we will see in detail in Section 2.2) is basically a consequence of the fact that the graviton is described by a massless spin-2 field. The fact that the Newtonian force is transverse is valid only because we took a test mass at rest, i.e. $u^\mu = (c, 0, 0, 0)$. If we consider test masses with non-zero velocity, the geodesic deviation has a longitudinal term, although suppressed by a factor v^2/c^2 with respect to the transverse term. For example, if we have two test masses with initial spatial separation $\boldsymbol{\xi} = (0, 0, \xi_z)$, both moving with velocity v along the x axis, so that $u^\mu = (1/\gamma)(c, v, 0, 0)$, and the GW propagates along the z direction, the equation of geodesic deviation gives

$$\ddot{\xi}_z = -v^2 R_{zxzx}\xi_z\,,$$

where the dot is the derivative with respect to coordinate time (recall that in this case $d\tau = \gamma dt$). For a GW propagating along z, $R_{zxzx} = -(1/2)\partial_z^2 h_{xx} = -(1/2c^2)\ddot{h}_+$, and therefore

$$\ddot{\xi}_z = \frac{v^2}{2c^2}\ddot{h}_+\xi_z\,,$$

so the relative displacement of two test masses in the direction of the GWs changes. This is easily understood by performing a boost with velocity $-v$; in the transformed frame the particles are at rest, but the propagation direction of the GW now has both a component along z and a component along x.

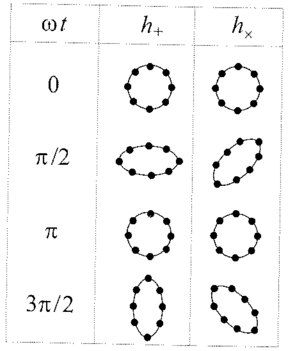

Fig. 1.1 The deformation of a ring of test masses due to the + and × polarization.

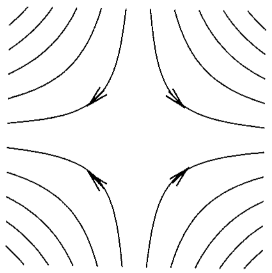

Fig. 1.2 The lines of force corresponding to the + polarization. The arrows show the direction of the force when $\sin\omega t$ is positive. The force reverses when $\sin\omega t$ is negative.

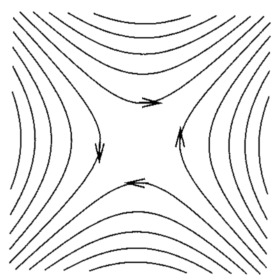

Fig. 1.3 The lines of force corresponding to the × polarization.

$$\delta y(t) = \frac{h_\times}{2} x_0 \sin\omega t\,. \tag{1.104}$$

The resulting deformation of a ring of test masses located in the (x,y) plane is shown in Fig. 1.1. From eq. (1.96) we see that

$$\partial_i F_i = \frac{m}{2} \ddot{h}_{ij}^{\rm TT} \delta_{ij}\,, \tag{1.105}$$

and this vanishes, because $h_{ij}^{\rm TT}$ is traceless. Thus, the Newtonian force (1.96) has vanishing divergence, $\nabla\cdot\mathbf{F} = 0$. We can have a pictorial representation of $\mathbf{F}$ drawing its lines of force in the (x,y) plane (defined so that at each point (x,y) they go in the direction of the force, and their density is proportional to the modulus $|\mathbf{F}|$ of the force). The condition $\nabla\cdot\mathbf{F} = 0$ then implies that there are no sources nor sinks for the field lines, just as for the magnetic field in classical electrodynamics. The lines of force in the (x,y) plane obtained from eq. (1.96), for the h_+ and for the $h_\times$ polarization, are shown in Figs. 1.2 and 1.3. The symmetry axes of these lines of force have a typical quadrupolar pattern, with the shape of a + and of a × sign, respectively, and this is the origin of the denominations "plus" and "cross" polarizations. Observe that Fig. 1.3 is obtained from Fig. 1.2 by performing a rotation of 45 degrees, in agreement with eqs. (1.49) and (1.50).

1.4 The energy of GWs

Our next task is to understand the energy and momentum carried by gravitational waves. The fact that GWs do indeed carry energy and momentum is already clear from the discussion of the interaction of GWs with test masses presented above. We have seen that, in the proper detector frame, an incoming GW sets in motion a ring of test masses initially at rest (and, in fact, the action of the waves on nearby test masses can even be described in terms of a Newtonian force, see eq. (1.96)), so GWs impart kinetic energy to these masses. If, for instance, we connect these masses together with a loose spring with friction, this kinetic energy will be dissipated into heat. Thus, GWs can do work, and conservation of energy requires that the kinetic energy acquired by the test masses must necessarily come from the energy of the GWs. To get the explicit expression of the energy–momentum tensor of GWs we can follow two different routes, one more geometrical and the other more field-theoretical:

(1) Since, according to general relativity, any form of energy contributes to the curvature of space-time, we can ask whether GWs are themselves a source of space-time curvature.

(2) We can treat linearized gravity as any other classical field theory, and apply Noether's theorem, the standard field-theoretical tool that answers this question.

In this section we pursue the former approach, while in Section 2.1 we discuss the latter, and we will see that they both lead to the same answer.

1.4.1 Separation of GWs from the background

To discuss whether GWs curve the background space-time we must broaden our setting. Until now, we have linearized the Einstein equations expanding around the flat metric $\eta_{\mu\nu}$. In this setting the definition of GWs is relatively clear: the background space-time is flat, and the small fluctuations around it have been called "gravitational waves". The term "waves" is justified by the fact that, in a suitable gauge, $h_{\mu\nu}$ indeed satisfies a wave equation. However, to study whether GWs generate a curvature, we cannot define them as perturbation over the *flat* metric $\eta_{\mu\nu}$, otherwise we exclude from the beginning the possibility that GWs curve the background space-time. Rather, we must allow the background space-time to be dynamical, which means that we would like to define GWs as perturbations over some curved, dynamical, background metric $\bar{g}_{\mu\nu}(x)$, and write[16]

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x)\,, \qquad |h_{\mu\nu}| \ll 1\,. \tag{1.106}$$

[16] The condition $|h_{\mu\nu}| \ll 1$ assumes that we are using a coordinate system where the diagonal elements of $\bar{g}_{\mu\nu}$ are $O(1)$, on the region of space-time in which we are interested.

However, a problem arises immediately. How do we decide which part of $g_{\mu\nu}$ is the background and which is the fluctuations? In principle, in eq. (1.106) we can move x-dependent terms from $h_{\mu\nu}$ to $\bar{g}_{\mu\nu}$ or viceversa. The problem did not arise in linearized theory, where the background metric was chosen once and for all to be the constant flat-space metric $\eta_{\mu\nu}$.

As we will see in this section, this problem is not just an abstract issue of principle. On the contrary, the answer to this question allows us to understand properties of GWs such as their energy–momentum tensor, and to get rid of ambiguities concerning whether GWs can be "gauged away" or not.

In the most general setting, there is no unambiguous way to perform a separation of the type (1.106). The total metric $g_{\mu\nu}(x)$ can receive contributions which change, in space and in time, on all possible scales due, for example, to the time-varying Newtonian gravitational fields of nearby masses in movement. The situation is quite similar to that of waves in the sea. In principle, there is no unambiguous way to state which part of the vertical movement of the surface of the water belongs to a given wave, and which part belongs to a "background" originated by the incoherent superposition of perturbations of varied origin. Nevertheless, there are obviously situations where a description of the perturbation of the sea surface in terms of waves is useful, at least at the level of an effective description.

In particular, a natural splitting between the space-time background and gravitational waves arises when there is a clear separation of scales. For example, a natural distinction occurs if, in some coordinate system, we can write the metric as in eq. (1.106), where $\bar{g}_{\mu\nu}$ has a typical scale

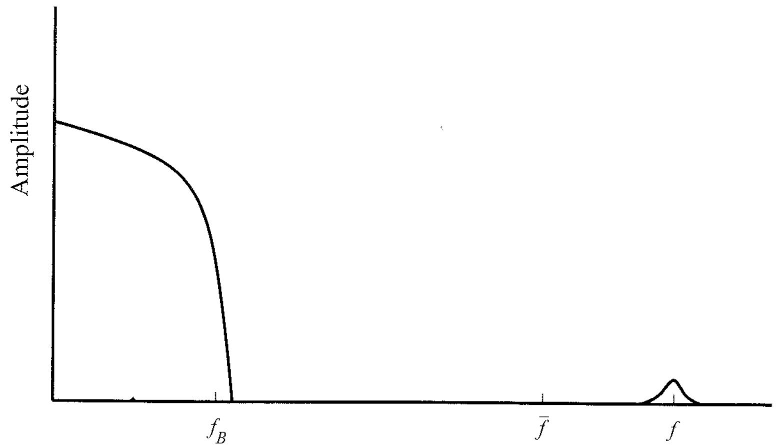

Fig. 1.4 A situation that allows us to separate the metric into a low-frequency background and a small high-frequency perturbation. The background is defined as the part with frequencies $f \ll \bar{f}$ and the GW as the part with $f \gg \bar{f}$. This definition is largely independent of the precise value of $\bar{f}$.

of spatial variation L_B, on top of which small amplitude perturbations are superimposed, characterized by a wavelength λ such that

$$\lambdabar \ll L_B\,, \tag{1.107}$$

where $\lambdabar = \lambda/(2\pi)$ is the reduced wavelength.[17] In this case $h_{\mu\nu}$ has the physical meaning of small ripples on a smooth background. Alternatively, a natural distinction can be made in frequency space, if $\bar{g}_{\mu\nu}$ has frequencies up to a maximum value f_B, while $h_{\mu\nu}$ is peaked around a frequency f such that

$$f \gg f_B\,. \tag{1.108}$$

In this case $h_{\mu\nu}$ is a high-frequency perturbation of a static or slowly varying background. The situation (1.108) is illustrated in Fig. 1.4. We will see below that in this case $h_{\mu\nu}$, in a suitable gauge, obeys a wave equation, and as a consequence its characteristic wavelength and frequency, λ and f, are related by $\lambda = c/f$. However, the scales L_B and f_B that characterize the background are a priori unrelated, so the conditions (1.107) and (1.108) are independent, and it suffices that one of them be satisfied.

We can now ask two questions:

- How this high-frequency (or short wavelength) perturbation propagates in the background space-time with metric $\bar{g}_{\mu\nu}$. The answer to this question will justify the fact that the perturbation $h_{\mu\nu}$ is called a gravitational "wave".
- How this perturbation affects the background metric itself. The answer to this question will allow us to assign an energy–momentum tensor to GWs.

[17] For a function $f(x)$ oscillating as e^{ikx} with $k = 2\pi/\lambda = 1/\lambdabar$, the typical length-scale is $\lambdabar$ rather than λ, in the sense that $|df/dx| = (1/\lambdabar)|f|$.

In the next subsections we will address these two questions. First we remark that, traditionally, the separation of the metric into a smooth background plus fluctuations is discussed using the condition (1.107), and the method is called the *short-wave expansion*. It should be observed, however, that from the point of view of GW detectors, the condition (1.108) is fulfilled instead. Consider for instance a GW with a frequency $f \sim 10^2 - 10^3$ Hz, corresponding to a reduced wavelength $\lambdabar \simeq$ 500–50 km, which are typical GWs that can be searched by ground-based detectors. The Earth's gravitational potential is *not* spatially smooth over a scale of tens of kms, compared to the GW perturbation. On the contrary, fluctuations in the metric due to local density variations, mountains, etc. are many orders of magnitude bigger than the expected GWs: the Newtonian gravitational potential at the surface of the Earth is in fact $|h_{00}| = 2GM_{\oplus}/(R_{\oplus}c^2) \sim 10^{-9}$, while, as we will see, GWs arriving on Earth are expected to have at most $h \sim 10^{-21}$ so even a spatial variation of just one part in 10^{12} due to local inhomogeneities, is large compared to the expected GWs.

On the other hand, these Newtonian gravitational fields are essentially static, and it is much more difficult to find important *temporal* variations at large frequency scales, e.g. at $f \sim 1$ kHz, since it requires relatively large masses moving at these frequencies. A distinction between background and gravitational waves based on the condition $f \gg f_B$ becomes therefore possible.

Indeed, ground-based GW detectors have a size which is much smaller than the wavelength of the GWs that they are searching. A GW with frequency $f \sim 10^2$–10^3 Hz has a reduced wavelength $\lambdabar \sim$ 500–50 km, which is much bigger than the size of the detector. Therefore, GW detectors do not monitor *spatial* variations of the gravitational field on length-scales $L \gg \lambdabar$. Rather, their output is a time series, which is analyzed in Fourier space looking for *temporal* variations in their output induced by a passing GW. As we will see in Chapter 9, after suitable isolation, the residual noise due to seismic motion and Newtonian gravitational fields is important only at lower frequencies, say below $O(10)$ Hz. Therefore, we are actually searching for fast temporal variations in the detector output due to GWs, over a background which is slowly varying in time.

1.4.2 How GWs curve the background

We therefore consider the situation in which, in some reference frame, we can separate the metric into a background plus fluctuations, as in eq. (1.106), and this separation is based on the fact that there is a clear distinction of scales either in space, in which case eq. (1.107) applies, or in time, in which case eq. (1.108) applies.

As discussed above, our aim is to understand how the perturbation $h_{\mu\nu}$ propagates, and how it affects the background space-time. To address these questions, we begin by expanding the Einstein equations around the background metric $\bar{g}_{\mu\nu}$. In the expansion we have two small parameters: one is the typical amplitude $h \equiv O(|h_{\mu\nu}|)$, and the second is either $\bar{\lambda}/L_B$ or f_B/f, depending on whether eq. (1.107) or eq. (1.108) applies. The situation in which $\bar{\lambda}/L_B \ll 1$ and the situation in which $f_B/f \ll 1$ can be treated in parallel, with the appropriate change of notation, and we will refer generically to both cases as the short-wave expansion.

As a first step, we expand to quadratic order in $h_{\mu\nu}$. It is convenient to cast the Einstein equations in the form

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \tag{1.109}$$

where $T_{\mu\nu}$ is the energy–momentum tensor of matter and T its trace, and then we expand the Ricci tensor to $O(h^2)$,

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \ldots, \tag{1.110}$$

where $\bar{R}_{\mu\nu}$ is constructed with $\bar{g}_{\mu\nu}$ only, $R^{(1)}_{\mu\nu}$ is linear in $h_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$ is quadratic in $h_{\mu\nu}$. The crucial observation now is the following. The quantity $\bar{R}_{\mu\nu}$ is constructed from $\bar{g}_{\mu\nu}$ and therefore contains only low-frequency modes.[18] $R^{(1)}_{\mu\nu}$ by definition is linear in $h_{\mu\nu}$ and therefore contains only high-frequency modes. $R^{(2)}_{\mu\nu}$ is quadratic in $h_{\mu\nu}$ and therefore contains *both* high and low frequencies: for instance, in a quadratic term $\sim h_{\mu\nu}h_{\rho\sigma}$ a mode with a high wave-vector $\mathbf{k}_1$ from $h_{\mu\nu}$ can combine with a mode with a high wave-vector $\mathbf{k}_2 \simeq -\mathbf{k}_1$ from $h_{\rho\sigma}$ to give a low wave-vector mode. Therefore the Einstein equations can be split into two separate equations for the low- and high-frequency parts,

[18] To be more precise, we should take into account that $\bar{R}_{\mu\nu}$ is non-linear in $\bar{g}_{\mu\nu}$. If $\bar{k}$ separates the low frequency from the high frequency modes, then $\bar{g}_{\mu\nu}$ has only modes up to a typical wave-vector $k_B \simeq 2\pi/L_B$ with $k_B \ll \bar{k}$. The Christoffel symbols of the background are quadratic in the background metric and therefore have modes up to $2k_B$. Terms quadratic in the Christoffel symbols, such as those which appear in the definition of the Ricci tensor, therefore have modes up to $\sim 4k_B$. In any case, if the separation of scales between the background and the GW is clear-cut, we still have $4k_B \ll \bar{k}$. In this sense $\bar{R}_{\mu\nu}$ contains only low-frequency modes.

$$\bar{R}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\rm Low} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\rm Low}, \tag{1.111}$$

and

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\rm High} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\rm High}. \tag{1.112}$$

The superscript "Low" denotes the projection on the low momenta (i.e. long wavelengths) or on the low frequencies, depending on whether eq. (1.107) or eq. (1.108) applies, and similarly for the superscript "High".

The explicit expression for $R^{(1)}_{\mu\nu}$ is computed in Problem 1.1, and is

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left(\bar{D}^\alpha\bar{D}_\mu h_{\nu\alpha} + \bar{D}^\alpha\bar{D}_\nu h_{\mu\alpha} - \bar{D}^\alpha\bar{D}_\alpha h_{\mu\nu} - \bar{D}_\nu\bar{D}_\mu h\right), \tag{1.113}$$

where $\bar{D}_\mu$ is the covariant derivative with respect to the background metric. At quadratic order one finds, after some long algebra,

$$\begin{aligned} R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[&\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma}) \\ &+ h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma}) \\ &+ (\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\Big]. \end{aligned} \tag{1.114}$$

In the next subsections we will closely examine eqs. (1.111) and (1.112). We will see that, from eq. (1.111), we can understand what the energy–momentum tensor of GWs is, while eq. (1.112) is a wave equation that describes the propagation of $h_{\mu\nu}$ on the background space-time.

First we discuss, from the vantage point of the expansion over a generic curved background, why the expansion over flat space-time presented in Section 1.1 cannot be promoted to a systematic expansion. Consider first the situation in which there is no external matter, $T_{\mu\nu} = 0$. In eqs. (1.111) and (1.112) we have equated terms of different orders in the h-expansion. The reason is that we have a second small expansion parameter, which is $\bar{\lambda}/L_B$ (or f_B/f; as usual, the two cases can be treated in parallel with just a change of notation, and for definiteness we use $\bar{\lambda}/L_B$), which can compensate for the smallness in h. The relative strength of these parameters is therefore fixed by the Einstein equations themselves. We use the notation $h = O(|h_{\mu\nu}|)$, while we take $\bar{g}_{\mu\nu} = O(1)$ (in a limited region of space, we can always set $\bar{g}_{\mu\nu} = O(1)$ with a suitable rescaling of the coordinates).

Since we set $T_{\mu\nu} = 0$, we see from eq. (1.111) that $\bar{R}_{\mu\nu}$ is determined only by $[R^{(2)}_{\mu\nu}]^{\rm Low}$. From the explicit expression (1.114) we see that $R^{(2)}_{\mu\nu}$ is a sum of terms of order $(\partial h)^2$ and of terms of order $h\partial^2 h$. Let us anticipate that, when we compute the projection onto the low modes, these two terms give contributions which are of the same order of magnitude, and one finds that $[R^{(2)}_{\mu\nu}]^{\rm Low}$ is of order $(\partial h)^2$ (compare with eqs. (1.125) and (1.133) below). Then, in order of magnitude, eq. (1.111) in the absence of matter fields reads

$$\bar{R}_{\mu\nu} \sim (\partial h)^2, \tag{1.115}$$

and expresses the fact that the derivatives of the perturbation $h_{\mu\nu}$ affect the curvature of the background metric $\bar{g}_{\mu\nu}$. The scale of variation of $\bar{g}_{\mu\nu}$ is L_B, while that of h is $\bar{\lambda}$; therefore, in order of magnitude,

$$\partial\bar{g}_{\mu\nu} \sim \frac{1}{L_B}, \tag{1.116}$$

(recall that we took $\bar{g}_{\mu\nu} = O(1)$), while

$$\partial h \sim \frac{h}{\bar{\lambda}}. \tag{1.117}$$

Since the background curvature $\bar{R}_{\mu\nu}$ is constructed from the second derivatives of the background metric, eq. (1.116) implies that

$$\bar{R}_{\mu\nu} \sim \partial^2\bar{g}_{\mu\nu} \sim \frac{1}{L_B^2}. \tag{1.118}$$

while eq. (1.117) gives $(\partial h)^2 \sim (h/\bar{\lambda})^2$. Therefore eq. (1.115) gives the relation

$$\frac{1}{L_B^2} \sim \left(\frac{h}{\bar{\lambda}}\right)^2, \tag{1.119}$$

that is,

$$h \sim \frac{\lambdabar}{L_B}\,, \qquad \text{(curvature determined by GWs)}\,. \tag{1.120}$$

Consider now the opposite limit where $T_{\mu\nu}$ is non-vanishing, and the contribution of GWs to the background curvature is negligible compared to the contribution of matter sources. In this case the total background curvature will be much bigger than the contribution of GWs, $1/L_B^2 \sim h^2/\lambdabar^2 +$ (matter contribution) $\gg h^2/\lambdabar^2$, i.e.

$$h \ll \frac{\lambdabar}{L_B}\,, \qquad \text{(curvature determined by matter)}\,. \tag{1.121}$$

At this point we can understand why the linearized approximation of Section 1.1 cannot be extended beyond linear order. If we force the background metric to be $\eta_{\mu\nu}$, we are actually forcing $1/L_B$ to be strictly equal to zero, and therefore any arbitrarily small, but finite, value of h necessarily violates the condition $h \lesssim \lambdabar/L_B$, and the expansion in powers of h has no domain of validity. This means that the linearized expansion of the classical theory cannot be promoted to a systematic expansion, and if we want to compute higher-order corrections we cannot insist on a flat background metric.

We can also understand from eqs. (1.120) and (1.121) that the notion of GW is well defined only for small amplitudes, $h \ll 1$. If h becomes of order one, eqs. (1.120) and (1.121) tell us that $\lambdabar/L_B$ also becomes at least of order one. Since the separation between $\lambdabar$ and L_B is at the basis of the definition of GWs, when h becomes of order one the distinction between GWs and background vanishes. In a general context, there is nothing like "a GW of arbitrary amplitude".[19]

We consider now eq. (1.111). When there is a clear-cut separation between the length-scale $\lambdabar$ of the GWs and the length-scale L_B of the background, there is a simple way to perform the projection on the long-wavelength modes: we introduce a scale $\bar{l}$ such that $\lambdabar \ll \bar{l} \ll L_B$, and we average over a spatial volume with side $\bar{l}$. In this way, modes with a wavelength of order L_B remain unaffected, because they are basically constant over the volume used for the averaging, while modes with a reduced wavelength of order $\lambdabar$ are oscillating very fast and average to zero. Similarly, if $h_{\mu\nu}$ is a high-frequency perturbation of a quasi-static background, we can introduce a time-scale $\bar{t}$ which is much larger than the period $1/f$ of the GW and much smaller than the typical time-scale $1/f_B$ of the background, and average over this time $\bar{t}$, i.e. over several periods of the GW. We can therefore write eq. (1.111) as

$$\bar{R}_{\mu\nu} = -\langle R^{(2)}_{\mu\nu}\rangle + \frac{8\pi G}{c^4}\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\rangle\,, \tag{1.122}$$

where $\langle ...\rangle$ denotes a spatial average over many reduced wavelengths $\lambdabar$, if eq. (1.107) applies, and a temporal average over several periods $1/f$ of the GW, if rather eq. (1.108) applies.

[19] In special cases one can find *exact* wave-like solutions of the full non-linear Einstein equations, see, e.g. Misner, Thorne and Wheeler (1973), Section 35.9, and then there is no need to perform a separation between the background and the waves. However, it would be hopeless to look for exact solutions for the gravitational waves emitted by realistic astrophysical sources.

In the context of general relativity and gravitational-wave physics, the usefulness of introducing some averaging procedure was understood in the 1960s. To put it into a broader theoretical framework, it is useful to realize that what we have done is basically a special case of a general technique, which is known as a renormalization group transformation, and which is nowadays one of the most important tools both in quantum field theory and in statistical physics. The basic idea is to start from the fundamental equations of a theory and to "integrate out" the fluctuations that take place on a length-scale smaller than l, in order to obtain an effective theory that describes the physics at the length-scale l. These renormalization group transformations can be performed in coordinate space, which is the language that we used above; in momentum space, integrating out the high-momentum modes, in order to get the corresponding low-energy effective action; or in frequency space, in order to eliminate the fast temporal variations and to obtain the effective dynamics of the slowly varying degrees of freedom.

We now define an effective energy–momentum tensor of matter, that we denote by $\bar{T}^{\mu\nu}$, from

$$\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\rangle = \bar{T}^{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}\,, \tag{1.123}$$

where $\bar{T} = \bar{g}_{\mu\nu}\bar{T}^{\mu\nu}$ is the trace. By definition, $\bar{T}^{\mu\nu}$ is a purely low-frequency (or low-momentum) quantity, and is a smoothed form of the matter energy–momentum tensor $T_{\mu\nu}$; for instance, when the separation has been done on the basis of the condition $\lambdabar \ll L_B$, we can visualize it as a "macroscopic" (with respect to the scale $\lambdabar$) version of the energy–momentum tensor, while $T_{\mu\nu}$ is the fundamental ("microscopic") quantity.[20]

We also define the quantity $t_{\mu\nu}$ as

$$\boxed{t_{\mu\nu} = -\frac{c^4}{8\pi G}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\rangle\,,} \tag{1.125}$$

where

$$R^{(2)} = \bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}\,, \tag{1.126}$$

and we define its trace as[21]

$$\begin{aligned} t &= \bar{g}^{\mu\nu}t_{\mu\nu} \\ &= +\frac{c^4}{8\pi G}\langle R^{(2)}\rangle\,. \end{aligned} \tag{1.127}$$

To go from the first to the second line, in eq. (1.127), we used the fact that $\bar{g}^{\mu\nu}\langle R^{(2)}_{\mu\nu}\rangle = \langle \bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}\rangle$ since $\bar{g}^{\mu\nu}$ by definition is a purely low-frequency quantity, as well as the obvious identity $\bar{g}^{\mu\nu}\bar{g}_{\mu\nu} = 4$. Inserting eq. (1.127) into eq. (1.125) (and using again the fact that $\bar{g}_{\mu\nu}$ is constant under the averaging procedure, so $\langle \bar{g}_{\mu\nu}R^{(2)}\rangle = \bar{g}_{\mu\nu}\langle R^{(2)}\rangle$) we see that

$$-\langle R^{(2)}_{\mu\nu}\rangle = \frac{8\pi G}{c^4}\left(t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}t\right)\,. \tag{1.128}$$

[20] In a typical situation, the fundamental energy–momentum tensor $T^{\mu\nu}$ generated by a macroscopic matter distribution will already be quite smooth, so it will be approximately constant on the scale used for averaging. In this case

$$\begin{aligned} \langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\rangle &\simeq T_{\mu\nu} - \frac{1}{2}\langle g_{\mu\nu}\rangle T \\ &= T_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}T\,, \end{aligned} \tag{1.124}$$

and therefore $\bar{T}_{\mu\nu} \simeq T_{\mu\nu}$. However, the definition (1.123) copes with the most general situation.

[21] Observe that, since $R^{(2)}_{\mu\nu}$ is already quadratic in $h_{\mu\nu}$, we have

$$\begin{aligned} g^{\mu\nu}R^{(2)}_{\mu\nu} &= (\bar{g}^{\mu\nu} + h^{\mu\nu})R^{(2)}_{\mu\nu} \\ &= \bar{g}^{\mu\nu}R^{(2)}_{\mu\nu} + O(h^3) \end{aligned}$$

and, since we are working up to $O(h^2)$, it is irrelevant whether we define the traces of $R^{(2)}_{\mu\nu}$ and of $t_{\mu\nu}$ contracting with $g^{\mu\nu}$ or with $\bar{g}^{\mu\nu}$.

So, we can rewrite eq. (1.122) as

$$\bar{R}_{\mu\nu} = \frac{8\pi G}{c^4}\left(t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}t\right) + \frac{8\pi G}{c^4}\left(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}\right), \qquad (1.129)$$

or, in an equivalent way,

$$\boxed{\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \frac{8\pi G}{c^4}\left(\bar{T}_{\mu\nu} + t_{\mu\nu}\right).} \qquad (1.130)$$

This can be appropriately called the "coarse-grained" form of the Einstein equations. These equations determine the dynamics of $\bar{g}_{\mu\nu}$, which is the long-wavelength (or low-frequency) part of the metric, in terms of the long-wavelength (or, respectively, low-frequency) part of the matter energy–momentum tensor, $\bar{T}_{\mu\nu}$, and of a tensor $t_{\mu\nu}$ which does not depend on the external matter but only on the gravitational field itself, and is quadratic in $h_{\mu\nu}$.[22]

[22]Recall however that we limited ourselves to an expansion of $R_{\mu\nu}$ up to quadratic order in $h_{\mu\nu}$, so all higher-order non-linearities in $h_{\mu\nu}$ have been neglected. We will come back to these non-linear terms in Section 2.2.3 and especially in Chapter 5.

We can then summarize the results of this analysis as follows.

- At a "microscopic" level, there is no fundamental distinction between a background metric and fluctuations over it. The gravitational field is described by all its modes, and its dynamics is fully accounted for by the Einstein equations (1.3).
- If some fluctuations $h_{\mu\nu}$ are clearly distinguishable from the background because their typical length-scale $\lambda\!\!\!^-$ is much smaller than the typical length-scale L_B that characterizes the spatial variations of the background, it becomes useful to introduce a "macroscopic" level of description, i.e. an approximate description which is valid at a length-scale $\bar{l}$, such that $\lambda\!\!\!^- \ll \bar{l}$ (but still $\bar{l} \ll L_B$). This is obtained "integrating out" the short-wavelength degrees of freedom, which, in practice, can be obtained by performing a spatial average of the Einstein equations over a box of size $\bar{l}$, i.e. over several wavelengths $\lambda\!\!\!^-$.
  If the separation between fluctuations and background is based on the condition $f_B \ll f$ instead, we integrate out the fast-varying degrees of freedom, performing a temporal average over several periods $1/f$, and we are left with an effective dynamics for the slowly varying degrees of freedom.
- The result of this procedure (which, basically, is a renormalization group transformation) is summarized by eq. (1.130), together with the definitions of $t_{\mu\nu}$ and $\bar{T}_{\mu\nu}$ given in eqs. (1.123) and (1.125). The left-hand side of eq. (1.130) is the Einstein tensor for the slowly varying metric $\bar{g}_{\mu\nu}$. On the right-hand side we find, not surprisingly, a smoothed version of the matter energy–momentum tensor, $\bar{T}_{\mu\nu}$.
  The most interesting aspect of eq. (1.130), however, is that it shows that the effect of GWs on the background curvature is formally identical to that of matter with energy–momentum tensor $t^{\mu\nu}$. We are therefore able to assign an energy–momentum tensor to GWs.

- It is useful to observe that $t_{\mu\nu}$ comes out automatically in an averaged form. This averaging procedure is not something that is imposed by hand afterwards. It comes out this way because, to derive the effect of GWs on the background, one is passing from a fundamental, "microscopic", description, to a coarse-grained, "macroscopic" description.

1.4.3 The energy–momentum tensor of GWs

We now compute explicitly $t_{\mu\nu}$, using eq. (1.125) with $R^{(2)}_{\mu\nu}$ given in eq. (1.114). We are interested in the energy and momentum carried by the GWs at large distances from the source (e.g. at the position of the detector), where we can approximate the background space-time as flat. In this case we can simply replace $\bar{D}^\mu \to \partial^\mu$ in eq. (1.114), so we get

$$\begin{aligned} R^{(2)}_{\mu\nu} = \frac{1}{2}\Big[& \frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} - h^{\alpha\beta}\partial_\nu\partial_\beta h_{\alpha\mu} - h^{\alpha\beta}\partial_\mu\partial_\beta h_{\alpha\nu} \\ & + h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + \partial^\beta h^\alpha_\nu \partial_\beta h_{\alpha\mu} - \partial^\beta h^\alpha_\nu \partial_\alpha h_{\beta\mu} - \partial_\beta h^{\alpha\beta}\partial_\nu h_{\alpha\mu} \\ & + \partial_\beta h^{\alpha\beta}\partial_\alpha h_{\mu\nu} - \partial_\beta h^{\alpha\beta}\partial_\mu h_{\alpha\nu} - \frac{1}{2}\partial^\alpha h \partial_\alpha h_{\mu\nu} + \frac{1}{2}\partial^\alpha h \partial_\nu h_{\alpha\mu} \\ & + \frac{1}{2}\partial^\alpha h \partial_\mu h_{\alpha\nu}\Big]. \end{aligned} \qquad (1.131)$$

As we saw in Section 1.2, the 4×4 symmetric matrix $h_{\mu\nu}$ has 10 degrees of freedom, out of which eight are gauge modes and two are physical modes. Correspondingly, in $t_{\mu\nu}$ one can have in principle contributions both from the physical modes and from the gauge modes. The left-hand side of eq. (1.130), i.e. the Einstein tensor of the background metric $\bar{g}_{\mu\nu}$, is of course a quantity that depends on the coordinate system, since it is a tensor. Thus, in principle there is nothing wrong if, on the right-hand side, we have both physical contributions and coordinate-dependent contributions, i.e contributions from gauge modes. The issue is how to distinguish the contribution to $t_{\mu\nu}$ due to the physical modes from the contribution of the gauge modes. The former will give the energy–momentum tensor of the GWs, and describe physical effects that cannot be gauged away, while the latter will be associated with ripples in space-times that are due to the choice of the coordinate system, and that can be made to vanish with an appropriate gauge choice.

The most straightforward way to get the contribution of the physical modes is to make use of the Lorentz gauge condition (1.18). This immediately eliminates four spurious degrees of freedom, leaving us with the two physical degrees of freedom contained in $h^{\rm TT}_{ij}$ and the four gauge modes ξ_μ which satisfy $\Box\xi_\mu = 0$, as discussed in Section 1.2. We also choose the ξ_μ so that $h = 0$ (so that only three independent gauge modes remain). Then $\bar{h}_{\mu\nu} = h_{\mu\nu}$ and the Lorentz gauge condition becomes $\partial^\mu h_{\mu\nu} = 0$.

We can now drastically simplify $R^{(2)}_{\mu\nu}$ in eq. (1.131) observing that, inside the spatial or temporal average, the space-time derivative ∂_μ can be integrated by parts, neglecting the boundary term.[23] Performing in-

[23]On generic functions, an integration by parts of ∂_t is possible only if we have performed an integral over time, while an integration by parts of ∂_i requires a spatial integral. Recall however that in the Lorentz gauge, outside the source, the equation of motion is a simple wave equation $\Box h_{\mu\nu} = 0$. So, for a solution propagating for instance in the z direction, all quantities are functions of the combination $x^0 - z$, where $x^0 = ct$. In expressions such as $\int dz\, g(x^0 - z)\partial_0 f(x^0 - z)$ we can replace $\partial_0 f$ with $-\partial_z f$, integrate ∂_z by parts and then replace again $\partial_z g$ with $-\partial_0 g$. Therefore, for solutions of the wave equation, a spatial average allows us to integrate by parts not only the spatial derivative but even the time derivative, and similarly for a time average.
Observe also that, in the integration by parts, the boundary terms vanish only when the size of the box used for the integration is infinitely larger than $\lambda\!\!\!^-$. A more precise statement is that the non-zero terms are of higher order in $\lambda\!\!\!^-/L_B$. However, we will only need the result to leading order.

tegrations by parts and making use of the gauge conditions $\partial^\mu h_{\mu\nu} = 0$ and $h = 0$ and of the equation of motion $\Box h_{\alpha\beta} = 0$, it is immediate to see that all terms in eq. (1.131) collapse to zero except the first two, which are related to each other by an integration by parts, giving

$$\langle R^{(2)}_{\mu\nu}\rangle = -\frac{1}{4}\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle\,, \tag{1.132}$$

while $\langle R^{(2)}\rangle$ vanishes upon integration by parts and using the equation of motion $\Box h_{\alpha\beta} = 0$. Recalling the factor $-c^4/(8\pi G)$ from eq. (1.125), we finally find

$$\boxed{t_{\mu\nu} = \frac{c^4}{32\pi G}\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle\,.} \tag{1.133}$$

We can now verify that the residual gauge modes ξ_μ do not contribute to this expression. In fact, under the gauge transformation (1.8), the variation of $t_{\mu\nu}$ is

$$\begin{aligned}\delta t_{\mu\nu} &= \frac{c^4}{32\pi G}\,[\langle \partial_\mu h_{\alpha\beta}\partial_\nu(\delta h^{\alpha\beta})\rangle + (\mu\leftrightarrow\nu)]\\ &= \frac{c^4}{32\pi G}\,[\langle \partial_\mu h_{\alpha\beta}\partial_\nu(\partial^\alpha\xi^\beta + \partial^\beta\xi^\alpha)\rangle + (\mu\leftrightarrow\nu)]\\ &= \frac{c^4}{16\pi G}\,[\langle \partial_\mu h_{\alpha\beta}\partial_\nu\partial^\alpha\xi^\beta\rangle + (\mu\leftrightarrow\nu)]\,,\end{aligned} \tag{1.134}$$

and this vanishes since, inside $\langle\ldots\rangle$, we can integrate by parts ∂^α, and then we can use the Lorentz condition $\partial^\alpha h_{\alpha\beta} = 0$.[24] Therefore $t_{\mu\nu}$ depends only on the physical modes $h^{\rm TT}_{ij}$, and we can simply replace $h_{\mu\nu}$ in eq. (1.133) with the metric in the TT gauge. In particular, the gauge-invariant energy density is

$$t^{00} = \frac{c^2}{32\pi G}\langle \dot h^{\rm TT}_{ij}\dot h^{\rm TT}_{ij}\rangle\,, \tag{1.135}$$

(where the dot denotes $\partial_t = (1/c)\partial_0$) or, in terms of the amplitudes h_+ and $h_\times$,

$$\boxed{t^{00} = \frac{c^2}{16\pi G}\langle \dot h_+^2 + \dot h_\times^2\rangle\,.} \tag{1.136}$$

For a plane wave traveling along the z direction, $h^{\rm TT}_{ij}$ is a function of $t - z/c$, and therefore $t^{01} = t^{02} = 0$ while $\partial_z h^{\rm TT}_{ij} = -\partial_0 h^{\rm TT}_{ij} = +\partial^0 h^{\rm TT}_{ij}$ and therefore

$$t^{03} = t^{00}\,. \tag{1.137}$$

An alternative way of extracting the gauge-invariant part from $t_{\mu\nu}$ is to start from the full expression (1.131) without performing any prior gauge fixing, and consider its variation under a linearized gauge transformation (1.8), where now ξ_μ are generic, rather than being constrained to satisfy $\Box\xi_\mu = 0$. Then, with straightforward algebra one finds that

$$t_{\mu\nu} \to t_{\mu\nu} + \partial_\rho U^\rho_{\mu\nu}\,, \tag{1.138}$$

[24] The fact that this result holds only to leading order in $\lambda\!\!\!^{-}/L_B$ (see Note 23) is not surprising, since it is only in this limit that the notion of GW is well defined. If $\lambda\!\!\!^{-}/L_B$ approaches one, the distinction between the background and the perturbation fades away, and correspondingly one can no longer assign a gauge-invariant energy–momentum tensor to the perturbations.

with $U^\rho_{\mu\nu}$ some tensor. The additional term is a total divergence, and we would like to trow it away inside the average, as we have done above. Here however we must be careful because, since we have not fixed the Lorentz gauge, the metric now does not satisfy a simple wave equation such as $\Box h_{\mu\nu} = 0$. Thus, the argument discussed in Note 23, which allowed us to integrate by parts ∂_ρ inside a temporal average, or inside a spatial average, no longer goes through. However, we can integrate by parts ∂_μ inside a *space-time* average, that we denote as $\langle\langle\ldots\rangle\rangle$. Then $\langle\langle\partial_\rho U^\rho_{\mu\nu}\rangle\rangle$ vanishes and $\langle\langle t_{\mu\nu}\rangle\rangle$ is gauge invariant (again to leading order in $\lambda\!\!\!^{-}/L_B$). Thus, an equivalent way to single out the gauge-invariant part of $t_{\mu\nu}$ is to average it over space-time, and the result gives again eq. (1.133).[25]

Finally, observe that in eq. (1.130) the left-hand side is covariantly conserved with respect to $\bar D^\mu$, i.e. $\bar D^\mu(\bar R_{\mu\nu} - \frac{1}{2}\bar g_{\mu\nu}\bar R) = 0$, because of the Bianchi identity. Therefore, we have

$$\boxed{\bar D^\mu(\bar T_{\mu\nu} + t_{\mu\nu}) = 0\,.} \tag{1.140}$$

The fact that the covariantly conserved quantity is the sum of $\bar T_{\mu\nu}$ and $t_{\mu\nu}$, rather than each one separately, reflects the fact that there is in general exchange of energy and momentum between the matter sources and GWs. At large distances from the source the metric approaches the flat-space metric, so $\bar D^\mu$ approaches ∂^μ, while outside the source $\bar T_{\mu\nu} = 0$. Then, far from the sources, eq. (1.140) reduces to

$$\partial^\mu t_{\mu\nu} = 0\,. \tag{1.141}$$

The energy flux

Having obtained the energy-momentum tensor carried by the GWs, it is now straightforward to compute the corresponding energy flux, i.e. the energy of GWs flowing per unit time through a unit surface at a large distance from the source. We start from the conservation of the energy–momentum tensor, $\partial_\mu t^{\mu\nu} = 0$, which implies that

$$\int_V d^3x\,(\partial_0 t^{00} + \partial_i t^{i0}) = 0\,, \tag{1.142}$$

where V is a spatial volume in the far region, bounded by a surface S. The GW energy inside the volume V is

$$E_V = \int_V d^3x\, t^{00}\,, \tag{1.143}$$

so eq. (1.142) can be written as

$$\begin{aligned}\frac{1}{c}\frac{dE_V}{dt} &= -\int_V d^3x\,\partial_i t^{0i}\\ &= -\int_S dA\, n_i t^{0i}\,,\end{aligned} \tag{1.144}$$

[25] To make the proof simpler we worked at large distance from the source, where the background metric can be taken as flat. The argument can however be repeated in a generic background $\bar g_{\mu\nu}$, although it becomes technically more involved. In this case one makes use of the fact that, inside a space-time average, to lowest order in $\lambda\!\!\!^{-}/L_B$: (i) Covariant divergences can be integrated by parts discarding the boundary term. In particular, expressions such as $\langle\langle\bar D_\rho U^\rho_{\mu\nu}\rangle\rangle$ vanish. (ii) Covariant derivative commutes. Then one finds, from the full expression (1.114), that

$$t_{\mu\nu} \to t_{\mu\nu} + \bar D_\rho U^\rho_{\mu\nu}\,, \tag{1.139}$$

and therefore $\langle\langle t_{\mu\nu}\rangle\rangle$ is gauge invariant, to leading order in $\lambda\!\!\!^{-}/L_B$. A further technical subtlety is that, in curved space, the sum of tensors at different points in space-time is not a tensor, so the result of integrating a tensor is also not a tensor. Thus, before integrating over d^4x, one must carry the tensors $t_{\mu\nu}(x)$ back to a single common point using parallel transport along geodesics. Details can be found in the Appendix of Isaacson (1968b), and references therein.
In principle, the same parallel transport procedure should be applied to the spatial and to the temporal averages that we introduced in Section 1.4.2. However, we will always end up computing these averages very far from the sources, where the background space-time can be taken as flat.

where n^i is the outer normal to the surface and dA is the surface element.[26] Furthermore, outside the source, we can impose the TT gauge. Let S be a spherical surface at a large distance r from the source. Its surface element is $dA = r^2 d\Omega$, and its normal $\hat{n} = \hat{r}$ is the unit vector in the radial direction. Then eq. (1.144) gives

$$\frac{dE_V}{dt} = -c \int dA\, t^{0r}\,, \tag{1.145}$$

where

$$t^{0r} = \frac{c^4}{32\pi G} \langle \partial^0 h_{ij}^{\rm TT} \frac{\partial}{\partial r} h_{ij}^{\rm TT} \rangle\,. \tag{1.146}$$

A GW propagating radially outward, at sufficiently large distances r, has the general form

$$h_{ij}^{\rm TT}(t,r) = \frac{1}{r} f_{ij}(t - r/c)\,, \tag{1.147}$$

where $f_{ij}(t - r/c)$ is some function of retarded time $t_{\rm ret} = t - r/c$. We will prove this result in Section 3.1, but it is in fact completely analogous to the result for electromagnetic waves. Therefore

$$\frac{\partial}{\partial r} h_{ij}^{\rm TT}(t,r) = -\frac{1}{r^2} f_{ij}(t - r/c) + \frac{1}{r}\frac{\partial}{\partial r} f_{ij}(t - r/c)\,. \tag{1.148}$$

On a function of the combination $t - r/c$ we have

$$\frac{\partial}{\partial r} f_{ij}(t - r/c) = -\frac{1}{c}\frac{\partial}{\partial t} f_{ij}(t - r/c)\,, \tag{1.149}$$

and therefore

$$\begin{aligned} \frac{\partial}{\partial r} h_{ij}^{\rm TT}(t,r) &= -\partial_0 h_{ij}^{\rm TT}(t,r) + O(1/r^2) \\ &= +\partial^0 h_{ij}^{\rm TT}(t,r) + O(1/r^2)\,. \end{aligned} \tag{1.150}$$

Then, from eq. (1.146), we see that at large distances, $t^{0r} = +t^{00}$ (which could also have been derived more simply from eq. (1.137), observing that an observer sitting at large distances from the source sees a plane wavefront), and the energy inside the volume V satisfies

$$\frac{dE_V}{dt} = -c \int dA\, t^{00}\,. \tag{1.151}$$

The fact that E_V decreases means that the outward-propagating GW carries away an energy flux

$$\begin{aligned} \frac{dE}{dAdt} &= +c\, t^{00} \\ &= \frac{c^3}{32\pi G} \langle \dot{h}_{ij}^{\rm TT} \dot{h}_{ij}^{\rm TT} \rangle\,, \end{aligned} \tag{1.152}$$

or, writing the surface element $dA = r^2 d\Omega$,

$$\boxed{\frac{dE}{dt} = \frac{c^3 r^2}{32\pi G} \int d\Omega\, \langle \dot{h}_{ij}^{\rm TT} \dot{h}_{ij}^{\rm TT} \rangle\,.} \tag{1.153}$$

[26] More precisely, we take as volume V a spherical shell centered on the source but far away from it, so that both its inner boundary S_1 and its outer boundary S_2 are in the far region, where the gravitational field is given simply by gravitational waves. Then in eq. (1.143) we can limit ourselves to the energy–momentum tensor t^{00} of GWs, neglecting the energy–momentum tensor of the quasi-static gravitational fields, as well as the energy–momentum tensor of matter. Therefore the time derivative of E_V is given by two terms: the energy flowing in through S_1 minus the energy flowing out from S_2. We are interested in the energy flux through a unit surface at a given distance from the source (say, in the energy flowing through a unit surface of our detector), which for definiteness we choose to be on the outer surface S_2, so in the following we simply take $S = S_2$.

In terms of h_+ and $h_\times$, we can rewrite the result as

$$\frac{dE}{dAdt} = \frac{c^3}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle\,. \tag{1.154}$$

The total energy flowing through dA between $t = -\infty$ and $t = +\infty$ is therefore[27]

$$\frac{dE}{dA} = \frac{c^3}{16\pi G} \int_{-\infty}^{\infty} dt\, \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle\,. \tag{1.155}$$

As discussed in the previous section, in the situations relevant for GW detectors the average $\langle \ldots \rangle$ in eq. (1.155) is a purely temporal average, over a few periods. Then in eq. (1.155) we can first perform the integral over dt from $-\infty$ to $+\infty$, eliminating therefore any time dependence, and the subsequent temporal average is just the average of a constant. Therefore the average in eq. (1.155) can be omitted, and

$$\frac{dE}{dA} = \frac{c^3}{16\pi G} \int_{-\infty}^{\infty} dt\, \left(\dot{h}_+^2 + \dot{h}_\times^2 \right)\,. \tag{1.156}$$

Inserting the plane wave expansion of $h_{+,\times}(t)$, given in eqs. (1.52) and (1.48), we get

$$\begin{aligned} \frac{dE}{dA} &= \frac{c^3}{16\pi G} \int_{-\infty}^{\infty} df\, (2\pi f)^2 \left(|\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right) \\ &= \frac{\pi c^3}{4G} \int_{-\infty}^{\infty} df\, f^2 \left(|\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right)\,. \end{aligned} \tag{1.157}$$

Since the integrand is even under $f \to -f$, we can restrict it to physical frequencies, $f > 0$, writing

$$\frac{dE}{dA} = \frac{\pi c^3}{2G} \int_0^{\infty} df\, f^2 \left(|\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right)\,. \tag{1.158}$$

Therefore

$$\frac{dE}{dAdf} = \frac{\pi c^3}{2G} f^2 \left(|\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right)\,. \tag{1.159}$$

We will always use the convention that the energy spectrum dE/df is the quantity that gives the total energy when it is integrated over the positive frequencies, rather than between $-\infty$ and $+\infty$. Writing $dA = r^2 d\Omega$, and integrating over a sphere surrounding the source, we find the energy spectrum

$$\boxed{\frac{dE}{df} = \frac{\pi c^3}{2G} f^2 r^2 \int d\Omega \left(|\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2 \right)\,.} \tag{1.160}$$

In the same way we can compute the flux of momentum. The momentum of the GWs inside a spherical shell V at large distances from the source is given by

$$P_V^k = \frac{1}{c} \int_V d^3x\, t^{0k}\,. \tag{1.161}$$

[27] The integration over t from $-\infty$ to $+\infty$ is necessary if we want to resolve all possible frequencies. In an experiment one will integrate a signal only over a certain time interval Δt and one has a corresponding resolution in frequency $\Delta f \simeq 1/\Delta t$.

Considering again a GW propagating radially outward, and repeating the same steps leading from eq. (1.142) to eq. (1.154), we get

$$c\partial_0 P^k_V = \int_V d^3x\, \partial_0 t^{0k} = -\int_S dA\, t^{0k}\,, \tag{1.162}$$

and therefore the momentum flux carried away by the outward-propagating GW is

$$\frac{dP^k}{dAdt} = +t^{0k}\,. \tag{1.163}$$

Inserting the expression (1.133) for t^{0k}, we get

$$\boxed{\frac{dP^k}{dt} = -\frac{c^3}{32\pi G} r^2 \int d\Omega\, \langle \dot{h}^{\rm TT}_{ij} \partial^k h^{\rm TT}_{ij}\rangle\,.} \tag{1.164}$$

Observe that, if t^{0k} is odd under a parity transformation $\mathbf{x} \to -\mathbf{x}$, then the angular integral vanishes.

1.5 Propagation in curved space-time

In the last section we have examined the consequences of the low-modes equation, eq. (1.111). We have seen that it determines the dynamics of the background metric $\bar{g}_{\mu\nu}$, and that it allows us to identify the energy–momentum tensor of $h_{\mu\nu}$.

We now turn our attention to the high-mode equation (1.112). First of all, we examine it in the limiting case of no external matter, $T_{\mu\nu} = 0$, so eq. (1.112) becomes

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\rm High}\,. \tag{1.165}$$

We are interested in the leading term in $\lambda\!\!\!^{-}/L_B$ (or in f_B/f; for definiteness we use $\lambda\!\!\!^{-}/L_B$). We therefore first perform an order of magnitude estimates of $R^{(1)}_{\mu\nu}$ and of $[R^{(2)}_{\mu\nu}]^{\rm High}$. In principle, in the short-wave expansion we have two small parameters, $h \equiv O(h_{\mu\nu})$ and $\lambda\!\!\!^{-}/L_B$. Recall however, from eq. (1.120), that when $T_{\mu\nu} = 0$ the Einstein equations fix these two scales to the same order of magnitude. Therefore, in this case we have a single small parameter, that we denote by ϵ,

$$\epsilon = O(h) = O(\lambda\!\!\!^{-}/L_B)\,. \tag{1.166}$$

To simplify notation, we use units $L_B = 1$ when we estimate the order of magnitude of the various terms, so that $h \sim \lambda\!\!\!^{-} \sim \epsilon$. From eq. (1.113), the leading term of $R^{(1)}_{\mu\nu}$ is

$$R^{(1)}_{\mu\nu} \sim \partial^2 h \sim \frac{h}{\lambda\!\!\!^{-2}} \sim \frac{1}{\epsilon} \tag{1.167}$$

while

$$R^{(2)}_{\mu\nu} \sim \partial^2 h^2 \sim \frac{h^2}{\lambda\!\!\!^{-2}} \sim 1\,. \tag{1.168}$$

So $[R^{(2)}_{\mu\nu}]^{\rm High}$ is at most $O(1)$ and can be neglected in eq. (1.165), compared to the leading term of $R^{(1)}_{\mu\nu}$, which is $O(1/\epsilon)$. Thus, if we limit ourselves to the leading term, eq. (1.112) simply becomes

$$[R^{(1)}_{\mu\nu}]_{1/\epsilon} = 0\,, \tag{1.169}$$

where $[\ldots]_{1/\epsilon}$ means that we must extract the $O(1/\epsilon)$ part. Equation (1.169) can be written explicitly as

$$\eta^{\rho\sigma}(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\nu\partial_\mu h_{\rho\sigma} - \partial_\rho\partial_\sigma h_{\mu\nu}) \simeq 0\,, \tag{1.170}$$

since the $O(1/\epsilon)$ part is obtained substituting the covariant derivatives with ordinary derivatives, and at the same time $\bar{g}^{\rho\sigma}$ in front of the parenthesis can be substituted with $\eta^{\rho\sigma}$, again to leading order in ϵ. This is just a propagation equation for the field $h_{\mu\nu}$ in a flat background, and it is the same equation that governs the propagation in the linearized theory discussed in Section 1.1, so we can again introduce $\bar{h}_{\mu\nu} = h_{\mu\nu} - (1/2)\eta_{\mu\nu}h$, impose the Lorentz gauge, and eq. (1.170) is nothing but

$$\Box \bar{h}_{\mu\nu} \simeq 0\,, \tag{1.171}$$

where $\Box = \partial_\mu\partial^\mu$ is the flat space d'Alembertian. So, this is the same as eq. (1.24) with the matter energy–momentum tensor $T_{\mu\nu} = 0$. We therefore discover that the high-frequency equation (1.112) is a wave equation for $h_{\mu\nu}$. We find that the propagation of GWs at $O(h)$ is the same as in the linearized theory because we considered the limit in which GWs are the only source of curvature. We now turn to the more interesting case in which external matter is present and dominates the curvature, so the low-frequency equation (1.111) becomes

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} \simeq \frac{8\pi G}{c^4}\bar{T}_{\mu\nu}\,. \tag{1.172}$$

To expand the high-frequency equation (1.112), we recall from eq. (1.121) that in this case $h \ll \lambda\!\!\!^{-}/L_B \ll 1$, so the expansions in h and in $\lambda\!\!\!^{-}/L_B$ are different. We keep only the terms linear in h (terms quadratic in h, in a typical situation involving GWs, are utterly negligible), and we expand the result in powers of $\lambda\!\!\!^{-}/L_B$. If we limit only to the leading and next-to-leading order in $\lambda\!\!\!^{-}/L_B$, eq. (1.112) becomes simply[28]

$$R^{(1)}_{\mu\nu} = 0\,. \tag{1.174}$$

Now $\bar{g}_{\mu\nu}$ is determined by $T_{\mu\nu}$ and is not close to flat, so $R^{(1)}_{\mu\nu}$ is a fully covariant quantity with respect to a non-trivial background metric. As we show in Problem 1.1, in a curved background the equation $R^{(1)}_{\mu\nu} = 0$, written explicitly, reads

$$\bar{g}^{\rho\sigma}(\bar{D}_\rho\bar{D}_\nu h_{\mu\sigma} + \bar{D}_\rho\bar{D}_\mu h_{\nu\sigma} - \bar{D}_\nu\bar{D}_\mu h_{\rho\sigma} - \bar{D}_\rho\bar{D}_\sigma h_{\mu\nu}) = 0\,. \tag{1.175}$$

The discussion of this equation parallels exactly that of Section 1.1,

[28] In fact, in eq. (1.112), $[R^{(2)}_{\mu\nu}]^{\rm High}$ is negligible with respect to $R^{(1)}_{\mu\nu}$, because it has one more power of h. To estimate the order of magnitude of $(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)^{\rm High}$, we observe that, if $T_{\mu\nu}$ is smooth, as we expect for macroscopic matter, its high-frequency part will come from the fact that the energy–momentum tensor $T_{\mu\nu}$ depends in general on the metric $g_{\mu\nu}$, and therefore will have a high-frequency component $O(h)$. Besides, also $g_{\mu\nu}T = (\bar{g}_{\mu\nu} + h_{\mu\nu})T$ has a high-frequency part $O(h)$ which comes from multiplying $\bar{g}_{\mu\nu}$ with the $O(h)$ high-frequency part of T, and another high-frequency part $O(h)$ which comes from multiplying $h_{\mu\nu}$ with the low-frequency part of T. So,

$$\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\rm High} = O(h/L_B^2)\,. \tag{1.173}$$

Instead, $R^{(1)}_{\mu\nu} \sim \partial^2 h \sim h/\lambda\!\!\!^{-2}$. Then $(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)^{\rm High}$ is smaller than $R^{(1)}_{\mu\nu}$ by a factor $O(\lambda\!\!\!^{-2}/L_B^2)$ and, to leading and next-to-leading order in $\lambda\!\!\!^{-}/L_B$, it does not contribute.

with $\eta_{\mu\nu}$ replaced by $\bar{g}_{\mu\nu}$. The equation becomes simpler introducing $h = \bar{g}^{\mu\nu} h_{\mu\nu}$ and[29]

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} h\,. \tag{1.176}$$

[29]There is a slight notational clash here. The bar over $g_{\mu\nu}$ denotes the background metric, while over $h_{\mu\nu}$ it denotes the combination (1.176).

We now impose the gauge condition

$$\boxed{\bar{D}^{\nu}\bar{h}_{\mu\nu} = 0\,,} \tag{1.177}$$

that we still call the Lorentz gauge. In this gauge the equation $R^{(1)}_{\mu\nu} = 0$ becomes

$$\bar{D}^{\rho}\bar{D}_{\rho}\bar{h}_{\mu\nu} + 2\bar{R}_{\mu\rho\nu\sigma}\bar{h}^{\rho\sigma} - \bar{R}_{\mu\rho}\bar{h}_{\nu}^{\ \rho} - \bar{R}_{\nu\rho}\bar{h}_{\mu}^{\ \rho} = 0\,. \tag{1.178}$$

Outside the matter sources, where $\bar{T}_{\mu\nu} = 0$, the Einstein equation for the background, eq. (1.172), tells us that $\bar{R}_{\mu\nu} = 0$. More precisely, using eq. (1.111), $\bar{R}_{\mu\nu}$ gets a contribution only from the term $[R^{(2)}_{\mu\nu}]^{\rm Low}$, so $\bar{R}_{\mu\nu} = O(h^2/\bar{\lambda}^2)$. Then, to linear order in h we can drop the terms $\bar{R}_{\mu\rho}\bar{h}_{\nu}^{\ \rho}$ and $\bar{R}_{\nu\rho}\bar{h}_{\mu}^{\ \rho}$ in eq. (1.178). Furthermore, $\bar{R}_{\mu\rho\nu\sigma}\bar{h}^{\rho\sigma} = O(h/L_B^2)$ while $\bar{D}^{\rho}\bar{D}_{\rho}\bar{h}_{\mu\nu} = O(h/\bar{\lambda}^2)$. Thus, since we have already restricted ourselves to the leading term and next-to-leading term in $\bar{\lambda}/L_B$ (see Note 28) we simply have

$$\boxed{\bar{D}^{\rho}\bar{D}_{\rho}\bar{h}_{\mu\nu} = 0\,.} \tag{1.179}$$

Equations (1.177) and (1.179) determine the propagation of GWs in the curved background, in the limit $\bar{\lambda} \ll L_B$. In conclusion we find that, after separating the Einstein equations into a low-frequency part and a high-frequency part, the low-frequency part describes the effect of GWs and of external matter on the background space-time, while the high-frequency part gives a wave equation in curved space, which describes the propagation of $h_{\mu\nu}$. This curved-space equation can be solved using the eikonal approximation of geometric optics, as we now discuss.

1.5.1 Geometric optics in curved space

Electromagnetic waves

We first recall how geometric optics works for electromagnetic waves, in a curved space with metric $\bar{g}_{\mu\nu}$.[30] The action of the electromagnetic field in this curved space is

[30]We follow Misner, Thorne and Wheeler (1973), Section 22.5.

$$S = -\frac{1}{4}\int d^4x\,\sqrt{-\bar{g}}\;\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}\,F^{\mu\nu}F^{\alpha\beta}\,, \tag{1.180}$$

and its variation gives the equations of motion

$$\bar{D}_{\mu}(\bar{D}^{\mu}A^{\nu} - \bar{D}^{\nu}A^{\mu}) = 0 \tag{1.181}$$

(we raise and lower the indices with $\bar{g}_{\mu\nu}$), which generalize the flat-space pair of Maxwell equations $\partial_{\mu}F^{\mu\nu} = 0$. One imposes on the four-vector potential A^{μ} the curved-space generalization of the Lorentz gauge,

$$\bar{D}_{\mu}A^{\mu} = 0\,. \tag{1.182}$$

From the definition of covariant derivative, $\bar{D}_{\mu}\bar{D}^{\nu}A^{\mu} = \bar{D}^{\nu}\bar{D}_{\mu}A^{\mu} + R^{\nu}_{\ \mu}A^{\mu}$, where $R^{\nu}_{\ \mu}$ is the Ricci tensor. The term $\bar{D}^{\nu}\bar{D}_{\mu}A^{\mu}$ vanishes because of the gauge condition, so eq. (1.181) becomes

$$\bar{D}^{\rho}\bar{D}_{\rho}A^{\mu} - R^{\mu}_{\ \rho}A^{\rho} = 0\,. \tag{1.183}$$

Geometric optics is valid when $\bar{\lambda}$ is much smaller than the other length-scales in the problem. So we must have $\bar{\lambda} \ll L_B$, with L_B the typical scale of variation of the background metric, and also $\bar{\lambda} \ll L_c$, where L_c is the characteristic length-scale over which the amplitude, polarization or wavelength of the electromagnetic field change substantially. In particular, $\bar{\lambda}$ must be much smaller than the curvature radius of the wavefront.

Under these conditions, we can use the eikonal approximation, which consists in looking for a solution with a phase θ rapidly varying, i.e. θ changes on the scale $\bar{\lambda}$, while the amplitude changes only on the scale L_B or L_c (whichever is smaller), so it is slowly varying. To perform the expansion systematically it is convenient to write

$$A^{\mu}(x) = [a^{\mu}(x) + \varepsilon b^{\mu}(x) + \varepsilon^2 c^{\mu}(x) + \ldots]e^{i\theta(x)/\varepsilon}\,, \tag{1.184}$$

where ε is a fictitious parameter, to be finally set equal to unity, that reminds us that a term to which a factor ϵ^n is attached, is of order $(\bar{\lambda}/L)^n$, where L is the smallest between L_B and L_c. An expansion of the form (1.184) is just an ansatz, and its validity is verified by substituting it in the equations.

Since $R^{\mu}_{\ \rho}A^{\rho} = O(A/L_B^2)$, where A is the typical amplitude of A^{μ}, while $\bar{D}^{\rho}\bar{D}_{\rho}A^{\mu} = O(A/\bar{\lambda}^2)$, to leading and next-to-leading order in $\bar{\lambda}/L_B$ we can neglect $R^{\mu}_{\ \rho}A^{\rho}$, and the equation of motion is simply

$$\bar{D}^{\rho}\bar{D}_{\rho}A^{\mu} = 0\,. \tag{1.185}$$

Defining the wave-vector $k_{\mu} \equiv \partial_{\mu}\theta$ and plugging the ansatz (1.184) into eq. (1.182) we get, to lowest order,

$$\bar{g}_{\mu\nu}k^{\mu}a^{\nu} = 0\,. \tag{1.186}$$

From eq. (1.185) we get instead, to lowest order,

$$\bar{g}_{\mu\nu}k^{\mu}k^{\nu} = 0\,. \tag{1.187}$$

This is known as the *eikonal equation*. From this it also follows that $0 = \bar{D}_{\nu}(k_{\mu}k^{\mu}) = 2k^{\mu}\bar{D}_{\nu}k_{\mu}$ (recall again that indices are raised and lowered with $\bar{g}_{\mu\nu}$). Since θ is a scalar, and on a scalar the covariant derivatives commute, we have $\bar{D}_{\nu}\partial_{\mu}\theta = \bar{D}_{\nu}\bar{D}_{\mu}\theta = \bar{D}_{\mu}\bar{D}_{\nu}\theta = \bar{D}_{\mu}\partial_{\nu}\theta$, so we can interchange the indices, $\bar{D}_\nu k_\mu = \bar{D}_\mu k_\nu$, and the condition $k^\mu \bar{D}_\nu k_\mu = 0$ becomes

$$k^\mu \bar{D}_\mu k_\nu = 0\,. \tag{1.188}$$

This is the geodesic equation in the space-time of the background metric $\bar{g}_{\mu\nu}$,[31] so eq. (1.188) states that the curves orthogonal to the surfaces of constant phase (the "rays" of the geometric optics approximation) travel along the null geodesics of $\bar{g}_{\mu\nu}$.

[31] Writing $k^\mu = dx^\mu/d\lambda$, where λ is the affine parameter along the geodesic, eq. (1.188) gives the geodesic equation, in the form

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0\,.$$

To next-to-leading order in ε, eq. (1.185) gives

$$2k_\rho \bar{D}^\rho a^\mu + (\bar{D}^\rho k_\rho) a^\mu = 0\,. \tag{1.189}$$

It is convenient to introduce the real scalar amplitude $a = (a^\mu a^*_\mu)^{1/2}$ and the polarization vector e^μ defined from $a^\mu = a e^\mu$, so $e^\mu e^*_\mu = 1$. An equation for the scalar amplitude is obtained observing that, on the one hand, one has the trivial identity $k^\mu \partial_\mu (a^2) = 2ak^\mu \partial_\mu a$. On the other hand, on a scalar such as a^2, ∂_μ can be replaced by $\bar{D}_\mu$, so $k^\mu \partial_\mu (a^2) = k^\mu \bar{D}_\mu (a^\rho a^*_\rho) = -(\bar{D}^\rho k_\rho) a^2$, where we used eq. (1.189). Comparing these two results, we get an equation for the scalar amplitude,

$$k^\mu \partial_\mu a = -\frac{1}{2}(\bar{D}_\mu k^\mu) a\,. \tag{1.190}$$

Finally, to obtain an equation for e^μ, we substitute $a^\mu = a e^\mu$ into eq. (1.189) and we use eq. (1.190). This gives

$$k^\rho \bar{D}_\rho e^\mu = 0\,. \tag{1.191}$$

Expanding the equations to still higher orders we could determine the corrections $b_\mu, c_\mu, \dots$ to the amplitude in terms of a_μ. Equations (1.186), (1.187), (1.188), (1.190) and (1.191) are the fundamental results of the geometric optics of electromagnetic waves in curved space. Equations (1.187) and (1.188) states that light rays (or photons, in a quantum language) travel along the null geodesics of $\bar{g}_{\mu\nu}$. Equation (1.186) states that the polarization vector e^μ is orthogonal to the propagation direction, $k_\mu e^\mu = 0$, and eq. (1.191) states that it is parallel-transported along the null geodesics. Finally, eq. (1.190) expresses (in the quantum language) the conservation of the number of photons in the limit of geometric optics. This can be seen rewriting it in the form

$$\bar{D}^\mu (a^2 k^\mu) = 0\,. \tag{1.192}$$

This shows that the current $j^\mu = a^2 k^\mu$ is covariantly conserved. Its associated conserved charge, according to the Noether theorem that we will recall in Section 2.1.1, is the integral of $a^2 k^0$ over a spatial surface at constant time. In a plane wave, the energy density is proportional to $|\mathbf{E}|^2 + |\mathbf{B}|^2 = 2|\mathbf{E}|^2$. In the gauge $A_0 = 0$, the electric field is $\mathbf{E} = \partial_0 \mathbf{A}$, so its amplitude is $k^0 a$, and the energy density is proportional to $(k^0 a)^2$. Since each photon carries an energy k^0, we see that $k^0 a^2$ is proportional to the number density of photons, so eq. (1.192) expresses the fact that, in the limit of geometric optics, the number of photons is conserved.

Gravitational waves

We can now discuss the eikonal approximation for GWs. We make the ansatz

$$\bar{h}_{\mu\nu}(x) = [A_{\mu\nu}(x) + \varepsilon B_{\mu\nu}(x) + \dots] e^{i\theta(x)/\varepsilon}\,. \tag{1.193}$$

Again we define $k_\mu = \partial_\mu \theta$, and we write $A_{\mu\nu} = A e_{\mu\nu}$ where the polarization tensor $e_{\mu\nu}$ is normalized as $e^{\mu\nu} e^*_{\mu\nu} = 1$, and A is the scalar amplitude. Substituting this ansatz into eqs. (1.177) and (1.179) and repeating basically the same steps as for the electromagnetic case, we find that k_μ still obeys eqs. (1.187) and (1.188), so gravitons travel along the null geodesic of $\bar{g}_{\mu\nu}$. Just as for photons, the scalar amplitude satisfies

$$k^\mu \partial_\mu A = -\frac{1}{2}(\bar{D}_\mu k^\mu) A\,, \tag{1.194}$$

which can be written as $\bar{D}^\mu (A^2 k^\mu) = 0$, and gives the conservation of the number of gravitons. Finally, the polarization tensor satisfies

$$k^\nu e_{\mu\nu} = 0\,, \tag{1.195}$$

$$k^\rho \bar{D}_\rho e_{\mu\nu} = 0\,, \tag{1.196}$$

so it is transverse and is parallel-propagated along the null geodesics.

Since gravitons propagate along null geodesics, just as photons, their propagation through curved space-time is the same as the propagation of photons, as long as geometric optics applies. For instance, they suffer gravitational deflection when passing near a massive body, with the same deflection angle as photons, and they undergo the same redshift in a gravitational potential.[32]

[32] The redshift of gravitons in a FRW cosmological model will be discussed explicitly in Section 4.1.4.

One practical difference concerning the lensing of gravitational and electromagnetic waves is however worth observing. Both type of waves can in principle be lensed by a large mass situated between the source and the observer. When the different images of the source cannot be resolved we are in the regime of microlensing, where we have a single image which is magnified. The amplification factor $\mathcal{A}$ in the energy density, computed within geometric optics, is[33]

[33] See e.g. Binney and Merrifield (1998).

$$\mathcal{A} = \frac{u^2+2}{u\sqrt{u^2+4}} \simeq \frac{1}{u}\,, \tag{1.197}$$

where $u = \beta/\theta_E$, β is the angle of the source with respect to the observer-lens axis (see Fig. 1.5), and θ_E is the Einstein angle, given by

$$\theta_E^2 = \frac{2R_S d_{\rm SL}}{d_{\rm OL}(d_{\rm SL} + d_{\rm OL})}\,, \tag{1.198}$$

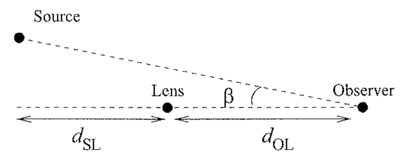

Fig. 1.5 The geometry for the lensing of gravitational or electromagnetic waves.

where R_S is the Schwarzschild radius of the lens and $d_{\rm SL}$ and $d_{\rm OL}$ are the source–lens and observer–lens distances, respectively. The second equality in eq. (1.197) holds when $u \ll 1$, i.e. when the source, lens and observer are well aligned, and in this case the amplification factor is large. In fact, it even becomes formally infinite if $u = 0$, i.e. when the source, lens and observer are perfectly aligned. However, the geometric

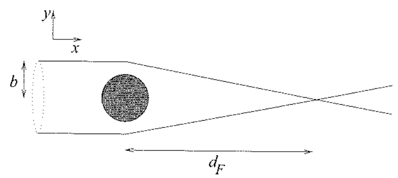

Fig. 1.6 The focusing of GWs from a source.

optics approximation breaks down near the caustics, i.e. when the light rays coming from the source cross each other, since there the scale of variation of the wavefront is no longer small compared to $\lambdabar$.

The actual behavior near the caustics can be obtained taking into account diffraction (or, in a quantum language, the uncertainty principle). In order of magnitude the effect can be estimated as follows. Consider a circular ring of rays, which is part of a plane wavefront, arriving with impact parameter b on a star of mass M which acts as a gravitational lens, as in Fig. 1.6. The deflection angle due to the gravitational field of the star is given by the classical Einstein result, $\theta = 2R_S/b$, where $R_S = 2GM/c^2$ is the Schwarzschild radius of the star. The whole circular ring of rays with impact parameter b is then focused on a single point, at a focal distance d_F given by $b/d_F = \tan\theta \simeq \theta$, i.e.

$$d_F \simeq \frac{b^2}{2R_S}\,. \tag{1.199}$$

The fact that a one-dimensional surface is focused onto a point is responsible for the formal infinite enhancement of the luminosity. In practice, however, diffraction forbids such a perfect focusing. Indeed, when we state that a photon has impact parameter b, we are implicitly assuming that the error Δy on its transverse position is smaller than b. Then, by the Heisenberg principle, it has an uncertainty on the transverse momentum $\Delta k_y \gtrsim \hbar/b$, and an angular spreading

$$\Delta\theta_s \simeq \frac{\Delta k_y}{k_x} \gtrsim \frac{\lambdabar}{b}\,. \tag{1.200}$$

Propagating to a distance d_F this induces a transverse spread

$$\Delta y_s \simeq d_F \Delta\theta_s \gtrsim \frac{\lambdabar b}{2R_S}\,. \tag{1.201}$$

The focusing is substantial only if $\Delta y_s \ll b$, which gives

$$\lambdabar \ll 2R_S\,. \tag{1.202}$$

For a lens with a mass of order $M_\odot$, this means that a substantial focusing is possible only for waves with $\lambdabar \ll O(6)$ km, i.e. a frequency $f \gg O(10)$ kHz.[34] For electromagnetic waves in the visible spectrum, this condition is very well satisfied, and microlensing is indeed commonly observed. For GWs, however, we will see that no astrophysical source is expected to produce waves with frequencies much larger that $O(10)$ kHz, and no significant amplification can be obtained for these waves from typical stellar-mass lenses.

[34] A more accurate estimate can be obtained taking into account the detailed internal structure of the lens, see Bontz and Haugan (1981).

1.5.2 Absorption and scattering of GWs

Finally, GWs are insensitive to absorption and scattering, during their propagation from astrophysical sources to the observer, because of the smallness of the gravitational cross-section. To make a quantitative

estimate, recall that the mean free path l of a particle scattering off an ensemble of target with number density n and cross-section σ, is given by

$$l = \frac{1}{n\sigma}\,. \tag{1.203}$$

For a graviton of energy E, the scattering cross-section off a target of mass m is (using units $\hbar = c = 1$) $\sigma \sim G_N^2 s$, where s is the square of the center-of-mass energy.[35] Consider for instance the scattering of a graviton with four-momentum $k^\mu = (E,0,0,E)$ off a nucleon at rest, with four-momentum $p^\mu = (m_n,0,0,0)$, where the mass $m_n \sim 1$ GeV. Then the square of the center-of-mass energy is $s = -(p+k)^2 = m_n^2 + 2m_n E$. For all astrophysically plausible values of the GW frequency $f_{\rm gw}$, $E = \hbar\omega_{\rm gw}$ is totally negligible with respect to m_n. For instance, if $f_{\rm gw} = 1$ kHz, we have $\hbar\omega_{\rm gw} = O(10^{-21})$ GeV; so $s \simeq m_n^2$ and

$$\sigma \sim G_N^2 m_n^2\,. \tag{1.204}$$

We can now compare the absorption of electromagnetic and of gravitational waves, for instance from the Sun. For photons in a neutral plasma, such as the Sun, the most important process is the Thompson scattering on electrons, which has a cross-section $\sigma = 8\pi\alpha^2/(3m_e^2)$, where $\alpha \simeq 1/137$ is the fine-structure constant, and we use units $\hbar = c = 1$. Inserting the numerical values, we see that the Thomson cross-section for scattering of photons on electrons is larger than the gravitational cross-section for scattering of gravitons on nucleons, eq. (1.204), by a huge factor $O(10^{80})$. The number density n_e of electrons in the Sun (which is relevant for computing the electromagnetic mean free path due to electron–photon scattering) is about the same as the number density of protons, relevant to compute the gravitational mean free path due to proton–graviton scattering, so the mean-free path for gravitons in the Sun is larger by a factor $O(10^{80})$ compared to that of photons! Using the value of n_e of the Sun, one finds that the photon mean free path inside the Sun is $O(1)$ cm. Using this value of l, one can show that a photon produced by thermonuclear reactions in the Sun core takes about 3×10^4 yr to reach the surface of the Sun and finally escape.[36] For a graviton the mean free path inside the Sun is $O(10^{80})$ cm, which is huge even compared to the observable size of the Universe (consider that 1 Gpc $\simeq 3 \times 10^{27}$ cm). Therefore, for a GW the Sun is completely transparent.

Significant absorption of GWs can take place if the wave impinges on a black hole. In this case, we can use the result for the capture cross section of a relativistic particle by a black hole with Schwarzschild radius R_S, $\sigma = (27/4)\pi R_S^2$.[37] Another possibility is that the GW impinges on a neutron star, just with the right frequency to excite one of its normal modes. In this case, the wave interacts coherently with the neutron star (while, in the above estimate of scattering in the Sun, we computed the incoherent scattering off the single protons). However, as we see in microlensing experiments, the probability that a compact object lies on the path from an astrophysical source to the Earth is very small.

[35] This can be shown most easily using the field-theoretical methods of Chapter 2, observing that the graviton-matter-matter vertex is proportional to $G_N^{1/2}$. In the Feynman diagram for the graviton-matter scattering there are two such vertices, so the amplitude is proportional to G_N and the cross-section to G_N^2. The dependence on s is then fixed by dimensional arguments observing that, in units $\hbar = c = 1$, G_N is an inverse mass squared, as well as from Lorentz invariance, that dictates that the energy dependence is through the Lorentz-invariant quantity s. This is the same result that holds (at energies $E \ll M_W$) for neutrinos, with the Fermi constant G_F replacing Newton's constant G_N.

[36] See e.g. Exercise 1.2 of Maggiore (2005).

[37] See e.g. Landau and Lifshitz, Vol. II (1979), Section 102.

1.6 Solved problems

Problem 1.1. Linearization of the Riemann tensor in curved space

In this problem we compute the Riemann and Ricci tensors, linearized to first order in $h_{\mu\nu}$ over a generic curved background $\bar{g}_{\mu\nu}$. The inversion of $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ is $g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + O(h^2)$. Then it is straightforward to find that, to $O(h)$, the linearization of the Christoffel symbol gives

$$\Gamma^{\mu}_{\nu\rho} = \bar{\Gamma}^{\mu}_{\nu\rho} + \frac{1}{2}\bar{g}^{\mu\sigma}\left(\bar{D}_{\nu}h_{\rho\sigma} + \bar{D}_{\rho}h_{\nu\sigma} - \bar{D}_{\sigma}h_{\nu\rho}\right) . \tag{1.205}$$

The calculation of the Riemann tensor at a given point x is enormously simplified if we perform it in a coordinate system where $\bar{\Gamma}^{\mu}_{\nu\rho}(x) = 0$ (paying attention, of course, to the fact that the derivatives of $\bar{\Gamma}^{\mu}_{\nu\rho}$ are non-zero!) We see from eq. (1.205) that in this frame $\Gamma^{\mu}_{\nu\rho} = O(h)$ and therefore the terms $\sim \Gamma\Gamma$ in the Riemann tensor are $O(h^2)$, so they do not contribute to $O(h)$, and we simply have $R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}_{\nu\rho} + O(h^2)$. Furthermore, substituting eq. (1.205) for $\Gamma^{\mu}_{\nu\rho}$, we only need to keep the terms where the derivative acts on the background Christoffel symbols $\bar{\Gamma}^{\mu}_{\nu\rho}$, since background Christoffel symbols on which no derivative act give zero. Since $\bar{\Gamma}^{\mu}_{\nu\rho}(x) = 0$, in the final expression (i.e. after having performed all derivatives) we are free to write derivatives as covariant derivatives with respect to the background, and we obtain an expression valid in all coordinate systems.

Then, in the frame where $\bar{\Gamma}^{\mu}_{\nu\rho}(x) = 0$, we get

$$\begin{aligned} R_{\mu\nu\rho\sigma} = \bar{R}_{\mu\nu\rho\sigma} + \tfrac{1}{2}\big(&\bar{D}_{\rho}\bar{D}_{\nu}h_{\mu\sigma} + \bar{D}_{\sigma}\bar{D}_{\mu}h_{\nu\rho} - \bar{D}_{\rho}\bar{D}_{\mu}h_{\nu\sigma} - \bar{D}_{\sigma}\bar{D}_{\nu}h_{\mu\rho} \\ &+ h_{\mu}{}^{\tau}\bar{R}_{\tau\nu\rho\sigma} - h_{\nu}{}^{\tau}\bar{R}_{\tau\mu\rho\sigma}\big) . \end{aligned} \tag{1.206}$$

We have performed the computation in a special frame. Since, however, the final result is expressed in terms of covariant quantities, this expression holds in any frame. The linearization of the Ricci tensor is then obtained using $R_{\mu\nu} = g^{\alpha\beta}R_{\alpha\mu\beta\nu} = (\bar{g}^{\alpha\beta} - h^{\alpha\beta})(\bar{R}_{\alpha\mu\beta\nu} + R^{(1)}_{\alpha\mu\beta\nu})$, so that the part linear in $h_{\mu\nu}$ is $R^{(1)}_{\mu\nu} = \bar{g}^{\alpha\beta}R^{(1)}_{\alpha\mu\beta\nu} - h^{\alpha\beta}\bar{R}_{\alpha\mu\beta\nu}$. Then we get[38]

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left(\bar{D}^{\alpha}\bar{D}_{\mu}h_{\nu\alpha} + \bar{D}^{\alpha}\bar{D}_{\nu}h_{\mu\alpha} - \bar{D}^{\alpha}\bar{D}_{\alpha}h_{\mu\nu} - \bar{D}_{\nu}\bar{D}_{\mu}h\right) , \tag{1.207}$$

where $h = \bar{g}^{\alpha\beta}h_{\alpha\beta}$. Observe that $\bar{D}_{\nu}\bar{D}_{\mu}h = \bar{D}_{\nu}\partial_{\mu}h = \partial_{\nu}\partial_{\mu}h - \bar{\Gamma}^{\rho}_{\nu\mu}\partial_{\rho}h$ is symmetric under the exchange $\mu \leftrightarrow \nu$, and therefore $R^{(1)}_{\mu\nu}$ is also symmetric, as it should be.

If the background is flat, $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, the covariant derivatives become ordinary derivatives and therefore commute. If we impose the Lorentz gauge condition $\partial^{\mu}\bar{h}_{\mu\nu} = 0$, i.e $\partial^{\mu}h_{\mu\nu} = (1/2)\partial_{\nu}h$, the linearized Ricci tensor takes a very simple form,

$$R^{(1)}_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} , \qquad \text{(flat background).} \tag{1.208}$$

[38] We make use of the identity

$$[\bar{D}_{\nu}, \bar{D}_{\alpha}]h_{\mu}{}^{\alpha} = h^{\alpha\beta}\bar{R}_{\alpha\mu\beta\nu} - h_{\mu}{}^{\tau}\bar{R}_{\tau\nu} ,$$

which follows from the definition of covariant derivative, see e.g. Weinberg (1972), eq. (6.5.3). Pay attention to the fact that Weinberg has the opposite sign convention for the Riemann tensor.

Problem 1.2. Gauge transformation of $h_{\mu\nu}$ and $R^{(1)}_{\mu\nu\rho\sigma}$

In the text we showed that, when the background space-time is $\eta_{\mu\nu}$, the resulting linearized theory has a gauge symmetry given by eq. (1.8), and that the linearized Riemann tensor $R^{(1)}_{\mu\nu\rho\sigma}$ is gauge-invariant. It is interesting to see how these results are modified when the background space-time is curved.

Under the coordinate transformation $x^{\mu} \to x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$, the usual transformation law of the metric is

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}g_{\rho\sigma}(x) . \tag{1.209}$$

Writing $g'_{\mu\nu}(x) = g'_{\mu\nu}(x' - \xi)$ and expanding to first order in ξ, we have

$$g'_{\mu\nu}(x) \simeq g'_{\mu\nu}(x') - \xi^{\rho}\partial_{\rho}g'_{\mu\nu} . \tag{1.210}$$

Combining this with eq. (1.209) we get

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x) - (D_{\mu}\xi_{\nu} + D_{\nu}\xi_{\mu}) , \tag{1.211}$$

where the covariant derivative is

$$D_{\mu}\xi_{\nu} = \partial_{\mu}\xi_{\nu} - \Gamma^{\rho}_{\mu\nu}\xi_{\rho} . \tag{1.212}$$

Equation (1.211) gives the lowest-order term in the small parameter $|D_{\mu}\xi_{\nu}|$. Similar to what we have done in linearized theory, we restrict ourselves to $|D_{\mu}\xi_{\nu}| \lesssim h$, where $h = O(|h_{\mu\nu}|)$. Since $\Gamma^{\rho}_{\mu\nu} = O(\partial g_{\mu\nu}) = O(1/L_B)$, where L_B is the typical variation scale of the background metric, the condition $|D_{\mu}\xi_{\nu}| \lesssim h$ means that both

$$|\partial\xi| \lesssim h , \quad \text{and} \quad \xi \lesssim hL_B , \tag{1.213}$$

must be satisfied. (We use the notation $\xi = O(|\xi^{\mu}|)$.) In the case of a flat background metric, discussed in Section 1.1, we have $L_B = \infty$ and therefore we found only the condition $|\partial\xi| \lesssim h$.

A generic function ξ^{μ} has both low-frequency and high-frequency modes, without a clear separation between them, and therefore in the transformed metric $g'_{\mu\nu}(x)$ the separation between the background and the GW in general disappears. It is therefore more useful to restrict ourselves to functions ξ^{μ} which maintain a clear-cut separation between low- and high-frequencies. In particular, we can consider a function ξ^{μ} that has only high-frequency modes. We observe that

$$D_{\mu}\xi_{\nu} + D_{\nu}\xi_{\mu} = (\bar{D}_{\mu}\xi_{\nu} + \bar{D}_{\nu}\xi_{\mu}) + \xi^{\rho}(\bar{D}_{\mu}h_{\nu\rho} + \bar{D}_{\nu}h_{\mu\rho} - \bar{D}_{\rho}h_{\mu\nu}) , \tag{1.214}$$

where we have used the expansion (1.205) for the Christoffel symbol. Since $\bar{\Gamma}^{\mu}_{\nu\rho}$ is a purely low-frequency term, and ξ^{μ} has only high frequencies, the terms $\bar{D}_{\mu}\xi_{\nu}$ are purely high-frequency, and therefore contribute to the transformation of $h_{\mu\nu}$ rather than of $\bar{g}_{\mu\nu}$.[39] Therefore under such a transformation we have $\bar{g}'_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x)$ and

$$\boxed{h'_{\mu\nu}(x) \simeq h_{\mu\nu}(x) - (\bar{D}_{\mu}\xi_{\nu} + \bar{D}_{\nu}\xi_{\mu}) ,} \tag{1.215}$$

under the condition that ξ^{μ} contains only high-frequencies and that it satisfies

$$|\bar{D}_{\mu}\xi_{\nu}| \leqslant |h_{\mu\nu}| . \tag{1.216}$$

[39] Note that, since $|\partial\xi| \lesssim h$ and $|\xi| \lesssim hL_B$, the terms $\xi\bar{D}h$ in eq. (1.214) are at most $O(h^2 L_B/\lambda\!\!\!^{-})$. When the curvature is dominated by matter we found in Section 1.4.2 that $h \ll \lambda\!\!\!^{-}/L_B$, and therefore in eq. (1.214) the term $\xi\bar{D}h$ is negligible with respect to the term $\bar{D}_{\mu}\xi_{\nu}$. In the opposite limit of curvature dominated by GWs we have $h \sim \lambda\!\!\!^{-}/L_B$ and the term $\xi\bar{D}h$ becomes of the same order as $D_{\mu}\xi_{\nu}$, but not larger; therefore, even in this limit, the order-of-magnitude estimates given below are not affected when neglecting the term $\xi\bar{D}h$.

Equation (1.215) has the form of a gauge transformation for a symmetric tensor field $h_{\mu\nu}$ on a curved space described by $\bar{g}_{\mu\nu}$. In this sense, we have a gauge theory for a spin-2 field in a curved background. We next compute how the linearized Riemann tensor transforms under this gauge transformation. We write

$$R_{\mu\nu\rho\sigma} = \bar{R}_{\mu\nu\rho\sigma} + R^{(1)}_{\mu\nu\rho\sigma} + O(h^2)\,, \tag{1.217}$$

where $\bar{R}_{\mu\nu\rho\sigma}$ is the Riemann tensor of the background and we raise and lower indices with $\bar{g}_{\mu\nu}$. The explicit calculation performed in Problem 1.1 gives

$$2R^{(1)}_{\mu\nu\rho\sigma} = \bar{D}_\rho\bar{D}_\nu h_{\mu\sigma}+\bar{D}_\sigma\bar{D}_\mu h_{\nu\rho}-\bar{D}_\rho\bar{D}_\mu h_{\nu\sigma}-\bar{D}_\sigma\bar{D}_\nu h_{\mu\rho}+h_\mu{}^\tau\bar{R}_{\tau\nu\rho\sigma}-h_\nu{}^\tau\bar{R}_{\tau\mu\rho\sigma}\,, \tag{1.218}$$

which generalizes eq. (1.13) to a curved background. Under the gauge transformation (1.215) the variation of $R^{(1)}_{\mu\nu\rho\sigma}$ is given by

$$\begin{aligned}\delta R^{(1)}_{\mu\nu\rho\sigma} &= \xi^\tau\bar{D}_\tau\bar{R}_{\mu\nu\rho\sigma} + \bar{R}_{\tau\nu\rho\sigma}\bar{D}_\mu\xi^\tau - \bar{R}_{\tau\mu\rho\sigma}\bar{D}_\nu\xi^\tau\\ &+\bar{R}_{\mu\nu\tau\rho}\bar{D}_\sigma\xi^\tau - \bar{R}_{\mu\nu\tau\sigma}\bar{D}_\rho\xi^\tau\,.\end{aligned} \tag{1.219}$$

Therefore, *if the background is not flat, $R^{(1)}_{\mu\nu\rho\sigma}$ is no longer gauge invariant* under the gauge transformations of linearized theory. However, let us estimate the order of magnitude of the various terms. For the background we have

$$\bar{R}_{\mu\nu\rho\sigma} \sim \partial^2\bar{g}_{\mu\nu} \sim \frac{1}{L_B^2}\,, \qquad \bar{D}_\tau\bar{R}_{\mu\nu\rho\sigma} \sim \partial^3\bar{g}_{\mu\nu} \sim \frac{1}{L_B^3}\,, \tag{1.220}$$

Equation (1.216) gives instead $\xi \sim hL_B$, $\bar{D}\xi \sim h$ and we therefore see that

$$\delta R^{(1)}_{\mu\nu\rho\sigma} \sim \frac{h}{L_B^2}\,. \tag{1.221}$$

This means that the variation of $R^{(1)}_{\mu\nu\rho\sigma}$ is much smaller than $R^{(1)}_{\mu\nu\rho\sigma}$ itself, since

$$R^{(1)}_{\mu\nu\rho\sigma} \sim \partial^2 h \sim \frac{h}{\lambda\!\!\!^{-2}} \tag{1.222}$$

and therefore

$$\boxed{\delta R^{(1)}_{\mu\nu\rho\sigma} \sim \frac{\lambda\!\!\!^{-2}}{L_B^2}R^{(1)}_{\mu\nu\rho\sigma}\,.} \tag{1.223}$$

We conclude that $R^{(1)}_{\mu\nu\rho\sigma}$ is approximately gauge-invariant in the limit $\lambda\!\!\!^{-}/L_B \ll 1$. More precisely, its leading term in an expansion in powers of $\lambda\!\!\!^{-}/L_B$ is gauge-invariant. Therefore, in the limit $\lambda\!\!\!^{-}/L_B \ll 1$, which was used from the very beginning to define $h_{\mu\nu}$, we can see $h_{\mu\nu}$ as a gauge field, with a gauge-invariant field-strength tensor given by the leading terms of $R^{(1)}_{\mu\nu\rho\sigma}$, as obtained from eq. (1.218).

Further reading

- Classical textbooks on general relativity are Weinberg (1972), Misner, Thorne and Wheeler (1973), and Landau and Lifshitz, Vol. II (1979). Among the more recent books, we suggest Hartle (2003) (at a rather introductory level, and with a very physical approach), and Straumann (2004) (more advanced).
- For discussions of freely falling frames, Riemann and Fermi normal coordinates, TT frame and the proper detector frame, see Misner, Thorne and Wheeler (1973), Sections 13.6 and 37.2, Hartle (2003), Section 8.4, and Thorne (1983, 1987). The metric of an accelerated, rotating observer is computed to quadratic order in x^i by Ni and Zimmermann (1978).
- The energy–momentum tensor of GWs and the short-wave expansion are discussed by Isaacson (1968a, 1968b), Misner, Thorne and Wheeler (1973), and Thorne (1987). The fact that performing a space-time average one obtains a gauge-invariant energy–momentum tensor was already discussed in Arnowitt, Deser and Misner (1961).
- The geometric optics approximation in curved space-time is discussed in Isaacson (1968a, 1968b) and Misner, Thorne and Wheeler (1973) (see in particular Section 22.5 for photons, and Section 35.14 and Exercise 35.15 for gravitational waves). See also Thorne (1983, 1987). Diffraction and lensing of GWs is discussed in Bontz and Haugan (1980) and in Section 2.6.1 of Thorne (1983).
- A definition of GWs based on the asymptotics of the gravitational field at null infinity was given by Bondi, van der Burg and Metzner (1962) and Sachs (1962), and was also important historically for resolving the controversy on the reality of gravitational radiation. A geometric description of the asymptotic fall-off of radiative solutions, using the notion of asymptotically simple space-time, was given by Penrose (1963, 1965).
- The development of the concept of gravitational wave has a very interesting history. The notion even predates Einstein general relativity (the term "gravitational wave" was used by Poincaré as early as 1905, referring to the fact that, even in a gravitational theory, the interaction must propagate at a finite speed). With the advent of general relativity, gravitational waves were introduced by Einstein in 1916. However, the existence of physical effects associated with them has been questioned many times, with Einstein himself changing his mind more then once. Eddington is associated to the ironic remark that "gravitational waves propagate at the speed of thought", implying that they are gauge artifact. (Actually, he was referring only to the transverse-longitudinal and longitudinal-longitudinal components of h_{ij}, which, indeed, are pure gauge modes. Concerning the transverse-transverse part, he rather showed that it carries energy, and even corrected an erroneous factor of two in Einstein's early version of the quadrupole formula.) The controversy on the existence of GWs was not settled until the early 1960s. A very interesting book on the history of the research in GWs is Kennefick (2007), see also the review article Kennefick (1997).