Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures

By JERRY L. KAZDAN* and F. W. WARNER*

1. Introduction

In previous work [11], [13] we have considered the problem of describing the set of Gaussian curvatures (if $\dim M = 2$) and scalar curvatures (if $\dim M \geqq 3$) on a compact, connected, but not necessarily orientable manifold M (see also [12] for the case of open manifolds). We refer the reader to the introductions of [11] and [13] for relevant background and literature.

In this paper we give a unified proof that the obvious sign condition demanded by the Gauss Bonnet Theorem, namely,

$$\text{(1.1)} \qquad \begin{aligned} &\text{(a)}\ K \text{ positive somewhere if } \chi(M) > 0\,, \\ &\text{(b)}\ K \text{ change sign unless} \equiv 0 \text{ if } \chi(M) = 0\,, \\ &\text{(c)}\ K \text{ negative somewhere if } \chi(M) < 0\,, \end{aligned}$$

is sufficient for a smooth function K on a given compact 2-manifold to be the Gaussian curvature of some metric. Our proof recovers the earlier results on existence of metrics with prescribed Gaussian curvature in [4], [7], [11], [17] as well as completing the hitherto unresolved case of the 2-sphere S^2. It also provides a new proof of, as well as additions to, some of the results on scalar curvature in [13].

We have attacked both Gaussian and scalar curvature questions by fixing a metric g on M with associated Laplacian Δ and Gaussian (respectively scalar) curvature k if $\dim M = 2$ (resp. $\dim M \geqq 3$), and seeking the desired metric g_1 having curvature K as being pointwise conformal to g, $g_1 = e^{2u}g$ (resp. $g_1 = u^{4/(n-2)}g$, $u > 0$). Since g_1 is to have Gaussian (resp. scalar) curvature K, then u must satisfy the non-linear elliptic partial differential equation

$$\text{(1.2)} \qquad -e^{-2u}(\Delta u - k) = K \qquad (\dim M = 2)\,,$$

$$\text{(1.3)} \qquad -u^{(n+2)/(2-n)}\left(\frac{4(n-1)}{n-2}\Delta u - ku\right) = K\,, \qquad u > 0\ (\dim M = n \geqq 3)\,.$$

In view of our non-existence theorems for these equations in [11], [13], which

* Supported in part by NSF Grants GP 28976X and GP 29258.

showed in general that one cannot realize all curvature functions pointwise conformal to some given metric, we are led to try to realize K as the Gaussian (resp. scalar) curvature of a metric conformally equivalent to g, that is, of the form $(\varphi^{-1})^*(e^{2u}g)$ (resp. $(\varphi^{-1})^*(u^{4/(n-2)}g)$) for some diffeomorphism φ of M. Thus one seeks diffeomorphisms φ (depending on K) so that the equations

$$-e^{-2u}(\Delta u - k) = K \circ \varphi \,, \tag{1.4}$$

$$-u^{(n+2)/(2-n)}\left(\frac{4(n-1)}{(n-2)}\Delta u - ku\right) = K \circ \varphi \tag{1.5}$$

have solutions. Introduction of the auxiliary diffeomorphism φ gives additional flexibility which enables us to solve (1.4) or (1.5) in situations where (1.2) or (1.3) are unsolvable. These equations are both of the form $T(u) = f \circ \varphi$, where T is a nonlinear elliptic operator. Our approach to (1.4) and (1.5) consists of the following ingredients:

(i) *Inverse Function Theorem.* If $T(u_0) = f_0$, and if the linearization T' of T is invertible at u_0, then one can solve $Tu = f$ for f sufficiently close to f_0 in L_p where $p > \dim M$.

(ii) *An Approximation Theorem* which gives necessary and sufficient conditions for a given function f_0 to be in the L_p closure of the orbit of a continuous function f under the group of diffeomorphisms of M. We then apply step (i) to $f \circ \varphi$. L_p existence is needed in (i) because the Approximation Theorem requires L_p.

(iii) *Perturbation Theorem.* To complicate matters, in many natural cases $T'(u_0)$ is not invertible. Therefore we need a perturbation argument which asserts that for the special cases of T in (1.4) and (1.5) there is a function u_1 arbitrarily close to u_0 such that $T'(u_1)$ is invertible.

This procedure allows us to obtain definitive results for the question of existence of metrics with prescribed curvatures and conformal equivalence. However it does not yield the more subtle and delicate facts of [11], [13], [17] concerning curvatures of pointwise conformal metrics, i.e., solvability of (1.2) and (1.3).

Section 2 contains the L_p Approximation Theorem and Section 3 the version we need of the Inverse Function Theorem. The Perturbation Theorem is contained in Section 4. Applications to Gaussian curvatures of both compact and open manifolds are in Section 5 (the reader may find it helpful to begin with this section). Scalar curvature is treated in Section 6. Other applications of this method are in Section 7.

For convenience, we assume throughout that the curvature candidate K is infinitely differentiable, although our methods extend immediately to functions K that are only Hölder continuous, in which case the metrics obtained have Hölder continuous second derivatives.

Notation. $H_{k,p}(M)$ will denote the space of functions whose derivatives up to order k are in $L_p(M)$, $p \geqq 1$, where we define $L_p(M)$ with respect to some smooth positive measure. The norm on $H_{k,p}(M)$ is written $\|\ \|_{k,p}$, while the norm on $L_p(M)$ is $\|\ \|_p$. $\|\ \|_\infty$ will denote the uniform norm, so the standard norm on $C^k(M)$ is written $\|\ \|_{k,\infty}$. $\langle\ ,\ \rangle$ will denote the inner product in $L_2(M)$. If M has a Riemannian metric, then dV will denote the element of volume, $L_p(M)$ will be defined with this volume element, and Δ will be the associated Laplacian (the sign convention we use gives $\Delta u = u_{xx} + u_{yy}$ for the standard metric on $\mathbf{R}^2$).

2. An approximation theorem

Given a continuous function f in $L_p(M)$ and a function g in $L_p(M)$ on a connected manifold M (not necessarily compact), we determine when there is a diffeomorphism φ of M such that $f \circ \varphi$ closely approximates g in $L_p(M)$, $1 \leqq p < \infty$. In other words, we wish to describe the L_p-closure of the orbit O_f of f under the group of diffeomorphisms, where diffeomorphisms act on $L_p(M)$ by composition. This paper will only use the case of M compact and g continuous. We wish to thank J. Munkres for his suggestions for the proof of this theorem.

THEOREM 2.1. *Assume* $\dim M = n \geqq 2$ *and let* $f \in C(M) \cap L_p(M)$. *Then a function* $g \in L_p(M)$ *is in the* L_p*-closure of* O_f *if and only if the range of* g *is contained in the range of* f, *that is,* $\inf f \leqq g(x) \leqq \sup f$ *for almost all* x *in* M.

Proof. The necessity is obvious. For the sufficiency, let $\varepsilon > 0$ be given. We will find a diffeomorphism φ of M such that $\|f \circ \varphi - g\|_p < \varepsilon$. Since continuous functions are dense in $L_p(M)$, we can assume that $g \in C(M) \cap L_p(M)$ and still satisfies the range condition. There is a connected open set $E \subset M$ whose closure $\bar{E}$ is compact and such that

$$\int_{M-E} |f - g|^p \, dV < \varepsilon^p/4 \ . \tag{2.2}$$

There is also a locally finite triangulation of M so fine that g is nearly constant in each simplex that intersects E. If Δ_i denotes an n-simplex that intersects E and $M_1 = \cup \Delta_i$, we specifically require that

$$\max_{x,y \in \Delta_i} |g(x) - g(y)| < \delta \ ,$$

where $2\delta = \varepsilon/(4 \operatorname{vol} M_1)^{1/p}$. Let $g(y_1) = \max g(x)$ and $g(y_2) = \min g(x)$ for x in

$\bar{M}_1$. The condition on the ranges gives the existence of points x_1, x_2 in M such that $f(x_1) > g(y_1) - \delta/2$, and $f(x_2) < g(y_2) + \delta/2$. There is a diffeomorphism φ_1 of M such that $\varphi_1(y_j) = x_j$ and such that (2.2) holds with f replaced by $f \circ \varphi_1$ (if M is compact, one can let $E = M_1 = M$ and it is unnecessary to introduce φ_1). Let $b_i \in \operatorname{Int}(\Delta_i)$. The continuity of $f \circ \varphi_1$ then guarantees that there exist disjoint open sets $V_i \subset M_1$ such that

$$|f \circ \varphi_1(y) - g(b_i)| < \delta$$

for $y \in V_i$ and for each i. We next choose a neighborhood Ω of the $(n-1)$-skeleton of M_1, disjoint from the b_i, so small that

$$(\max_{\bar{M}_1} |f \circ \varphi_1| + \max_{\bar{M}_1} |g|)^p \operatorname{vol} \Omega < \varepsilon^p/4 .$$

For each i, let U_i be a neighborhood of b_i, disjoint from Ω, and choose open sets O_1 and O_2 so that

$$M_1 - \Omega \subset O_1 \subset \bar{O}_1 \subset O_2 \subset \bar{O}_2 \subset M_1 - (n-1)\text{-skeleton} .$$

There is a diffeomorphism φ_2 of M such that $\varphi_2 \mid M - M_1 = \mathrm{id}$ and such that, $\varphi_2(U_i) \subset V_i$ for each i (here $\dim M \geqq 2$ is needed), and there is a diffeomorphism φ_3 of M which satisfies $\varphi_3 \mid M - O_2 = \mathrm{id}$ and $\varphi_3(O_1 \cap \Delta_i) \subset U_i$ for each i. Let $\varphi = \varphi_1 \circ \varphi_2 \circ \varphi_3$. Then, since (2.2) holds with f replaced by $f \circ \varphi_1$, we find that

$$\begin{aligned} \|f \circ \varphi - g\|_p^p &= \left(\int_{M-M_1} + \int_\Omega + \int_{M_1-\Omega}\right)(|f \circ \varphi - g|^p \, dV) \\ &< \frac{\varepsilon^p}{4} + \frac{\varepsilon^p}{4} + \sum_i \int_{O_1 \cap \Delta_i} |f \circ \varphi(y) - g(b_i) + g(b_i) - g(y)|^p \, d \\ &< \frac{\varepsilon^p}{2} + \sum_i 2^{p+1}\delta^p \operatorname{Vol}(\Delta_i) = \varepsilon^p . \end{aligned}$$

Q.E.D.

Remark. It is easy to construct examples showing the theorem is false if $\dim M = 1$.

3. An inverse function theorem

Let $T(u)$ be a second order quasi-linear differential operator on a compact manifold M, so in a local coordinate system

$$T(u) = \sum a^{ij}(x, u, u_1, \cdots, u_n)u_{ij} + b(x, u, u_1, \cdots, u_n) , \tag{3.1}$$

which we write as $T(u) = \sum a^{ij}(u)u_{ij} + b(u)$, where $u_i = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/\partial x_i \partial x_j$, and a^{ij} and b are smooth functions of their variables. The derivative (or linearization) of T with respect to u at a given function u is then the second order linear differential operator

$$
\begin{aligned}
T'(u)v &= \frac{d}{dt} T(u + tv)\Big|_{t=0} \\
&= \sum \{a^{ij}(u)v_{ij} + [a^{ij}_{u_k}(u)v_k + a^{ij}_u(u)v]u_{ij} + b_{u_k}(u)v_k\} + b_u(u)v ,
\end{aligned} \tag{3.2}
$$

where the subscripts denote partial differentiation.

The classical inverse function theorem leads one to believe that if $T(u)=f$ and if the linear map $T'(u)$ is invertible, then one can solve $T(v) = g$ for all g sufficiently near f. This is in fact true for (3.1) acting on a certain Sobolev space if u is an elliptic solution of $T(u) = f$, i.e., if the linear map $T'(u)$ defined by (3.2) is elliptic (this is the case precisely when the matrix $a^{ij}(u)$ is positive- or negative-definite). To apply the inverse function theorem for Banach spaces one needs several ingredients.

The first is a consequence of the Sobolev inequality [5] which asserts that if $p > n = \dim M$, then

$$
\|\varphi\|_{1,\infty} \leqq c\|\varphi\|_{2,p} \tag{3.3}
$$

for some constant c independent of φ. This shows that convergence in $H_{2,p}$ implies uniform convergence of the functions and their first derivatives, which in turn implies that if $\varphi \in H_{2,p}$ then $\varphi \in C^1$. We shall always assume $p > \dim M$ in this section. Then, since the operator T of (3.1) is quasilinear, it is a continuous map from $H_{2,p}$ to L_p. Moreover, the map $u \to T'(u)$ is continuous in the sense that, given any $\varepsilon > 0$, there is a $\delta > 0$ such that if $\|u - u_0\|_{2,p} < \delta$, then

$$
\|T'(u)v - T'(u_0)v\|_p < \varepsilon\|v\|_{2,p}
$$

for any $v \in H_{2,p}$ (by use of a partition of unity, one need only prove this locally, in which case it follows easily using (3.3)).

The second standard fact one needs is the fundamental L_p estimate for a second order linear elliptic operator L with smooth coefficients on M. It says that if kernel $(L) = 0$, then for any $\varphi \in H_{2,p}$ one has

$$
\|\varphi\|_{2,p} \leqq \gamma\|L\varphi\|_p , \tag{3.4}
$$

where the constant γ is independent of φ (see [1], [2] for a local version, and [11] for remarks on the extension to compact manifolds). From the fundamental inequality (3.4) one shows ([5]) that $L: H_{2,p} \to L_p$ is bijective with a continuous inverse, that is, L is invertible. Our theorem is now just the classical inverse function theorem for Banach spaces with the exception of the regularity statement which follows from a standard inductive argument [2].

THEOREM 3.5. *Let T be as in* (3.1) *and let $u_0 \in H_{2,p}(M)$, $p > \dim M$. Assume that*

(i) *$a^{ij}(u)$ and $b(u)$ are smooth functions of their variables for u near u_0, that is, for $\|u - u_0\|_{1,\infty} < 2r$ with $r > 0$ some constant;*

(ii) *the linear operator $Lv = T(u_0)v$: $H_{2,p} \to L_p$ is elliptic with* $\ker L = 0$ (and therefore is invertible).

Then there is a constant $\eta > 0$ such that if $\|f - f_0\|_p < \eta$, then there is a $u \in H_{2,p}$ such that $T(u) = f$ and $\|u - u_0\|_{2,p} < r$. Moreover, u is C^∞ in any open set where f is C^∞.

4. A perturbation theorem

For our application to Gaussian curvatures on a compact 2-manifold, we must solve the equation (1.2), which we write as $T_1(u) = K$. The derivative of T_1 with respect to u at the function $u_0 = 0$ is the linear elliptic operator

$$T_1'(0)v \equiv -\Delta v - 2kv \ . \tag{4.1}$$

In order to apply the Inverse Function Theorem (3.5) at $u_0 = 0$ the operator $T_1'(0)$ must be invertible. However, in many cases of geometric interest—such as the standard metric on S^2 with curvature $k = 1$, or the standard metric on the torus with $k = 0$ — $T_1'(0)$ is not invertible. A similar difficulty arises for the equation (1.3) concerning scalar curvature. While this difficulty does not preclude the possibility of solving $T(u) = K$, it is ominous. In fact, as we have shown in [11] and [13], there are indeed natural situations in which one can not solve the non-linear equations (1.2) and (1.3).

We shall circumvent this difficulty by showing that there is a function u_0 arbitrarily close to zero such that $T_1'(u_0)$ is invertible, in fact, we will show that $T_1'(u)$ is invertible for an open dense set of functions u.

In addition to $T_1(u) = K$, we will let (cf. (1.3))

$$T_2(u) = -u^{-a}(\alpha\Delta u - ku) \ , \qquad u > 0 \ , \tag{4.2}$$

where $a > 1$ and $\alpha > 0$ are constants and $k \in C^\infty$ is a prescribed function, and will restrict T_2 to act only on positive functions u. For T_1 and T_2 we find, if $v \in C^2$,

$$T_1'(u)v = -e^{-2u}[\Delta v - 2(\Delta u - k)v] = -e^{-2u}A_1(u)v \ , \tag{4.3}$$

and

$$T_2'(u)v = -\alpha u^{-a}\left[\Delta v - \left(a\frac{\Delta u}{u} - (a-1)\frac{k}{\alpha}\right)v\right] = -\alpha u^{-a}A_2(u)v \ , \tag{4.4}$$

where the linear elliptic operators $A_j(u)$ defined by (4.3) and (4.4) have the property that $\ker A_j(u) = \ker T_j(u)$, $j = 1, 2$, and, moreover, that the $A_j(u)$ are self-adjoint. Here and in what follows, T will refer to T_1 or T_2 as A refers to A_1 or A_2 respectively. When referring to T_2, T will act only on positive functions.

PERTURBATION THEOREM 4.5. *The linear elliptic operator* $T_1'(u): H_{2,p} \to L_p$ (*respectively* $T_2'(u)$) *is invertible on an open dense set of functions* $u \in C^2(M)$ (*resp. positive functions* $u \in C^2(M)$).

This will follow from a preliminary lemma.

PERTURBATION LEMMA 4.6. *If* $T'(u)$ *is not invertible, then there is a function* $z \in C^\infty$ *such that*

$$\dim \ker T'(u + tz) \leqq \dim \ker T'(u) - 1$$

for all $t \neq 0$, $|t|$ *sufficiently small.*

Proof of Perturbation Theorem (assuming the lemma). Since $T'(u)$ is linear elliptic, it is invertible if and only if $0 = \ker T'(u) = \ker A(u)$, that is, if 0 is not an eigenvalue of $A(u)$. Since $A(u)$ is self-adjoint, its spectrum depends continuously on u [10, V. Thm. 4.10], which yields the openness assertion. Denseness follows from repeated applications of the Perturbation Lemma. Q.E.D.

Proof of Perturbation Lemma. Say $\dim \ker A(u) = N \geqq 1$, so $\lambda = 0$ is an eigenvalue of multiplicity N. Given $z \in C^\infty$, for $|t|$ sufficiently small, the self-adjoint operator $A(u + tz)$ depends analytically on t; therefore so do the spectrum and eigenfunctions of $A(u + tz)$ [10, VII, § 4.8]. Thus there are orthonormal functions $\varphi_i(t)$ and numbers $\lambda_i(t)$, $i = 1, \cdots, N$ depending analytically on t for $|t|$ small such that $\varphi_i(0)$ are an orthonormal basis for $\ker A(u)$, $\lambda_i(0) = 0$ and such that

$$A(u + tz)\varphi_i(t) = \lambda_i(t)\varphi_i(t)\ , \qquad i = 1, \cdots, N\ . \tag{4.7}$$

We will show that for an appropriate function $z \in C^\infty$, we can make $\lambda_1(t) \neq 0$ for $0 \neq t$, $|t|$ sufficiently small; this will prove that $\dim \ker A(u + tz) \leqq N - 1$ for these values of t. Our procedure will be to show that one can either make the first derivative $\lambda_1'(0) \neq 0$, or else make $\lambda_1'(0) = 0$ but $\lambda_1''(0) \neq 0$. Here we use a prime to denote differentiation with respect to t and will write $\lambda = \lambda_1(0)$, $\lambda' = \lambda_1'(0)$, $\lambda'' = \lambda_1''(0)$, $\varphi' = \varphi_1'(0)$ etc., and for $v \in C^\infty$.

$$A'v = \left.\frac{dA(u + tz)v}{dt}\right|_{t=0}, \qquad A''v = \left.\frac{d^2A(u + tz)v}{dt^2}\right|_{t=0}.$$

To begin, we derive the standard formulas for λ', φ', and λ''. Differentiating (4.7) with respect to t and setting $t = 0$ we find, since $\lambda(0) = 0$,

$$A'\varphi + A\varphi' = \lambda'\varphi\ ; \tag{4.8}$$

$$A''\varphi + 2A'\varphi' + A\varphi'' = \lambda''\varphi + 2\lambda'\varphi'\ . \tag{4.9}$$

Since $\|\varphi(t)\|_2=1$, we see that $\langle\varphi, \varphi'\rangle=0$. Also $A\varphi=0$ implies that $\langle\varphi, A\varphi'\rangle = \langle A\varphi, \varphi'\rangle = 0$ and similarly $\langle\varphi, A\varphi''\rangle=0$. Therefore taking the inner product of (4.8) and (4.9) with φ, we obtain the formulas

$$\lambda' = \langle\varphi, A'\varphi\rangle \ , \tag{4.10}$$

$$\lambda'' = \langle\varphi, A''\varphi\rangle + 2\langle\varphi, A'\varphi'\rangle \ . \tag{4.11}$$

Beginning here, we discuss the operators A_1 and A_2 separately, although the reasoning is quite similar.

Case (i): For A_1 we find, for any $v \in C^\infty$, that

$$A_1'(u)v = -2(\Delta z)v \ , \qquad A_1''(u)v = 0 \ .$$

Thus

$$\lambda' = -2\langle\varphi, (\Delta z)\varphi\rangle = -2\langle\varphi^2, \Delta z\rangle \ .$$

If φ is not a constant, pick $z = \Delta(\varphi^2)$. Then $\lambda' = -2\langle\Delta(\varphi^2), z\rangle < 0$.

If $\varphi = c$ is a constant, then $\lambda' = 0$ for all z so we examine λ''. Pick a nontrivial $z \perp 1$. Now $c = \varphi \in \ker A_1(u)$ so $\Delta u - k = 0$ and $A_1(u)v = \Delta v$. Therefore from (4.8),

$$\Delta\varphi' = A_1\varphi' = -A_1'\varphi = 2(\Delta z)c \ .$$

Since $\langle\varphi', \varphi\rangle = 0$ this shows that $\varphi' = 2cz$. Substituting into (4.11) we conclude that

$$\lambda'' = 2\langle c, A_1'(2cz)\rangle = -8c^2\langle z, \Delta z\rangle = 8c^2 \|\nabla z\|^2 \ .$$

Thus, if φ is a constant and z is not a constant, then $\lambda'' > 0$.

Case (ii): For $A_2(u)$, $u > 0$, we find, for any $v \in C^\infty$, that

$$A_2'(u)v = -\frac{aB(z)v}{u} \ , \qquad A_2''(u)v = \frac{2azB(z)v}{u^2} \ , \tag{4.12}$$

where

$$Bz = \Delta z - \left(\frac{\Delta u}{u}\right)z \ .$$

Thus

$$\lambda' = -a\left\langle\varphi, \frac{B(z)\varphi}{u}\right\rangle = -a\left\langle\frac{\varphi^2}{u}, Bz\right\rangle .$$

If $\varphi^2/u \notin \ker B$, let $z = B(\varphi^2/u)$, yielding $\lambda' < 0$ as in Case (i).

On the other hand, if $\varphi^2/u \in \ker B$, we must work harder since $\lambda' = 0$ for all z. Recall that for any second order self-adjoint linear scalar elliptic operator, the eigenfunction associated with the lowest eigenvalue does not change sign (cf. [13, after (2.3)]) and all other eigenfunctions change sign.

Clearly $u \in \ker B$. Since $u > 0$, this shows that zero is the lowest eigenvalue of B. Thus $\dim \ker B = 1$ and u is a basis for $\ker B$. Consequently $\varphi^2/u = c^2 u$, that is, $\varphi = cu$, for some constant $c \neq 0$. Since $\varphi \in \ker A_2(u)$, this implies that $\alpha\Delta u - ku = 0$, so $A_2(u)v = \Delta v - kv/\alpha = Bv$. Hence, from (4.8),

$$B\varphi' = A_2\varphi' = -A_2'\varphi = a\frac{B(z)\varphi}{u} = acBz . \tag{4.13}$$

Choose a nontrivial function $z \perp \ker B$. Then from (4.13) $\varphi' = acz$. Equation (4.11) then yields

$$\begin{aligned}\lambda'' &= \left\langle \varphi^2, \frac{2azB(z)}{u^2} \right\rangle - 2a^2c\left\langle \varphi, \frac{zB(z)}{u} \right\rangle \\ &= 2ac^2(1-a)\langle z, Bz\rangle ,\end{aligned}$$

which is not zero for any nontrivial $z \perp \ker B$, by the variational characterization of the lowest eigenvalue of B. Q.E.D.

Remark. Under suitable hypotheses one can prove Theorem 4.5 for more general operators, such as $T(u) = a(u)\Delta u + b(x, u)$, where a and b depend analytically on u. To be more specific, if a and a' do not change sign and if $\lambda = 0$ is an eigenvalue but not the first eigenvalue of the linear elliptic operator $T'(u_0)$, then there are u's arbitrarily close to u_0 such that $T'(u)$ is invertible. There is difficulty if $\lambda = 0$ is the lowest eigenvalue of $T'(u_0)$ (which we had to circumvent for the special cases in Theorem 4.5) since $T'(u)$ may not be invertible for any u near u_0, as the following example shows. Let $T(u) = \Delta u/u$ for $u > 0$ (this is essentially $T_2(u)$ with $a = 1$). Then

$$T'(u)z = \frac{1}{u}\big(\Delta z - (\Delta u/u)z\big) .$$

However for any $u > 0$, one finds $u \in \ker T'(u)$, so $\dim \ker T'(u) = 1$ for all u. In this example, it is also easy to see that if one can solve $T(u) = f$ for some f, then one can solve $T(u) = f + \alpha$ if and only if α is an eigenvalue of $-\Delta u - fu$, that is, for very few constants α. Thus if $f \in \operatorname{Im}(T)$ then no open set about f is in $\operatorname{Im}(T)$.

5. Gaussian curvature

The two theorems in this section are the main results of this paper. Different proofs have been given elsewhere if $\chi(M) < 0$ [11, Thm. 11.6], $\chi(M) = 0$ [11, Thm. 6.2], and if $M = P^2$ [17].

THEOREM 5.1. *Let (M, g) be a compact, connected, 2-dimensional Riemannian manifold having constant Gaussian curvature. A function $K \in C^\infty(M)$ is the Gaussian curvature of a metric conformally equivalent to g if* *and only if K satisfies the sign condition* (1.1).

Proof. Since the necessity of (1.1) is obvious, we assume K satisfies (1.1) and will find a diffeomorphism φ of M and a function u such that u satisfies (1.4), which we write as $T_1(u) = K \circ \varphi$. The case $K \equiv$ const is trivially solved by setting u equal to an appropriate constant. Thus, we may assume that K is not a constant. First, consider the special case where K satisfies

$$\min K < k < \max K , \tag{5.2}$$

where k is the constant curvature of g. The Perturbation Theorem 4.5 shows that given any $\varepsilon > 0$, there is a $u_1 \in C^\infty$ so close to $u_0 = 0$ that

$$\| T_1(u_1) - k \|_\infty = \| T_1(u_1) - T_1(0) \|_\infty < \varepsilon ,$$

and such that $T_1'(u_1)$ is invertible. Because of (5.2) we can pick ε so small that

$$\min K < T_1(u_1) < \max K . \tag{5.3}$$

Choose $p > 2$. The Inverse Function Theorem 3.5 asserts that there is an $\eta > 0$ such that if $f \in C^\infty$ and $\| f - T_1(u_1) \|_p < \eta$, then there is a $u \in C^\infty$ such that $T_1(u) = f$. However the Approximation Theorem 2.1 shows, in view of (5.3), that there is a diffeomorphism φ of M such that $\| K \circ \varphi - T_1(u_1) \|_p < \eta$. Thus one can solve $T_1(u) = K \circ \varphi$.

We now remove assumption (5.2). The sign condition (1.1) implies that K and k have the same sign at some point of M. Thus there is a constant $c > 0$ such that the function cK satisfies (5.2). The above shows one can solve $T_1(u) = cK \circ \varphi$, for some diffeomorphism φ. But then $v = u + (\log c)/2$ is a solution of $T_1(v) = K \circ \varphi$. Q.E.D.

Remark 5.4. The Perturbation Theorem 4.5 is not needed if $k < 0$, that is, if $\chi(M) < 0$ since then $T_1'(0)$ is already invertible. Similarly, it is not needed for P^2 with the standard metric inherited from S^2, since there $T_1'(0)v = -\Delta v - 2v$ is invertible on $C^\infty(P^2)$ (but not on $C^\infty(S^2)$) since $T_1'(0)$ is invertible on the set of functions in $C^\infty(S^2)$ that are also orthogonal to the first order spherical harmonics.

Remark 5.5. The standard metrics on compact 2-manifolds with $\chi(M) \geqq 0$ have constant curvature, and it is a well known classical fact that the compact 2-manifolds with $\chi(M) < 0$ admit metrics of constant curvature. This latter fact also follows from [11, Thm. 11.6] or it can easily be obtained by the methods of this paper. One uses a connectedness argument to prove that every strictly negative K is a Gaussian curvature. Openness follows from the Inverse Function Theorem 3.5 applied to (1.2), whereas closedness follows from easy estimates of uniform bounds on solutions of (1.2) obtained by routine use of the maximum principle. For details of a similar argument see the proof of Lemma 6.3.

Our assumption that g has constant curvature can be removed by an additional argument. Since it uses somewhat different techniques, we do not digress to present it here.

THEOREM 5.6. *Let M be a compact connected 2-dimensional manifold. A function $K \in C^\infty(M)$ is the Gaussian curvature of a metric on M if and only if K satisfies the sign condition* (1.1).

This is, of course, obvious in view of Theorem 5.1.

Remark 5.7. One can also obtain our results [12, Thm. 3.1] on Gaussian curvatures on open manifolds by the methods of this paper thereby giving a unified proof of the entire Gaussian curvature problem. In place of Proposition 2.1 of [12] one uses Theorems 3.5 and 4.5 and in place of Proposition 2.6 of [12] one uses a slightly modified version of Theorem 2.1 in which one assumes that M is compact, $f \in L_p(M)$ and is continuous on some open set U, g is continuous, and $\inf_U f \leqq g \leqq \sup_U f$.

6. Scalar curvature

In order to determine the set of C^∞ functions on a compact connected manifold M that are scalar curvatures of some metric, we put a metric g on M with scalar curvature k and determine the set $CE(g)$ of C^∞ functions that are scalar curvatures of metrics conformally equivalent to g. The key ingredient is the following lemma.

LEMMA 6.1. *Given $K \in C^\infty(M)$, if there is a constant $c > 0$ such that* $\min K < ck < \max K$ *then $K \in CE(g)$.*

Proof. It is sufficient to find a diffeomorphism φ of M and a positive function u satisfying (1.5), which we write as $T(u) = K \circ \varphi$. The proof now proceeds just as that of Theorem 5.1, first treating the case $c = 1$ by showing that $T'(u_1)$ is invertible for some u_1 arbitrarily close to $u_0 = 1$, and appealing to the Inverse Function Theorem 3.5 and Approximation Theorem 2.1. The general case is reduced to the case $c = 1$ by multiplying u in (1.5) by an appropriate positive constant. Q.E.D.

For the next theorem, let $Lu = -\alpha\Delta u + ku$, where $\alpha = 4(n-1)/(n-2)$, and let $\lambda_1(g)$ be the first eigenvalue of L. To motivate the following, recall that if g and g_1 are conformally equivalent metrics, then $\lambda_1(g)$ and $\lambda_1(g_1)$ have the same sign, so the sign of $\lambda_1(g)$ is a conformal invariant [13, Thm. 3.2].

THEOREM 6.2.

(a) *If* $\lambda_1(g) < 0$, *then* $CE(g)$ *is precisely the set of* C^∞ *functions that are negative somewhere on* M.

(b) *If* $\lambda_1(g) = 0$, *then* $CE(g)$ *is precisely the set of* C^∞ *functions that either change sign or are identically zero on* M.

(c) *If* $\lambda_1(g) > 0$, *then* $CE(g)$ *contains the set of* C^∞ *functions* K *for which there is a constant* $c > 0$ *such that* $\min K < ck < \max K$. *Moreover, if there is a constant in* $CE(g)$, *then* $CE(g)$ *is precisely the set of* C^∞ *functions that are positive somewhere on* M.

Remarks. Part (a) is also proved (differently) as Theorem 3.3 of [13]. For part (c), it is not known if $\lambda_1(g) > 0$ implies there is a constant in $CE(g)$; this is the only part of Yamabe's question [19] still unresolved. However, $\lambda_1(g) > 0$ does imply there is a positive function in $CE(g)$ [13, Prop. 3.8].

Proof. (a) If $\lambda_1(g) < 0$, then the following Lemma 6.3 shows that there is a metric g_1 pointwise conformal to g having scalar curvature $k_1 = -1$. If K is negative somewhere, there is a constant $c > 0$ such that $\min K < -c < \max K$. Thus $K \in CE(g_1) = CE(g)$ by Lemma 6.1.

Conversely, if $K \in CE(g)$, there is a diffeomorphism Φ of M and a positive solution u of $Lu = (K \circ \Phi)u^a$ (where $a = (n+2)/(n-2)$). If $\varphi > 0$ is the eigenfunction of L corresponding to $\lambda_1(g)$, then

$$\lambda_1\langle \varphi, u\rangle = \langle L\varphi, u\rangle = \langle \varphi, Lu\rangle = \langle \varphi, (K\circ\Phi)u^a\rangle ,$$

from which it follows that K must be negative somewhere if $\lambda_1(g) < 0$.

(b) If $\lambda_1(g) = 0$, let $\varphi > 0$ be the corresponding eigenfunction of L. Then $T\varphi = 0$, so there is a metric g_1 pointwise conformal to g having scalar curvature $k_1 = 0$. The proof is completed as in part (a).

(c) This follows immediately from Lemma 6.1. Q.E.D.

We shall now demonstrate how, using methods of this paper, one can show that there is a metric g_1 pointwise conformal to g having scalar curvature $k_1 = -1$ if $\lambda_1(g) < 0$. Let $N = \{f \in C^\infty(M): f < 0\}$ with the uniform topology, and let T_2 denote the operator (4.2) with $\alpha = 4(n-1)/(n-2)$ and $a = (n+2)/(n-2)$ (actually, all that is necessary for the following lemma is $a > 1$ and $\alpha > 0$).

LEMMA 6.3. $N \subset \operatorname{Im} T_2$ *if and only if* $\lambda_1(g) < 0$.

Proof. Let $\varphi > 0$ be an eigenfunction associated with λ_1. If $-1 \in \operatorname{Im} T_2$, say $T_2(u) = -1$ for some $u > 0$. Then $Lu = -u^a$ and $L\varphi = \lambda_1\varphi$, so $\lambda_1 < 0$ from the inequality

$$0 < \langle \varphi, u^a\rangle = -\langle \varphi, Lu\rangle = -\langle L\varphi, u\rangle = -\lambda_1\langle \varphi, u\rangle .$$

For the converse, let $B = N \cap \operatorname{Im} T_2$. N is clearly connected. B is non-empty since $T_2(\varphi) = \lambda_1\varphi^{1-a} < 0$. In view of the Inverse Function Theorem 3.5, B will be open in N if for any $u > 0$ we show that $T_2(u) \in B$ implies that $\ker A_2(u) = 0$, where $A_2(u)$ is defined by (4.4). Let μ_1 be the first eigenvalue of $-A_2(u)$ and let $\psi > 0$ be the corresponding eigenfunction. It is enough to show that $\mu_1 > 0$ since then 0 is not an eigenvalue of $A_2(u)$. But $\mu_1 > 0$ since $u > 0$, $\psi > 0$, and

$$\begin{aligned}\mu_1\langle u, \psi\rangle &= -\langle u, A_2(u)\psi\rangle = -\langle A_2(u)u, \psi\rangle \\ &= (a-1)\left\langle \Delta u - \frac{k}{\alpha}, \psi\right\rangle = \frac{-(a-1)}{\alpha}\langle T_2(u)u^a, \psi\rangle > 0 .\end{aligned}$$

It remains to show that B is closed in N. Say $f_j \in B$ and $f_j \to f \in N$, uniformly. Then there are u_j such that $T_2(u_j) = f_j$. Let $w_j = \log(u_j/\varphi)$, where $\varphi > 0$ is the eigenfunction associated with the eigenvalue λ_1 of L. Then

$$\alpha\Delta w_j + \alpha\nabla w_j\cdot(\nabla w_j + 2(\nabla\varphi)/\varphi) = \lambda_1 - f_j\varphi^{a-1}e^{(a-1)w_j} .$$

By considering the maxima and minima of w_j, one finds that $0 < m_1 \leqq u_j \leqq m_2$ for some constants m_1 and m_2 independent of j. Inequality (3.4) with $p > \dim M$ shows that

$$\begin{aligned}\|u_j\|_{2,p} &\leqq \text{const}\,\|\Delta u_j - u_j\|_p \\ &= \text{const}\,\|(1/\alpha)(ku_j - f_ju_j^a) - u_j\|_p \leqq \text{const} ,\end{aligned}$$

where the constants are independent of j. Thus, by the Rellich-Kondrashov compactness theorem there is a subsequence of the u_j's, which we re-index as u_j, that converges strongly in $H_{1,p}$ to some $u \in H_{2,p}$. Moreover, since $p > \dim M$, by the Sobolev inequality (3.3), $u_j \to u$ uniformly. This sequence is Cauchy in $H_{2,p}$ too since

$$\begin{aligned}\|u_i - u_j\|_{2,p} &\leqq \text{const}\,\|(\Delta u_i - u_i) - (\Delta u_j - u_j)\|_p \\ &\leqq \text{const}\,\|(u_i - u_j) + (1/\alpha)(ku_i - ku_j + f_iu_i^a - f_ju_j^a)\|_\infty \longrightarrow 0 .\end{aligned}$$

Therefore, passing to the limit in $T_2(u_j) = f_j$, we find that $T_2(u) = f$, so $f \in B$. Q.E.D.

By using Theorem 6.2, it is possible to prove several results concerning which functions on M are scalar curvatures.

THEOREM 6.4. *Let M be a compact connected differentiable manifold with* $\dim M \geqq 3$.

(a) *Every element of $C^\infty(M)$ is a scalar curvature, if and only if M admits a metric with constant positive scalar curvature.*

(b) *Any function that is negative somewhere on M is a scalar curvature.*

Proof. Part (b) is an immediate consequence of Theorem 6.2 (a) and a theorem of Eliasson [6] and Aubin [3] which asserts that on any M, there is a metric with $\int_M k\,dV < 0$ which in turn implies $\lambda_1(g) < 0$ (use the variational characterization of λ_1 [13, Remark 2.4]). To prove part (a), note that if g is a metric with constant positive scalar curvature, then $\lambda_1(g) > 0$. As in (b) there is a metric g_1 with $\lambda_1(g_1) < 0$, so by a continuity argument [10, VII § 6, esp. 6.5] (see also [13, Thm. 3.9]) there is a metric g_0 with $\lambda_1(g_0) = 0$. The result now follows from Theorem 6.2. Q.E.D.

Part (b) is the only assertion that can be made for an arbitrary M since there are topological obstructions to having metrics with positive scalar curvature [16] (see also [8]) and to having metrics with zero scalar curvature [14]. We do not know of any manifold having a metric with zero scalar curvature that is known not to admit a metric of positive scalar curvature. In particular, we do not know if the torus T^n, $n \geqq 3$, admits a metric having positive scalar curvature.

An example of Theorem 6.4 (a) is that any C^∞ function on S^n, $n \geqq 3$, is the scalar curvature of some metric.

Remark 6.5. One can also obtain our result [13, Thm. 1.4] on scalar curvatures of open manifolds by the methods of this paper using the modifications outlined in Remark 5.7.

7. Further application

As an immediate consequence of Theorem 5.1 above and the solution to the Weyl embedding problem [18] we find that given any smooth strictly positive function K on S^2, there is a diffeomorphism φ of S^2, a compact strictly convex surface Σ in E^3, and a conformal diffeomorphism $\psi: \Sigma \longrightarrow S^2$ such that $K \circ \varphi \circ \psi$ is the Gaussian curvature of Σ. The possibility of this theorem with $\varphi =$ identity map, has been raised by L. Nirenberg. Partial results had previously been obtained by D. Koutroufiotis [15] and J. Moser [17], who both allow $\varphi =$ identity map, while we showed there are K's for which the result is not true unless one includes the additional diffeomorphism φ [11, see after Thm. 8.8].

UNIVERSITY OF PENNSYLVANIA

BIBLIOGRAPHY

[1] S. AGMON, The L_p Approach to the Dirichlet Problem, Ann. Scuola Norm. Sup. Pisa **13** (1959), 405–448.

[2] ———, A. DOUGLIS and L. NIRENBERG, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, Comm. Pure Appl. Math. **12** (1959), 623–727.

[3] T. AUBIN, Metriques Riemanniennes et Courbure, J. Diff. Geo. **4** (1970), 383-424.
[4] MELVYN BERGER, On Riemannian structures of prescribed Gauss curvature for compact two-dimensional manifolds, J. Diff. Geo. **5** (1971), 325-332.
[5] L. BERS, F. JOHN and M. SCHECHTER, *Partial Differential Equations*, John Wiley & Sons, Inc., New York, 1964.
[6] H. ELÍASSON, On variations of Metrics, Math. Scand. **29** (1971), 317-327.
[7] H. GLUCK, The generalized Minkowski problem in differential geometry in the large, Ann. of Math. **96** (2) (1972), 245-275.
[8] N. HITCHIN, The Space of Harmonic Spinors (to appear).
[9] G. JENSEN, The scalar curvature of left-invariant Riemannian metrics, Indiana U. Math. J. **20**, (12) (1971), 1125-1144.
[10] T. KATO, *Perturbation Theory for Linear Operators*, Grundlehren, Band 132, Springer-Verlag, New York (1966).
[11] JERRY L. KAZDAN and F. W. WARNER, Curvature functions for compact 2-manifolds, Ann. of Math. **99** (1974), 14-74.
[12] ———, Curvature functions for open 2-manifolds, Ann. of Math. **99** (1974), 203-219.
[13] ———, Scalar Curvature and Conformal Deformation of Riemannian Structure (to appear in J. Diff. Geo. 1975).
[14] ———, Prescribing Curvatures, (to appear in Proc. of Symposia in Pure Mathematics).
[15] D. KOUTROUFIOTIS, On Gaussian curvature and conformal mapping (to appear).
[16] A. LICHNEROWICZ, Spineurs harmoniques, C. R. Acad. Sci. Paris, **257** (1963), 7-9.
[17] J. MOSER, On a nonlinear problem in differential geometry, *Dynamical Systems* (M. Peixoto ed.), Academic Press, N.Y. (1973).
[18] L. NIRENBERG, The Weyl and Minkowski problems in differential geometry in the large, Comm. Pure and Appl. Math. **6** (1953), 337-394.
[19] H. YAMABE, On a deformation of Riemannian structures on compact manifolds, Osaka Math. J. **12** (1960), 21-37.

(Received August 31, 1973)