AUBIN'S LEMMA FOR THE YAMABE CONSTANTS OF INFINITE COVERINGS AND A POSITIVE MASS THEOREM

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ABSTRACT. Aubin's Lemma says that, if the *Yamabe constant* of a closed conformal manifold (M, C) is positive, then it is strictly less than the Yamabe constant of any of its non-trivial finite conformal coverings. We generalize this lemma to the one for the Yamabe constant of any (M_∞, C_∞) of its *infinite* conformal coverings, provided that $\pi_1(M)$ has a *descending chain of finite index subgroups tending to* $\pi_1(M_\infty)$. Moreover, if the covering M_∞ is *normal*, the limit of the Yamabe constants of the finite conformal coverings (associated to the descending chain) is equal to that of (M_∞, C_∞). For the proof of this, we also establish a version of positive mass theorem for a specific class of asymptotically flat manifolds with singularities.

1. INTRODUCTION AND MAIN RESULTS

There is a natural differential-topological invariant, called the *Yamabe invariant*, which arises from a variational problem for the functional E below on a given closed smooth n-manifold M with $n \geq 3$. It is well known that a Riemannian metric on M is *Einstein* if and only if it is a critical point of the normalized Einstein-Hilbert functional E on the space $\mathcal{M}(M)$ of all Riemannian metrics on M

$$E : \mathcal{M}(M) \to \mathbb{R}, \quad g \mapsto E(g) := \frac{\int_M R_g d\mu_g}{\mathrm{Vol}_g(M)^{(n-2)/n}}$$

Here, $R_g, d\mu_g$ and $\mathrm{Vol}_g(M)$ denote respectively the scalar curvature, the volume element of g and the volume of (M, g).

Because the restriction of E to any conformal class

$$[g] := \{e^{2f} \cdot g \mid f \in C^\infty(M)\}$$

is bounded from below, we can consider the following conformal invariant (called the *Yamabe constant* of $[g]$)

$$\begin{aligned} Y(M,[g]) &:= \inf\{E(\tilde{g}) \mid \tilde{g} = u^{4/(n-2)} \cdot g \in [g],\ u \in C^\infty_+(M)\} \\ &= \inf\left\{ Q_{(M,g)}(u) := \frac{\int_M (\alpha_n |\nabla u|^2 + R_g u^2) d\mu_g}{\left(\int_M u^{2n/(n-2)}\ d\mu_g\right)^{(n-2)/n}} \;\middle|\; u \in C^\infty(M),\ u \not\equiv 0 \right\}, \end{aligned}$$

where $\alpha_n := \frac{4(n-1)}{n-2} > 0$ and

$$C^\infty_+(M) := \{u \in C^\infty(M) \mid u > 0\}.$$

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A remarkable theorem [39, 37, 8, 28, 33] (cf. [9, 26, 34]) of Yamabe, Trudinger, Aubin, and Schoen asserts that each conformal class $[g]$ contains metrics $\check{g}$, called *Yamabe metrics*, for which

$$Y(M,[g]) = E(\check{g}).$$

A first variation argument shows that these metrics $\check{g}$ must have constant scalar curvature

$$R_{\check{g}} \equiv Y(M,[g]) \cdot \mathrm{Vol}_{\check{g}}(M)^{-2/n}.$$

Hence, we call a conformal class $[g]$ *positive* if $Y(M,[g]) > 0$.

Let $\mathcal{C}(M)$ denote the space of all conformal classes on M. The study of the second variation of E done in [24, 30] (cf. [11]) leads naturally to the definition of the following differential-topological invariant

$$Y(M) := \sup_{C\in\mathcal{C}(M)} \inf_{g\in C} E(g).$$

This invariant is called the *Yamabe invariant* of M and it was introduced independently by O. Kobayashi [21] and Schoen [29] (see also [22, 30]). In other words, the Yamabe invariant $Y(M)$ of M is the supremum of the scalar curvatures of unit-volume Yamabe metrics on M. We remark also that, for any M and $C \in \mathcal{C}(M)$, Aubin [8] (cf. [9, 26]) proved the following fundamental inequality

$$Y(M,C) \leq Y(S^n,[g_{\mathbb{S}}]) = n(n-1)\mathrm{Vol}_{g_{\mathbb{S}}}(S^n)^{2/n}, \tag{1}$$

where $g_{\mathbb{S}}$ denotes the standard metric of constant curvature one on the standard n-sphere S^n (for each n). This implies that

$$Y(M) = \sup_{C\in\mathcal{C}(M)} Y(M,C) \leq Y(S^n) = n(n-1)\mathrm{Vol}_{g_{\mathbb{S}}}(S^n)^{2/n}.$$

We will denote $\mathbf{Y}_n := Y(S^n,[g_{\mathbb{S}}]) = n(n-1)\mathrm{Vol}_{g_{\mathbb{S}}}(S^n)^{2/n}$.

In the present article, we will study the Yamabe constants of conformal coverings of a given closed conformal manifold (M,C). When $Y(M,C) \leq 0$, the maximum principle implies that any two constant scalar curvature metrics in C are identical up to a scaling (cf. [9]). Thus, for a finite k-fold conformal covering $(\widetilde{M},\widetilde{C})$,

$$Y(\widetilde{M},\widetilde{C}) = k^{2/n} \cdot Y(M,C).$$

On the other hand, when $Y(M,C) > 0$, we can not expect any similar explicit relations between $Y(M,C)$ and $Y(\widetilde{M},\widetilde{C})$ (cf. [21, 30]). One reason is that, in a given positive conformal class, the uniqueness for unit-volume metrics of constant scalar curvature does not hold. However, Aubin [7, Lemma 2 and Theorem 6] proved the following (see [5, Lemma 3.6] for details):

Aubin's Lemma. *Let (M,C) be a closed positive conformal n-manifold with $n \geq 3$ and $(\widetilde{M},\widetilde{C})$ a non-trivial finite conformal covering of (M,C). Then,*

$$Y(M,C) < Y(\widetilde{M},\widetilde{C}).$$

This lemma was one of the crucial ingredients in [5], where it was used in order to prove, for instance, that

$$Y(\mathbb{RP}^3\#(S^2\times S^1)) \leq Y(\mathbb{RP}^3,[h_0]), \tag{2}$$

where h_0 denotes the standard metric of constant curvature one on $\mathbb{RP}^3$.

We now explain briefly how this inequality is obtained. First, we remark that the *inverse mean curvature flow technique* developed in [12] can not be applied directly to determine the Yamabe invariant $Y(\mathbb{RP}^3\#(S^2\times S^1))$. Instead, take any metric g with $Y(M,[g])>0$ on

$$M:=\mathbb{RP}^3\#(S^2\times S^1)$$

and a smooth loop c in $S^2\times S^1$ whose homology class $[c]$ generates $H_1(S^2\times S^1;\mathbb{Z})$. We regard the loop c as a loop in M. Let (M_k,g_k) and (M_∞,g_∞) denote, respectively, the k-fold Riemannian covering and the *normal* (cf. [16]) infinite Riemannian covering of (M,g) associated to $[c]\in H_1(M;\mathbb{Z})$. Note that, topologically,

$$M_k=\#kM\#(S^2\times S^1)\quad\text{and}\quad M_\infty=\#_1^\infty M\#(S^2\times\mathbb{R}).$$

We can now apply Aubin's Lemma to the coverings $M_k\to M$ for all $k\geq 2$ and conclude that

$$Y(M,[g])<Y(M_k,[g_k]).$$

After some crucial arguments, the inverse mean curvature flow technique can then be applied on each M_k for all sufficiently large k in order to conclude that, for any small $\varepsilon>0$, there exists k_0 such that the following inequality holds (see [5] for details)

$$Y(M_k,[g_k])\leq Y(\mathbb{RP}^3,[h_0])+\varepsilon\qquad\text{for all}\quad k\geq k_0. \tag{3}$$

Using the above inequalities, we obtain

$$Y(M,[g])\leq Y(\mathbb{RP}^3,[h_0]),$$

and hence the desired result (2) holds.

In the argument sketched above, we did not compute the Yamabe constant of $(M_\infty,[g_\infty])$. However, the argument in (3) was inspired by the following *working hypothesis*:

$$\lim_{k\to\infty}Y(M_k,[g_k])=Y(M_\infty,[g_\infty])\leq Y(\mathbb{RP}^3,[h_0]). \tag{4}$$

This observation leads naturally to the problem of extending Aubin's Lemma to the case of the Yamabe constants of infinite conformal coverings. Furthermore, since some sort of noncompact conformal manifolds appear as various kinds of limits of closed conformal manifolds, the study of the Yamabe constants of noncompact conformal manifolds is useful and indispensable for the study of Yamabe invariants of closed conformal manifolds (cf. [1, 2, 3, 4, 6, 36]). In this article, we will present some extensions to Aubin's Lemma which will include, as a particular case, the first equality in the above working hypothesis (4). Therefore, by combining this with the inequality (3), the second inequality in (4) follows.

Before starting our main results (i.e., Theorem 1.2 and Theorem 1.4 below), we need to introduce some definitions first. For a given open conformal n-manifold (X,C) and a metric $g\in C$ (possibly incomplete), the *Yamabe constant* $Y(X,C)$ of (X,C) is also defined by

$$Y(X,C):=\inf\left\{\,Q_{(X,g)}(f)\;\middle|\;f\in C_c^\infty(X),f\not\equiv 0\right\}.$$

Note that $Y(X,C)$ does not depend on the choice of $g\in C$ (cf. [33]), and hence it is a conformal invariant of (X,C). Moreover, Aubin's argument used in the proof of the inequality (1) is still valid for any noncompact conformal manifold (cf. [26, Lemma 3.4]) and so

$$Y(X,C)\leq\mathbf{Y}_n.$$

Definition 1.1. Let G be an infinite group and H $(\subset G)$ a subgroup of infinite index. Let $\{G_i\}_{i\geq 1}$ be an infinite sequence of subgroups of G. We shall call $\{G_i\}_{i\geq 1}$ a *descending chain of finite index subgroups tending to* H if it satisfies the following:

(i) Each G_i is a finite index subgroup of G with $G_i \supset H$.
(ii) $G = G_1 \supsetneq G_2 \supsetneq \cdots \supsetneq G_i \supsetneq G_{i+1} \supsetneq \cdots$.
(iii) $\bigcap_{i=1}^{\infty} G_i = H$.

The following theorem corresponds to an analogue of Aubin's lemma for the Yamabe constants of *infinite* conformal coverings. We identify each $\pi_1(M_k)$ and $\pi_1(M_\infty)$ with their projections to $\pi_1(M)$ in the theorem below.

Theorem 1.2. *Let (M, C) be a closed positive conformal n-manifold with $n \geq 3$. Let $(M_\infty, C_\infty) \to (M, C)$ be an infinite conformal covering such that $\pi_1(M)$ has a descending chain of finite index subgroups tending to $\pi_1(M_\infty)$. Then,*

$$Y(M, C) < Y(M_\infty, C_\infty).$$

Moreover, if the covering M_∞ is normal, then

$$\lim_{k\to\infty} Y(M_k, C_k) = Y(M_\infty, C_\infty)$$

for any family $\{(M_k, C_k)\}_{k\geq 1}$ of finite conformal coverings of (M, C) satisfying $(M_1, C_1) = (M, C)$ and such that $\{\pi_1(M_k)\}_{k\geq 1}$ is a descending chain of finite index subgroups of $\pi_1(M)$ tending to $\pi_1(M_\infty)$.

Remark 1.3. There is one well understood example of the above assertion worked out by Kobayashi [21] and Schoen [30] (cf. [4]). The infinite universal Riemannian covering

$$(S^{n-1} \times \mathbb{R}^1, g_{\mathbb{S}} + dt^2) \to (S^{n-1} \times S^1(1), g_{\mathbb{S}} + dt^2)$$

is normal and

$$\lim_{r\nearrow\infty} Y(S^{n-1} \times S^1(r), [g_{\mathbb{S}} + dt^2]) = Y(S^{n-1} \times \mathbb{R}^1, [g_{\mathbb{S}} + dt^2]),$$

where $S^1(r) := [0, 2\pi r]/\sim$.

When $Y(M_\infty, C_\infty) = \mathbf{Y}_n$, the proof of the second assertion in Theorem 1.2 requires the following positive mass theorem.

Theorem 1.4 (cf. [2], [4])**.** *Let (M_∞, g_∞) be a noncompact Riemannian n-manifold $(n \geq 3)$ which is a normal infinite Riemannian covering of a closed Riemannian n-manifold (M, g) with $Y(M, [g]) > 0$ and such that $\pi_1(M)$ has a descending chain of finite index subgroups tending to $\pi_1(M_\infty)$. Let G_∞ be the normalized minimal positive Green's function on M_∞ with pole at a fixed $p_\infty \in M_\infty$ for the conformal Laplacian* (cf. [33, 2])

$$\mathcal{L}_{g_\infty} := -\frac{4(n-1)}{n-2}\Delta_{g_\infty} + R_{g_\infty}.$$

Set

$$g_{\infty,AF} := G_\infty^{\frac{4}{n-2}} g_\infty \quad on \quad M_\infty - \{p_\infty\},$$

which is a scalar-flat, asymptotically flat metric on $M_\infty - \{p_\infty\}$. Assume that either $3 \leq n \leq 5$, or $(M_\infty, [g_\infty])$ is conformally flat near p_∞. Then, $(M_\infty - \{p_\infty\},\ g_{\infty,AF})$ has nonnegative mass

$$\mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}) \geq 0.$$

Moreover if $\mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}) = 0$*, then* $(M_\infty, [g_\infty])$ *is simply connected and locally conformally flat. Furthermore if* $\mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}) = 0$ *and* $n \geq 4$*, then it is a simply connected domain in* $(S^n, [g_{\mathrm{s}}])$*.*

The following result does not hold without some assumption on M_∞ since $Y(\Omega, [g_{\mathrm{s}}]\big|_\Omega) = \mathbf{Y}_n$ for any non-simply connected domain Ω in S^n (see [33, Lemma 2.1]).

Corollary 1.5. *Let* (M_∞, C_∞) *be a noncompact positive conformal* n*-manifold with* $n \geq 3$ *which is a normal infinite conformal covering of a closed positive conformal* n*-manifold* (M, C) *and such that* $\pi_1(M)$ *has a descending chain of finite index subgroups tending to* $\pi_1(M_\infty)$*. Assume that*

$$Y(M_\infty, C_\infty) = \mathbf{Y}_n.$$

Then, (M_∞, C_∞) *is simply connected and locally conformally flat. Moreover if* $n \geq 4$*, it is a simply connected domain in* $(S^n, [g_{\mathrm{s}}])$*.*

This corollary gives an affirmative partial answer to the following problem, which has been proposed by Kobayashi [23].

Problem. *Assume that a noncompact conformal* n*-manifold* (X, C) *with* $n \geq 3$ *statisfies*

$$Y(X, C) = \mathbf{Y}_n.$$

Then, is (X, C) *locally conformally flat ?*

Remark 1.6. From the final resolution of the Yamabe problem due to Schoen [28], [33], any compact conformal manifold (M, C) satisfying $Y(M, C) = \mathbf{Y}_n$ is conformally equivalent to $(S^n, [g_{\mathrm{s}}])$, and so (M, C) must be locally conformally flat. Before this result, Aubin [8] has proved that any closed conformal manifold (M, C) with $n \geq 6$ which is not locally conformally flat satisfies

$$Y(M, C) < \mathbf{Y}_n.$$

His method is still valid for any noncompact conformal manifold (cf. [2, Proposition 6.6]) and so an affirmative answer to the above problem has been known for $n \geq 6$.

In the next section, we prove Theorem 1.2. One of the main difficulties in the proof arises from the fact that the infinite covering M_∞ in Theorem 1.2 might have *infinitely many ends*. A key ingredient to overcome the difficulty is a construction of a family of nice cut-off functions on M_∞, which is a modification of the construction done in [5, Sections 5 and 7]. One corollary of Theorem 1.2 will include that the first equality in the working hypothesis (4) holds (Corollary 2.7).

In Section 3, we prove Theorem 1.4 and Corollary 1.5. As it was mentioned before, Corollary 1.5 is needed in order to prove Theorem 1.2 in the case $Y(M_\infty, C_\infty) = \mathbf{Y}_n$. The core of the proof of Theorem 1.4 is the following. The mass of the asymptotically flat manifold $(M_\infty - \{p_\infty\}, g_{\infty,AF})$ is the limit of the positive masses of asymptotically flat manifolds arising from closed positive conformal manifolds.

Finally, we will also show another extension to Aubin's Lemma in the last section. Namely, Aubin's lemma for the Yamabe constants of finite conformal coverings of a *noncompact* positive conformal manifold.

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2. Proof of Theorem 1.2

In this section, we first prove the first assertion of Theorem 1.2, that is, the following strict inequality:

$$Y(M,C) < Y(M_\infty, C_\infty).$$

Proof of the first assertion of Theorem 1.2. Take a unit-volume Yamabe metric $g \in C$, and consider its lift $g_\infty \in C_\infty$ to M_∞. Note that $R_{g_\infty} = R_g = Y(M,C) > 0$. Moreover note that, the Sobolev embedding $W^{1,2}(M_\infty; g_\infty) \hookrightarrow L^{2n/(n-2)}(M_\infty; g_\infty)$ combined with the positivity of R_{g_∞} implies that $Y(M_\infty, C_\infty) > 0$. Take also $p_\infty \in M_\infty$. Since $\pi_1(M)$ has a descending chain of finite index subgroups tending to $\pi_1(M_\infty)$, there exists a sequence of finite Riemannian coverings $\{(M_k, g_k)\}_{k\geq 1}$ of (M,g) satisfying $(M_1, g_1) = (M,g)$, $\bigcap_{k=1}^\infty \pi_1(M_k) = \pi_1(M_\infty)$, and the following:

(i) M_∞ is an infinite covering of each M_k.
(ii) M_{k+1} is a non-trivial finite covering of M_k for $k \geq 1$.

From Aubin's Lemma, we have

$$Y(M,[g]) < Y(M_2,[g_2]) < \cdots < Y(M_k,[g_k]) < Y(M_{k+1},[g_{k+1}]) < \cdots \leq \mathbf{Y}_n,$$

and hence the limit of $\{Y(M_k,[g_k])\}_{k\geq 1}$ always exists. Therefore, by taking a suitable subsequence if necessary, it is enough to consider only the subsequence for the proof. From this, we can also assume the following:

(iii) For $k \geq 2$, there exists a fundamental domain of M_k in M_∞ containing p_∞ such that its closure $\widehat{M}_k$ $(\subset M_\infty)$ satisfies

$$\{x \in M_\infty \mid \mathrm{dist}_{g_\infty}(x, p_\infty) \leq k\} \subsetneq \widehat{M}_k \qquad \text{and} \qquad \widehat{M}_k \subsetneq Int\big(\widehat{M}_{k+1}\big),$$

where $Int\big(\widehat{M}_{k+1}\big)$ denotes the interior of $\widehat{M}_{k+1}$.

We can assume, without loss of generality, that $\partial\widehat{M}_k$ is piecewise smooth (possibly disconnected) and that $\partial\widehat{M}_{k+1}, \partial\widehat{M}_k$ have the same projection on M_k for all $k \geq 2$. Denote by $\widehat{M}$ $(\subset M_\infty)$ the closure of a fundamental domain of M containing p_∞ which satisfies $\widehat{M} \subsetneq \widehat{M}_2$ and such that both $\partial\widehat{M}_2$ and $\partial\widehat{M}$ have the same projection on M. Take and fix a sequence of the closures $\{\widehat{\Omega}_k\}_{k\geq 1}$ of domains in M_∞ with smooth boundary $\partial\widehat{\Omega}_k$ satisfying

$$\widehat{M}_{k-1} \subsetneq \widehat{\Omega}_k \subset \widehat{M}_k,$$

where $\widehat{M}_0 := \{p_\infty\}$ for the sake of convenience.

When $Y(M_\infty, C_\infty) = \mathbf{Y}_n$, it follows from Aubin's Lemma and the inequality (1) that

$$Y(M,C) < Y(M_2,[g_2]) \leq \mathbf{Y}_n = Y(M_\infty, C_\infty).$$

Hence, we assume that $Y(M_\infty, C_\infty) < \mathbf{Y}_n$ from now on.

By the definition of $Y(M_\infty, C_\infty)$ and the condition (iii), there exists $k_0 \in \mathbb{N}$ such that

$$Q_k := \inf_{\varphi \in C_c^\infty\left(Int(\widehat{\Omega}_k)\right)} Q_{(M_\infty, g_\infty)}(\varphi) < \mathbf{Y}_n \qquad \text{for all} \quad k \geq k_0$$

and that

$$Q_{k_0} > Q_{k_0+1} > \cdots > Q_{k_0+i} > Q_{k_0+i+1} > \cdots , \qquad \lim_{k\to\infty} Q_k = Y(M_\infty, C_\infty).$$

By combining the inequality $Q_k < \mathbf{Y}_n$ with the same argument as the one used on closed conformal manifolds (cf. [9, 26, 34]), the Yamabe problem of Dirichlet-type can be solved for each $(\widehat{\Omega}_k, C_\infty|_{\widehat{\Omega}_k})$. Namely, there exists $\psi_k \in C^\infty(\widehat{\Omega}_k)$ such that

- $Q_{(M_\infty, g_\infty)}(\psi_k) = Q_k$.
- $\psi_k > 0$ in $Int(\widehat{\Omega}_k)$ and $\psi_k = 0$ on $\partial\widehat{\Omega}_k$.

We denote the zero extension of ψ_k to M_∞ also by $\psi_k \in C^{0,1}(M_\infty) \cap W^{1,2}(M_\infty; g_\infty)$. Because $\psi_k|_{\partial\widehat{\Omega}_k} = 0$, ψ_k can also be regard as a function on the closed manifold M_k. Then,

$$Y(M_k, [g_k]) \leq Q_{(M_k, g_k)}(\psi_k) = Q_{(M_\infty, g_\infty)}(\psi_k) = Q_k,$$

and hence

$$\limsup_{k\to\infty} Y(M_k, [g_k]) \leq \lim_{k\to\infty} Q_k = Y(M_\infty, C_\infty).$$

From Aubin's Lemma, we obtain

$$Y(M, C) < Y(M_2, [g_2]) \leq \lim_{k\to\infty} Y(M_k, [g_k]) = \limsup_{k\to\infty} Y(M_k, [g_k]) \leq Y(M_\infty, C_\infty).$$

This completes the proof of the first assertion. □

Next, we prove the second assertion of Theorem 1.2. The same notation as in the previous proof is used.

We start by reviewing some indispensable properties (Lemma 2.1) for the Green's functions of the conformal Laplacians on (M_∞, g_∞) and (M_k, g_k) (for $k \geq 1$). After this, we prove the second assertion modulo some auxiliary lemmas and Corollary 1.5, whose proofs will be given in the end of this section and in the next section respectively. Those auxiliary lemmas will be stated precisely during the proof of the second assertion.

For simplicity, we often identify a set in M_∞ with its projection on each M_k and a tensor (including a function) on M_k with its lift to M_∞. For instance, g_k denotes both the Riemannian metric on M_k (given in the above) and its lift to M_∞.

Let G_∞ be the *normalized minimal positive Green's function* on M_∞ for the *conformal Lapalacian*

$$\mathcal{L}_{g_\infty} := -\alpha_n \Delta_{g_\infty} + R_{g_\infty}$$

with pole at p_∞ and G_k the normalized (positive) Green's function on M_k for the conformal Lapalacian $\mathcal{L}_{g_k}$ with pole at p_∞, respectively (see [33, Section 1], [2, Section 6] for the existence and details). Namely, for instance G_∞, the following properties hold

$$G_\infty > 0 \quad \text{on} \quad M_\infty - \{p_\infty\}, \qquad \mathcal{L}_{g_\infty} G_\infty = c_n \cdot \delta_{p_\infty} \quad \text{on} \quad M_\infty,$$

$$\lim_{x\to p_\infty} \mathrm{dist}_{g_\infty}(x, p_\infty)^{n-2} G_\infty(x) = 1 \quad \cdots \quad (\textit{normalization}),$$

and that if G'_∞ is another normalized positive Green's function in the above sense, then

$$G'_\infty \geq G_\infty \quad \text{on} \quad M_\infty - \{p_\infty\} \quad \cdots \quad (\textit{minimality}).$$

Here, $c_n > 0$ and δ_{p_∞} stand respectively for a specific universal positive constant and the Dirac δ-function at p_∞. Set

$$g_{\infty,AF} := G_\infty^{\frac{4}{n-2}} g_\infty \quad \text{on} \quad M^*_\infty := M_\infty - \{p_\infty\}$$

and

$$g_{k,AF} := G_k^{\frac{4}{n-2}} g_k \quad \text{on} \ \ M_k^* := M_k - \{p_\infty\}.$$

Then, these $g_{\infty,AF}$ and $g_{k,AF}$ define scalar-flat, asymptotically flat metrics on M_∞^* and M_k^*, respectively (cf. [26]). Note that the asymptotically flat manifold $(M_\infty^*, g_{\infty,AF})$ has singularities created by the ends of M_∞^*.

For simplicity, we set the following notation for the rest of this article. Any quantity Q that depends on g_k or $g_{k,AF}$ (resp. g_∞ or $g_{\infty,AF}$) will be denoted by Q_k or $Q_{k,AF}$ (resp. Q_∞ or $Q_{\infty,AF}$).

The Yamabe constant $Y(M_\infty, C_\infty)$ can be rewritten as

$$\begin{aligned} Y(M_\infty, C_\infty) &= Y(M_\infty^*, [g_{\infty,AF}]) \\ &= \inf\left\{ \frac{\int_{M_\infty^*} \alpha_n |\nabla u|^2_{\infty,AF} d\mu_{\infty,AF}}{\left(\int_{M_\infty^*} u^{\frac{2n}{n-2}} d\mu_{\infty,AF}\right)^{\frac{n-2}{n}}} \ \middle| \ u \not\equiv 0 \ \text{ in } \ C_c^\infty(M_\infty^*) \right\} \\ &= \inf\left\{ \frac{\int_{M_\infty^*} \alpha_n |\nabla u|^2_{\infty,AF} d\mu_{\infty,AF}}{\left(\int_{M_\infty^*} u^{\frac{2n}{n-2}} d\mu_{\infty,AF}\right)^{\frac{n-2}{n}}} \ \middle| \ u \not\equiv 0 \ \text{ in } \ W^{1,2}(M_\infty^*; g_{\infty,AF}) \right\}. \end{aligned}$$

From now on, we assume that the covering $M_\infty \to M$ is normal. Let

$$\mathcal{G} := \pi_1(M)/\pi_1(M_\infty) \qquad \text{and} \qquad \mathcal{G}_k := \pi_1(M_k)/\pi_1(M_\infty) \subset \mathcal{G}$$

denote the groups of deck transformations for the normal covering $M_\infty \to M$ and for the normal covering $M_\infty \to M_k$ respectively. We identify $\pi_1(M_\infty)$ with its pojections to $\pi_1(M)$ and $\pi_1(M_k)$. With having this in mind, the standard arguments for G_k and G_∞ in [33, 17, 2] (combined with the *normality* of each covering $M_\infty \to M_k$) imply the following lemma.

Lemma 2.1.

(i) *For all $k \geq 1$,*

$$G_k = \sum_{\gamma \in \mathcal{G}_k} G_\infty \circ \gamma \quad \text{on} \quad M_\infty.$$

(ii) *For all $k \geq 2$,*

$$0 < G_\infty < G_k < G_1,$$

and G_k converges unifomly in C^ℓ (for every $\ell \geq 1$) to G_∞ on every compact subset of M_∞^*.

(iii) *For every compact set K of M_∞^*,*

$$\lim_{j\to\infty} \sup\{G_\infty(\gamma(x)) \mid x \in K, \gamma \in W_j\} = 0,$$

where $W_j := \{\gamma \in \mathcal{G} \mid \mathrm{dist}_{g_\infty}(\gamma(p_\infty), p_\infty) \geq j\}$.

(iv) *For any open set O containig p_∞, there exists a constant L independent of k such that, for all $k \geq 1$,*

$$|\nabla G_k|_k \leq L G_k \quad \text{on} \quad M_k - O.$$

(v)

$$\int_{M_\infty} G_\infty d\mu_\infty < \infty.$$

Proof of the second assertion of Theorem 1.2.
Case 1. First, we consider the case when $Y(M_\infty, C_\infty) < \mathbf{Y}_n$. Let $v_k \in C^\infty_+(M_k)$ be a Yamabe minimizer (with respect to g_k) satisfying $||v_k||_{L^{\frac{2n}{n-2}}(M_k;g_k)} = 1$. Arguing like in the first assertion of Theorem 1.2, we obtain from Aubin's Lemma that

$$Y(M_k, [g_k]) < \lim_{\ell\to\infty} Y(M_\ell, [g_\ell]) \le Y(M_\infty, C_\infty) < \mathbf{Y}_n \qquad \text{for all} \quad k \ge 1.$$

By combining this uniformly strict inequality for $Y(M_k, [g_k])$ with the normality of the covering $M_\infty \to M$, a similar renormalization argument to the one used in the proof of [34, Chapter 5, Theorem 2.1] and [2, Theorem 6.1] implies the following: There exists a positive constant K (independent of k) such that

$$v_k \le K \quad \text{on} \quad M_k \quad \text{for all} \quad k \ge 1. \tag{5}$$

Hence, the elliptic estimates combined with (5) and $||v_k||_{L^{\frac{2n}{n-2}}(M_k;g_k)} = 1$ imply

$$|\nabla v_k|_{g_k} \le K' \quad \text{on} \quad M_k \quad \text{for all} \quad k \ge 1, \tag{6}$$

where K' is also independent of k. Combining the above inequality (6) with $||v_k||_{L^{\frac{2n}{n-2}}(M_k;g_k)} = 1$ and $Y(M_k, [g_k]) > Y(M, [g])$, we can find a positive number μ_0 (independent of k) satisfting the following: For each k sufficiently large, there exists at most μ_0 distinct deck transformations

$$\{\gamma_{k_\lambda}\}_{k_\lambda \in \Lambda_k} \subset \mathcal{G} \quad (|\Lambda_k| \le \mu_0)$$

of the covering $M_\infty \to M$ such that

$$\left\{ x \in \widehat{M}_k \ (\cong M_k) \ \middle| \ v_k(x)^{\frac{4}{n-2}} > \frac{\alpha_n}{2 \cdot Y(M_k, [g_k])} \right\} \ \subset \ \bigcup_{k_\lambda \in \Lambda_k} \gamma_{k_\lambda}(\widehat{M}) \ \subset \ \widehat{M}_k. \tag{7}$$

For reasons of simplicity, we will assume that either $\Lambda_k = \phi$ or $\{\gamma_{k_\lambda}\}_{k_\lambda \in \Lambda_k} = \{id\}$ from now on. Hence,

$$\left\{ x \in \widehat{M}_k \ (\cong M_k) \ \middle| \ v_k(x)^{\frac{4}{n-2}} > \frac{\alpha_n}{2 \cdot Y(M_k, [g_k])} \right\} \ \subset \ \widehat{M}. \tag{8}$$

Using these properties for v_k, we obtain the following.

Lemma 2.2. *There exist positive constants r_0, L_0 (independent of k) and k_0 such that*

$$v_k^2 \le L_0 G_k \quad \textit{on} \quad M_k - B_{r_0}(p_\infty; g_k) \quad \textit{for all} \quad k \ge k_0.$$

Remark 2.3. For the sake of the general case (7), we remark the following: Replacing $\widehat{M}_k$ by the closure of another fundamental domain of M_k in M_∞ if necessary, we can assume that

$$\mathrm{dist}_{g_\infty}\left(\partial \widehat{M}_k,\ \cup_{k_\lambda \in \Lambda_k} \gamma_{k_\lambda}(\widehat{M})\right) \ge r_k,$$

where $r_k \nearrow \infty$ when $k \to \infty$. For each $k_\lambda \in \Lambda_k$, set

$$G_{k,k_\lambda} := \Sigma_{\gamma \in \mathcal{G}_k} (G_\infty \circ \gamma_{k_\lambda}^{-1}) \circ \gamma \qquad \text{on} \quad M_\infty.$$

Then, G_{k,k_λ} is the normalized Green's function on M_k with pole at $\gamma_{k_\lambda}(p_\infty) \in M_k$. Replacing respectively G_k in Lemma 2.2 by

$$\widetilde{G}_k := \Sigma_{k_\lambda \in \Lambda_k} G_{k,k_\lambda}$$

and the cut-off function $\eta_{k,\delta}$ (giving in the proof of Lemma 2.4 below) by

$$\widetilde{\eta}_{k,\delta} := \min\Big\{1,\ \max_{k_\lambda\in\Lambda_k}\{\frac{(G_{k,k_\lambda}-\delta)_+}{\delta}\}\Big\} \qquad \text{on} \quad M_k - \{\cup_{k_\lambda\in\Lambda_k}\gamma_{k_\lambda}(p_\infty)\},$$

then we can obtain similar results to Lemma 2.2 and Lemmas 2.4, 2.5 below.

We now return to the proof. Recall that

$$Y(M_k,[g_k]) = Q_{(M_k,g_k)}(v_k).$$

Every function v_k is positive on M_∞ and so it is not an admissible function for the functional $Q_{(M_\infty,g_k)}$ restricted on $Int(\widehat{M}_k)$ (denote it by $Q_{(\widehat{M}_k,g_k)}$) because it does not vanish on the boundary $\partial\widehat{M}_k$ of $\widehat{M}_k$ ($\subset M_\infty$). Therefore, if we want to get an admissible function for $Q_{(\widehat{M}_k,g_k)}$, we need to multiply v_k by some cut-off function vanishing on the boundary $\partial\widehat{M}_k$. This is a delicate issue because, in general, the multiplication by a cut-off function makes the Yamabe quotient $Q_{(\widehat{M}_k,g_k)}(\ \cdot\ )$ increase. Moreover, since M_∞ has generally infinitely many ends, the number of the components of the boundary $\partial\widehat{M}_k$ increases rapidly as $k\to\infty$. This also complicates the construction of appropriate cut-off functions. Fortunately, using Lemma 2.1, we can construct a nice cut-off function on each $\widehat{M}_k^* := \widehat{M}_k - \{p_\infty\}$ with respect to the asymptotically flat metric $g_{k,AF} = G_k^{\frac{4}{n-2}}g_k$, and not with respect to the lifting metric g_k. The reason is that our construction heavily depends on the properties for G_∞ and G_k in Lemma 2.1, particularly on the property (v). For this reason, analysis on $(\widehat{M}_k^*, g_{k,AF})$ has definitely an advantage.

Lemma 2.4. *There exists a sequence of Lipschitz functions* $\eta_k (0\le\eta_k\le 1)$ *on the asymptotically flat manifolds* $\big(\widehat{M}_k^*, g_{k,AF}\big)$ *with boundary such that*

$$\eta_k \equiv 0 \quad \text{on} \ \ \partial\widehat{M}_k^*, \qquad \eta_k\equiv 1 \quad \text{outside a compact set of } \widehat{M}_k^*,$$

and for all k *sufficiently large*

$$\text{(9)}\qquad \begin{cases} \int_{\widehat{M}_k^*} G_k^{-1}|\nabla\eta_k|^2_{k,AF}d\mu_{k,AF} = o(1), \\ \int_{\{x\in\widehat{M}_k^*|\nabla\eta_k\neq 0\}} G_k^{-\frac{n}{n-2}}d\mu_{k,AF} = o(1) \qquad as \quad k\to\infty. \end{cases}$$

Set $\varphi_k := v_k\cdot G_k^{-1}\in C^\infty(\widehat{M}_k^*)$ and $u_k := \eta_k\cdot\varphi_k \in C^{0,1}(\widehat{M}_k^*)$ for $k\ge 1$. Then, using Lemmas 2.2 and Lemma 2.4, we have the following.

Lemma 2.5.

$$Q_{(\widehat{M}_k^*,g_{k,AF})}(u_k) \le Y(M_k,[g_k]) + o(1) \quad as \quad k\to\infty,$$

where

$$Q_{(\widehat{M}_k^*,g_{k,AF})}(u_k) = \frac{\int_{\widehat{M}_k^*}\alpha_n|\nabla u_k|^2_{k,AF}d\mu_{k,AF}}{\Big(\int_{\widehat{M}_k^*}u_k^{\frac{2n}{n-2}}d\mu_{k,AF}\Big)^{\frac{n}{n-2}}}.$$

We can now completes the proof of Case 1. From the construction of $\eta_k\cdot v_k$ and u_k, we obtain

$$Y(M_\infty,C_\infty) \le Q_{(M_\infty,g_\infty)}(\eta_k\cdot v_k) = Q_{(\widehat{M}_k,g_k)}(\eta_k\cdot v_k) = Q_{(\widehat{M}_k^*,g_{k,AF})}(u_k) \quad \text{for all } k\ge 1.$$

From this and Lemma 2.5, we obtain

$$\text{(10)}\qquad Y(M_\infty,C_\infty) \le \liminf_{k\to\infty} Y(M_k,[g_k]).$$

Applying the first assertion of Theorem 1.2 to each infinite conformal covering $(M_\infty, C_\infty) \to (M_k, [g_k])$, we also obtain

$$Y(M_k, [g_k]) < Y(M_\infty, C_\infty) \quad \text{for all} \quad k \geq 1.$$

These inequalities imply that

$$\limsup_{k\to\infty} Y(M_k, [g_k]) \leq Y(M_\infty, C_\infty). \tag{11}$$

It then follows from (10), (11) that

$$\lim_{k\to\infty} Y(M_k, [g_k]) = Y(M_\infty, C_\infty).$$

This completes the proof of Case 1.

Case 2. We consider the case when $Y(M_\infty, C_\infty) = \mathbf{Y}_n$. In this case, the renormalization argument in Case 1 does not work, and hence we can not get a similar uniform estimate to (5) for all v_k. However, combining the following approximation lemma (see [3, Proposition 3.4] for the proof) with Corollary 1.5, we will be able to overcome this difficulty.

Lemma 2.6. *Let (X, h) be a complete Riemannian manifold (possibly noncompact) with $0 < L_1 \leq R_h \leq L_2$ for some positive constants L_1, L_2. Let $\{h_j\}_{j\geq 1}$ be a sequence of complete metrics on X satisfying*

$$h_j \to h \quad \text{as} \quad j \to \infty$$

with respect to the uniform C^2-norm on (X, h). Then,

$$Y(X, [h_j]) \to Y(X, [h]) \quad \text{as} \quad j \to \infty.$$

From Corollary 1.5, we have that (M_∞, C_∞) is locally conformally flat and hence, the same property holds for (M, C). In other words, if a metric g' on M is not locally conformally flat, then $Y(M_\infty, [g'_\infty]) < \mathbf{Y}_n$ for its lifting g'_∞ to M_∞. Take a sequence $\{g(j)\}_{j\geq 1}$ of metrics on M, each of which is not a locally conformally flat metric, such that

$$g(j) \to g \quad \text{on} \quad M \quad \text{as} \quad j \to \infty$$

with respect to the uniform C^2-norm on (M, g). Let $g(j)_k$ and $g(j)_\infty$ denote respectively the liftings of $g(j)$ to M_k and M_∞. We now apply Lemma 2.6 to the metric g_∞ and to the sequence $\{g(j)_\infty\}_{j\geq 1}$ in order to obtain that for any $\varepsilon > 0$ there exists j_1 such that

$$\left| Y(M_\infty, [g(j)_\infty]) - Y(M_\infty, C_\infty) \right| < \varepsilon \cdot Y(M_\infty, C_\infty) \quad \text{for} \quad j \geq j_1.$$

Since each $g(j)_k$ for $k \geq 1$ is a lifting metric of $g(j) \in \mathcal{M}(M)$, the sequence $\{g(j)_k\}_{j\geq 1}$ C^2-converges to g_k uniformly in j for all k. Then, Lemma 2.6 also implies the following: There exists j_2 such that

$$\left| Y(M_k, [g(j)_k]) - Y(M_k, [g_k]) \right| < \varepsilon \cdot Y(M_k, [g_k]) \quad \text{for} \quad j \geq j_2, \ k \geq 1.$$

Set $j_0 := \max\{j_1, j_2\}$. Since $g(j_0)$ is not a locally conformally flat metric, then the result of Case 1 implies that, for all k sufficiently large

$$\left| Y(M_k, [g(j_0)_k]) - Y(M_\infty, [g(j_0)_\infty]) \right| < \varepsilon \cdot \mathbf{Y}_n,$$

where $j_0 := \max\{j_1, j_2\}$. These three inequalities imply that, for all k sufficiently large

$$\left| Y(M_k, [g_k]) - Y(M_\infty, C_\infty) \right| < 3\varepsilon \cdot \mathbf{Y}_n,$$

and hence

$$\lim_{k\to\infty} Y(M_k, [g_k]) = Y(M_\infty, C_\infty).$$

This completes the proof of Case 2, and thus that of the second assertion of Theorem 1.2. □

Corollary 2.7. *Let (M, C) be a closed positive conformal n-manifold with a loop c whose homology class $[c] \in H_1(M; \mathbb{Z})$ is of infinite order. Let (M_k, C_k) and (M_∞, C_∞) denote respectively the normal k-fold conformal covering and the normal infinite conformal covering of (M, C) associated to $[c]$. Then,*

$$Y(M, C) < Y(M_\infty, C_\infty) \quad \textit{and} \quad \lim_{k\to\infty} Y(M_k, C_k) = Y(M_\infty, C_\infty).$$

Proof. Consider a subsequence $\{(M_{2^j}, C_{2^j})\}_{j\geq 0}$ of $\{(M_k, C_k)\}_{k\geq 1}$. We can apply Theorem 1.2 directly to the normal infinite conformal covering (M_∞, C_∞) and to the subsequence $\{(M_{2^j}, C_{2^j})\}_{j\geq 0}$ in order to conclude that

$$Y(M, C) < Y(M_\infty, C_\infty).$$

We can not apply Theorem 1.2 directly because $M_{k+1} \to M_k$ is not a covering for $k \geq 2$. However, in order to prove

$$\limsup_{k\to\infty} Y(M_k, C_k) \leq Y(M_\infty, C_\infty), \tag{12}$$

the following property for $\{M_k\}_{k\geq 1}$ is enough (see the proof of the first assertion of Theorem 1.2): There exists a fundamental domain of M_k (for every $k \geq 1$) in M_∞ containing p_∞ such that its closure $\widehat{M}_k$ $(\subset M_\infty)$ satisfies

$$\{x \in M_\infty \mid \mathrm{dist}_{g_\infty}(x, p_\infty) \leq \nu(k)\} \subsetneq \widehat{M}_k, \qquad \widehat{M}_k \subsetneq Int\big(\widehat{M}_{k+1}\big),$$

$\partial\widehat{M}_k$ is piecewise smooth, and $\partial\widehat{M}_{k+1}, \partial\widehat{M}_k$ have the same projection on M_k for all $k \geq 1$. Here, $\nu(k)$ is a sequence of positive numbers going to infinity with $\nu(k) \leq \nu(k+1)$ for all $k \geq 1$.

Assume that $Y(M_\infty, C_\infty) < \mathbf{Y}_n$. Then, by considering the quantities Q_k defined in the proof of Theorem 1.2, the following still holds:

$$Y(M_k, C_k) \leq Q_k \quad \text{for} \quad k \geq 1 \qquad \text{and} \qquad \lim_{k\to\infty} Q_k = Y(M_\infty, C_\infty).$$

Hence, there exists a small constant $\delta_0 > 0$ such that

$$Y(M_k, C_k) \leq Y(M_\infty, C_\infty) + \delta_0 < \mathbf{Y}_n \quad \text{for all} \quad k \quad \text{sufficiently large.}$$

This property combined with the normality of the covering $M_\infty \to M$ implies (see (10) in the proof of Theorem 1.2)

$$Y(M_\infty, C_\infty) \leq \liminf_{k\to\infty} Y(M_k, C_k). \tag{13}$$

Thus, when $Y(M_\infty, C_\infty) < \mathbf{Y}_n$, it follows from (12) and (13) that

$$\lim_{k\to\infty} Y(M_k, C_k) = Y(M_\infty, C_\infty).$$

The case $Y(M_\infty, C_\infty) = \mathbf{Y}_n$ can be treated in the same way as in the proof of Theorem 1.2. This completes the proof of Corollary 1.5. □

The rest of this section is devoted to the proofs of Lemmas 2.2, 2.4 and 2.5.

Proof of Lemma 2.2. First recall that for $k \geq 1$

$$\begin{cases} \Delta_{g_k} G_k = \frac{R_{g_k}}{\alpha_n} G_k \quad \text{on} \quad M_k^*, \\ \Delta_{g_k} v_k = \frac{R_{g_k}}{\alpha_n} v_k - \frac{Y(M_k,[g_k])}{\alpha_n} v_k^{\frac{n+2}{n-2}} \quad \text{on} \quad M_k. \end{cases} \tag{14}$$

The property for v_k presented in (8) implies the existence of positive constants r_0, k_1 such that

$$v_k^{\frac{4}{n-2}} \leq \frac{\alpha_n}{2 \cdot Y(M_k,[g_k])} \qquad \text{on} \quad M_k - B_{r_0}(p_\infty; g_k) \quad \text{for all} \quad k \geq k_1.$$

Thus, we obtain from this inequality and the second equation in (14) that

$$\begin{aligned} \Delta_{g_k} v_k^2 &\geq \frac{R_{g_k}}{\alpha_n} v_k^2 + \frac{R_{g_k}}{\alpha_n} v_k^2 \Big(1 - \frac{2 \cdot Y(M_k,[g_k])}{\alpha_n} v_k^{\frac{4}{n-2}} \Big) \\ &\geq \frac{R_{g_k}}{\alpha_n} v_k^2 \qquad \text{on} \quad M_k - B_{r_0}(p_\infty; g_k) \quad \text{for all} \quad k \geq k_1. \end{aligned}$$

Set

$$f_k := v_k^2 - L_0 G_k \quad \text{on} \quad M_k - B_{r_0}(p_\infty; g_k),$$

where L_0 is a positive constant to be fixed later.

It follows from the above inequality for v_k^2 and the first equation in (14) that for all $k \geq k_1$

$$\Delta_{g_k} f_k \geq \frac{R_{g_k}}{\alpha_n} f_k \quad \text{on} \quad M_k - B_{r_0}(p_\infty; g_k).$$

Then, the maximum principle implies that for all $k \geq k_1$

$$\sup_{M_k - B_{r_0}(p_\infty; g_k)} f_k \leq \sup_{\partial B_{r_0}(p_\infty; g_k)} f_k.$$

Since $\{v_k\}_{k\geq 1}$ (*resp.* $\{G_k\}_{k \geq 1}$) are uniformly bounded from above (*resp.* from below) on $\overline{B_{r_0}(p_\infty; g_k)}$, there exist large constants L_0 and k_2 such that

$$\sup_{\partial B_{r_0}(p_\infty; g_k)} f_k \leq 0 \quad \text{for all} \quad k \geq k_2.$$

These inequalities imply that for all $k \geq k_0 := \max\{k_1, k_2\}$

$$v_k^2 \leq L_0 G_k \quad \text{on} \quad M_k - B_{r_0}(p_\infty; g_k).$$

□

Proof of Lemma 2.4. We will modify the argument in [5, Sections 5 and 7]. For any $\delta > 0,\ k \geq 1$, set

$$\eta_{k,\delta} := \min \Big\{ 1, \frac{(G_k - \delta)_+}{\delta} \Big\} \quad \text{on} \quad M_k^*,$$

where $(G_k - \delta)_+$ denotes the nonnegative part of the function $G_k - \delta$.

We start by showing the following: For any $\varepsilon > 0$, there exist a small $\delta_0 = \delta_0(\varepsilon) > 0$ and a large integer $k_0 = k_0(\varepsilon)$ such that, for any $k \geq k_0$,

$$\eta_{k,\delta_0} \equiv 0 \quad \text{on} \quad \partial \widehat{M}_k^*, \qquad \eta_{k,\delta_0} \equiv 1 \quad \text{outside a compact set of } \widehat{M}_k^* \tag{15}$$

and

$$\begin{cases} \int_{\widehat{M}_k^*} G_k^{-1} |\nabla \eta_{k,\delta_0}|^2_{k,AF} d\mu_{k,AF} \leq \varepsilon, \\ \int_{\{x \in \widehat{M}_k^* | \nabla \eta_{k,\delta_0} \neq 0\}} G_k^{-\frac{n}{n-2}} d\mu_{k,AF} \leq \varepsilon. \end{cases}$$

From Lemma 2.1-(iii), (v), we can choose $\delta_0 = \delta_0(\varepsilon) > 0$ so that

$$\int_{G_\infty^{-1}\left([\delta_0, 2\delta_0]\right)} G_\infty d\mu_\infty \le \frac{\varepsilon}{8L^2} \qquad \text{and} \qquad \int_{G_\infty^{-1}\left([\delta_0, 2\delta_0]\right)} G_\infty^{\frac{n}{n-2}} d\mu_\infty \le \frac{\varepsilon}{2}, \tag{16}$$

where L is the constant given in Lemma 2.1-(iv). From Lemma 2.1-(iii), there exists a large integer $k_0 = k_0(\varepsilon)$ such that for all $k \ge k_0$

$$\partial \widehat{M}_k \subset G_k^{-1}\left([0, \delta_{k,\delta_0}]\right).$$

This implies that the cut-off functions η_{k,δ_0} (for $k \ge k_0$) satisfy the condition (15). Combining the estimate (16) with Lemma 2.1-(ii), (iv), (replacing k_0 by another large integer if necessary) we obtain for all $k \ge k_0$

$$\begin{aligned}
\int_{\widehat{M}_k^*} G_k^{-1} |\nabla \eta_{k,\delta_0}|_{k,AF}^2 d\mu_{k,AF} &= \int_{M_k^*} G_k^{-1} |\nabla \eta_{k,\delta_0}|_{k,AF}^2 d\mu_{k,AF} \\
&= \int_{G_k^{-1}\left([\delta_0, 2\delta_0]\right)} G_k |\nabla \eta_{k,\delta_0}|_k^2 d\mu_k \\
&\le 4L^2 \int_{G_k^{-1}\left([\delta_0, 2\delta_0]\right)} G_k d\mu_k \le \varepsilon, \\
\int_{\{x \in \widehat{M}_k^* \,|\, \nabla \eta_{k,\delta_0} \ne 0\}} G_k^{-\frac{n}{n-2}} d\mu_{k,AF} &\le \int_{G_k^{-1}\left([\delta_0, 2\delta_0]\right)} G_k^{\frac{n}{n-2}} d\mu_k \le \varepsilon.
\end{aligned}$$

From the above argument, we can choose two sequences $\{\delta_0(1/j)\}_{j\ge 1}, \{k_0(1/j)\}_{j\ge 1}$ satisfying

$$\begin{aligned}
&\delta_0(1) > \delta_0(1/2) > \cdots > \delta_0(1/j) > \delta_0(1/(j+1)) > \cdots \searrow 0, \\
&k_0(1) < k_0(1/2) < \cdots < k_0(1/j) < k_0(1/(j+1)) < \cdots \nearrow \infty.
\end{aligned}$$

Then, we can define the desired sequence $\{\eta_k\}_{k\ge 1}$ of cut-off functions by

$$\eta_k := \begin{cases} \eta_{k,\delta_0(1)} & \text{for} \quad k_0(1) \le k < k_0(1/2), \\ \quad \cdots & \\ \eta_{k,\delta_0(1/j)} & \text{for} \quad k_0(1/j) \le k < k_0(1/(j+1)), \\ \quad \cdots & \end{cases}$$

This completes the proof. □

Before proving Lemma 2.5, we make the following remark. In [5, Theorem 4.1], we constructed a sequence $\{u_k\}_{k\ge 1}$ of *approximate* Yamabe minimizers satisfying $u_k \le 1$ for all k. In the present case, we only have the inequality $u_k = \eta_k (v_k \cdot G_k^{-1}) \le L_0^{\frac{1}{2}} G_k^{-\frac{1}{2}}$. Nevertheless, due to the asymptotic estimates (9) in Lemma 2.4, this inequality is enough for our purposes.

Proof of Lemma 2.5. Replacing L_0 given in Lemma 2.2 by a large constant if necessary, we obtain that for all $k \ge 1$

$$v_k^2 \le L_0 G_k \quad \text{on} \quad M_k^*. \tag{17}$$

This inequality follows from the estimate (5), Lemma 2.1-(ii), (iii), Lemma 2.2 and the following normalizations

$$\lim_{x\to p_\infty} \mathrm{dist}_{g_\infty}(x,p_\infty)^{n-2}G_\infty(x)=1, \quad \lim_{x\to p_\infty} \mathrm{dist}_{g_k}(x,p_\infty)^{n-2}G_k(x)=1 \quad \text{for } k\geq 1.$$

Recall that

$$\text{(18)} \qquad \begin{cases} \int_{\widehat{M}_k^*} \varphi_k^{\frac{2n}{n-2}} d\mu_{k,AF}=1, \\ \alpha_n \int_{\widehat{M}_k^*} |\nabla\varphi_k|^2 d\mu_{k,AF} = Q_{(\widehat{M}_k^*, g_{k,AF})}(\varphi_k)=Y(M_k,[g_k]). \end{cases}$$

From Lemma 2.1-(iv), Lemma 2.4, (17) and (18), we have

$$\begin{aligned} \alpha_n \int_{\widehat{M}_k^*} |\nabla u_k|^2_{k,AF} d\mu_{k,AF} &\leq \alpha_n \Bigg\{ \int_{\widehat{M}_k^*} |\nabla\varphi_k|^2_{k,AF} d\mu_{k,AF} \\ &+ 2\sqrt{\int_{\widehat{M}_k^*} |\nabla\varphi_k|^2_{k,AF} d\mu_{k,AF}} \sqrt{\int_{\widehat{M}_k^*} \varphi_k^2 |\nabla\eta_k|^2_{k,AF} d\mu_{k,AF}} + \int_{\widehat{M}_k^*} \varphi_k^2 |\nabla\eta_k|^2_{k,AF} d\mu_{k,AF} \Bigg\} \\ &\leq Y(M_k,[g_k]) \\ &+ 2\sqrt{\alpha_n L_0 Y(M_k,[g_k]) \int_{\widehat{M}_k^*} G_k^{-1} |\nabla\eta_k|^2_{k,AF} d\mu_{k,AF}} + L_0 \int_{\widehat{M}_k^*} G_k^{-1} |\nabla\eta_k|^2_{k,AF} d\mu_{k,AF} \\ &= Y(M_k,[g_k]) + o(1) \qquad \text{as } k\to\infty. \end{aligned}$$

On the other hand, from Lemma 2.4, (17) and (18), we also have

$$\begin{aligned} \int_{\widehat{M}_k^*} u_k^{\frac{2n}{n-2}} d\mu_{k,AF} &\geq \int_{\widehat{M}_k^*} \varphi_k^{\frac{2n}{n-2}} d\mu_{k,AF} - \int_{\{x\in\widehat{M}_k^* \mid \nabla\eta_k\neq 0\}} \varphi_k^{\frac{2n}{n-2}} d\mu_{k,AF} \\ &= 1 - L_0^{\frac{n}{n-2}} \int_{\{x\in\widehat{M}_k^* \mid \nabla\eta_k\neq 0\}} G_k^{-\frac{n}{n-2}} d\mu_{k,AF} \\ &= 1+o(1) \qquad \text{as } k\to\infty. \end{aligned}$$

Therefore,

$$Q_{(\widehat{M}_k^*, g_{k,AF})}(u_k) = \frac{\int_{\widehat{M}_k^*} \alpha_n |\nabla u_k|^2_{k,AF} d\mu_{k,AF}}{\left(\int_{\widehat{M}_k^*} u_k^{\frac{2n}{n-2}} d\mu_{k,AF} \right)^{\frac{n}{n-2}}} \leq Y(M_k,[g_k]) + o(1) \quad \text{as } k\to\infty,$$

and this completes the proof. □

3. Proof of Theorem 1.4 and Corollary 1.5

In this section, we prove Theorem 1.4 and Corollary 1.5.

Proof of Theorem 1.4. First note that, under the condition that either $3\leq n\leq 5$, or $(M_\infty,[g_\infty])$ is conformally flat near p_∞, the mass $\mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF})$ is well-defined [10] (cf. [26]). We also remark that for a conformal metric

$$\widetilde{g_\infty} = u^{\frac{4}{n-2}} g_\infty$$

on $M_\infty^* = M_\infty - \{p_\infty\}$, the corresponging Green's function $\widetilde{G_\infty}$ is given by

$$\widetilde{G_\infty} = u(p_\infty)^{-1} u^{-1} G_\infty.$$

Then, the corresponging asymptotically flat metric $\widetilde{g_\infty}_{,AF} = \widetilde{G_\infty}^{\frac{4}{n-2}} \widetilde{g_\infty}$ and its mass $\mathfrak{m}_{\mathrm{ADM}}(\widetilde{g_\infty}_{,AF})$ are given by

$$\widetilde{g_\infty}_{,AF} = u(p_\infty)^{-\frac{4}{n-2}} g_{\infty,AF}, \qquad \mathfrak{m}_{\mathrm{ADM}}(\widetilde{g_\infty}_{,AF}) = u(p_\infty)^{-2} \mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}),$$

respectively. Therefore, we may assume that g is a unit volume Yamabe metric on M with $R_g \equiv Y(M,[g]) > 0$.

We use the same notations as in the proof of Theorem 1.2 for $\{(M_k, g_k)\}_{k\geq 1}$, $\{\widehat{M}_k\}_{k\geq 1}$, $\{\widehat{\Omega}_k\}_{k\geq 1}$, $\{G_k\}_{k\geq 1}$ and $\{g_{k,AF}\}_{k\geq 1}$. From the Positive Mass Theorem proved by Schoen-Yau [30, 31, 32] and Lohkamp [27] (cf. [10, 26, 34, 35, 38]), each mass $\mathfrak{m}_{\mathrm{ADM}}(g_{k,AF})$ is positive $\big($since $(M_k,[g_k]) \not\cong (S^n,[g_{\mathbb{S}}])\big)$. Then, using a similar argument to that in the proof of [2, Theorem 6.13] combined with the properties for G_∞, G_k in Lemma 2.1, we obtain (cf. [4])

$$\lim_{k\to\infty} \mathfrak{m}_{\mathrm{ADM}}(g_{k,AF}) = \mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}).$$

Therefore, the positivity of every mass $\mathfrak{m}_{\mathrm{ADM}}(g_{k,AF})$ implies

$$\mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}) \geq 0.$$

In the rest of this proof, we assume that $\mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}) = 0$. Set

$$\widehat{\Omega}_2^* := \widehat{\Omega}_2 - \{p_\infty\} \quad \big(\ \widehat{M}^* \subsetneq \widehat{\Omega}_2^* \subset \widehat{M}_2^* \subset\ M_\infty^*\ \big).$$

Modifying the argument in [28, Lemma 3], [26, Lemma 10.7], we obtain the following. (The argument in the corresponding result [2, Proposition 6.14] is not sufficient without the changes we describe below.)

Lemma 3.1. *Under the assumption that* $\mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}) = 0$*,* $g_{\infty,AF}$ *is Ricci-flat on* $\widehat{\Omega}_2^*$*.*

Proof. For any symmetric 2-tensor $h = (h_{ij})$ on M_∞ with compact support in the interior $Int\big(\widehat{\Omega}_2^*\big)$ of $\widehat{\Omega}_2^*$, we define a smooth family $\{g_\infty^t\}$ of smooth metrics by

$$g_\infty^t := g_\infty + tG_\infty^{-\frac{4}{n-2}} h \quad \text{on} \quad M_\infty$$

for small t $(-\varepsilon < t < \varepsilon)$. Since $\widehat{\Omega}_2 \subset \widehat{M}_k$ for any $k \geq 2$, we can identify $g_\infty^t|_{\widehat{M}_k}$ with the metric $g_k + tG_\infty^{-\frac{4}{n-2}} h$ on M_k (for $k \geq 2$), and then denote its lifting to M_∞ by $g_\infty^{k,t}$. By Lemma 2.6, there exists sufficiently small $\varepsilon_0 > 0$ such that $Y(M_\infty,[g_\infty^t]) > 0$, $Y(M_\infty,[g_\infty^{k,t}]) > 0$ and $Y(M_k,[g_k + tG_\infty^{-\frac{4}{n-2}} h]) > 0$ for any t $(-\varepsilon_0 < t < \varepsilon_0)$, $k \geq 2$. Note that, since $g_\infty^{k,t} = g_\infty^t = g_\infty$ on a small neighborhood of p_∞, the metrics $g_\infty^{k,t}$, g_∞^t are locally comformally flat near p_∞ provided that $n \geq 6$. Applying the first assertion of Theorem 1.4 to each normal infinite Riemannian covering $(M_\infty,\ g_\infty^{k,t}) \to (M_k,\ g_k + tG_\infty^{-\frac{4}{n-2}} h)$, we obtain

$$\mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}^{k,t}) \geq 0 \quad \text{for all} \quad t\ (-\varepsilon_0 < t < \varepsilon_0) \quad \text{and} \quad k \geq 2. \tag{19}$$

For each $k \geq 2$ and t $(-\varepsilon_0 < t < \varepsilon_0)$, let $G_\infty^{k,t}$ denote the normalized minimal positive Green's function on M_∞ for $\mathcal{L}_{g_\infty^{k,t}}$ with pole at p_∞. Replacing ε_0 by a smaller positive constant if necessary, we may assume that, for all $k \geq 2$ and t $(-\varepsilon_0 < t < \varepsilon_0)$,

$$2R_{g_\infty} \geq R_{g_\infty^{k,t}} \geq \frac{1}{2} R_{g_\infty} \ \Big(= \frac{1}{2} Y(M,[g]) > 0\ \Big) \quad \text{on} \quad M_\infty, \tag{20}$$

$$Y(M_k,[g_\infty^{k,t}]) \geq \frac{1}{2}Y(M_k,[g_k]) > 0. \tag{21}$$

From (21), Aubin's Lemma and the first assertion of Theorem 1.2, we have

$$Y(M_\infty,[g_\infty^{k,t}]) > \frac{1}{2}Y(M,[g]) > 0. \tag{22}$$

The sublemma we present next will have been proved at the end of this section.

Sublemma 3.2. *For any t $(-\varepsilon_0 < t < \varepsilon_0)$, there exists uniquely a normalized minimal positive Green's function G_∞^t for $\mathcal{L}_{g_\infty^t}$ with pole at p_∞ such that*

$$\mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}^t) = \lim_{k\to\infty}\mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}^{k,t}). \tag{23}$$

Hence, it follows from (19) and (23) that

$$\mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}^t) \geq 0 \quad \text{for all} \;\; t\ (-\varepsilon_0 < t < \varepsilon_0).$$

Similarly to [26, Lemma 10.7], the mass zero condition $\mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}) = 0$ implies

$$0 = \frac{d}{dt}\Big|_{t=0}\mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}^t) = \int_{\widehat{\Omega}_2^*}\langle \mathrm{Ric}_{g_{\infty,AF}},\ h\rangle_{\infty,AF}\, d\mu_{\infty,AF},$$

where $\mathrm{Ric}_{g_{\infty,AF}}$ denotes the Ricci curvature of $g_{\infty,AF}$. This holds for all compactly supported h in $Int(\widehat{\Omega}_2^*)$. Therefore $\mathrm{Ric}_{g_{\infty,AF}} = 0$ on $\widehat{\Omega}_2^*$, and then this completes the proof of Lemma 3.1. □

From the existence result in [10] (cf. [2, Lemma 6.17]), there exist *harmonic coordinates near infinity* $x = (x^1,\cdots,x^n)$ on $(\widehat{\Omega}_2^*,\ g_{\infty,AF})$. Namely, (x^i) are smooth functions on $\widehat{\Omega}_2^*$ which give asymptotically flat coordinates near infinity for $(\widehat{\Omega}_2^*,\ g_{\infty,AF})$, and for which

$$\Delta_{g_{\infty,AF}}x^i = 0 \quad \text{on} \;\; \widehat{\Omega}_2^*, \qquad \frac{\partial x^i}{\partial \nu} = 0 \quad \text{on} \;\; \partial\widehat{\Omega}_2^*.$$

Here, ν is the outword unit normal vector field normal to $\partial\widehat{\Omega}_2^*$ with respect to $g_{\infty,AF}$. We now apply the Bochner technique. The harmonicity of (x^i) implies that $\{dx^i\}$ are harmonic 1-forms on $(\widehat{\Omega}_2^*,\ g_{\infty,AF})$. The Bochner formula for the 1-forms $\{dx^i\}$, combined with the conditions $\frac{\partial x^i}{\partial\nu} = 0$ on $\partial\widehat{\Omega}_2^*$ and $\mathrm{Ric}_{g_{\infty,AF}} = 0$ on $\widehat{\Omega}_2^*$, implies that (cf. [10, Theorem 4.4], [26, Proposition 10.2])

$$\mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}) = a_n\sum_{i=1}^n\int_{\widehat{\Omega}_2^*}|\nabla dx^i|^2 d\mu_{\infty,AF},$$

where $a_n > 0$ is a specific universal positive constant depending only on n. Then, combining the condition $\mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}) = 0$ with the above equation, we obtain that the 1-forms $\{dx^i\}$ are parallel on $\widehat{\Omega}_2^*$ with respect to $g_{\infty,AF}$. Since the coframe $\{dx^i\}$ is orthonormal at infinity, $\{dx^i\}$ is a parallel orthonormal coframe everywhere on $(\widehat{\Omega}_2^*,\ g_{\infty,AF})$. This implies that the map $x = (x^1,\cdots,x^n) : (\widehat{\Omega}_2^*,\ g_{\infty,AF}) \to (\mathbb{R}^n, g_{\mathbb{E}})$ is a local isometry, where $g_{\mathbb{E}}$ stands for the Euclidean metric on $\mathbb{R}^n$. Hence, $g_{\infty,AF}$ is locally conformally flat on $\widehat{\Omega}_2^*$. Therefore, g_∞ is locally conformally flat everywhere on M_∞ since it is the lifting of $g = g_\infty|_{\widehat{M}}$.

Since g_∞ is the lifiting of g on the closed manifold M, g_∞ is a complete metric with $R_{g_\infty} \equiv Y(M,[g]) > 0$ and with bounded Ricci curvature. Then, the results in [33, Proposition 3.3 and Proposition 4.4] combined with the locally conformally flatness of g_∞ and with $\mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}) = 0$ imply that $(M_\infty, [g_\infty])$ is a simply connected domain in $(S^n, [g_{\mathbb{S}}])$ if $n \geq 4$.

For the rest of this proof, we assume that $n = 3$ (and that $\mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}) = 0$). By combining [14, Theorem 8.1] (cf. [18]) with $Y(M, [g]) > 0$, (replacing M by its orientable double covering if necessary) M can be decomposed uniquely into *prime* closed 3-manifolds

$$M = N_1 \# \cdots \# N_{\ell_1} \# \ell_2 (S^1 \times S^2),$$

where $\pi_1(N_j)$ is finite for $j = 1, \cdots, \ell_1$ and ℓ_1, ℓ_2 are nonnegative integers. By the C-prime decomposition theorem for closed locally conformally flat manifolds [19, 20], the locally conformally flat manifold $(M, [g])$ can be decomposed as

$$(M, [g]) = (N_1, \check{C}_1) \# \cdots \# (N_{\ell_1}, \check{C}_{\ell_1}) \# (S^1 \times S^2, \check{\check{C}}_1) \# \cdots \# (S^1 \times S^2, \check{\check{C}}_{\ell_2}),$$

where each $\check{C}_i$ and $\check{\check{C}}_j$ are flat conformal structures on N_i and $S^1 \times S^2$ respectively. Then, Kuiper's Theorem [25] implies that each $(N_i, \check{C}_i)$ is a non-trivial quotient of $(S^3, [g_{\mathbb{S}}])$. After taking an appropriate finite covering M' of M, we have

$$M' = \# \ell (S^1 \times S^2) \quad \text{for some} \quad \ell \geq 1.$$

Consider the (infinite) universal covering $\widetilde{M}$ of M and denote the lift of the metric g by $\widetilde{g}$. Then, $\widetilde{M} \to M'$ is also the (infinite) universal covering of M' such that $\pi_1(M')$ has a decending chain of finite index subgroups tending to $\pi_1(\widetilde{M}) = \{e\}$. Let g' be the lifting of g to M'. Applying the first assertion of Theorem 1.4 to the normal infinite Riemannian covering $(\widetilde{M}, \widetilde{g}) \to (M', g')$, we have that

$$\mathfrak{m}_{\mathrm{ADM}}(\widetilde{g}_{AF}) \geq 0.$$

Since both $\widetilde{M} \to M$ and $M_\infty \to M$ are coverings of M, then there exists a unique universal covering $\mathcal{P} : \widetilde{M} \to M_\infty$. Since $(M_\infty, [g_\infty])$ is locally conformally flat, we can take a metric $h \in [g_\infty]$ on M_∞ which is flat near p_∞. With respect to this metric h, take Euclidean coordinates $x = (x^1, \cdots, x^n)$ around p_∞ with $x(p_\infty) = 0$. Note that

$$\mathfrak{m}_{\mathrm{ADM}}(h_{AF}) = c\ \mathfrak{m}_{\mathrm{ADM}}(g_{\infty,AF}) = 0,$$

where $c > 0$ is some positive constant. Then, around p_∞, the normalized minimal positive Green's function G_∞ can be expressed by

$$G_\infty(x) = \frac{1}{|x|} + O(|x|), \qquad |x| = \mathrm{dist}_h(p_\infty, x). \tag{24}$$

Let $\widetilde{h}$ be the lifting of h to $\widetilde{M}$. Fix a point in $\mathcal{P}^{-1}(p_\infty)$ and denote it also by $p_\infty \in \widetilde{M}$. Using the lifting coordinates of x around $p_\infty \in \widetilde{M}$, we can express the normalized minimal positive Green's function $\widetilde{G}$ for $\mathcal{L}_{\widetilde{h}}$ with pole at p_∞ by

$$\widetilde{G}(x) = \frac{1}{|x|} + A + O(|x|), \qquad |x| = \mathrm{dist}_{\widetilde{h}}(p_\infty, x),$$

where $A = c_0\ \mathfrak{m}_{\mathrm{ADM}}(\widetilde{h}_{AF})$ for some universal positive constant $c_0 > 0$. From the nonnegativity of $\mathfrak{m}_{\mathrm{ADM}}(\widetilde{g}_{AF})$, we have

$$A = c_0\ \mathfrak{m}_{\mathrm{ADM}}(\widetilde{h}_{AF}) = c_0 c\ \mathfrak{m}_{\mathrm{ADM}}(\widetilde{g}_{AF}) \geq 0.$$

Since the universal covering $\widetilde{M} \to M_\infty$ is normal, we can apply Lemma 2.1-(i) to G_∞ and $\widetilde{G}$ (even if M_∞ is noncompact). Then,

$$G_\infty(x) = \sum_{\gamma \in \mathcal{G}} (\widetilde{G} \circ \gamma)(x) = \frac{1}{|x|} + A + \sum_{\gamma \neq \mathrm{id}} (\widetilde{G} \circ \gamma)(x) + O(|x|), \tag{25}$$

where $|x| = \mathrm{dist}_{\widetilde{h}}(p_\infty, x)$ and $\mathcal{G} = \pi_1(M_\infty)$ denotes the group of deck transformations for the covering $\widetilde{M} \to M_\infty$. Hence, it follows from (24) and (25) that

$$A + \sum_{\gamma \neq \mathrm{id}} (\widetilde{G} \circ \gamma)(p_\infty) = 0.$$

Since $A \geq 0$ and $\widetilde{G} > 0$, we obtain that $A = 0$ and $\pi_1(M_\infty) = \{e\}$. Thus, $M_\infty = \widetilde{M}$ is simply connected and this completes the proof of Theorem 1.4. □

Proof of Corollary 1.5. When $n \geq 6$, the assumption $Y(M_\infty, C_\infty) = \mathbf{Y}_n$ combined with Remark 1.6 implies that (M_∞, C_∞) is locally conformally flat. Then, for any dimension $n \geq 3$, Theorem 1.4 implies that, for the lifting $g_\infty \in C_\infty$ of a metric $g \in C$ to M_∞,

$$\mathfrak{m}_{\mathrm{ADM}}(g_{\infty, AF}) \geq 0.$$

If $\mathfrak{m}_{\mathrm{ADM}}(g_{\infty, AF}) > 0$, a similar argument to that of [28, Theorem 1] and [34, Chapter 5, Theorem 4.1] implies that $Y(M_\infty, C_\infty) < \mathbf{Y}_n$. This is a contradiction. Hence,

$$\mathfrak{m}_{\mathrm{ADM}}(g_{\infty, AF}) = 0.$$

We can now apply Theorem 1.4 and obtain the desired result. □

Proof of Sublemma 3.2. Firstly, we prove some uniform estimates for $G_\infty^{k,t}$ $(k \geq 2,\ -\varepsilon_0 < t < \varepsilon_0)$. Each Green's function $G_\infty^{k,t}$ belongs to $L^1(M_\infty^*; g_\infty^{k,t})$ because of Lemma 2.1-(v). More precisely, using the arguments as in the proof of [33, Proposition 2.4-(ii)], we obtain that

$$\int_{M_\infty^*} G_\infty^{k,t} d\mu_{g_\infty^{k,t}} \leq \frac{c_n}{\min_{M_\infty} R_{g_\infty^{k,t}}} \quad \text{for} \quad k \geq 2,\ t\ (-\varepsilon_0 < t < \varepsilon_0),$$

where c_n is the positive constant given in Section 2. From this and the inequality (20), we have the following uniform estimate

$$\int_{M_\infty^*} G_\infty^{k,t} d\mu_{g_\infty^{k,t}} \leq \frac{2c_n}{Y(M,[g])} \quad \text{for} \quad k \geq 2,\ t\ (-\varepsilon_0 < t < \varepsilon_0). \tag{26}$$

We can assume, without loss of generality, that $\mathrm{dist}_{g_\infty}(\widehat{M}_1, \partial \widehat{M}_2) \geq 2$, and that

$$\mathrm{dist}_{g_\infty^t}(\widehat{M}_1, \partial \widehat{M}_2) \geq 1, \quad \mathrm{dist}_{g_\infty^{k,t}}(\widehat{M}_1, \partial \widehat{M}_2) \geq 1 \quad \text{for} \quad k \geq 2,\ t\ (-\varepsilon_0 < t < \varepsilon_0).$$

Recall that each $G_\infty^{k,t}$ satisfies the linear elliptic equation $\mathcal{L}_{g_\infty^{k,t}} G_\infty^{k,t} = 0$ on M_∞^*. Applying the standard L^p-estimates and the estimate [17, Proposition 1.2.7] to this equation, we obtain the existence of some positive constant K (independent of k and t) for which

$$||G_\infty^{k,t}||_{W^{2,\frac{n+1}{2}}(M_\infty - \widehat{M}_2; g_\infty^{k,t})} \leq K ||G_\infty^{k,t}||_{L^1(M_\infty - \widehat{M}_1; g_\infty^{k,t})}.$$

This inequality was derived using (20), the facts that g_∞ is the lifting metric of g on the closed manifold M and that $g_\infty^{k,t}$ is C^2-close to g_∞ uniformly on M_∞ (independently of k and t). It then follows from this estimate and (26) that

$$(27)\qquad ||G_\infty^{k,t}||_{W^{2,\frac{n+1}{2}}(M_\infty - \widehat{M}_2;g_\infty^{k,t})} \leq \frac{2Kc_n}{Y(M,[g])} \quad \text{for} \ \ k \geq 2,\ t\ (-\varepsilon_0 < t < \varepsilon_0).$$

Similarly, we also obtain that

$$(28)\qquad ||G_\infty^{k,t}||_{W^{1,2}(M_\infty - \widehat{M}_2;g_\infty^{k,t})} \leq \frac{2K'c_n}{Y(M,[g])} \quad \text{for} \ \ k \geq 2,\ t\ (-\varepsilon_0 < t < \varepsilon_0),$$

where K' is a positive constant independent of k and t. Combining the estimates (27) and (28) with the Sobolev embedding theorems, we have

$$(29)\qquad G_\infty^{k,t} \leq K_1 ||G_\infty^{k,t}||_{W^{2,\frac{n+1}{2}}(M_\infty - \widehat{M}_2;g_\infty^{k,t})} \leq \frac{2KK_1c_n}{Y(M,[g])} \quad \text{on} \ \ M_\infty - \widehat{M}_2,$$

$$(30)\qquad ||G_\infty^{k,t}||_{L^{\frac{2n}{n-2}}(M_\infty - \widehat{M}_2;g_\infty^{k,t})} \leq K_2 ||G_\infty^{k,t}||_{W^{1,2}(M_\infty - \widehat{M}_2;g_\infty^{k,t})} \leq \frac{2K'K_2c_n}{Y(M,[g])}$$

for all $k \geq 2$ and t $(-\varepsilon_0 < t < \varepsilon_0)$, where K_1, K_2 are some positive constants independent of k and t.

Now using the Moser iteration technique (cf. [2, Proposition 5.8]) and the uniform estimates (22), (29), (30), we have the following decay estimates for $G_\infty^{k,t}$:

$$(31)\qquad G_\infty^{k,t}(x) \leq \frac{K_0}{Y(M,[g])^{\frac{n-2}{4}}} \cdot \frac{1}{r(x)^{\frac{n-2}{2}}} \qquad \text{on} \ \ M_\infty - \widehat{M}_2,$$

where $r(x) := \mathrm{dist}_{g_\infty}(x, p_\infty)$ and K_0 is a positive constant independent of k and t.

Secondly, for each t $(-\varepsilon_0 < t < \varepsilon_0)$, we construct a normalized minimal positive Green's function for $\mathcal{L}_{g_\infty^t}$ with pole at p_∞. Since $R_{g_\infty^t} > 0$, we have for every $k \geq 2$ the existence of a unique normalized positive Green's function $G_k^t \in C_+^\infty\big(Int(\widehat{\Omega}_k^*)\big) \cap C^0\big(\widehat{\Omega}_k^*\big)$ such that

$$\begin{cases} \mathcal{L}_{g_\infty^t} G_k^t = c_n \delta_{p_\infty} & \text{in} \ \ Int(\widehat{\Omega}_k), \\ G_k^t = 0 & \text{on} \ \ \partial\widehat{\Omega}_k. \end{cases}$$

From the standard removable singularities theorem, for each $\ell > k \geq 2$, the difference $(G_\ell^t - G_k^t) \in C^\infty\big(Int(\widehat{\Omega}_k) - \{p_\infty\}\big)$ extends smoothly on $Int(\widehat{\Omega}_k)$. Then, the maximum principle implies that for any $k \geq 2$

$$G_k^t \leq G_{k+1}^t \leq G_{k+2}^t \ \cdots \quad \text{on} \ \ \widehat{\Omega}_k^*.$$

Since $g_\infty^{k,t} = g_\infty^t$ on $\widehat{M}_k$ and $\widehat{\Omega}_k \subset \widehat{M}_k$, the standard removable singularities theorem and the maximum principle also imply that for any $k \geq 2$

$$G_k^t \leq G_\infty^{k,t} \quad \text{on} \ \ \widehat{\Omega}_k^*,$$

and hence from (29) and (31)

$$G_k^t \leq G_\infty^{2,t} + L \quad \text{on} \ \ \partial\widehat{M}_2,$$

$$(32)\qquad G_k^t(x) \leq \frac{K_0}{Y(M,[g])^{\frac{n-2}{4}}} \cdot \frac{1}{r(x)^{\frac{n-2}{2}}} \quad \text{on} \ \ \widehat{\Omega}_k - \widehat{M}_2,$$

where L is a nonnegative constant independent of k and t. The maximum principle implies again that, for any $k \geq 3$,

$$G_k^t \leq G_\infty^{2,t} + L \quad \text{on} \quad \widehat{M}_2^*. \tag{33}$$

Then, by Harnack's convergence theorem and the uniform estimates (32), (33), there exists a normalized positive Green's function G_∞^t for $\mathcal{L}_{g_\infty^t}$ with pole at p_∞ such that the sequence $\{G_k^t\}_{k\geq 2}$ converges uniformly to G_∞^t on each compact subset of M_∞^* and that

$$G_\infty^t(x) \leq \frac{K_0}{Y(M,[g])^{\frac{n-2}{4}}} \cdot \frac{1}{r(x)^{\frac{n-2}{2}}} \quad \text{on} \quad M_\infty - \widehat{M}_2. \tag{34}$$

The minimality and the uniqueness of G_∞^t follow at once from its construction and the maximum principle.

Finally, we prove the equality (23). By applying the maximum principle for the operator $\mathcal{L}_{g_\infty^{k,t}} = \mathcal{L}_{g_\infty^t}$ on $\widehat{M}_k$ to each function $(G_\infty^t - G_\infty^{k,t}) \in C^\infty(M_\infty)$, it follows from the uniform decay estimates (31), (34) that

$$\sup_{\widehat{M}_2} |G_\infty^t - G_\infty^{k,t}| = o(1) \qquad \text{as} \quad k \to \infty. \tag{35}$$

From the theory of *conformal normal coordinates* [26], [13], [15], there exist a conformal metric $\widetilde{g} = u^{\frac{4}{n-2}} g$ on M and small open neighborhoods U, V $(U \subset V)$ of p_∞ such that

$$\begin{cases} \det(\widetilde{g}_{ij}) \equiv 1 \quad \text{in } \widetilde{g}\text{−normal coordinates at } p \text{ on } U, \\ u \equiv 1 \quad \text{on} \quad M - V, \\ spt(h) \cap V = \phi, \end{cases} \tag{36}$$

where $spt(h)$ is the support of the given symmetric 2-tensor h and we identify M with $\widehat{M}$ $(\subset M_\infty)$. Let $\widetilde{g}_\infty$, $\widetilde{g}_\infty^t$ and $\widetilde{g}_\infty^{k,t}$ denote the corresponding metrics on M_∞. Recall that $g_\infty^{k,t} = g_\infty^t = g_\infty$ on V, which implies that $\widetilde{g}_\infty^{k,t} = \widetilde{g}_\infty^t = \widetilde{g}_\infty$ on V. Since the Green's functions $\widetilde{G}_\infty^t$ and $\widetilde{G}_\infty^{k,t}$ corresponding to $\widetilde{g}_\infty^t$ and $\widetilde{g}_\infty^{k,t}$ are given by

$$\widetilde{G}_\infty^t = u(p_\infty)^{-1} u^{-1} G_\infty^t \quad \text{and} \quad \widetilde{G}_\infty^{k,t} = u(p_\infty)^{-1} u^{-1} G_\infty^{k,t} \quad \text{on} \quad V$$

respectively, a similar estimate to (35) holds for $|\widetilde{G}_\infty^t - \widetilde{G}_\infty^{k,t}|$. Hence, we may assume that g itself satisfies the same properties as (36).

Besides $g_\infty^{k,t} = g_\infty^t = g_\infty$ on V, recall that g_∞ is locally comformally flat near p_∞ provided that $n \geq 6$. Then $G_\infty^t, G_\infty^{k,t}$ have the following expansions in fixed g_∞-normal coordinates $x = (x^1, \cdots, x^n)$ at p_∞ (cf. [26, Lemma 6.4]):

$$G_\infty^t(x) = \frac{1}{|x|^{n-2}} + A^t + O(|x|), \qquad G_\infty^{k,t}(x) = \frac{1}{|x|^{n-2}} + A^{k,t} + O(|x|),$$

where A^t and $A^{k,t}$ are some constants. It follows from (35) that

$$\lim_{k\to\infty} A^{k,t} = A^t,$$

and hence we obtain the desired equality (23). □

4. Another Extension

We start this section by proving another analogue of Aubin's Lemma.

Proposition 4.1. *Let (X, C) be a noncompact positive conformal n-manifold with $n \geq 3$ and $(\widetilde{X}, \widetilde{C})$ a non-trivial finite conformal covering. Then,*

$$Y(X, C) \leq Y(\widetilde{X}, \widetilde{C}). \tag{37}$$

Remark 4.2. The above inequality (37) can not be improved into a *strict* inequality in this setting. For instance, let Ω be the outside domain of a thin tubular neighborhood of an embedded S^{n-2} in S^n. Then, Lemma 2.1 in [33] implies that

$$Y(\Omega, [g_{\mathbb{S}}]\big|_{\Omega}) = \mathbf{Y}_n.$$

Hence, combining the inequality in Proposition 4.1 with the equality above, we see that the Yamabe constant of any finite conformal covering of $(\Omega, [g_{\mathbb{S}}]\big|_{\Omega})$ is always equal to $Y(\Omega, [g_{\mathbb{S}}]\big|_{\Omega})$.

Under some additional assumptions, the *strict* inequality in (37) holds.

Theorem 4.3. *Let (X, C) be a noncompact positive conformal n-manifold with $n \geq 3$ and $(\widetilde{X}, \widetilde{C})$ a non-trivial finite conformal covering. Assume that (X, C) is a normal infinite conformal covering of a closed positive conformal n-manifold $(M, \check{C})$ and that $\pi_1(M)$ has a descending chain of finite index subgroups tending to $\pi_1(X)$. Then,*

$$Y(X, C) < Y(\widetilde{X}, \widetilde{C}),$$

where we identify $\pi_1(X)$ with its projection to $\pi_1(M)$.

Proof of Proposition 4.1. When $Y(\widetilde{X}, \widetilde{C}) = \mathbf{Y}_n$, the assertion is obvious since

$$Y(X, C) \leq \mathbf{Y}_n = Y(\widetilde{X}, \widetilde{C}).$$

Hence, we assume that $Y(\widetilde{X}, \widetilde{C}) < \mathbf{Y}_n$.

Let $h \in C$ be a complete metric on X and denote by $\widetilde{h} \in \widetilde{C}$ its lifting to $\widetilde{X}$. Denote the number of *sheets* of the covering $(\widetilde{X}, \widetilde{C}) \to (X, C)$ by k (≥ 2). For every $i \geq 1$, there exist a relatively compact domain Ω_i with smooth boundary and a function $f_i \in C_c^\infty(\widetilde{X})$ with $||f_i||_{L^{\frac{2n}{n-2}}(\widetilde{X};\widetilde{h})} = 1$ whose support $spt(f_i)$ is contained in Ω_i and such that

$$Q_{(\widetilde{X}, \widetilde{h})}(f_i) \leq Y(\widetilde{X}, \widetilde{C}) + \frac{1}{i}.$$

We may assume that

$$\Omega_1 \subset \Omega_2 \subset \cdots \Omega_i \subset \Omega_{i+1} \subset \cdots, \qquad \cup_{i=1}^\infty \Omega_i = \widetilde{X}.$$

Take a positive integer i_0 satisfying

$$Y(\widetilde{X}, \widetilde{C}) + \frac{1}{i} < \mathbf{Y}_n \quad \text{for all } \ i \geq i_0. \tag{38}$$

For every $i \geq 1$, take a relatively compact domain V_i in X with smooth boundary satisfying

$$\mathcal{P}(\Omega_i) \subset V_i, \qquad V_1 \subset V_2 \subset \cdots V_i \subset V_{i+1} \subset \cdots,$$

where $\mathcal{P}$ denotes the covering map of $\widetilde{X} \to X$. For every $i \geq 1$, define a relatively compact domain W_i in $\widetilde{X}$ with smooth boundary by

$$W_i := \mathcal{P}^{-1}(V_i).$$

Note that

$$W_i = \mathcal{P}^{-1}\big(\mathcal{P}(W_i)\big), \qquad spt(f_i) \subset \Omega_i \subset W_i.$$

Similarly to the proof of the first assertion of Theorem 1.2, the strict inequality (38) implies the existence of $u_i \in C^\infty\big(\overline{W_i}\big)$ with

$$||u_i||_{L^{\frac{2n}{n-2}}(\overline{W_i};\widetilde{h})} = 1$$

and such that, for each $i \geq i_0$,

$$(39) \qquad \begin{cases} Q_{(\widetilde{X},\widetilde{h})}(u_i) = \inf\big\{Q_{(\widetilde{X},\widetilde{h})}(f) \;\big|\; f \in C_c^\infty(W_i),\ f \not\equiv 0\big\} =: q_i, \\ \mathcal{L}_{\widetilde{h}} u_i = q_i \cdot u_i^{\frac{n+2}{n-2}} \quad \text{on } \ \overline{W_i}, \\ u_i > 0 \quad \text{in } \ W_i, \qquad u_i = 0 \quad \text{on } \ \partial W_i. \end{cases}$$

We denote the zero extension of u_i to $\widetilde{X}$ by $u_i \in C^{0,1}(\widetilde{X}) \cap W^{1,2}(\widetilde{X};\widetilde{h})$. Note that

$$(40) \qquad Y(\widetilde{X},\widetilde{C}) \leq q_i \leq Q_{(\widetilde{X},\widetilde{h})}(f_i) \leq Y(\widetilde{X},\widetilde{C}) + \frac{1}{i} \quad \text{for all } \ i \geq i_0.$$

We now prove the desired result using the approximate solutions $\{u_i\}_{i\geq i_0}$. For any $x \in \widetilde{X}$, set

$$\{x_1, \cdots, x_k\} := \mathcal{P}^{-1}\big(\mathcal{P}(x)\big) \quad \text{with } \ x_1 = x.$$

Define positive functions v_i and $v_i^{\langle p\rangle}$ (for each $p > 0$) on $\widetilde{X}$ by

$$v_i(x) := \sum_{\alpha=1}^k u_i(x_\alpha) \qquad \text{and} \qquad v_i^{\langle p\rangle}(x) := \sum_{\alpha=1}^k u_i(x_\alpha)^p.$$

These functions satisfy

$$v_i \equiv 0, \quad v_i^{\langle p\rangle} \equiv 0 \qquad \text{on } \ \widetilde{X} - W_i.$$

For any *evenly covered* open set U $(\subset X)$ for $\mathcal{P}$, set

$$\{U_1, \cdots, U_k\} := \mathcal{P}^{-1}(U) \ (\subset \widetilde{X}).$$

Moreover, all the sets are isometric and so we can find, for instance, k isometries

$$\gamma_\alpha : U_1 \to U_i \quad \text{for } \ \alpha = 1, \cdots, k$$

satisfying $\mathcal{P} \circ \gamma_\alpha = \mathcal{P}$. As a result, it is possible to express v_i and $v_i^{\langle p\rangle}$ on U_1 as

$$v_i = \sum_{\alpha=1}^k u_i \circ \gamma_\alpha, \qquad v_i^{\langle p\rangle} = \sum_{\alpha=1}^k (u_i \circ \gamma_\alpha)^p,$$

and hence $v_i, v_i^{\langle p\rangle} \in C^\infty\big(\overline{W_i}\big) \cap C^{0,1}(\widetilde{X})$. We can use the local expression described above in order to find k distinct isometries

$$\gamma_\alpha : \widetilde{X} - \mathcal{S} \ \longrightarrow \ \widetilde{X} - \mathcal{S} \quad \text{for } \ \alpha = 1, \cdots, k,$$

where $\mathcal{S}$ is a piecewise smooth $(n-1)$-submanifold (possibly empty) satisfying $\mathcal{S} = \mathcal{P}^{-1}\big(\mathcal{P}(\mathcal{S})\big)$. Hence, if we set $u_{i,\alpha} := u_i \circ \gamma_\alpha$, we have that on $\widetilde{X} - \mathcal{S}$

$$v_i = \sum_{\alpha=1}^k u_{i,\alpha} \qquad \text{and} \qquad v_i^{\langle p\rangle} = \sum_{\alpha=1}^k u_{i,\alpha}^p,$$

from which it follows that

$$\int_{\widetilde{X}} v_i^{\langle p\rangle} d\mu_{h_k} = k \int_{\widetilde{X}} u_i^p d\mu_{h_k} \qquad \text{and} \qquad \mathcal{L}_{\widetilde{h}} v_i = q_i \cdot v_i^{\langle \frac{n+2}{n-2} \rangle} = q_i \cdot \sum_{\alpha=1}^{k} u_{i,\alpha}^{\frac{n+2}{n-2}} \quad \text{on } \overline{W_i}.$$

Define w_i to be the function on X whose lift to $\widetilde{X}$ is v_i. Combining the property (39) with a similar argument to the one used in [7, Lemma 2 and Theorem 6], we obtain that (cf. [5, Lemma 3.6])

$$Q_{(X,h)}(w_i) < q_i.$$

Then, this and the inequality (40) imply that

$$Y(X,C) \le Q_{(X,h)}(w_i) < q_i \le Y(\widetilde{X},\widetilde{C}) + \frac{1}{i} \quad \text{for all } \; i \ge i_0,$$

and hence

$$Y(X,C) \le Y(\widetilde{X},\widetilde{C}).$$

□

Proof of Theorem 4.3. We start by noting that $Y(X,C) < \mathbf{Y}_n$ because, otherwise, we could apply Corollary 1.5 to the normal infinite conformal covering $(X,C) \to (M,\check{C})$ and conclude that X is simply connected. This contradicts that $\widetilde{X}$ is a non-trivial covering of X.

When $Y(\widetilde{X},\widetilde{C}) = \mathbf{Y}_n$, the assertion is obvious. Hence, we assume that $Y(\widetilde{X},\widetilde{C}) < \mathbf{Y}_n$.

Since $\pi_1(X)$ is a normal subgroup of $\pi_1(M)$ and $\pi_1(\widetilde{X})$ is a finite index subgroup of $\pi_1(X)$, the normalizer $\mathcal{N} = \mathcal{N}_{\pi_1(M)}\big(\pi_1(\widetilde{X})\big)$ of $\pi_1(\widetilde{X})$ in $\pi_1(M)$ is a finite index subgroup of $\pi_1(M)$. Although the covering $\widetilde{X} \to M$ may not be normal, the following holds: There exists a compact subset K of $\widetilde{X}$ such that, for any $x \in \widetilde{X}$ there exists a deck transformation $\gamma \in \mathcal{N}/\pi_1(\widetilde{X})$ satisfying $\gamma(x) \in K$. Therefore, this fact and the strict inequality $Y(\widetilde{X},\widetilde{C}) < \mathbf{Y}_n$ imply that the renormalization argument in the proof of [34, Chapter 5, Theorem 2.1] and [2, Theorem 6.1] is still valid on $(\widetilde{X},\widetilde{C})$.

Let $h \in C$ be a complete metric on X and denote its lift to $\widetilde{X}$ by $\widetilde{h} \in \widetilde{C}$. The renormalization argument implies the existence of both a positive constant L and a positive function $v \in C^\infty_+(\widetilde{X})$ with $||v||_{L^{\frac{2n}{n-2}}(\widetilde{X};\widetilde{h})} = 1$ such that

$$\begin{cases} Q_{(\widetilde{X},\widetilde{h})}(v) = Y(\widetilde{X},\widetilde{C}), \\ \mathcal{L}_{\widetilde{h}} v = Y(\widetilde{X},\widetilde{C}) \cdot v^{\frac{n+2}{n-2}} \quad \text{on } \; \widetilde{X}, \\ v(x) \le L r^{-\frac{n-2}{2}} \quad \text{for all } \; x \in \widetilde{X} \; \text{ with } \; r := \mathrm{dist}_{\widetilde{h}}(x,p_0) \ge 1, \end{cases}$$

where $p_0 \in \widetilde{X}$ is a fixed point. Then, by arguing like in the proof of [7, Lemma 2 and Theorem 6], it follows straightforwardly that (see also [5, Lemma 3.6] for details)

$$Y(X,C) < Y(\widetilde{X},\widetilde{C}).$$

□

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