

# Perelman's Invariant, Ricci Flow, and the Yamabe Invariants of Smooth Manifolds

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## Abstract

In his study of Ricci flow, Perelman introduced a smooth-manifold invariant called  $\bar{\lambda}$ . We show here that, for completely elementary reasons, this invariant simply equals the Yamabe invariant, alias the sigma constant, whenever the latter is non-positive. On the other hand, the Perelman invariant just equals  $+\infty$  whenever the Yamabe invariant is positive.

Let  $M$  be a smooth compact manifold of dimension  $n \geq 3$ . Perelman's celebrated work on Ricci flow [12, 13] led him to consider the functional which associates to every Riemannian metric  $g$  the least eigenvalue  $\lambda_g$  of the elliptic operator  $4\Delta_g + s_g$ , where  $s_g$  denotes the scalar curvature of  $g$ , and  $\Delta = d^*d = -\nabla \cdot \nabla$  is the positive-spectrum Laplace-Beltrami operator associated with  $g$ . In other words,  $\lambda_g$  can be expressed in terms of Raleigh quotients as

$$\lambda_g = \inf_u \frac{\int_M [s_g u^2 + 4|\nabla u|^2] d\mu}{\int_M u^2 d\mu}$$

where the infimum is taken over all smooth, real-valued functions  $u$  on  $M$ .

One of Perelman's remarkable observations is that the scale-invariant quantity  $\lambda_g V_g^{2/n}$  is non-decreasing under the Ricci flow, where  $V_g = \int_M d\mu_g$  denotes the total volume of  $(M, g)$ . This led him to consider the differential-topological invariant obtained by taking the supremum of this quantity over the space of all Riemannian metrics [13, 6]:

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**Definition 1** Let  $M$  be a smooth compact  $n$ -manifold,  $n \geq 3$ . Perelman's  $\bar{\lambda}$  invariant of  $M$  is defined to be

$$\bar{\lambda}(M) = \sup_g \lambda_g V_g^{2/n}$$

where the supremum is taken over all smooth metrics  $g$  on  $M$ .

The rôle of the scalar curvature and Laplacian in defining  $\lambda_g$  might immediately make one wonder whether this invariant might somehow be related to the Yamabe problem. Recall that, as was conjectured by Yamabe [21], and later proved by Trudinger, Aubin, and Schoen [2, 3, 11, 18, 20], every conformal class on any smooth compact manifold contains a metric of constant scalar curvature. If  $M$  is a smooth compact manifold of dimension  $n \geq 3$ , and if

$$\gamma = [g] = \{vg \mid v : M \rightarrow \mathbb{R}^+\},$$

is the conformal class of an arbitrary metric, such a metric  $\hat{g}$  can in fact be constructed by minimizing the normalized total-scalar-curvature functional

$$\hat{g} \mapsto \frac{\int_M s_{\hat{g}} d\mu_{\hat{g}}}{\left(\int_M d\mu_{\hat{g}}\right)^{\frac{n-2}{n}}},$$

among all metrics conformal to  $g$ . Indeed, by setting  $\hat{g} = u^{4/(n-2)}g$ , this expression can be rewritten as

$$\frac{\int_M s_{\hat{g}} d\mu_{\hat{g}}}{\left(\int_M d\mu_{\hat{g}}\right)^{\frac{n-2}{n}}} = \frac{\int_M [s_g u^2 + 4 \frac{n-1}{n-2} |\nabla u|^2] d\mu_g}{\left(\int_M u^{2n/(n-2)} d\mu_g\right)^{(n-2)/n}},$$

and the proof proceeds by showing that there is a smooth positive function  $u$  which minimizes the right-hand expression. In particular, each conformal class  $\gamma$  has an associated number  $Y_\gamma$ , called its *Yamabe constant*, obtained by setting

$$Y_\gamma = \inf_{g \in \gamma} \frac{\int_M s_g d\mu_g}{\left(\int_M d\mu_g\right)^{\frac{n-2}{n}}};$$

the content of the Trudinger-Aubin-Schoen theorem is exactly that this number is actually realized as the constant scalar curvature of some unit-volume metric in  $\gamma$ . A constant-scalar-curvature metric of this type is called a *Yamabe minimizer*. It is not difficult to show that any Riemannian metric with

$s = \text{const} \leq 0$  is a Yamabe minimizer, and that, moreover, the Yamabe minimizer  $g \in \gamma$  is unique (up to constant rescaling) whenever  $Y_\gamma \leq 0$ . The situation is much more complicated when  $Y_\gamma > 0$ , but it is still not difficult to see that if  $g$  is a metric for which  $s$  has a fixed sign (positive, zero, or negative) everywhere on  $M$ , then this sign necessarily agrees with that of the number  $Y_{[g]}$ .

Yamabe's work was apparently motivated by the hope of constructing Einstein metrics via a variational approach. This idea eventually led Kobayashi [7] and Schoen [19] to independently introduce the smooth manifold invariant

$$\mathcal{Y}(M) = \sup_{\gamma} Y_\gamma = \sup_{\gamma} \inf_{g \in \gamma} \frac{\int_M s_g d\mu_g}{\left(\int_M d\mu_g\right)^{\frac{n-2}{n}}}.$$

By construction, this is a diffeomorphism invariant of  $M$ , and is now commonly known as the *Yamabe invariant* of  $M$ ; note, however, that Schoen called  $\mathcal{Y}(M)$  the *sigma constant*, and that this terminology is still preferred by some authors. Notice that  $\mathcal{Y}(M) \leq 0$  iff  $M$  does not admit metrics of positive scalar curvature, and that, when this happens,  $\mathcal{Y}(M)$  is simply the supremum of the scalar curvatures of unit-volume constant-scalar-curvature metrics on  $M$ .

The fact that there is some fundamental relation between the Yamabe invariant  $\mathcal{Y}(M)$  and Perelman's  $\bar{\lambda}$  invariant was probably first pointed out by Anderson [1]. More recently, an e-print by Fang and Zhang [4] computed the Perelman invariant for a large class of 4-manifolds where the Yamabe invariants had already been computed by the present authors [9, 5] and others [14, 15], and, as was later emphasized by Kotschick [8], their answers exactly agree with those previously discovered in the Yamabe case. The point of this brief note is to observe that this was no mere matter of coincidence:

**Theorem A** *Suppose that  $M$  is a smooth compact  $n$ -manifold,  $n \geq 3$ . Then*

$$\bar{\lambda}(M) = \begin{cases} \mathcal{Y}(M) & \text{if } \mathcal{Y}(M) \leq 0, \\ +\infty & \text{if } \mathcal{Y}(M) > 0. \end{cases}$$

In fact, this will follow easily once we clearly understand the behavior of  $\lambda V^{2/n}$  on each individual conformal class.

**Proposition 1** *Suppose that  $\gamma$  is a conformal class on  $M$  which does not contain a metric of positive scalar curvature. Then*

$$Y_\gamma = \sup_{g \in \gamma} \lambda_g V_g^{2/n}$$

**Proof.** Let  $g \in \gamma$ , and let  $\hat{g} = u^{4/(n-2)}g$  be the Yamabe minimizer in  $\gamma$ . Then

$$0 \geq Y_\gamma = \frac{\int_M [s_g u^2 + 4 \frac{n-1}{n-2} |\nabla u|^2] d\mu_g}{\left(\int_M u^{2n/(n-2)} d\mu_g\right)^{(n-2)/n}}.$$

Thus

$$\begin{aligned} \lambda_g \int u^2 d\mu &\leq \int [s u^2 + 4 |\nabla u|^2] d\mu \\ &\leq \int \left[ s u^2 + 4 \frac{n-1}{n-2} |\nabla u|^2 \right] d\mu \\ &= Y_\gamma \left( \int u^{2n/(n-2)} d\mu \right)^{(n-2)/n} \\ &\leq Y_\gamma V^{-2/n} \int u^2 d\mu \end{aligned}$$

where, since  $Y_\gamma \leq 0$ , the last step is an the application of the Hölder inequality

$$\int f_1 f_2 d\mu \leq \left( \int |f_1|^p d\mu \right)^{1/p} \left( \int |f_2|^q d\mu \right)^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

with  $f_1 = 1$ ,  $f_2 = u^2$ ,  $p = n/2$ , and  $q = n/(n-2)$ . Moreover, equality holds precisely when  $u$  is constant — which is to say, precisely when  $g$  has constant scalar curvature.

Since this shows that

$$\lambda_g V^{2/n} \leq Y_\gamma$$

for every  $g \in \gamma$ , and since equality occurs if  $g$  is the Yamabe minimizer, it follows that

$$\sup_{g \in \gamma} \lambda_g V_g^{2/n} = Y_\gamma,$$

exactly as claimed. ■

We now need make only one more simple observation:

**Lemma 2** *If  $M$  carries a metric with  $s > 0$ , then  $\bar{\lambda}(M) = +\infty$ .*

**Proof.** Given such a manifold  $M$  and any smooth non-constant function  $f : M \rightarrow \mathbb{R}$ , Kobayashi [7] has shown that there exists a unit-volume metric on  $M$  with  $s = f$ . In particular, given any real number  $L$ , there is a unit-volume metric  $g_L$  on  $M$  with  $s > L$  everywhere. But for such a metric,  $\lambda > L$  and  $V = 1$ . Thus, taking  $L \rightarrow \infty$ ,  $\bar{\lambda}(M) = \sup_g \lambda_g V_g^{2/n} = +\infty$ .  $\blacksquare$

Theorem A now follows immediately. Indeed, if  $\mathcal{Y}(M) > 0$ ,  $M$  admits a metric with  $s > 0$ , and Lemma 2 therefore tells us that  $\bar{\lambda}(M) = +\infty$ . Otherwise, no conformal class contains a metric of positive scalar curvature, and Proposition 1 therefore tells us that each constant-scalar-curvature metric maximizes  $\lambda V^{2/n}$  in its conformal class. Given an arbitrary maximizing sequence  $\hat{g}_j$  for  $\lambda V^{2/n}$ , we may thus, by conformal rescaling, construct a new maximizing sequence  $g_j$  consisting of unit-volume constant-scalar-curvature metrics. But for any such sequence, the numbers  $s_{g_j}$  may be viewed either as  $\{\lambda_{g_j} V_{g_j}^{2/n}\}$  or as  $\{Y_{[g_j]}\}$ . Thus the suprema over the space of all Riemannian metrics of  $Y_{[g]}$  and  $\lambda_g V_g^{2/n}$  must precisely coincide.

Now, there is a substantial literature [5, 9, 10, 14, 15, 16] concerning manifolds of non-positive Yamabe invariant, and the exact value of the invariant is moreover known for large numbers of such manifolds. By virtue of Theorem A, all of these facts about  $\mathcal{Y}(M)$  may therefore immediately be interpreted as instead pertaining to  $\bar{\lambda}(M)$ .

On the other hand, we have also seen that the Perelman invariant jumps to positive infinity whenever there is a metric of positive scalar curvature. By contrast, the Yamabe invariant always remains finite; indeed, one of Aubin's fundamental contributions to the theory of the Yamabe problem is the fact that  $\mathcal{Y}(M) \leq \mathcal{Y}(S^n)$  for any smooth compact  $n$ -manifold  $M$ . This systematic discrepancy fundamentally reflects the fact that constant-scalar-curvature metrics are generally not minimizers in the positive case [7, 18], and that, moreover, constant-scalar-curvature metrics of arbitrarily high energy exist in profusion [17] in this setting. Thus, while we have found a fascinating link between Perelman's invariant and the Yamabe problem, this observation actually teaches us nothing at all precisely in the regime where the Yamabe problem is the most technically difficult and, in some respects, still the most poorly understood.

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