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Proc. R. Soc. Lond. A 1996 **452**, 945-952
doi: 10.1098/rspa.1996.0047

On the existence of C^∞ solutions to the asymptotic characteristic initial value problem in general relativity

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The asymptotic characteristic initial value problem for Einstein's vacuum field equations is treated. It is shown that C^∞ solutions exist and are unique for C^∞ initial values. The proof is based on Friedrich's regular conformal vacuum field equations and Rendall's method of reducing the characteristic to an ordinary initial value problem.

1. Introduction

In this paper we shall solve the asymptotic characteristic initial value problem for Einstein's vacuum field equations. The initial values (in our case C^∞ functions) are given on an incoming null hypersurface $\mathcal{N}$ and on a part of past null infinity $\mathcal{I}^-$, which intersect in a two-dimensional space-like surface $\mathcal{Z}$ diffeomorphic to S^2. (We have chosen this 'traditional' topology for $\mathcal{Z}$, but the proof applies equally to cases where $\mathcal{Z}$ has a more complicated topology.) This problem arises in the study of far fields of isolated systems, in radiation problems and in the case of black holes, where horizons are present. A technical advantage of the characteristic initial value problem is that the constraints are reduced to a hierarchy of ordinary differential or algebraic equations, which are easier to solve than the elliptic constraints of the traditional Cauchy problem.

The first existence proof is due to Friedrich (1982), who has shown that, in the case of analytic initial values, an analytic solution of the asymptotic characteristic initial value problem exists and is unique in some neighbourhood of the intersection $\mathcal{Z}$. Analytic functions are, however, not general enough for the treatment of all physical problems. In the case of analytic initial values, a change of the data on a part of the initial surface does not influence the solution only in the domain of dependence of this region, i.e. analytic solutions are not suitable to exhibit causality. Therefore, it is important to consider the more general problem where the initial values are smooth, that is C^∞. These are reasonable in most physical problems. We shall prove in this paper the following theorem.

Theorem 1.1. *Given a C^∞ 'reduced initial value' set t_r (see (2.14) and (2.15)) on $\mathcal{N}$ and $\mathcal{I}^-$, there exists a unique C^∞ solution of the conformal vacuum field equations*

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Proc. R. Soc. Lond. A (1996) **452**, 945–952
Printed in Great Britain

$R_{\mu\nu}(\Omega^{-2}g_{\kappa\lambda}) = 0$ (see (2.2)) in a certain neighbourhood of $\mathcal{Z}$, which implies the given data on $\mathcal{N}$ and $\mathcal{I}^-$.

This result, and similar others, (for a review, see Friedrich 1992) show that Penrose's concept of asymptotic flatness (see Penrose 1963, 1965) is compatible with the field equations, and they provide a strong base for numerical (approximation) methods.

The conformal Einstein vacuum field equations are (at least formally) singular at null infinity. Friedrich (1981*a*, *b*) has shown that instead of these equations one can consider a regular system, the 'reduced conformal vacuum field equations'. This system is first-order quasilinear and symmetric hyperbolic. For such equations, Rendall (1990) has given a general method to reduce the characteristic initial value problem to the ordinary Cauchy problem, for which the existence and uniqueness of solutions has been established (see Taylor 1993). Our proof is based essentially on his method.

In § 2 we shall briefly review the geometry of the problem and the known results regarding the field equations (the details can be found, for example, in Friedrich 1981*a*). In § 3 Rendall's method will be applied to quite general symmetric hyperbolic systems. The last section contains the proof of the theorem stated above. We shall show first that the reduced conformal vacuum field equations, together with the 'complete initial-value set' (see (2.7)) on $\mathcal{N}$ and $\mathcal{I}^-$, determine all the derivatives of a possible solution along these surfaces. Second, we shall demonstrate in detail how Rendall's method, which was formulated for $R^+ \times R^+ \times R^2$ manifolds, is applicable to the present problem, which has the topology $R^+ \times R^+ \times S^2$.

Our conventions and notations are the same as those of Friedrich (1981). We shall use the spin frame formalism of Newman & Penrose (1962) without further explanation. Phrases such as 'on $\mathcal{N}$' ('on $\mathcal{M}$', etc.) will always refer to suitable neighbourhoods of $\mathcal{Z}$ on $\mathcal{N}$ (on $\mathcal{M}$, etc.).

2. The conformal vacuum field equations

A solution to the asymptotic characteristic initial value problem is a triplet $(\mathcal{M}, \Omega, g_{\mu\nu})$, where $(\mathcal{M}, g_{\mu\nu})$ is an 'unphysical spacetime' with a regular Lorentz metric $g_{\mu\nu}$ of signature $(+,-,-,-)$. The manifold $\mathcal{M}$ with boundary is diffeomorphic to $R^+ \times R^+ \times S^2$. The boundary consists of an incoming null hypersurface $\mathcal{N}$ and past null infinity $\mathcal{I}^-$; both have $R^+ \times S^2$ topology. The conformal factor Ω is a regular function on $\mathcal{M}$ and satisfies the conditions

$$\Omega \equiv 0, \quad d\Omega \neq 0, \text{ on } \mathcal{I}^-, \quad \Omega > 0 \text{ on } \mathcal{M} \setminus \mathcal{I}^-. \tag{2.1}$$

On the 'physical spacetime' $(\mathcal{M} \setminus \mathcal{I}^-, \Omega^{-2}g_{\mu\nu})$ the vacuum Einstein field equations hold:

$$R_{\mu\nu}(\Omega^{-2}g_{\kappa\lambda}) = 0. \tag{2.2}$$

The pair $(\Omega, g_{\mu\nu})$ is subject to gauge transformations; for an arbitrary positive function Θ, the new triplet $(\mathcal{M}, \Theta\Omega, \Theta^2 g_{\mu\nu})$ also satisfies the above conditions and determines the same physical spacetime as $(\mathcal{M}, g_{\mu\nu})$. In order to restrict the gauge we impose the condition

$$R(g_{\mu\nu}) = 0. \tag{2.3}$$

These two equations (equipped with suitable initial conditions) are to be solved for the metric $g_{\mu\nu}$ and the conformal factor Ω on the whole of $\mathcal{M}$.

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A coordinate system and a null tetrad $\{e_{aa'}{}^{\mu}\}$ satisfying

$$g_{\mu\nu}e_{aa'}{}^{\mu}e_{bb'}{}^{\nu} = \epsilon_{ab}\bar{\epsilon}_{a'b'} \tag{2.4 a}$$

are fixed as follows. First choose on $\mathcal{Z} = \mathcal{N} \cap \mathcal{I}^-$ a coordinate system $\{x^A\}$, (the index A will always take the values $\{3,4\}$) and a complex null-frame $e_{01'}{}^{\mu}, e_{10'}{}^{\mu} = \bar{e}^{\mu}_{10'}$, for which (2.4 a) holds. Let $e_{11'}{}^{\mu}$ be the null generators along $\mathcal{I}^-$, and the coordinate x^1, which vanishes on $\mathcal{Z}$, a parameter of the integral curves of $e_{11'}{}^{\mu}$ on $\mathcal{I}^-$. Let the vector fields $e_{11'}{}^{\mu}$, $e_{01'}{}^{\mu}$ and $e_{10'}{}^{\mu}$ be parallely transported along the null generators of $\mathcal{I}^-$. Let us consider the $\{x^1 = \text{const.}\}$ null hypersurfaces in $\mathcal{M}$. The remaining null vector $e_{00'}{}^{\mu}$ can be chosen as a generator of these hypersurfaces, i.e. $e_{00'}{}^{\mu} = g^{\mu\nu}\partial_\nu x^1$. Let x^2 be a parameter along the integral curves of $e_{00'}{}^{\mu}$ which vanishes on $\mathcal{I}^-$. The vectors $e_{11'}{}^{\mu}$, $e_{01'}{}^{\mu}$, $e_{10'}{}^{\mu}$ are parallely propagated along these curves; the coordinates $\{x^A\}$ are constant along both classes of the null generators considered. In this special coordinate system, where $\mathcal{N}$: $\{x^1 = 0\}$ and $\mathcal{I}^-$: $\{x^2 = 0\}$, the tetrad defined above has the components

$$\left.\begin{aligned} e_{00'}{}^{\mu} = \delta^{\mu}_{2}, \quad e_{11'}{}^{\mu} = \delta^{\mu}_{1} + U\delta^{\mu}_{2} + X^A\delta^{\mu}_{A}, \\ e_{01'}{}^{\mu} = \omega\delta^{\mu}_{2} + \xi^A\delta^{\mu}_{A}, \quad e_{10'}{}^{\mu} = \bar{\omega}\delta^{\mu}_{2} + \bar{\xi}^A\delta^{\mu}_{A}, \end{aligned}\right\} \tag{2.4 b}$$

where U and X^A are real, ω and ξ^A are complex functions obeying

$$U = X^A = \omega = 0, \quad \text{on } \mathcal{I}^-. \tag{2.4 c}$$

The spin coefficients satisfy

$$\left.\begin{aligned} \Gamma_{11'01} = \Gamma_{11'11} = 0, \quad \text{on } \mathcal{I}^-, \\ \Gamma_{11'00} = \bar{\Gamma}_{01'0'1'} + \Gamma_{01'01}, \quad \Gamma_{10'00} = \bar{\Gamma}_{01'0'0'}, \quad \Gamma_{00'ab} = 0, \quad \text{on } \mathcal{M}. \end{aligned}\right\} \tag{2.5}$$

Next to these expressions, which follow from the above choice of the coordinates and the tetrad, there are also quantities which are determined on the initial surfaces through the special choice (2.3) of the conformal factor. These are the conditions

$$\left.\begin{aligned} \Sigma_{aa'} = e_{00'}(\Omega)\epsilon_a{}^0\bar{\epsilon}_{a'}{}^{0'}, \quad \Phi_{111'1'} = 0, \quad \text{on } \mathcal{I}^-, \\ \Phi_{000'0'} = 0, \quad \text{on } N, \\ e_{00'}(\Omega) = 1, \quad \Gamma_{10'00} = \Gamma_{01'11} = 0, \quad \text{on } Z. \end{aligned}\right\} \tag{2.6}$$

Equation (2.2) is (at least formally) singular on $\mathcal{I}^-$. Friedrich (1981a, b) has shown, however, that the problem can be reformulated as a first-order regular system for the unknown

$$t = (\Omega, \Sigma_{aa'}, s, e_{aa'}{}^{\mu}, \Gamma_{aa'bc}, \varphi_{abcd}, \Phi_{aba'b'}), \tag{2.7}$$

which consists of, respectively, the conformal factor, its gradient $\Sigma_{aa'} = \nabla_{aa'}\Omega$ and $s = \frac{1}{4}\nabla_{aa'}\nabla^{aa'}\Omega$, the components of the Newman–Penrose null tetrad, the spin coefficients, the conformally rescaled Weyl spinor $\varphi_{abcd} = \Omega^{-1}\psi_{abcd}$ (due to the arguments of Penrose (1963, 1965) that the Weyl tensor vanishes on $\mathcal{I}^-$, this is a regular quantity) and the Ricci spinor. ∇_μ denotes the Levi–Civita connection with respect to the metric $g_{\mu\nu}$. In the sequel, we also need the expressions for the torsion and curvature spinors in terms of the tetrad and spin coefficients, respectively:

$$\begin{aligned} t_{aa'}{}^{bb'}{}_{cc'}e_{bb'}{}^{\mu} = \epsilon^{be}(\Gamma_{aa'ec}e^{\mu}{}_{bc'} - \Gamma_{cc'ea}e^{\mu}{}_{ba'}) \\ + \bar{\epsilon}^{b'e'}(\bar{\Gamma}_{aa'e'c'}\bar{e}^{\mu}_{cb'} - \bar{\Gamma}_{cc'e'a'}\bar{e}^{\mu}_{ab'}) + e_{cc'}(e^{\mu}{}_{aa'}) - e_{aa'}(e^{\mu}{}_{cc'}), \end{aligned} \tag{2.8}$$

$$\begin{aligned} r_{abcc'dd'} &= e_{dd'}(\Gamma_{cc'ab}) - e_{cc'}(\Gamma_{dd'ab}) \\ &\quad + \epsilon^{ts}(\Gamma_{dd'at}\Gamma_{cc'sb} + \Gamma_{td'ab}\Gamma_{cc'sd} - \Gamma_{cc'at}\Gamma_{dd'sb} - \Gamma_{tc'ab}\Gamma_{dd'sc}) \\ &\quad + \bar{\epsilon}^{t's'}(\Gamma_{dt'ab}\bar{\Gamma}_{cc's'd'} - \Gamma_{ct'ab}\bar{\Gamma}_{dd's'c'}) - t_{cc'}{}^{ss'}{}_{dd'}\Gamma_{ss'ab} \end{aligned} \tag{2.9}$$

and the decomposition of the latter into its Weyl and Ricci parts. It is convenient to write

$$R_{abcc'dd'} = -\Omega\varphi_{abcd}\bar{\epsilon}_{c'd'} - \Phi_{abc'd'}\epsilon_{cd} \tag{2.10}$$

and to consider the equalities of the left-hand sides of the last two equations as part of the following overdetermined first-order system for t:

$$\left.\begin{aligned} k_{aa'} = \nabla_{aa'}\Omega - \Sigma_{aa'} = 0, \quad q_{aa'bb'} = \nabla_{aa'}\Sigma_{bb'} + \Omega\Phi_{aba'b'} - s\epsilon_{ab}\bar{\epsilon}_{a'b'} = 0, \\ p_{aa'} = \nabla_{aa'}s + \Phi_{aba'b'}\Sigma^{bb'} = 0, \quad t_{aa'}{}^{bb'}{}_{cc'}e_{bb'}{}^{\mu} = 0, \\ \Delta_{abcc'dd'} = r_{abcc'dd'} - R_{abcc'dd'} = 0, \\ h_{abcd'} = \nabla^{f}{}_{d'}\varphi_{abcf} = 0, \quad L_{abcd'} = \nabla_{a}{}^{f'}\Phi_{bcd'f'} - \varphi_{abcf}\Sigma^{f}{}_{d'} = 0. \end{aligned}\right\} \tag{2.11}$$

This system is equivalent to the above equations (2.2) and (2.3), provided that the gauge condition (2.3) holds at least at one point of $\mathcal{M}$. One can separate system (2.11) into two subsystems, namely the equations

$$\left.\begin{aligned} k_{00'} = 0, \quad q_{00'bb'} = 0, \quad p_{00'} = 0, \quad t_{00'}{}^{bb'}{}_{dd'}e_{bb'}{}^{\mu} = 0, \\ \Delta_{ab00'dd'} = 0, \quad h_{abc0'} = 0, \quad L_{0bcd'} = 0, \end{aligned}\right\} \tag{2.12}$$

which are constraints on the hypersurface $\mathcal{N}$, and

$$\left.\begin{aligned} &\left.\begin{aligned} k_{cc'} &= 0, \\ q_{cc'bb'} &= 0, \\ p_{cc'} &= 0, \end{aligned}\right\} \quad cc' \neq 00' \\ &\left.\begin{aligned} t_{cc'}{}^{bb'}{}_{dd'}e_{bb'}{}^{\mu} &= 0, \\ \Delta_{abcc'dd'} &= 0, \end{aligned}\right\} \quad cc' \neq 00', \quad dd' \neq 00' \\ &h_{abc1'} = 0, \\ &L_{1bcd'} = 0, \end{aligned}\right\} \tag{2.13}$$

which are similar constraints on past null infinity $\mathcal{I}^{-}$. On the initial surfaces, these equations form a hierarchy of ordinary differential equations (and algebraic expressions). This means that the initial values for all the components of the unknown (2.7), that is a 'complete initial value set' t_0, cannot be given freely, but one can calculate it with the help of (2.12) and (2.13) from a 'reduced initial value set' t_{r}. The choice of the reduced initial value set is not unique; we can choose, for example, the data

$$\left.\begin{aligned} \Gamma_{01'00}, \quad &\text{on } \mathcal{I}^{-}, \\ \varphi_{0000}, \quad &\text{on } N, \\ \varphi_{0001}, \quad \varphi_{0011} + \bar{\varphi}_{0'0'1'1'}, \quad \xi^{A}, \quad &\text{on } Z, \end{aligned}\right\} \tag{2.14}$$

such that $-(\xi^A\bar{\xi}^B + \bar{\xi}^A\xi^B)$ is a metric conformal to the standard metric on the sphere.

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We can instead consider another system of quantities:

$$\left.\begin{array}{c} \varphi_{1111}, \quad \text{on } \mathcal{I}^-, \\ \varphi_{0000}, \quad \text{on } N, \\ \Gamma_{01'00}, \quad \Phi_{001'1'}, \quad \varphi_{0001}, \quad \varphi_{0011}+\bar{\varphi}_{0'0'1'1'}, \quad \xi^A, \quad \text{on } Z, \end{array}\right\} \tag{2.15}$$

which is equivalent to the set (2.14) (one can see this easily from the constraint equations (2.13)). The advantage of this second choice is that the data are symmetric with respect to the initial surfaces.

From the overdetermined system (2.11), Friedrich has extracted a first-order quasi-linear symmetric hyperbolic system, the 'reduced conformal vacuum field equations', that is the system

$$\begin{array}{c} k_{00'}=0, \quad q_{00'bb'}=0, \quad p_{00'}=0, \quad t_{00'}{}^{bb'}{}_{dd'}e_{bb'}{}^{\mu}=0, \\ \Delta_{ab00'dd'}=0, \quad -h_{1110'}=0, \quad -h_{0bc0'}+h_{1bc1'}=0, \quad h_{0001'}=0, \\ -L_{0bc1'}=0, \quad -L_{0bc0'}+L_{1bc1'}=0, \quad L_{1bc0'}=0. \end{array} \tag{2.16 a}$$

He has shown also that solutions of the above reduced conformal vacuum field equations, which coincide on $\mathcal{N}$ and on $\mathcal{I}^-$ with the initial data calculated from the constraint equations (2.12) and (2.13), are also solutions of the whole system of conformal vacuum field equations (2.8). The gauge condition (2.3) for the conformal factor is also satisfied.

The reduced conformal vacuum field equations (2.16 a) can be written in the concise form

$$A^\mu(x^\nu,t)\partial_\mu t + B(x^\nu,t)=0. \tag{2.16 b}$$

The hypersurfaces $\mathcal{N}$ and $\mathcal{I}^-$ are characteristics of the above system. The equation is symmetric hyperbolic; the matrices A^μ are hermitian, and there is a direction, in our case it can be chosen to be $e_{00'\mu}+e_{11'\mu}$, such that the matrix $A^\mu(e_{00'\mu}+e_{11'\mu})$ is positive definite. This last property means that the hypersurfaces with normal $e_{00'\mu}+e_{11'\mu}$ are space-like with respect to system (2.16 a). The equation can be extended to negative values of the coordinates x^1 and x^2. Taking into account the expression of the null tetrad (2.4 b), this means that the hypersurface $\mathcal{F}$: $\{x^1+x^2=0\}$ is space-like in a neighbourhood of the intersection $\mathcal{Z}$ of the initial surfaces.

3. Reduction to the ordinary Cauchy problem

Let us consider an equation of the type (2.16 b) on R^4 (in a neighbourhood of $\mathcal{N}_1\cap\mathcal{N}_2$, where $\mathcal{N}_1$ and $\mathcal{N}_2$ denote the initial hypersurfaces $\{x^2=0\}$ and $\{x^1=0\}$, respectively). Let t_0 be initial values, defined on $\mathcal{N}_1$ and $\mathcal{N}_2$, continuous on $\mathcal{N}_1\cup\mathcal{N}_2$ and their restrictions smooth on $\mathcal{N}_1$ and on $\mathcal{N}_2$. Moreover, let $\mathcal{N}_1$ and $\mathcal{N}_2$ be characteristics for (2.16 b) and t_0. Let us finally suppose that the system of equations obtained from (2.16 b) by formal differentiation with respect to the x^ν to all orders and then restricting to $\mathcal{N}_1\cup\mathcal{N}_2$, can be solved uniquely to give all formal (interior and exterior) derivatives of t_0 as smooth functions on $\mathcal{N}_1$ and $\mathcal{N}_2$. By the Whitney extension theorem (see Abraham & Robbin 1967), a smooth function t_1 exists on a neighbourhood of $\mathcal{N}_1\cap\mathcal{N}_2$, all of the derivations of which agree with those formal derivatives. This means that the function

$$\Delta = A^\mu(x^\nu,t_1)\partial_\mu t_1 + B(x^\nu,t_1) \tag{3.1}$$

vanishes to all orders on the characteristic surfaces and the function

$$\delta = \begin{cases} 0, & x^1 > 0, \quad x^2 > 0, \\ \Delta, & \text{elsewhere}, \end{cases} \tag{3.2}$$

is smooth in a neighbourhood of $\mathcal{N}_1 \cap \mathcal{N}_2$, where t_1 exists.

Consider, then, the equation

$$A^\mu(x^\nu, t_1 + \tau)\partial_\mu(t_1 + \tau) + B(x^\nu, t_1 + \tau) = \delta \tag{3.3}$$

for the unknown τ, and its (ordinary) Cauchy problem with zero data on the initial surface $\mathcal{F}$: $\{x^1 + x^2 = 0\}$. This surface has a neighbourhood around $\mathcal{N}_1 \cap \mathcal{N}_2$ which is space-like with respect to the equation; thus, the given Cauchy problem has a unique solution τ on it in a neighbourhood $\mathcal{U}$ of $\mathcal{N}_1 \cap \mathcal{N}_2$. It is easy to see that τ is zero outside the intersection of $\mathcal{U}$ with the region $\{x^1 \geqslant 0, x^2 \geqslant 0\}$, and the function $t^* = t_1 + \tau$ is the unique solution of the characteristic initial value problem inside the intersection (for details, see Rendall 1990).

4. The existence proof

Using expression (2.4 *b*) of the null-tetrad and the explicit notation of Newman & Penrose (1962), the reduced conformal vacuum field equations (2.16 *a*) take the form

$$\left.\begin{aligned}
&\partial_2 t_i = f_i, \\
&\partial_2\varphi_4 - \bar\omega\partial_2\varphi_3 - \bar\xi^A\partial_A\varphi_3 = f_{21}, \\
&\partial_2\varphi_{j+1} + \partial_1\varphi_{j+1} + U\partial_2\varphi_{j+1} + X^A\partial_A\varphi_{j+1} - \omega\partial_2\varphi_{j+2} - \xi^A\partial_A\varphi_{j+2} \\
&\qquad -\bar\omega\partial_2\varphi_j - \bar\xi^A\partial_A\varphi_j = f_{22+j}, \\
&\partial_1\varphi_0 + U\partial_2\varphi_0 + X^A\partial_A\varphi_0 - \omega\partial_2\varphi_1 - \xi^A\partial_A\varphi_1 = f_{25}, \\
&\partial_2\Phi_{j2} - \omega\partial_2\Phi_{j1} - \xi^A\partial_A\Phi_{j1} = f_{26+j}, \\
&\partial_2\Phi_{j1} + \partial_1\Phi_{j1} + U\partial_2\Phi_{j1} + X^A\partial_A\Phi_{j1} - \bar\omega\partial_2\Phi_{j2} - \bar\xi^A\partial_A\Phi_{j2} \\
&\qquad -\omega\partial_2\Phi_{j0} - \xi^A\partial_A\Phi_{j0} = f_{29+j}, \\
&\partial_1\Phi_{j0} + U\partial_2\Phi_{j0} + X^A\partial_A\Phi_{j0} - \bar\omega\partial_2\Phi_{j1} - \bar\xi^A\partial_A\Phi_{j1} = f_{32+j},
\end{aligned}\right\} \tag{4.1}$$

where $i = \{1, \ldots, 20\}$, $j = \{0, 1, 2\}$ and t_l with $l = \{1, \ldots, 34\}$ means the lth non-zero component of the unknown t, namely

$$t = (t_l) = (\Omega; \Sigma_{aa'}; s; U, X^A, \omega, \xi^A; \tau, \gamma, \nu, \sigma, \beta, \mu, \rho, \lambda; \varphi_h; \Phi_{jk}), \tag{4.2}$$

where $h = \{0, \ldots, 4\}$, $\{j, k\} = \{0, 1, 2\}$ and the functions f_l on the right-hand side contain only linear and quadratic expressions of the components t_l. (Their precise form is unimportant from the point of view of the following proof.)

Having a complete initial data set t_0 (a solution of the constraint equations (2.12) and (2.13)), with the help of equations (4.1) we are able to calculate all the outgoing (transverse) derivatives which a solution has to have on the initial surfaces $\mathcal{N}$ and $\mathcal{I}^-$ (this means, in fact, that we are able to calculate all the derivatives). This we have to show in detail, because there is not any available criterion which would help us to decide whether this is true for a general first-order symmetric hyperbolic system.

Let us first consider past null infinity $\mathcal{I}^-$, where ∂_2 is transverse. The first transverse derivatives of all unknowns but φ_0 and Φ_{j0} are algebraically determined through t_0. Because of conditions (2.4 c), the 25th and the 32–34th equations (see the fourth and the last lines of (4.1)), evaluated on $\mathcal{I}^-$, do not, in fact, contain x^2-derivatives. After differentiation of these equations with respect to x^2, we get a system of linear first-order ordinary differential equations for $\partial_2\varphi_0$ and $\partial_2\Phi_{j0}$. The initial values for these quantities can be calculated on $\mathcal{Z}$, given t_0 on the initial surfaces. Let system (4.1) be modified so that the 25th and the 32–34th equations (see the fourth and the last lines of (4.1)) be replaced by their x^2-derivatives. Considering successively higher x^2-derivatives of this modified system, we can iteratively solve them, since they are always algabraic expansions and ordinary differential equations for the unknown transverse derivatives

On the other null hypersurface $\mathcal{N}$, the situation is quite similar. Here ∂_1 is transverse. Considering equations (4.1), we can see that here only $\partial_1\varphi_m$, $m \neq 4$ and $\partial_1\Phi_{ij}$, $j \neq 2$ are algebraically determined through the initial values t_0 (see the 22–25th and the 29–34th equations, that is the 3–4th and 6–7th lines of (4.1)). After calculating the x^1-derivatives of the remaining equations, we can also get here a system of linear first-order ordinary differential equations for the unknown first outgoing derivatives. Following the same method as in the case of $\mathcal{I}^-$, we can calculate all the higher x^1-derivatives step by step.

Let us consider an atlas $\{(U_1, \Phi_1); (U_2, \Phi_2)\}$ on $\mathcal{Z}$ (diffeomorphic to S^2) and the closed sets $V_i \subset U_i$ (here and in the following $i = \{1, 2\}$), which also cover $\mathcal{Z}$, that is $\mathcal{Z} \subset V_1 \cup V_2$. Define the two $C_0^\infty(R^2)$ functions $\eta^{(i)}$ as

$$\eta^{(i)}(x) = \begin{cases} 1, & x \in \Phi_i(V_i), \\ 0, & x \in R^2 \setminus \Phi_i(U_i). \end{cases} \tag{4.3}$$

Let $t_{\mathrm{r}}^{(i)}$ denote the restrictions of the reduced initial values (2.14) or (2.15) to U_i. Writing them as functions of the coordinates $\Phi_i(U_i)$, the functions $\eta^{(i)}t_{\mathrm{r}}^{(i)}$ define a C^∞ reduced initial value set on the hypersurfaces $\mathcal{N}_1 = R^+ \times \{0\} \times R^2$ and $\mathcal{N}_2 = \{0\} \times R^+ \times R^2$, which is identical with the original one on $R^+ \times \{0\} \times \Phi_i(V_i)$ and $\{0\} \times R^+ \times \Phi_i(V_i)$.

Solving the constraint equations (2.12) and (2.13) on $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively, we can get the complete initial value sets $t_0^{(i)}$. Using the procedure introduced at the beginning of this section, all the outgoing derivatives on $\mathcal{N}_1 \cup \mathcal{N}_2$ can be determined. So with the help of Rendall's method reviewed in the last section, we can construct the unique solutions $t^{*(i)}$ (both of them in a neighbourhood of $\mathcal{N}_1 \cap \mathcal{N}_2$). Their restrictions to the Cauchy development of $R^+ \times R^+ \times \Phi_i(V_i)$ are indeed (local) solutions to the original problem.

The last task is to show that they can be glued together to form a global solution on $\mathcal{Z}$. From the uniqueness of the local solutions $t^{*(i)}$ follows that their restrictions to a part (where both solutions exist) of the Cauchy development of $R^+ \times R^+ \times \Phi_i(V_1 \cap V_2)$ differ only by a coordinate transformation. This means that they can be identified on this region. The result is the unique C^∞ solution in a neighbourhood of $\mathcal{Z}$.

Now we have proved the theorem stated in the introduction. It is important to note that the proof applies even if the intersection surface $\mathcal{Z}$ has a more complicated topology. This means that the above theorem remains true without specifying the topology of the two-dimensional (orientable) manifold $\mathcal{Z}$.

Considering $\Omega \equiv 1$, we get as a special case the equations of the 'normal' vacuum characteristic initial value problem. The proof above is easily applicable to this case.

I thank B. Schmidt and H. Friedrich for suggesting the problem and for helpful discussions, J. Ehlers for careful reading of the manuscript.

The author of this paper is a member of the research groups OTKA F014196 and OTKA T016246.

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Received 26 June 1995; accepted 25 October 1995

PROCEEDINGS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES