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A simple proof of the recent generalizations of Hawking's black hole topology theorem

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Online at stacks.iop.org/CQG/25/162001**Abstract**

A key result in four-dimensional black hole physics, since the early 1970s, is Hawking's topology theorem assertion that the cross-sections of an 'apparent horizon', separating the black hole region from the rest of the spacetime, are topologically 2-spheres. Later, during the 1990s, by applying a variant of Hawking's argument, Gibbons and Woolgar could also show the existence of a genus-dependent lower bound for the entropy of topological black holes with negative cosmological constant. Recently, Hawking's black hole topology theorem, along with the results of Gibbons and Woolgar, has been generalized to the case of black holes in higher dimensions. Our aim here is to give a simple self-contained proof of these generalizations, which also makes their range of applicability transparent.

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1. Introduction

The notion of a trapped surface was introduced by Penrose [11]. In a four-dimensional spacetime the spacelike boundary, \mathcal{S} , of a three-dimensional spatial region is called a future trapped surface if gravity is so strong there that even the future and 'outwards' directed normal null rays starting at \mathcal{S} are dragged back so much that their expansion is non-positive everywhere on \mathcal{S} . Careful analysis justified that trapped surfaces necessarily occur whenever a sufficient amount of energy is concentrated in a small spacetime region [12].

Intuitively, a black hole region is considered to be a part of a spacetime from which nothing can escape. Therefore a black hole region is supposed to be a future set comprised by events that individually belong to some future trapped surface. The boundary of such a black hole region, referred to usually as the 'apparent horizon', \mathcal{H} , is then supposed to be comprised by marginal future trapped surfaces. As one of the most important recent results in black hole physics the existence of an 'apparent horizon' was proved in [1, 2]. More specifically, it was

shown there that given a strictly stable *marginally outer trapped surface* (MOTS) $\mathcal{S}_0 \subset \Sigma_0$ in a spacetime with reference foliation $\{\Sigma_t\}$, then there exists an open *tube*, $\mathcal{H} = \cup_t \mathcal{S}_t$, foliated by marginally outer trapped surfaces, with $\mathcal{S}_t \subset \Sigma_t$, through \mathcal{S}_0 . Let us merely mention, without getting into the details here, that the applied strict stability assumption is to exclude the appearance of future trapped surfaces in the complement of a black hole region.

Hawking's black hole topology theorem [9] is proven by demonstrating that whenever the dominant energy condition (DEC) holds a MOTS \mathcal{S} can be deformed, along the family of null geodesics transverse to the apparent horizon, yielding thereby—on contrary to the fact that \mathcal{S} is a MOTS—a future trapped surface in the complement of the black hole region, unless the Euler characteristic $\chi_{\mathcal{S}}$ of \mathcal{S} is positive. Whenever \mathcal{S} is a codimension-two surface in a four-dimensional spacetime the Euler characteristic and the 'genus', $g_{\mathcal{S}}$, of \mathcal{S} can be given, in virtue of the Gauss–Bonnet theorem, via the integral of the scalar curvature R_q of the metric q_{ab} induced on \mathcal{S} as

$$2\pi \chi_{\mathcal{S}} = 4\pi(1 - g_{\mathcal{S}}) = \frac{1}{2} \int_{\mathcal{S}} R_q \epsilon_q. \quad (1.1)$$

The main difficulty in generalizing Hawking's argument to the higher dimensional case originates from the fact that whenever \mathcal{S} is of dimension $s = n - 2 \geq 3$ in an n -dimensional spacetime, the integral of the scalar curvature $R^{(s)}$ by itself is not informative, as opposed to the case of $n = 4$, therefore the notion of Euler characteristic has to be replaced by the Yamabe invariant. The latter is known to be a fundamental topological invariant and is defined as follows. Denote by $[q]$ the conformal class of Riemannian metrics on \mathcal{S} determined by q_{ab} . It was conjectured by Yamabe, and later proved by Trudinger, Aubin and Schoen that to every conformal class on any smooth compact manifold there exists a metric \tilde{q}_{ab} of constant scalar curvature so that

$$R_{\tilde{q}} = Y(\mathcal{S}, [q]) \cdot \left(\int_{\mathcal{S}} \epsilon_{\tilde{q}} \right)^{-\frac{2}{s-2}}, \quad (1.2)$$

where the Yamabe constant $Y(\mathcal{S}, [q])$, associated with the conformal class $[q]$, is defined as

$$Y(\mathcal{S}, [q]) = \inf_{\hat{q} \in [q]} \frac{\int_{\mathcal{S}} R_{\hat{q}} \epsilon_{\hat{q}}}{\left(\int_{\mathcal{S}} \epsilon_{\hat{q}} \right)^{\frac{s-2}{s}}} = \inf_{u \in C^\infty(\mathcal{S}), u > 0} \frac{\int_{\mathcal{S}} [4 \frac{s-1}{s-2} (D^a u)(D_a u) + R_q u^2] \epsilon_q}{\left(\int_{\mathcal{S}} u^{\frac{2s}{s-2}} \epsilon_q \right)^{\frac{s-2}{s}}}. \quad (1.3)$$

In the later case, the metric $\hat{q} \in [q]$ can be given as $\hat{q}_{ab} = u^{\frac{4}{s-2}} q_{ab}$, and, moreover, D_a and R_q denote the covariant derivative operator and the scalar curvature associated with the metric q_{ab} on \mathcal{S} . The Yamabe invariant is then defined as

$$\mathcal{Y}(\mathcal{S}) = \sup_{[q]} Y(\mathcal{S}, [q]). \quad (1.4)$$

Some of the recent generalizations of Hawking's [9] black hole topology theorem, and also that of Gibbons' [7] and Woolgar's [15] results, proved by Galloway, Schoen, O'Murchadha and Cai, that are covered by [3, 4, 6] may then be formulated as.

Theorem 1.1. *Let (M, g_{ab}) be a spacetime of dimension $n \geq 4$ satisfying the Einstein equations*

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi T_{ab}, \quad (1.5)$$

with cosmological constant Λ and with matter subject to DEC. Suppose, furthermore, that \mathcal{S} is a strictly stable MOTS in a regular spacelike hypersurface Σ .

(1) If $\Lambda \geq 0$ then \mathcal{S} is of positive Yamabe type, i.e., $\mathcal{Y}(\mathcal{S}) > 0$.

(2) If $\mathcal{V}(\mathcal{S}) < 0$ and $\Lambda < 0$ then for the ‘area’ $A(\mathcal{S}) = \int_{\mathcal{S}} \epsilon_q$ of \mathcal{S} the inequality

$$A(\mathcal{S}) \geq \left(\frac{|\mathcal{V}(\mathcal{S})|}{2|\Lambda|} \right)^{\frac{2}{n-2}} \quad (1.6)$$

holds.

The significance of these results gets to be transparent if one recalls that in the first case, i.e., when $\mathcal{V}(\mathcal{S}) > 0$, \mathcal{S} cannot carry a metric of non-positive sectional curvature which immediately restricts the topological properties of \mathcal{S} [8]. Whereas, in the second case a lower bound on the ‘entropy’ of a black hole, which is considered to be proportional to the area $A(\mathcal{S})$ is provided by (1.6).

Before proceeding we would like to stress on an important conceptual point. Most of the quoted investigations of black holes, see [1–6], start by assuming the existence of a reference foliation $\{\Sigma_t\}$ of the spacetime by (partial) Cauchy surfaces Σ_t . In this respect it is worth recalling that by a non-optimal choice of $\{\Sigma_t\}$ one might completely miss a black hole region as it follows from the results of [14], where it was demonstrated that even in the extended Schwarzschild spacetime one may find a sequence of Cauchy surfaces which get arbitrarily close to the singularity such that neither of these Cauchy surfaces contains a future trapped surface. Hence, one of the motivations for the present work—besides providing a reduction of the complexity of the proof of theorem 1.1, and also a simultaneous widening of its range of applicability—was to carry out a discussion without making use of any reference foliation.

2. Preliminaries

As will be seen below, the simplicity of the presented argument allows the investigation of black holes essentially in arbitrary metric theory of gravity. Thereby, we do not restrict our considerations to either of the specific theories. Accordingly, a spacetime is assumed to be represented by a pair (M, g_{ab}) , where M is an n -dimensional ($n \geq 4$), smooth, paracompact, connected, orientable manifold endowed with a smooth Lorentzian metric g_{ab} of signature $(-, +, \dots, +)$. It is assumed that (M, g_{ab}) is time orientable and that a time orientation has been chosen.

The only restriction, concerning the geometry of the allowed spacetimes, is the following generalized version of DEC. A spacetime (M, g_{ab}) is said to satisfy the *generalized dominant energy condition* if there exists some smooth real function f on M such that for all future directed timelike vector t^a the combination $-[G^a_b t^b + f t^a]$ is a future directed timelike or null vector, where G_{ab} denotes the Einstein tensor $R_{ab} - \frac{1}{2}g_{ab}R$. It is straightforward to see that in Einstein’s theory of gravity, where g_{ab} is subject to (1.5), the generalized dominant energy condition, with the choice $f = \Lambda$, is equivalent to requiring the energy–momentum tensor, T_{ab} , to satisfy DEC [13].

To restrict our considerations to black hole spacetimes, we shall also assume the existence of future trapped surfaces in (M, g_{ab}) which are defined as follows. Let us consider a smooth orientable $(n-2)$ -dimensional compact manifold \mathcal{S} with no boundary in M . Let ℓ^a and k^a be smooth future directed null vector fields on \mathcal{S} scaled such that $k^a \ell_a = -1$, and which are also normal to \mathcal{S} , i.e., $g_{ab} \ell^a X^b|_{\mathcal{S}} = 0$ and $g_{ab} k^a X^b|_{\mathcal{S}} = 0$ for any vector field X^a tangent to \mathcal{S} . Consider then the null hypersurfaces generated by geodesics starting on \mathcal{S} with tangent ℓ^a and k^a . These null hypersurfaces are smooth in a neighbourhood of \mathcal{S} , and, by making use of the associated synchronized affine parametrizations of their null generators, the vector fields ℓ^a and k^a can be, respectively, extended to them. The level surfaces of the corresponding synchronized affine parametrizations do also provide foliations of these null hypersurfaces by

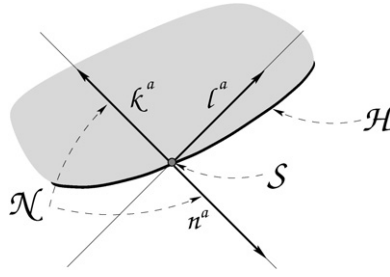


Figure 1. The black hole, represented by the shaded region, is bounded by horizon \mathcal{H} that is foliated by MOTS' homologous to \mathcal{S} .

$(n-2)$ -dimensional compact manifolds homologous to \mathcal{S} . Denote by ϵ_q the volume element associated with the metric, q_{ab} , induced on these $(n-2)$ -dimensional surfaces. Then the null expansions $\theta^{(\ell)}$ and $\theta^{(k)}$ with respect to ℓ^a and k^a are defined by

$$\mathfrak{L}_\ell \epsilon_q = \theta^{(\ell)} \epsilon_q, \quad \mathfrak{L}_k \epsilon_q = \theta^{(k)} \epsilon_q, \quad (2.1)$$

where \mathfrak{L}_ℓ and \mathfrak{L}_k denote the Lie derivatives with respect to the null vector fields ℓ^a and k^a .

The $(n-2)$ -dimensional surface \mathcal{S} is called *future trapped surface* if both the null expansions $\theta^{(\ell)}$ and $\theta^{(k)}$ are non-positive on \mathcal{S} . In the limiting case, i.e., whenever either of these null expansions (say $\theta^{(\ell)}$) vanishes on \mathcal{S} identically, \mathcal{S} is called *future marginally trapped surface*.

In case of a generic $(n-2)$ -dimensional surface the quasi-local concept of outward and inward directions is undetermined. Nevertheless, these concepts get to be well defined for non-minimal marginally trapped surfaces. It will be said that ℓ^a and k^a point *outwards* and *inwards*, respectively, provided that $\theta^{(\ell)} = 0$ and $\theta^{(k)} \leq 0$, and that $\theta^{(k)}$ is not identically zero. (If both $\theta^{(\ell)}$ and $\theta^{(k)}$ vanish identically \mathcal{S} is a minimal surface and the notions outwards and inwards become degenerate.)

To see that this quasi-local concept of ‘outer’ direction is not counter intuitive, consider the null hypersurface \mathcal{N} generated by null geodesics starting at the points of \mathcal{S} with tangent $n^a = -k^a$ (see figure 1). Since \mathcal{N} is smooth in a neighbourhood \mathcal{O} of \mathcal{S} it can be smoothly foliated by $(n-2)$ -dimensional surfaces, \mathcal{S}_u , defined as the $u = \text{const}$ cross-sections of \mathcal{N} , where u is the affine parameter along the generators of \mathcal{N} such that $n^a = (\partial/\partial u)^a$ and $u = 0$ on \mathcal{S} . Then, it seems natural to consider k^a as inward pointing if the ‘area’ $\mathcal{A}(\mathcal{S}_u) = \int_{\mathcal{S}_u} \epsilon_q$ of the cross-sections \mathcal{S}_u is non-decreasing in the direction of n^a which, in the case under consideration, follows from $\theta^{(k)} \leq 0$ as

$$\left. \frac{d\mathcal{A}(\mathcal{S}_u)}{du} \right|_{u=0} = \int_{\mathcal{S}} \mathfrak{L}_{-k} \epsilon_q = - \int_{\mathcal{S}} \theta^{(k)} \epsilon_q \geq 0. \quad (2.2)$$

Accordingly, \mathcal{S} is called *future marginally outer trapped surface* (MOTS) if $\theta^{(\ell)} = 0$ and $\theta^{(k)} \leq 0$ on \mathcal{S} .

In deriving our results we shall also apply a stability assumption. Before formulating it let us first recall that the above imposed conditions do not uniquely determine the pair of null vector fields ℓ^a and k^a on \mathcal{S} . In fact, together with ℓ^a and k^a any pair of null vector fields ℓ'^a and k'^a that is yielded by the boost transformation

$$k^a \rightarrow k'^a = e^v k^a, \quad \ell^a \rightarrow \ell'^a = e^{-v} \ell^a, \quad (2.3)$$

where $v : \mathcal{S} \rightarrow \mathbb{R}$ is an arbitrary smooth function on \mathcal{S} , will be suitable. It is well known, however, that the signs of $\theta^{(\ell)}$ and $\theta^{(k)}$ are invariant under such a positive rescaling of ℓ^a and k^a .

Suppose then that \mathcal{S} is a future MOTS with respect to a null normal ℓ^a . Then, \mathcal{S} will be called *strictly stably outermost* if there exists a rescaling of the type (2.3) so that $\mathbb{E}_k \theta^{(\ell')} \leq 0$ for the yielded vector fields ℓ'^a and k'^a , and also $\mathbb{E}_k \theta^{(\ell')} < 0$ somewhere on \mathcal{S} . Obviously, this definition is independent of the use of any sort of reference foliation. Moreover, as will be indicated below, it can be seen to be equivalent to the corresponding criteria applied in [1, 2].

3. The proof of the main result

The main argument of the present paper can then be given in the following simple geometric setup. We have already defined $n^a = -k^a$ to be a smooth past directed null vector field on \mathcal{S} that is also normal to \mathcal{S} . Similarly, the smooth null hypersurface \mathcal{N} , spanned by the $(n-2)$ -parameter congruence of null geodesics starting at \mathcal{S} with tangent n^a , and the affine parameter u along the geodesics, synchronized such that $u = 0$ on \mathcal{S} , have already been introduced. Denote by n^a the tangent field $(\partial/\partial u)^a$ on the null hypersurface \mathcal{N} , which is foliated by the smooth $u = \text{const}$ cross-sections, \mathcal{S}_u . Then, there exists a uniquely defined future directed null vector field ℓ^a on \mathcal{N} such that $g_{ab}n^a\ell^b = 1$, and that ℓ^a is orthogonal to each \mathcal{S}_u . Denote by r the affine parameter of the null geodesics determined by ℓ^a which are synchronized such that $r = 0$ on \mathcal{N} .

Since ℓ^a is, by construction, smooth on \mathcal{N} the null geodesics starting with tangent ℓ^a on \mathcal{N} do not meet within a sufficiently small open ‘elementary spacetime neighbourhood’ \mathcal{O} of \mathcal{S} . Extend, then, the function u from \mathcal{N} onto \mathcal{O} by keeping its value constant along the geodesics with tangent ℓ^a . Then the vector fields n^a and ℓ^a , defined so far only on \mathcal{N} , do also extend onto \mathcal{O} such that the relations $n^a = (\partial/\partial u)^a$ and $\ell^a = (\partial/\partial r)^a$ hold there, which do also imply that n^a and ℓ^a commute on \mathcal{O} . Note that \mathcal{O} is smoothly foliated by the two-parameter family of $(n-2)$ -dimensional $u = \text{const}, r = \text{const}$ level surfaces $\mathcal{S}_{u,r}$. The spacetime metric in \mathcal{O} can then always be given (see, e.g., [10] for a justification) as

$$g_{ab} = 2(\nabla_{(a}r - r\alpha\nabla_{(a}u - r\beta_{(a}\nabla_{b)}u + \gamma_{ab}, \quad (3.1)$$

where α, β_a and γ_{ab} are smooth fields on \mathcal{O} such that β_a and γ_{ab} are orthogonal to n^a and ℓ^a . Recall also that γ_{ab} and the positive definite metric q_{ab} on the $(n-2)$ -dimensional surfaces $\mathcal{S}_{u,r}$ are related as

$$q_{ab} = r^2\beta_c\ell_a\ell_b - 2r\beta_{(a}\ell_{b)} + \gamma_{ab}. \quad (3.2)$$

This latter relation implies that $q_{ab} = \gamma_{ab}$ on \mathcal{N} , represented by the $r = 0$ hypersurface in \mathcal{O} , i.e., γ_{ab} is the metric on the cross-sections \mathcal{S}_u of \mathcal{N} .

Since the vector fields n^a and ℓ^a are null and normal to the cross-sections \mathcal{S}_u the expansions of the associated null congruences at \mathcal{S}_u can be given as

$$\theta^{(n)}|_{\mathcal{S}_u} = \frac{1}{2}q^{ef}(\mathbb{E}_n q_{ef}) = \frac{1}{2}\gamma^{ef}(\mathbb{E}_n \gamma_{ef}) \quad \text{and} \quad \theta^{(\ell)}|_{\mathcal{S}_u} = \frac{1}{2}q^{ef}(\mathbb{E}_\ell q_{ef}) = \frac{1}{2}\gamma^{ef}(\mathbb{E}_\ell \gamma_{ef}), \quad (3.3)$$

where \mathbb{E}_n and \mathbb{E}_ℓ denote the Lie derivatives with respect to the vector fields n^a and ℓ^a , respectively, and (here and elsewhere) all the indices are raised and lowered with the spacetime metric g_{ab} . The second equalities in (2.1) follow from the fact that β_a and γ_{ab} are orthogonal to n^a and ℓ^a and also that n^a and ℓ^a commute in \mathcal{O} .

By making use of (3.1) and the definition of the Einstein tensor, it is straightforward to see that

$$G_{ab}n^a\ell^b = R_{ab}n^a\ell^b - \frac{1}{2}R_{ef}[2n^{(e}\ell^{f)} + \gamma^{ef}] = -\frac{1}{2}\gamma^{ef}R_{ef} \quad (3.4)$$

holds on \mathcal{N} . Then, in virtue of equation (82) of [10], and by the coincidence of q_{ab} and γ_{ab} and also of the projectors q^a_b , γ^a_b and p^a_b on \mathcal{N} (for the definition of p^a_b see (76) of [10]) we also have that on \mathcal{N}

$$G_{ab}n^a\ell^b = -\frac{1}{2}\left[-\gamma^{ab}(\mathfrak{L}_\ell\mathfrak{L}_n\gamma_{ab}) - \alpha\gamma^{ab}(\mathfrak{L}_\ell\gamma_{ab}) + R_q + D^a\beta_a - \frac{1}{2}\beta^a\beta_a + \gamma^{ab}\gamma^{cd}(\mathfrak{L}_\ell\gamma_{ac})(\mathfrak{L}_n\gamma_{bd}) - \frac{1}{2}\gamma^{ab}(\mathfrak{L}_\ell\gamma_{ab})\gamma^{cd}(\mathfrak{L}_n\gamma_{cd})\right], \quad (3.5)$$

where D_a and R_q denote the covariant derivative operator and the scalar curvature associated with the metric $q_{ab} = \gamma_{ab}$ on the $(n-2)$ -dimensional surfaces \mathcal{S}_u on \mathcal{N} . By making use of the fact that the vector fields n^a and ℓ^a commute in \mathcal{O} and that they are orthogonal to γ_{ab} a direct calculation justifies then the relation

$$-\gamma^{ab}(\mathfrak{L}_\ell\mathfrak{L}_n\gamma_{ab}) = -\mathfrak{L}_\ell(\gamma^{ab}\mathfrak{L}_n\gamma_{ab}) - \gamma^{ab}\gamma^{cd}(\mathfrak{L}_\ell\gamma_{ac})(\mathfrak{L}_n\gamma_{bd}). \quad (3.6)$$

Since n^a and ℓ^a commute we also have that

$$\mathfrak{L}_\ell\theta^{(n)} = \mathfrak{L}_n\theta^{(\ell)} \quad (3.7)$$

in \mathcal{O} , where the expansion $\theta^{(n)}$ is defined with respect to the volume element ϵ_q associated with the metric q_{ab} on the surfaces $\mathcal{S}_{u,r}$ as in (3.3). Similarly, it can be verified that

$$\mathfrak{L}_k\theta^{(\ell)} = \mathfrak{L}_n\theta^{(\ell)} \quad (3.8)$$

on \mathcal{N} , i.e., whenever $r = 0$, where k^a denotes the unique future directed null extension

$$k^a = -\left[n^a + \left(r\alpha + \frac{1}{2}r^2\beta^e\beta_e\right)\ell^a + r\beta^a\right] \quad (3.9)$$

of $k^a = -n^a$ on \mathcal{S} onto \mathcal{O} which is normal to the surfaces $\mathcal{S}_{u,r}$ and is scaled such that $k^a\ell_a = -1$ in \mathcal{O} .

Then, by making use of the vanishing of $\theta^{(\ell)}$ on \mathcal{S} , the above relations—in particular, equations (3.3), (3.5), (3.6) and (3.7)—imply that

$$\mathfrak{L}_n\theta^{(\ell)}|_{\mathcal{S}} = G_{ab}n^a\ell^b + \frac{1}{2}\left[R_q + D^a\beta_a - \frac{1}{2}\beta^a\beta_a\right]. \quad (3.10)$$

Since $-n^a$ and ℓ^a are both future directed null vector fields on \mathcal{N} , and also the generalized dominant energy condition holds, i.e., there exists a real function f on M such that the vector field $-[G^a_b\ell^b + f\ell^a]$ is future directed and causal, the inequality $G_{ab}n^a\ell^b + f \leq 0$ holds on \mathcal{N} . Finally, since \mathcal{S} was assumed to be a strictly stable MOTS, in virtue of (3.8), the null normals $n^a = -k^a$ and ℓ^a may be assumed, without loss of generality, to be such that $\mathfrak{L}_n\theta^{(\ell)}|_{\mathcal{S}} \geq 0$, and also that $\mathfrak{L}_n\theta^{(\ell)}|_{\mathcal{S}} > 0$ somewhere on \mathcal{S} .

To see that the stability condition applied here is equivalent to that used in [1, 2] note that $\beta_a = -q^e_a n_b \nabla_e \ell^b$ and it transforms under the rescaling (2.3) of the vector fields $k^a = -n^a$ and ℓ^a on \mathcal{S} as $\beta_a \rightarrow \beta'_a = \beta_a + D_a v$. By making use of the notation $\psi = e^{-2v}$ and $s_a = \frac{1}{2}\beta_a$, it can be verified then that

$$\mathfrak{L}_{\psi n'}\theta^{(\ell)}|_{\mathcal{S}} = -D^a D_a \psi + 2s^a D_a \psi + \frac{\psi}{2}\left[R_q + 2G_{ab}n^a\ell^b + 2D^a s_a - 2s^a s_a\right] \quad (3.11)$$

holds, which is exactly the expression ‘ $\delta_q\theta$ ’ given in lemma 3.1 of [2] whenever \mathcal{S} is a MOTS and the variation vector field q^a is chosen to be ‘ ψn^a ’. This justifies then that the strict stability conditions applied here and in [1, 2] (see, e.g., definition 5.1 and the discussion at the end of section 5 of [2] for more details) are equivalent.

In returning to the main stream of our argument note that whenever \mathcal{S} is a strictly stable MOTS and the generalized DEC holds then, in virtue of (3.10),

$$R_q + D^a\beta_a - \frac{1}{2}\beta^a\beta_a \geq 2f, \quad (3.12)$$

so that the inequality is strict somewhere on \mathcal{S} . Since q_{ab} is positive definite we also have that for any smooth function u on \mathcal{S}

$$u^2 D^a \beta_a = D^a(u^2 \beta_a) - 2u(D^a u)\beta_a \leq D^a(u^2 \beta_a) + 2(D^a u)(D_a u) + \frac{1}{2}u^2 \beta^a \beta_a. \quad (3.13)$$

Thus, multiplying (3.12) by u^2 , where $u > 0$ is arbitrary, we get, in virtue of (3.13), that

$$2(D^a u)(D_a u) + R_q u^2 + D^a(u^2 \beta_a) \geq 2f u^2, \quad (3.14)$$

so that the inequality is strict somewhere on \mathcal{S} .

To get the analogue of the first part of theorem 1.1 assume now that f is such that $f \geq 0$ throughout \mathcal{S} . Then, by taking into account the inequality $4\frac{s-1}{s-2} > 2$, which holds for any value of $s \geq 3$, we get from (3.14) that

$$\frac{\int_{\mathcal{S}} \left[4\frac{s-1}{s-2}(D^a u)(D_a u) + R_q u^2 \right] \epsilon_q}{\left(\int_{\mathcal{S}} u^{\frac{2s}{s-2}} \epsilon_q \right)^{\frac{s-2}{s}}} > 0, \quad (3.15)$$

for any smooth $u > 0$, i.e., $Y(\mathcal{S}, [q]) > 0$, which implies that \mathcal{S} is of positive Yamabe type.

Similarly, whenever the minimal value $f_{\min}^{\mathcal{S}}$ of f on \mathcal{S} is negative, on one hand,

$$2 \int_{\mathcal{S}} f u^2 \epsilon_q \geq -2|f_{\min}^{\mathcal{S}}| \int_{\mathcal{S}} u^2 \epsilon_q \quad (3.16)$$

while, on the other hand, by applying the Hölder inequality

$$\int_{\mathcal{S}} \phi_1 \phi_2 \epsilon_q \leq \left(\int_{\mathcal{S}} |\phi_1|^a \epsilon_q \right)^{\frac{1}{a}} \left(\int_{\mathcal{S}} |\phi_2|^b \epsilon_q \right)^{\frac{1}{b}}, \quad \frac{1}{a} + \frac{1}{b} = 1 \quad (3.17)$$

to the functions $\phi_1 = u^2$ and $\phi_2 = 1$ with $a = \frac{s}{s-2}$, $b = \frac{s}{2}$, we get that

$$\int_{\mathcal{S}} u^2 \epsilon_q \leq \left(\int_{\mathcal{S}} u^{\frac{2s}{s-2}} \epsilon_q \right)^{\frac{s-2}{s}} [\mathcal{A}(\mathcal{S})]^{\frac{2}{s}}. \quad (3.18)$$

The combination of (3.16) and (3.18), along with (3.14), justifies then that

$$\frac{\int_{\mathcal{S}} \left[4\frac{s-1}{s-2}(D^a u)(D_a u) + R_q u^2 \right] \epsilon_q}{\left(\int_{\mathcal{S}} u^{\frac{2s}{s-2}} \epsilon_q \right)^{\frac{s-2}{s}}} \geq -2|f_{\min}^{\mathcal{S}}| [\mathcal{A}(\mathcal{S})]^{\frac{2}{s}}. \quad (3.19)$$

Assuming finally that $\mathcal{Y}(\mathcal{S}) < 0$ we get that, for any conformal class $[q]$, the Yamabe constant $Y(\mathcal{S}, [q]) \leq \mathcal{Y}(\mathcal{S}) < 0$. This, along with (3.19), implies then

$$|\mathcal{Y}(\mathcal{S})| \leq |Y(\mathcal{S}, [q])| = -Y(\mathcal{S}, [q]) \leq 2|f_{\min}^{\mathcal{S}}| [\mathcal{A}(\mathcal{S})]^{\frac{2}{s}}, \quad (3.20)$$

which leads to the variant of the inequality (1.6) yielded by the replacement of Λ by $f_{\min}^{\mathcal{S}}$.

4. Final remarks

What has been proven in the previous section can be summarized as.

Theorem 4.1. *Let (M, g_{ab}) be a spacetime of dimension $n \geq 4$ in a metric theory of gravity. Assume that the generalized dominant energy condition, with smooth real function $f : M \rightarrow \mathbb{R}$, holds and that \mathcal{S} is a strictly stable MOTS in (M, g_{ab}) .*

- (1) *If $f \geq 0$ on \mathcal{S} then \mathcal{S} is of positive Yamabe type, i.e., $\mathcal{Y}(\mathcal{S}) > 0$.*
- (2) *If $\mathcal{Y}(\mathcal{S}) < 0$ and $f_{\min}^{\mathcal{S}} < 0$, where $f_{\min}^{\mathcal{S}}$ denotes the minimal value of f on \mathcal{S} , then*

$$\mathcal{A}(\mathcal{S}) \geq \left(\frac{|\mathcal{Y}(\mathcal{S})|}{2|f_{\min}^{\mathcal{S}}|} \right)^{\frac{s}{2}}. \quad (4.1)$$

We would like to mention that the argument of the previous section does also provide an immediate reduction of the complexity of the original proof of Hawking and that of Gibbons and Woolgar. To see this recall that, in virtue of (1.1), (1.3) and (1.4), $\mathcal{V}(\mathcal{S}) = 4\pi\chi_{\mathcal{S}}$, whenever \mathcal{S} is of dimension $s = 2$, and also that $f = \Lambda$, if attention is restricted to Einstein's theory with matter satisfying the dominant energy condition. Thereby, as an immediate consequence of theorem 4.1, we have that

$$\chi_{\mathcal{S}} > 0, \quad (4.2)$$

whenever $\Lambda \geq 0$ on \mathcal{S} , while

$$\mathcal{A}(\mathcal{S}) \geq \frac{4\pi(1 - g_{\mathcal{S}})}{|\Lambda|} \quad (4.3)$$

whenever both $\chi_{\mathcal{S}}$ and Λ are negative.

Clearly the above justification of theorem 4.1 is free of the use of any particular reference foliation of the spacetime. Note also that in the topological characterization of an n -2-dimensional strictly stable MOTS \mathcal{S} , only the quasi-local properties of the real function $F : M \rightarrow \mathbb{R}$ are important. In particular, as the conditions of theorem 4.1 do merely refer to the behaviour of f on \mathcal{S} it need not be bounded or have a characteristic sign throughout M . Similarly, it would suffice to require the generalized dominant energy condition to be satisfied only on \mathcal{S} .

Finally, we would also like to emphasize that theorem 4.1 provides a considerable widening of the range of applicability of the generalization of Hawking's black hole topology theorem, and also that of the results of Gibbons and Woolgar. As its conditions indicate, theorem 4.1 applies to any metric theory of gravity and the only restriction concerning the spacetime metric is manifested by the generalized dominant energy condition and by the assumption requiring the existence of a strictly stable MOTS. Accordingly, theorem 4.1 may immediately be applied in string theory or in various other higher dimensional generalizations of general relativity.

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