(I)

1. WKB for time dependent classical solutions
2. WKB for the bulk breathers in SG
3. WKB for the (pure) Neumann boundary breathers in BSG

(1) single scalar field $\Phi(x,t)$ (formal small imaginary part to make integrals converge)

$$\boxed{G(T) \equiv \mathrm{Tr}\left[\exp\left(-iHT/\hbar\right)\right] = \int \mathcal{D}\Phi(x,t)\,\exp\left\{\frac{i}{\hbar}S[\Phi(x,t)]\right\}}$$

$$S[\Phi] = \int_0^T dt \int dx\,\mathcal{L}(\Phi(x,t)) = \int_0^T dt\left[\frac{1}{2}\int dx\,\dot{\Phi}^2 - V[\Phi]\right] \qquad V[\Phi] = \int dx\left[\frac{1}{2}\Phi'^2 + U(\Phi)\right]$$

"Bohr-Sommerfeld condition" $\to$ periodic classical solutions $\Leftrightarrow$ bound states in QT

the propagator of the system

$$\boxed{\mathcal{G}(E) \equiv \mathrm{Tr}\,\frac{1}{E-\hat{H}} = \frac{i}{\hbar}\int_0^\infty dT\, G(T)\exp\left(\frac{iET}{\hbar}\right)}$$

poles of $\mathcal{G}(E) \Leftrightarrow$ bound states

suppose we have a classical solution, $\Phi_{cl}(x,t)$, depending on t as well! periodic with τ

$\hookrightarrow$ extremum of $S[\Phi]$

and expand $\Phi(x,t)$ around $\Phi_{cl}(x,t)$

compute $\mathcal{G}(E)$ using the Stationary Phase Approximation (both in the functional integral and in the ordinary one!)

$\hookrightarrow$ picks out the periodic classical solutions

trace

$$\xi(x,t) \equiv \Phi(x,t) - \Phi_{cl}(x,t)$$

Consider the "stability equation"

$\Phi_{cl}(x,t) + \xi(x,t)$ obey in linear order the classic

$$\boxed{\left(-\frac{\partial^2}{\partial t^2} - V''\right)\xi(x,t) \equiv \left[-\frac{\partial^2}{\partial t^2} + \nabla^2 - \frac{\partial^2 U}{\partial \Phi^2}\Big|_{\Phi_{cl}}\right]\xi(x,t) = 0}$$

since $\Phi_{cl}(x,t+\tau) = \Phi_{cl}(x,t)$

the solutions $\quad \xi_i(x,t+\tau) = e^{i\nu_i}\,\xi_i(x,t)$

in general $\exists$ an infinity of ξ_i and ν_i $(\nu_0 = 0)$

$\boxed{\nu_i = \nu_i(\tau)}$

Carry out the (Gaussian) functional integral over $\xi_i(x,t)$

(determinant complications!)

$$\mathcal{G}(E) = \frac{i}{\hbar}\sum_n \int_0^\infty dT \exp\left\{\frac{i}{\hbar}\left[ET + nS_{cl}\left(\frac{T}{n}\right)\right]\right\} \underbrace{\frac{T}{n}\left(-\frac{dE_{cl}}{dT}\right)^{1/2} \prod_{\nu_j > 0}\frac{1}{2i\sin\left(\frac{n\nu_j}{2}\right)}}_{\det}$$

$$= \frac{i}{\hbar}\sum_n \sqrt{n}\int_0^\infty d\tau\,\tau\left(-\frac{dE_{cl}}{d\tau}\right)^{1/2} \exp\left\{\frac{i}{\hbar}n\left[S_{cl}(\tau) + E\tau - \sum_j \sum_{p_j = 0}^\infty \left(p_j + \frac{1}{2}\right)\hbar\nu_j\right]\right\}$$

τ need not be T it may be T/n for any n

each contributes by $S_{cl}\left(\frac{T}{n}\right)$

SPA in the τ integral

$$-E_{cl} = \frac{\partial S_{cl}}{\partial \tau}$$

$$\boxed{E + \frac{\partial S_{cl}}{\partial \tau} - \sum_j \sum_{p_j} \hbar\left(p_j + \frac{1}{2}\right)\frac{\partial \nu_j}{\partial \tau} = 0}$$

II

$$\mathcal{G}(E) = \frac{i}{\hbar} \sum_{n,\{p\}} \tau(E_{cl}) \exp\left\{ \frac{in}{\hbar} W_{\{p\}}(E) \right\}$$

$\exists$ a pole in $\mathcal{G}(E)$ at $E = E_k$

$$\boxed{W_{\{p\}}(E) = S[E_{cl}] + E\,\tau(E_{cl}) - \sum_j \sum_{p_j} (p_j + \tfrac{1}{2})\, h\, \nu_j(E_{cl})}$$

if $\boxed{W_{\{p\}}(E_k) = 2\pi K \hbar}$

$\Downarrow$

quantization condition

infinite sum(s) regularization and renormalization is necessary!

finally in field theory

counterterms making the sums meaningfull

$$-\frac{\partial}{\partial \tau}\left[S_{cl}[\Phi_{cl}] + S_{ct}[\Phi_{cl}] - \sum (p_j + \tfrac{1}{2})\, h\, \nu_j(\Phi_{cl}) \right]_0^{\tau} = E$$

$$E_{cl}[\Phi_{cl}] + E_{ct}[\Phi_{cl}] + \sum (p_j + \tfrac{1}{2})\, h\, \frac{\partial \nu_j}{\partial \tau} = E$$

$$W_{\{p\}}(E) = S_{cl}[\Phi_{cl}] + S_{ct}[\Phi_{cl}] + E\,\tau(\Phi_{cl}) - \sum (p_j + \tfrac{1}{2})\, h\, \nu_j$$

$\rightarrow$ $E = E(\tau) \rightarrow \tau = \tau(E)$

$\downarrow$

gives the bound states

(2) bulk breather SG $\mathcal{L}_{SG} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^4}{\beta^2}\left(1 - \cos[\beta\varphi]\right)$

breather sol with period τ

$$\boxed{\Phi_\tau = \frac{4}{\beta} \operatorname{arctg}\left[\sqrt{\tilde{\tau}^2 - 1}\, \frac{\sin(mt/\tilde{\tau})}{\cosh\left(mx\sqrt{\tilde{\tau}^2-1}/\tilde{\tau}\right)} \right]}$$

$\tilde{\tau} = \frac{m\tau}{2\pi}$ dim. less period

$$S_{cl}[\Phi_\tau] = \int_0^\tau dt \int_{-\infty}^{+\infty} dx\, \mathcal{L} = \frac{32\pi}{\beta^2}\left[\arccos(1/\tilde{\tau}) - \sqrt{\tilde{\tau}^2 - 1} \right]$$

stability frequencies $\nu_0 = 0$ $\nu_1 = 0$ $\nu_{q_n} = \tau\sqrt{m^2 + q_n^2}$

$L q_n + \delta(q_n) = 2n\pi$

$\delta(q) = 4 \operatorname{arctg}\left(\frac{m\sqrt{\tilde{\tau}^2 - 1}}{q\,\tilde{\tau}} \right)$

$$V_{ct}[\Phi] = -\delta m^2 \frac{1}{\beta^2} \int_{-\infty}^{+\infty} dx \left[1 - \cos(\beta\Phi) \right] - E_{vac}$$

$E_{vac} = \frac{1}{2}\sum_{k_n} \sqrt{k_n^2 + m^2}$ $L k_n = 2n\pi$

$\forall p_j = 0$ (no higher excitations)

finite box (L) and momentum cut-off (Λ) finite result in $L, \Lambda \to \infty$

$$S_{ct}[\Phi_\tau] - \sum \frac{1}{2}\nu_i = -\frac{\beta^2}{8\pi} S_{cl}[\Phi_\tau] \quad \text{thus}$$

$$\underbrace{S_{cl}[\Phi_\tau] + S_{ct}[\Phi_\tau] - \frac{1}{2}\sum_i \nu_i} = \frac{32\pi}{\beta^2 8\pi}(8\pi - \beta^2)\left\{ \arccos(1/\tilde{\tau}) - \sqrt{\tilde{\tau}^2 - 1} \right\} = \frac{4}{p}\left\{ \arccos\left(\frac{1}{\tilde{\tau}}\right) - \sqrt{\tilde{\tau}^2 - 1} \right\}$$

thus $E = -\frac{d}{d\tau}\left(\quad\right) = \frac{4}{p}\frac{\sqrt{\tilde{\tau}^2-1}}{\tau} = \frac{4m\,2\pi}{p\,m\,2\pi}\frac{\sqrt{\tilde{\tau}^2-1}}{\tau} = \frac{2m}{\pi p}\frac{\sqrt{\tilde{\tau}^2-1}}{\tilde{\tau}}$ $\rightarrow$ $\arccos\frac{1}{\tilde{\tau}} = \arcsin\left(\frac{E\pi p}{2m}\right)$

$$W(E) = \frac{4}{p}\left[\arccos\left(\frac{1}{\tilde{\tau}}\right) - \sqrt{\tilde{\tau}^2-1} \right] + \frac{4}{p}\frac{\sqrt{\tilde{\tau}^2-1}}{\tau} = \frac{4}{p}\arccos\left(\frac{1}{\tilde{\tau}}\right) = \frac{4}{p}\arcsin\left(\frac{E\pi p}{2m}\right) \quad \boxed{W(E_n) = 2n\pi}$$

$$\boxed{E_n = 2\frac{m}{\pi p}\sin\left(n\pi\frac{p}{2}\right)} \qquad M_s = \frac{m}{\pi p}$$