Strings on $AdS \times S$, Dual Formulation of σ-models and Reduction

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Chiral $SU(2) \times SU(2)$ Model

The canonical framework for reducing non-linear sigma models is based on non-Abelian duality.
The chiral Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(\partial_\mu g^{-1}\partial^\mu g), \qquad g \in SU(2)$$

$$g(x) = e^{it^i \xi_i}, \qquad t^i = \frac{1}{2}\sigma^i$$

$$\mathrm{Tr}(t^i t^j) = \frac{1}{2}\delta^{ij}, \qquad i, j = 1, 2, 3$$

Current Formulation

The theory can be described in terms of currents

$$J_L^\mu = g\partial^\mu g^{-1}, \qquad g(x) \rightarrow ug(x)$$

$$J_\mu^R = g^{-1}\partial_\mu g, \qquad g(x) \rightarrow g(x)u$$

Both are conserved

$$\partial^\mu J_\mu^L = \partial J_\mu^R = 0$$

Also, the Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(J_\mu^L J_L^\mu) = -\frac{1}{2}(J_R^\mu J_\mu^R)$$

Either can be used for current formulation. We will use the right

$$J_\mu \equiv J_\mu^R$$

Currents as Dynamical Variables

Light-cone notation:

$$x^{\pm} = \frac{x^0 \pm x^1}{2}, \quad J_{\pm} = \frac{J_0 \pm J_1}{2}, \quad J^{\mp} = \frac{J^0 \mp J^1}{2},$$

(1) One has a current conservation

$$\partial^{\mu} J_{\mu} = \partial_+ J_- + \partial_- J_+ = 0$$

(2) and the Bianchi identity

$$\partial_{\mu} J_{\nu} - \partial_{\nu} J_{\mu} + [J_{\mu}, J_{\nu}] = 0$$

Standard description: solve the Bianchi identity; current conservation $\Rightarrow$ e.o.m.

Pseudo-Dual Formulation

Dual description: solve the conservation equation first:

$$\partial^\mu J_\mu = 0 \Rightarrow J_\mu = \epsilon_{\mu\nu}\partial^\nu \phi$$

Plugging into the Bianchi identity:

$$\partial^\mu \partial_\mu \phi - \frac{1}{2}\epsilon^{\mu\nu}[\partial_\mu \phi, \partial_\nu \phi] = 0$$

Pseudo-dual Lagrangian of Nappi [1]:

$$\mathcal{L}_{Nappi} = \frac{1}{2}\mathrm{Tr}(\partial^\mu \phi \partial_\mu \phi + \frac{1}{3}\phi \epsilon_{\mu\nu}[\partial^\mu \phi, \partial^\nu \phi]) \tag{1}$$

[1] C.R. Nappi, Phys. Rev. **D 21** 418(1980)

Dual Formulation

Start from the functional integral in terms of currents

$$Z = \int [dJ_\mu] \Pi \delta(\epsilon_{\mu\nu}[\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu]]) \exp\left(-\int J_\mu^2\right))$$

In the case of $SU(2)$

$$\mathcal{L}(J_\mu, \nu) = \frac{1}{2} J_\mu^2 - \psi(x) \cdot (\partial_\mu J_\nu + \frac{1}{2} J_\mu \times J_\nu) \epsilon_{\mu\nu}$$

where ψ is the Lagrange multiplier.

$$J_\mu = (1 - \psi^2)^{-1} [\epsilon_{\mu\nu} (\partial_\nu \psi - \psi(\psi_{,\nu} \cdot \psi)) - \psi_{,\mu} \times \psi]$$

Dual Lagrangian: [2]

$$\mathcal{L}(\psi) = -(1-\psi^2)^{-1} [(\partial\psi)^2 - (\psi \cdot \psi_{,\mu})(\psi \cdot \psi_{,\mu}) + \psi \cdot (\psi_{,\mu} \times \psi_{,\nu}) \epsilon_{\mu\nu}] \tag{2}$$

[2] B.E. Fridling and A. Jevicki, Phys. Lett. **B 134**, 70(1984).

Quantum Duality

At the quantum level [3] [4] the pseudo-dual representation fails!

The (nonlinear) dual representation can be checked to give identical results to the original sigma model:

- Higher Conservation Laws/No particle production
- Beta function (one, two loops, etc.)

[3] E. Fradkin and A. A. Tseytlin, Ann. Phys. **162**, 31(1985).

[4] J. Balog, P. Forgács, Z. Horváth, L. Palla Phys. Lett. B **388** 121(1996).

Symplectic Form

Note: $x^+ = \tau$.

Time evolution:

$$H = \frac{\partial}{\partial \tau}$$

Symplectic form:

$$\mathcal{L} = \mathrm{Tr}(\underbrace{\partial_+ \phi}_{\dot{\phi}} \cdot \underbrace{F(\phi, \partial_- \phi)}_{\pi}) \tag{3}$$

where $\partial_+ = \frac{\partial}{\partial x^+}$.

$S_2 = SU(2)/U(1) = CP_1$

The canonical framework for (S-G type/Pohlmeyer) reduction is the dual formulation.
Start with $g(x) \in SU(2)$,

$$J_\mu = g^{-1}\partial_\mu g = \sum_i t^i J^i_\mu$$

where

$$J^i_\mu = (g^{-1}\partial_\mu g)^i$$

For $i = 3$, the U(1) current,

$$(g^{-1}\partial_\mu g)^3 = A_\mu = J^3_\mu \qquad \text{will be gauged.}$$

The others, with $i = a = 1, 2$,

$$(g^{-1}\partial_\mu g)^a = \Pi^a_\mu \qquad \text{remain dynamical.}$$

U(1) transformation

J_0 generates $g(x)u(x)$ where

$$u(x) = e^{it^3\Lambda(x)} \in U(1)$$

$$A'_\mu = A_\mu + i\partial_\mu\Lambda(x)$$

Notation:

$$\Pi = \Pi^1 + i\Pi^2, \qquad \bar{\Pi} = \Pi^1 - i\Pi^2$$

Under $U(1)$ transformation:

$$\Pi' = e^{i\Lambda}\Pi, \qquad \bar{\Pi}' = e^{-i\Lambda}\bar{\Pi},$$

$$\mathcal{L} = -\frac{1}{2}\sum_{a=1}^{2} J^a_\mu J^{a\mu} = -\frac{1}{2}\sum_{a=1}^{2} \Pi^a_\mu \Pi^{a\mu} = -\frac{1}{2}\bar{\Pi}_\mu \Pi^\mu$$

is gauge invariant.

Bianchi identities

$$(\partial_\mu J_\nu - \partial_\nu J_\mu + [J_\mu, J_\nu])^i = 0$$

For $i = 3$,

$$\partial_\mu A_\nu - \partial_\nu A_\mu + (\bar{\Pi}_\mu \Pi_\nu - \bar{\Pi}_\nu \Pi_\mu) = 0$$

For $a, b = 1, 2$,

$$D_\mu \Pi^a_\nu \equiv \partial_\mu \Pi^a_\nu - i\epsilon_{ab} A_\mu \Pi^b_\nu$$

So that

$$\mathcal{L} = -\frac{1}{2}\bar{\Pi}_\mu \Pi^\mu + \psi \epsilon^{\mu\nu}(\partial_\mu A_\nu + \bar{\Pi}_\mu \Pi_\nu) + \lambda^a \epsilon^{\mu\nu}(D_\mu \Pi^a_\nu) \tag{4}$$

where ψ and λ^a are multipliers.

Light-cone notation

$$\Pi_\pm = \Pi_0 \pm \Pi_1, \qquad \bar{\Pi}_\pm = \bar{\Pi}_0 \pm \bar{\Pi}_1$$
$$A_\pm = A_0 \pm A_1, \qquad D_\pm = D_0 \pm D_1$$
$$\lambda = \lambda^1 + i\lambda^2, \qquad \bar{\lambda} = \lambda^1 - i\lambda^2$$

Finally, the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\bar{\Pi}_+\Pi_- + \bar{\Pi}_-\Pi_+) + \psi(\partial_- A_+ - \partial_+ A_- + (\Pi_+\bar{\Pi}_- - \Pi_-\bar{\Pi}_+)) - \lambda(D_+\bar{\Pi}_- - D_-\bar{\Pi}_+) - \bar{\lambda}(D_+\Pi_- - D_-\Pi_+) \tag{5}$$

Gauge Fixing and the sine-Gordon Equation

Fixing a Lorentz gauge,

$$\mathrm{Im}(\Pi_+\Pi_-) = 0$$

which leads to

$$\Pi_\pm = e^{\pm i\varphi}$$

In this gauge the equations of motion are

$$\begin{aligned}
A_\pm &= \pm\partial_\pm\varphi \\
\partial_\pm[e^{\pm i\varphi}\lambda] &= \pm(1 \pm i\psi)e^{\pm 2i\varphi} \\
\partial_\pm\psi &= \frac{i}{2}(e^{\pm i\varphi} - e^{\mp i\varphi}\bar{\lambda}) \\
\partial_+\partial_-\psi &= -(\psi\cos 2\varphi + \sin 2\varphi) \\
\partial_+\partial_-\varphi &= -\frac{1}{2}\sin(2\varphi)
\end{aligned} \tag{6}$$

finding the S-G equation.

Reduced Lagrangian

However, much like in the (pseudo) dual Nappi case one can not claim equivalence at the Lagrangian level.
Choose $A_+, \Pi_+, \bar{\Pi}_+$ as the Lagrange multipliers, the constraints are:

$$\begin{array}{l} \partial_- \psi + (\lambda \bar{\Pi}_- + \bar{\lambda} \Pi_-) = 0, \\ D_- \lambda + (1 + \psi) \Pi_- = 0, \\ D_- \bar{\lambda} + (1 - \psi) \bar{\Pi}_- = 0. \end{array}$$

The Lagrangian becomes

$$\begin{aligned} \mathcal{L} &= \psi \partial_+ \partial_- \varphi - \lambda \partial_+ e^{i\varphi} - \bar{\lambda} \partial_+ e^{-i\varphi} \\ &= (\partial_+ \varphi)(-\partial_- \psi - i\lambda e^{i\varphi} + i\bar{\lambda} e^{-i\varphi}) \end{aligned} \quad (7)$$

Lagrange multipliers

Here $\psi, \lambda, \bar{\lambda}$ are all functions of φ: $\psi(\varphi), \lambda(\varphi), \bar{\lambda}(\varphi)$
They can be solved by

$$\begin{aligned} \psi(\varphi) &= L_-^{-1} \cdot 2 \\ \lambda(\varphi) &= -D_-^{-1}(1+\psi(\varphi))\Pi_- \\ \bar{\lambda}(\varphi) &= -D_-^{-1}(1-\psi(\varphi))\bar{\Pi}_- \end{aligned}$$

where L_- is defined as

$$L_- = \partial_-(\partial_-\varphi)^{-1}(1+\partial_-)^2 + 4(\partial_-\varphi)\partial_-$$

Comments

- All the terms in (7) contain $\frac{\partial}{\partial \tau}$ and define the symplectic form.
- There are surface terms which define the Hamiltonian:

$$\mathcal{H} = \partial_-(2\psi A_+ + \frac{1}{2}\lambda\bar{\Pi}_+ + \frac{1}{2}\bar{\lambda}\Pi_+) \tag{8}$$

Symplectic Form

Consider the action of classical field theory [5]

$$S = \int d\tau d\sigma \{\Pi_+, (1 - i\psi)\Pi_-) + (\Pi_+, D_-\lambda) - (\Pi_-, D_+\lambda) \\ + \psi(\partial_+ A_- - \partial_- A_+)\}$$

The symplectic form read from the action is

$$\omega = \oint \{[(\delta\Pi_+, \delta\lambda) + \delta A_+ \delta\psi]dx^+ + [(\delta\Pi_-, \delta\lambda) + \delta A_- \delta\psi]dx^-\}$$

Plugging in the e.o.m., the symplectic form becomes

$$\omega = 2\int \{dx^+[\partial_+ \delta\psi\delta\varphi] - dx^-[\partial_- \delta\psi\delta\varphi]\} \tag{9}$$

[5] A. Mikhailov, hep-th/0511069.

Poisson structure

Denoting $q = \partial\varphi$ (here $\partial \equiv \partial_-$) and following Mikhailov, one works out the symplectic form

$$\omega = 4 \int dx^- \delta q L^{-1}(\partial q^{-1} \partial q^{-1} \partial + 4\partial)(L^T)^{-1} \delta q \qquad (10)$$

where

$$L \equiv \partial q^{-1}(1 + \partial^2) + 4q\partial$$
$$L^T \equiv -(1 + \partial^2) q^{-1} - 4\partial q$$

The Poisson structure is

$$\theta = \omega^{-1} = L^T (\partial q^{-1} \partial q^{-1} \partial + 4\partial)^{-1} L \qquad (11)$$

Poisson bracket

The usual Poisson bracket between q on the light cone is

$$\{q(x_1^-), q(x_2^-)\} = \theta\delta(x_1^- - x_2^-)$$

Another way:

$$\dot{q}(x^-) = \theta\frac{\delta H}{\delta q}(x^-) \tag{12}$$

As for the sine-Gordon model

$$\{q(x_1^-), q(x_2^-)\} = \delta'(x_1^- - x_2^-)$$

$$H = \int dx^- \cos 2\varphi$$

Plugging into (12)

$$\partial_+\varphi\partial_-\varphi = -\frac{1}{2}\sin 2\varphi \tag{13}$$

Poisson bracket

From (13), we know the standard Poisson structure

$$\theta_0 = \partial \tag{14}$$

Notice that θ can be written as

$$\theta = -(\theta_1 + \theta_0)\theta_1^{-1}(\theta_1 + \theta_0) \tag{15}$$

where θ_1 is the second Poisson structure of sine-Gordon model

$$\theta_1 = \partial^3 + 4\partial q \partial^{-1} q \partial \tag{16}$$

Conclusion

- First item
- Second item
- ...