Solitons and AdS String Solutions

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Outline

- Classical string solutions: spiky strings
- Spikes as sinh-Gordon solitons
- AdS string as a σ-model
- Inverse scattering method
- Infinite AdS string solutions
- Finite (closed) string solutions
- N-soliton (spike) construction
- Conclusion and Outlook

1. Motivation

- Semiclassical analysis of strings in $AdS \times S$ space-time is relevant for large $\lambda = g_{YM}^2 N$ (strong coupling) investigation of AdS/CFT;
- Computing gluon scattering amplitudes can be reduced to finding the minimal area of a classical string solution;
- Giant magnon solutions on $R \times S^2$ and $R \times S^3$ can be mapped to soliton solutions in sine-Gordon and complex sine-Gordon respectively.

GKP solution

Gubser, Klebanov and Polyakov [1] gave a first study of large (spin) angular momentum solutions in conformal gauge.

AdS$_3$ coordinates : $X^i = (t, \rho, \theta)$

metric : $ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\theta^2$

action : $A = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma G_{ij} \partial_\alpha X^i \partial^\alpha X^j$

Virasoro constraints :

$$T_{++} = \partial_+ X^i \partial_+ X^j G_{ij} = 0$$

$$T_{--} = \partial_- X^i \partial_- X^j G_{ij} = 0$$

Ansatz :

$$t = c\,\tau$$

$$\theta = c\,\omega\,\tau$$

where c is a constant to rescale the period of σ.

Assumption : $\rho = \rho\,(\sigma)$ ← rigid rotation

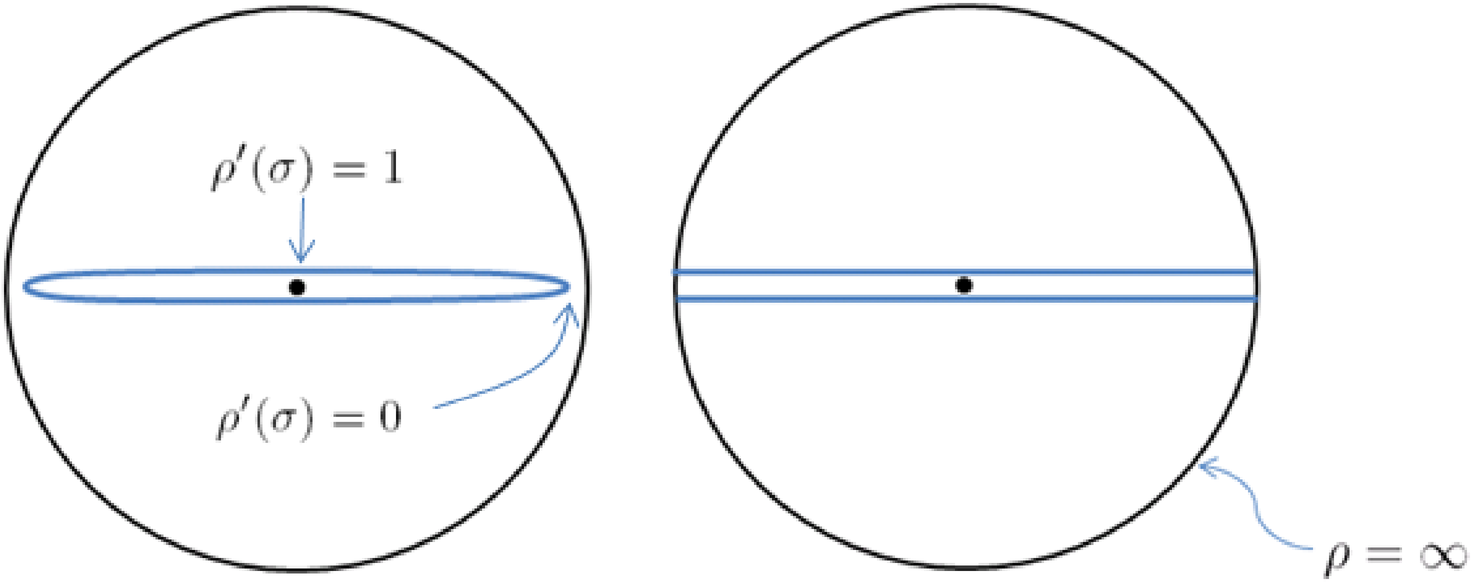
[1] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, hep-th/**0204051**.

Folded rotating string

Solution :

$$\rho'^2(\sigma) = c^2(\cosh^2\rho - \omega^2 \sinh^2\rho)$$

$$\Longrightarrow \quad \rho(\sigma) = \text{arccosh}(\text{nd}(\omega\sigma, \frac{1}{\omega}))$$



(a) w>1

(b) w=1

Energy calculation

$$E = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2L} d\sigma \cosh^2\rho = \frac{2\sqrt{\lambda}}{\pi}\Big[\frac{\omega}{\omega^2-1}E(\frac{1}{\omega})\Big],$$

$$S = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2L} d\sigma\, \omega \sinh^2\rho = \frac{2\sqrt{\lambda}}{\pi}\Big[\frac{\omega^2}{\omega^2-1}E(\frac{1}{\omega}) - K(\frac{1}{\omega})\Big].$$

where $\mathrm{E}(\frac{1}{\omega})$ and $\mathrm{K}(\frac{1}{\omega})$ are elliptic functions. Therefore,

$$E - \omega S = \frac{2\omega\sqrt{\lambda}}{\pi}[K(\frac{1}{\omega}) - E(\frac{1}{\omega})]$$

In the large S (spin angular momentum) limit, we have $\omega=1+2\eta$, where $\eta \ll 1$.

$$\mathrm{E}(\frac{1}{\omega}) \sim 1 + \eta \ln\frac{1}{\eta}, \qquad \mathrm{K}(\frac{1}{\omega}) \sim \frac{1}{2}\ln\frac{1}{\eta}$$

$$E = \frac{\sqrt{\lambda}}{2\pi}(\frac{1}{\eta} + \ln\frac{1}{\eta} + \cdots)$$

$$S = \frac{\sqrt{\lambda}}{2\pi}(\frac{1}{\eta} - \ln\frac{1}{\eta} + \cdots)$$

⇒

$$\boxed{E - S = \frac{\sqrt{\lambda}}{\pi}\ln(\frac{S}{\sqrt{\lambda}}) + \cdots}$$

Kruczenski's solution

Kruczenski [2] gave the spiky string solutions in physical gauge:

Ansatz :

$$t = \tau$$
$$\theta = \omega \tau + \sigma$$

rigid rotation :

$$\rho = \rho(\sigma)$$

Nambu-Goto action :

$$A = -\frac{\sqrt{\lambda}}{2\pi} \int d\tau d\sigma \sqrt{(\dot{X} X')^2 - \dot{X}^2 X'^2}$$

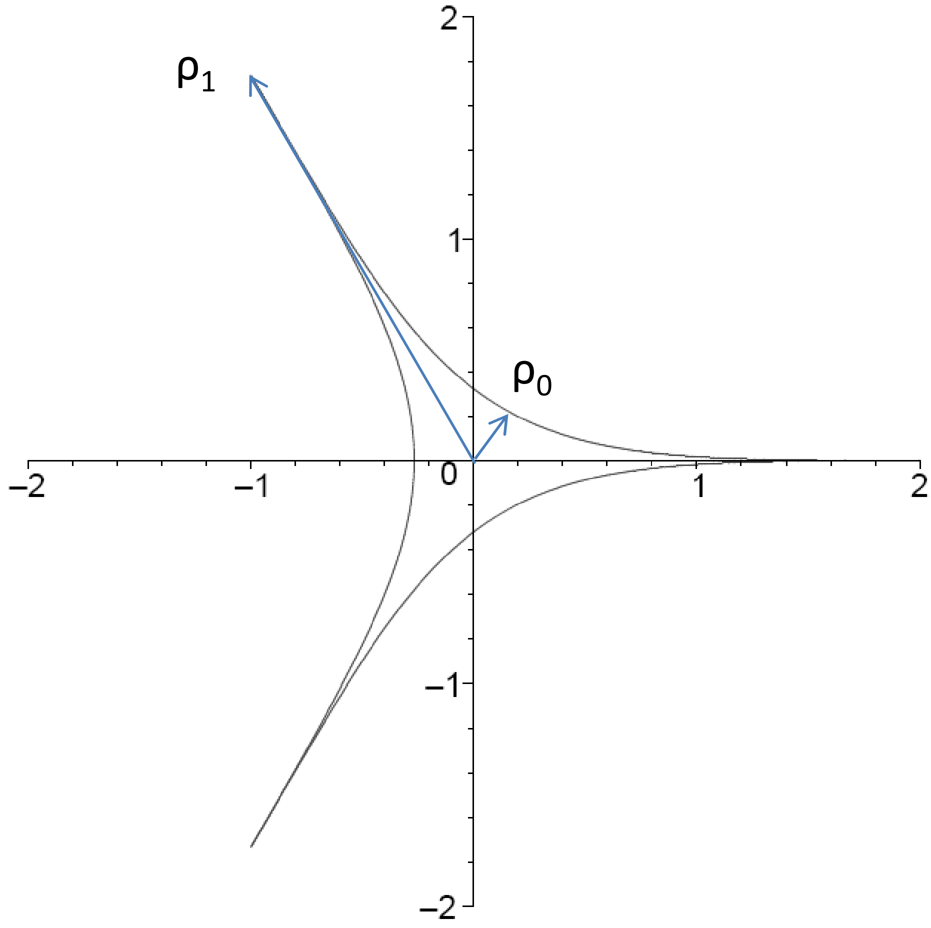
Spiky solution:

$$\rho'(\sigma) = \frac{1}{2} \frac{\sinh 2\rho}{\sinh 2\rho_0} \frac{\sqrt{\sinh^2 2\rho - \sinh^2 2\rho_0}}{\sqrt{\cosh^2 \rho - \omega^2 \sinh^2 \rho}}$$

where ρ_0 is the minimum value of ρ; the maximum value is arccoth $\omega = \rho_1$.

[2] M. Kruczenski, *Spiky strings and single trace operators in gauge theories*, hep-th/**0410226.**

Spiky strings in AdS



n=3 spiky configuration

Spikes: $$\theta_m = \frac{2\pi m}{n}$$

Energy calculation

$$E = \sqrt{\lambda}\frac{2n}{2\pi}\int_{\rho_0}^{\rho_1} d\rho \, \frac{\cosh^2\rho \sinh^2 2\rho - \omega^2 \sinh^2\rho \sinh^2 2\rho_0}{\sinh 2\rho\sqrt{(\cosh^2\rho - \omega^2\sinh^2\rho)(\sinh^2 2\rho - \sinh^2 2\rho_0)}}$$

$$S = \sqrt{\lambda}\frac{2n}{2\pi}\int_{\rho_0}^{\rho_1} d\rho \, \frac{\omega}{2}\frac{\sinh\rho}{\cosh\rho}\frac{\sqrt{\sinh^2 2\rho - \sinh^2 2\rho_0}}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}}$$

$$E - \omega S = \sqrt{\lambda}\frac{2n}{2\pi}\int_{\rho_0}^{\rho_1} d\rho \, \sinh 2\rho \frac{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}}{\sqrt{\sinh^2 2\rho - \sinh^2 2\rho_0}}$$

In the limit $\rho_1 \gg 1$ and $\rho_1 \gg \rho_0$, we have $\omega = \coth\rho_1 \to 1$

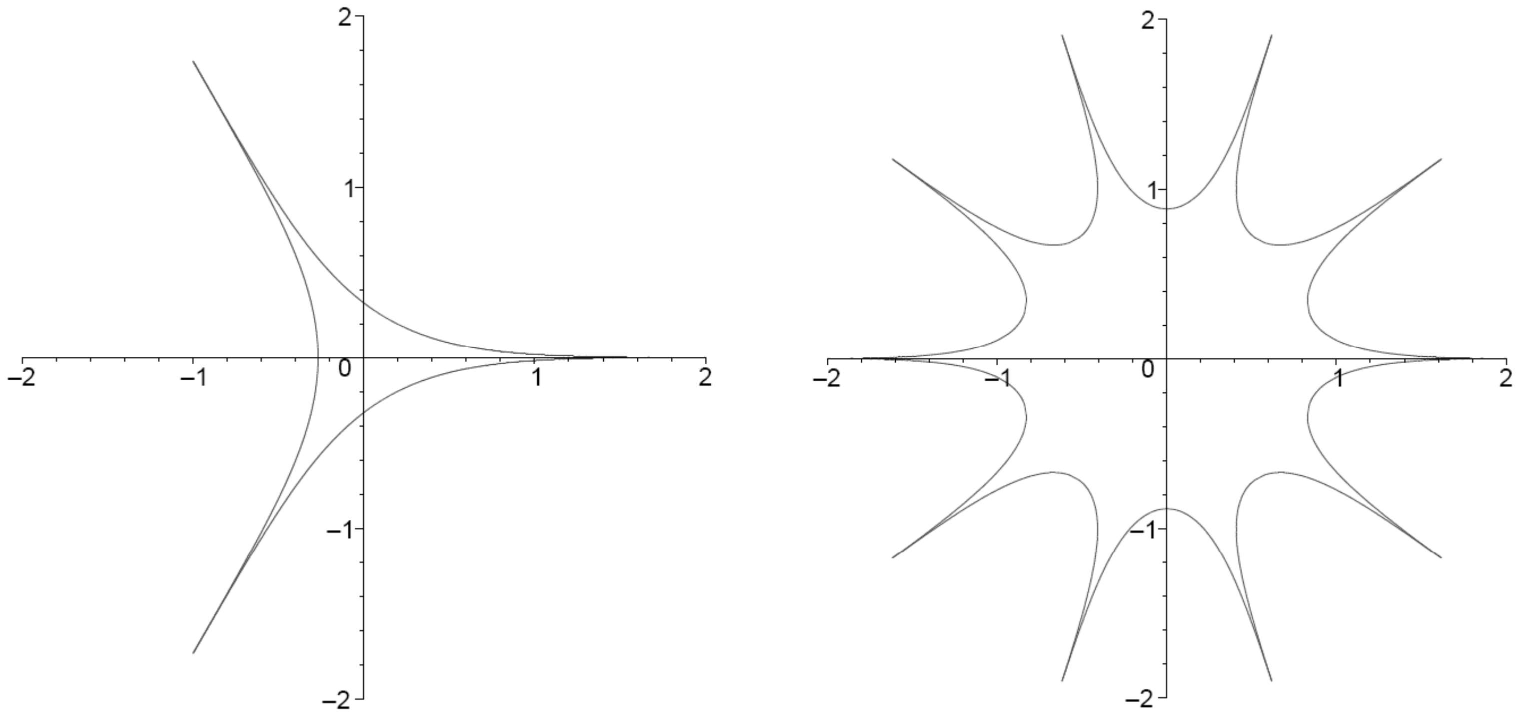
Large S energy of n-spike solution:

$$E - S = n\frac{\sqrt{\lambda}}{2\pi}\ln\frac{S}{\sqrt{\lambda}} + \cdots$$

n=2 agrees with the GKP solution:

$$E - S = \frac{\sqrt{\lambda}}{\pi}\ln\left(\frac{S}{\sqrt{\lambda}}\right) + \cdots$$

Spiky strings in AdS



- The main interest is to study the dynamics of spikes
- For this purpose, it is convenient to introduce the soliton picture
- We will show next the soliton picture of the GKP solution
- The same argument works for the Kruczenski n-spike solution

2. Spikes as sinh-Gordon solitons

Asymptotics near the turning point: GKP solution

$$\rho'^2 = \cosh^2\rho - \omega^2 \sinh^2\rho \sim \frac{1}{4}e^{2\rho}(1-\omega^2+(1+\omega^2)2e^{-2\rho})$$

Let $\omega = 1 + 2\eta$ where $\eta \ll 1$, then one gets

$$\rho'^2 \sim e^{2\rho}(e^{-2\rho} - \eta)$$

Denote $u = e^{-\rho}$, we have

$$u'^2 \sim u^2 - \eta$$

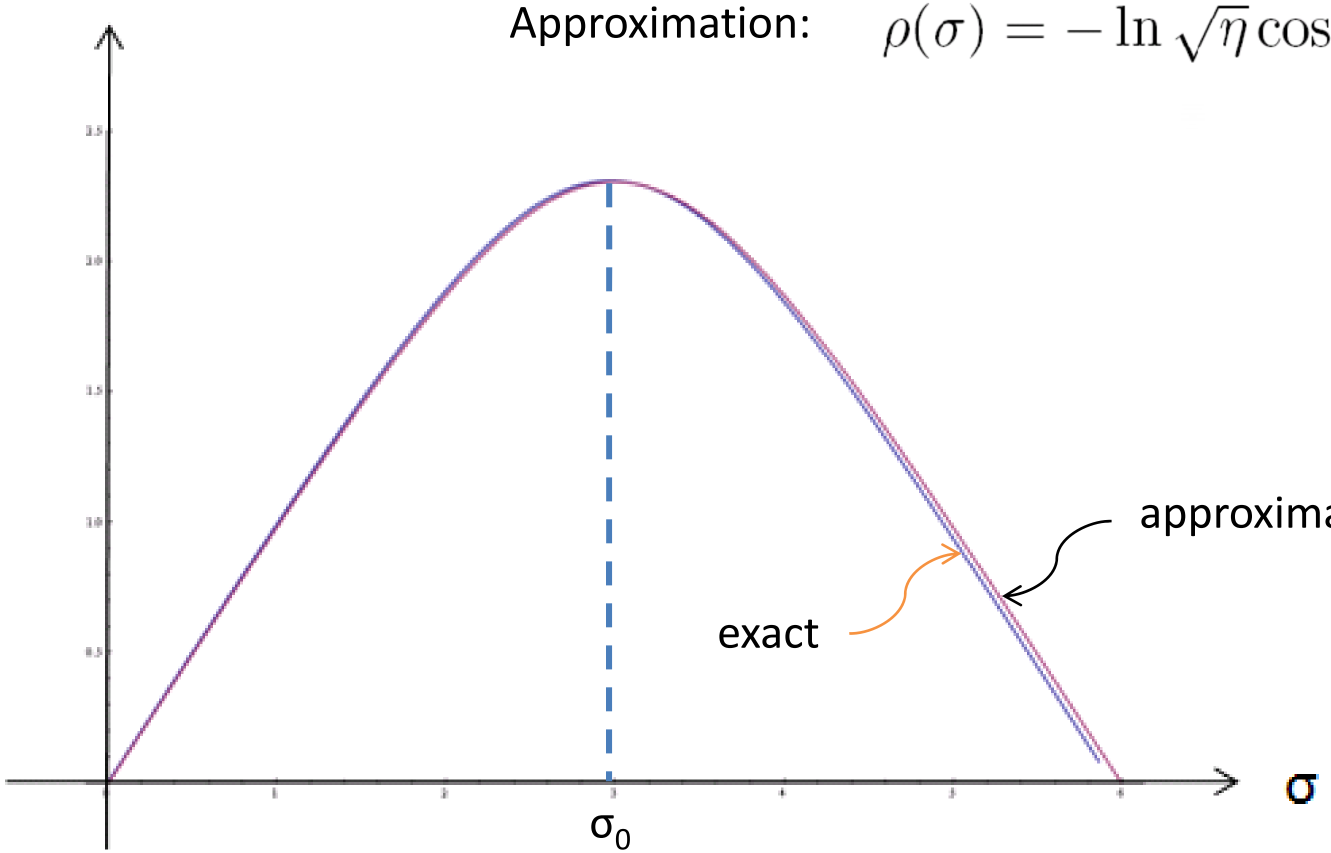
$$\Longrightarrow \quad \boxed{\rho(\sigma) = -\ln\sqrt{\eta}\cosh(\sigma - \sigma_0)}$$

Near-spike approximation

Exact solution: $\rho(\sigma) = \text{arccosh}(\text{nd}(\omega\sigma, \frac{1}{\omega}))$

Approximation: $\rho(\sigma) = -\ln\sqrt{\eta}\cosh(\sigma - \sigma_0)$

ρ (σ)



$\omega=1.02$

Relation to Sinh-Gordon soliton

One observes the correspondence with the sinh-Gordon soliton.

Define : $\alpha \equiv \ln(q_\xi \cdot q_\eta)$

where q being a AdS_3 string solution with signature: {-1, -1, +1, +1}.

One can check, that for the near turning point GKP solution,

$$\alpha = \ln(2{\rho'}^2) = \ln(2\tanh^2\sigma) = \ln 2 + \hat{\alpha}$$

satisfies the sinh-Gordon equation:

$$\hat{\alpha}_{\xi\eta} - 4\sinh\hat{\alpha} = 0$$

Therefore, the finite GKP solution is then a two-soliton configuration of sinh-Gordon system !

$$\xi = (\sigma + \tau)/2$$

$$\eta = (\sigma - \tau)/2$$

3. AdS string as a σ-model

We parameterize AdS_d with *d+1* embedding coordinates *q* subject to the constraint

$$q^2 = -q_{-1}^2 - q_0^2 + q_1^2 + q_2^2 + \cdots + q_{d-1}^2 = -1$$

Conformal gauge action :

$$A = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \left(\partial q \cdot \partial q + \lambda(\sigma, \tau)(q \cdot q + 1)\right)$$

where τ and σ are Minkowski worldsheet coordinates.

Equations of motion : $$q_{\xi\eta} - (q_\xi \cdot q_\eta) q = 0$$

Virasoro constraints : $$q_\xi^2 = q_\eta^2 = 0$$

$$\xi = (\sigma + \tau)/2 \qquad \partial_\xi = \partial_\sigma + \partial_\tau$$

$$\eta = (\sigma - \tau)/2 \qquad \partial_\eta = \partial_\sigma - \partial_\tau$$

Equivalence to sinh-Gordon model

Choose a basis : $e_i = (q, q_\xi, q_\eta, b_4, \cdots, b_{d+1})$

where i=1,2, … , d+1 and the vectors b_k with k=4,5, … , d+1 are orthonormal

$$b_k \cdot b_l = \delta_{kl}, \quad b_k \cdot q = b_k \cdot q_\xi = b_k \cdot q_\eta = 0$$

Define: $\alpha \equiv \ln(q_\xi \cdot q_\eta) \quad u_k \equiv b_k \cdot q_{\xi\xi} \quad v_k \equiv b_k \cdot q_{\eta\eta}$

The equations of motion are :

$$\alpha_{\xi\eta} - e^{\alpha} - e^{-\alpha} \sum_{i=4}^{d+1} u_i v_i = 0 \quad (u_i)_\eta = \sum_{j=4, j\neq i}^{d+1} u_j (b_j) \cdot (b_i)_\eta \quad (v_i)_\xi = \sum_{j=4, j\neq i}^{d+1} v_j (b_j) \cdot (b_i)_\xi$$

Generalized sinh-Gordon model [3].

d=2: Liouville equation d=3: sinh-Gordon equation d=4: B_2 Toda model

[3] H. J. de Vega and N. Sanchez, *Exact integrability of strings in D-dimensional dS spacetime*, PRD, **47**, 3394 (1993).

AdS_3 case in more details

$$u_\eta = 0 \Rightarrow u = u(\xi)$$

$$v_\xi = 0 \Rightarrow v = v(\eta)$$

$$\alpha_{\xi\eta} - e^{\alpha} - uve^{-\alpha} = 0 \qquad \hat{\alpha}_{\xi'\eta'} - 2\sinh\hat{\alpha} = 0$$

$$\frac{d\xi'}{d\xi} = \sqrt{u(\xi)} \qquad \frac{d\eta'}{d\eta} = \sqrt{-v(\eta)} \qquad \alpha(\xi,\eta) = \hat{\alpha}(\xi',\eta') + \frac{1}{2}\ln[-u(\xi)v(\eta)]$$

Now we express the derivatives of the basis vectors in terms of the basis itself :

$$\frac{\partial e_i}{\partial \xi} = A_{ij}(\xi,\eta)e_j, \qquad \frac{\partial e_i}{\partial \eta} = B_{ij}(\xi,\eta)e_j$$

we get :

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \alpha_\xi & 0 & u \\ e^{\alpha} & 0 & 0 & 0 \\ 0 & 0 & -ue^{-\alpha} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ e^{\alpha} & 0 & 0 & 0 \\ 0 & 0 & \alpha_\eta & v \\ 0 & -ve^{-\alpha} & 0 & 0 \end{pmatrix}$$

SO(2,2) symmetry

In order to see the explicit SO(2,2) symmetry, we choose an orthonormal basis

$$e_1 = b, \quad e_2 = \frac{q_\xi + q_\eta}{\sqrt{2}e^{\alpha/2}}, \quad e_3 = \frac{q_\xi - q_\eta}{\sqrt{2}ie^{\alpha/2}}, \quad e_4 = iq.$$

Then *A, B* matrices become

$$A = \begin{pmatrix} 0 & -\frac{u}{\sqrt{2}}e^{-\alpha/2} & \frac{iu}{\sqrt{2}}e^{-\alpha/2} & 0 \\ \frac{u}{\sqrt{2}}e^{-\alpha/2} & 0 & \frac{i}{2}\alpha_\xi & -\frac{i}{\sqrt{2}}e^{\alpha/2} \\ -\frac{iu}{\sqrt{2}}e^{-\alpha/2} & -\frac{i}{2}\alpha_\xi & 0 & \frac{1}{\sqrt{2}}e^{\alpha/2} \\ 0 & \frac{i}{\sqrt{2}}e^{\alpha/2} & -\frac{1}{\sqrt{2}}e^{\alpha/2} & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -\frac{v}{\sqrt{2}}e^{-\alpha/2} & -\frac{iv}{\sqrt{2}}e^{-\alpha/2} & 0 \\ \frac{v}{\sqrt{2}}e^{-\alpha/2} & 0 & -\frac{i}{2}\alpha_\eta & -\frac{i}{\sqrt{2}}e^{\alpha/2} \\ \frac{iv}{\sqrt{2}}e^{-\alpha/2} & \frac{i}{2}\alpha_\eta & 0 & -\frac{1}{\sqrt{2}}e^{\alpha/2} \\ 0 & \frac{i}{\sqrt{2}}e^{\alpha/2} & \frac{1}{\sqrt{2}}e^{\alpha/2} & 0 \end{pmatrix}.$$

4. Inverse Scattering Method

Remember the isometry :

$$SO(2,2) = SO(2,1) \times SO(2,1)$$

Introduce two commuting sets of SO(2,1) generators :

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_3, \quad [K_3, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_3, \quad [J_i, K_j] = 0,$$

Expand A, B matrices as

$$A = w^i_{1,(+)} J_i + w^i_{1,(-)} K_i, \qquad B = w^i_{2,(+)} J_i + w^i_{2,(-)} K_i,$$

with coefficients

$$\vec{w}_{1,(\pm)} = \Big(\frac{i}{2}\alpha_\xi, \frac{-i}{\sqrt{2}}(ue^{-\alpha/2} \mp e^{\alpha/2}), \frac{-i}{\sqrt{2}}(ue^{-\alpha/2} \pm e^{\alpha/2} \Big),$$

$$\vec{w}_{2,(\pm)} = \Big(\frac{-i}{2}\alpha_\eta, \frac{i}{\sqrt{2}}(ve^{-\alpha/2} \pm e^{\alpha/2}), \frac{-i}{\sqrt{2}}(ve^{-\alpha/2} \mp e^{\alpha/2} \Big).$$

Spinor representation

Remember SO(2,1)=SU(1,1), we can define two spinors as

$$\phi_\xi = w^i_{1,(+)}\sigma_i\phi = A_1\phi, \qquad \phi_\eta = w^i_{2,(+)}\sigma_i\phi = A_2\phi,$$
$$\psi_\xi = w^i_{1,(-)}\sigma_i\psi = B_1\psi, \qquad \psi_\eta = w^i_{2,(-)}\sigma_i\psi = B_2\psi.$$

where the matrices are given by

$$A_1 = \begin{pmatrix} \frac{-i}{2\sqrt{2}}(ue^{-\alpha/2}+e^{\alpha/2}) & \frac{i}{4}\alpha_\xi - \frac{1}{2\sqrt{2}}(ue^{-\alpha/2}-e^{\alpha/2}) \\ -\frac{i}{4}\alpha_\xi - \frac{1}{2\sqrt{2}}(ue^{-\alpha/2}-e^{\alpha/2}) & \frac{i}{2\sqrt{2}}(ue^{-\alpha/2}+e^{\alpha/2}) \end{pmatrix}, \quad B_1 = \begin{pmatrix} \frac{-i}{2\sqrt{2}}(ue^{-\alpha/2}-e^{\alpha/2}) & \frac{i}{4}\alpha_\xi - \frac{1}{2\sqrt{2}}(ue^{-\alpha/2}+e^{\alpha/2}) \\ -\frac{i}{4}\alpha_\xi - \frac{1}{2\sqrt{2}}(ue^{-\alpha/2}+e^{\alpha/2}) & \frac{i}{2\sqrt{2}}(ue^{-\alpha/2}-e^{\alpha/2}) \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \frac{-i}{2\sqrt{2}}(ve^{-\alpha/2}-e^{\alpha/2}) & -\frac{i}{4}\alpha_\eta + \frac{1}{2\sqrt{2}}(ve^{-\alpha/2}+e^{\alpha/2}) \\ \frac{i}{4}\alpha_\eta + \frac{1}{2\sqrt{2}}(ve^{-\alpha/2}+e^{\alpha/2}) & \frac{i}{2\sqrt{2}}(ve^{-\alpha/2}-e^{\alpha/2}) \end{pmatrix}. \quad B_2 = \begin{pmatrix} \frac{-i}{2\sqrt{2}}(ve^{-\alpha/2}+e^{\alpha/2}) & -\frac{i}{4}\alpha_\eta + \frac{1}{2\sqrt{2}}(ve^{-\alpha/2}-e^{\alpha/2}) \\ \frac{i}{4}\alpha_\eta + \frac{1}{2\sqrt{2}}(ve^{-\alpha/2}-e^{\alpha/2}) & \frac{i}{2\sqrt{2}}(ve^{-\alpha/2}+e^{\alpha/2}) \end{pmatrix}.$$

Then the string solution is given by:

$$\begin{aligned} q_{-1} &= \frac{1}{2}(\phi_1\psi_1^* - \phi_2\psi_2^*) + c.c. & \qquad q_0 &= \frac{i}{2}(\phi_1\psi_1^* - \phi_2\psi_2^*) + c.c. \\ q_1 &= \frac{1}{2}(\phi_2\psi_1 - \phi_1\psi_2) + c.c. & \qquad q_2 &= \frac{i}{2}(\phi_2\psi_1 - \phi_1\psi_2) + c.c. \end{aligned}$$

5. Infinite string solutions

Vacuum solution:

$$u = 2, v = -2, \alpha_0 = \ln 2,$$

Matrices :

$$A_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \quad A_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad B_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Spinors :

$$\phi_1 = e^{-i\tau} \quad \phi_2 = 0 \quad \psi_1 = \cosh\sigma \quad \psi_2 = -\sinh\sigma.$$

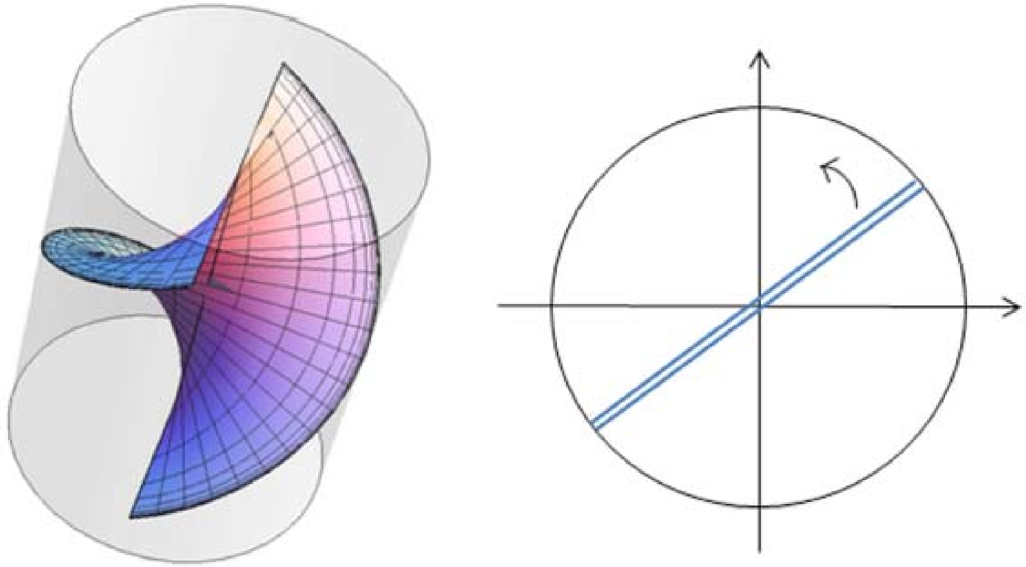
String solution :

$$q = \begin{pmatrix} \cosh\sigma\cos\tau \\ \cosh\sigma\sin\tau \\ \sinh\sigma\cos\tau \\ \sinh\sigma\sin\tau \end{pmatrix}$$

[4] A. Jevicki, K. Jin, C. Kalousios and A. Volovich, Generating AdS string solutions, arXiv : 0712.1193.

Vacuum solution

$$t = \tau$$
$$\theta = \tau$$
$$\rho = \sigma$$
$$\Downarrow$$
$$\omega = 1$$



(a) (b)

$$E = \frac{\sqrt{\lambda}}{\pi}\int_{-\Lambda}^{\Lambda} d\sigma \cosh^2\sigma \approx \frac{\sqrt{\lambda}}{4\pi}e^{2\Lambda},$$
$$S = \frac{\sqrt{\lambda}}{\pi}\int_{-\Lambda}^{\Lambda} d\sigma \sinh^2\sigma \approx \frac{\sqrt{\lambda}}{4\pi}e^{2\Lambda},$$

$$\Rightarrow \quad E - S \sim \frac{\sqrt{\lambda}}{\pi}\ln\frac{4\pi}{\sqrt{\lambda}}S$$

One-soliton solution

Sinh-Gordon : $$\alpha_s = \ln(2\tanh^2\sigma)$$

Spinors :

$$\phi_1 = e^{-i\tau}\cosh(\frac{1}{2}\ln\tanh\sigma),$$

$$\phi_2 = -e^{-i\tau}\sinh(\frac{1}{2}\ln\tanh\sigma),$$

$$\psi_1 = (\tau+i)\cosh(\frac{1}{2}\ln\sinh 2\sigma) - \tau\sinh(\frac{1}{2}\ln\sinh 2\sigma),$$

$$\psi_2 = -(\tau+i)\sinh(\frac{1}{2}\ln\sinh 2\sigma) + \tau\cosh(\frac{1}{2}\ln\sinh 2\sigma).$$

Linear !

String solution :

$$q_s = \frac{1}{2\sqrt{2}\cosh\sigma}\begin{pmatrix} 2\tau\cos\tau - \sin\tau(\cosh 2\sigma + 2) \\ 2\tau\sin\tau + \cos\tau(\cosh 2\sigma + 2) \\ -2\tau\cos\tau + \sin\tau\cosh 2\sigma \\ -2\tau\sin\tau - \cos\tau\cosh 2\sigma \end{pmatrix}$$

Energy of one-soliton solution

$$\mathcal{P}_t^\tau = \frac{\sqrt{\lambda}}{16\pi}(1 + 8\tau^2 + 4\cosh 2\sigma + \cosh 4\sigma)\,\mathrm{sech}^2\,\sigma \qquad \mathcal{P}_t^\sigma = \frac{\sqrt{\lambda}}{\pi}\tau\tanh\sigma$$

$$\mathcal{P}_\theta^\tau = \frac{\sqrt{\lambda}}{16\pi}(1 + 8\tau^2 - 4\cosh 2\sigma + \cosh 4\sigma)\,\mathrm{sech}^2\,\sigma \qquad \mathcal{P}_\theta^\sigma = \frac{\sqrt{\lambda}}{\pi}\tau\tanh\sigma$$

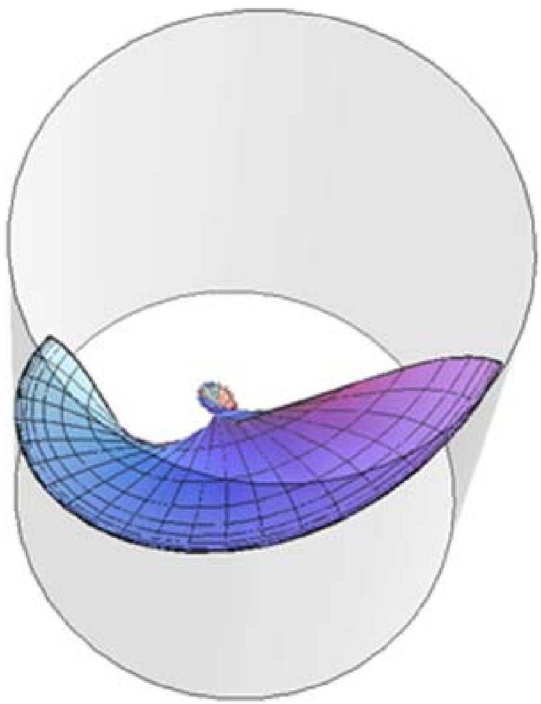
$$E = \int_{-\Lambda}^{\Lambda} d\sigma \mathcal{P}_t^\tau = \frac{\sqrt{\lambda}}{\pi}(\frac{1}{4}\sigma + \frac{1}{8}\sinh 2\sigma - \frac{1}{8}\tanh\sigma + \frac{1}{2}\tau^2\tanh\sigma)|_{-\Lambda}^{\Lambda} \approx \frac{\sqrt{\lambda}}{\pi}(\frac{1}{8}e^{2\Lambda} + \tau^2)$$

$$S = \int_{-\Lambda}^{\Lambda} d\sigma \mathcal{P}_\theta^\tau = \frac{\sqrt{\lambda}}{\pi}(-\frac{3}{4}\sigma + \frac{1}{8}\sinh 2\sigma + \frac{3}{8}\tanh\sigma + \frac{1}{2}\tau^2\tanh\sigma)|_{-\Lambda}^{\Lambda} \approx \frac{\sqrt{\lambda}}{\pi}(\frac{1}{8}e^{2\Lambda} + \tau^2)$$

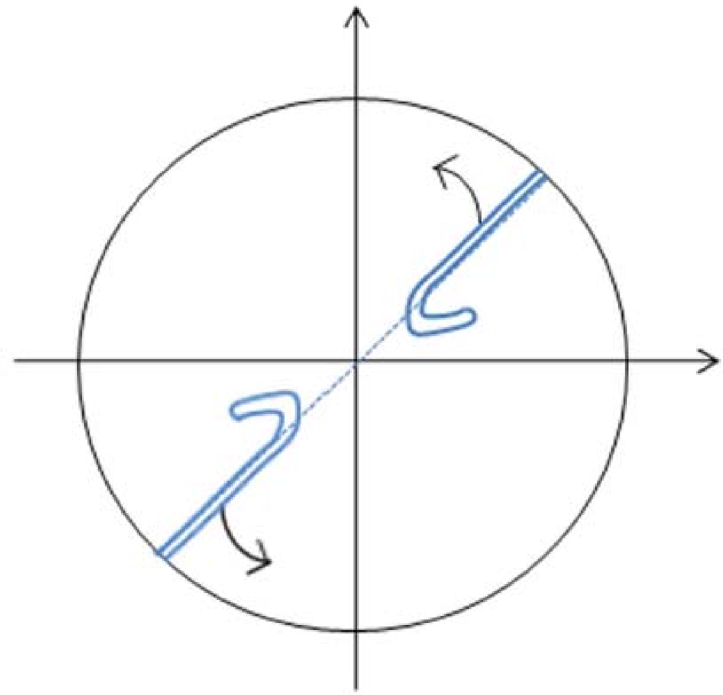
If we neglect the τ dependence since the exponential term increases much faster,

$$\Rightarrow \quad \boxed{E - S = \int_{-\Lambda}^{\Lambda} \frac{\sqrt{\lambda}}{2\pi}\cosh 2\sigma\,\mathrm{sech}^2\,\sigma\, d\sigma \sim \frac{\sqrt{\lambda}}{\pi}\ln\frac{8\pi}{\sqrt{\lambda}}S}$$

One-soliton solution



(a)



(b)

Other one-soliton solutions

$$\alpha_{\bar{s}} = \ln(2\coth^2\sigma) \quad \Longrightarrow \quad q_{\bar{s}} = \frac{1}{2\sqrt{2}\sinh\sigma}\begin{pmatrix} 2\tau\cos\tau + \sin\tau\cosh 2\sigma \\ 2\tau\sin\tau - \cos\tau\cosh 2\sigma \\ -2\tau\cos\tau - \sin\tau(\cosh 2\sigma - 2) \\ -2\tau\sin\tau + \cos\tau(\cosh 2\sigma - 2) \end{pmatrix}$$

$$\alpha'_{s} = \ln(2\tan^2\sigma) \quad \Longrightarrow \quad q'_{s} = \frac{1}{2\sqrt{2}\cos\sigma}\begin{pmatrix} 2\tau\cosh\tau - \sinh\tau\cos 2\sigma \\ 2\tau\sinh\tau - \cosh\tau(\cos 2\sigma + 2) \\ 2\tau\cosh\tau - \sinh\tau(\cos 2\sigma + 2) \\ -2\tau\sinh\tau + \cosh\tau\cos 2\sigma \end{pmatrix}$$

$$\alpha'_{\bar{s}} = \ln(2\cot^2\sigma) \quad \Longrightarrow \quad q'_{\bar{s}} = \frac{1}{2\sqrt{2}\sin\sigma}\begin{pmatrix} 2\tau\cosh\tau + \sinh\tau\cos 2\sigma \\ 2\tau\sinh\tau + \cosh\tau(\cos 2\sigma - 2) \\ -2\tau\cosh\tau - \sinh\tau(\cos 2\sigma - 2) \\ 2\tau\sinh\tau + \cosh\tau\cos 2\sigma \end{pmatrix}$$

Singular !

Two-soliton solution

sinh-Gordon : $$\boxed{\alpha_{ss} = \ln 2\left(\frac{v\cosh X - \cosh T}{v\cosh X + \cosh T}\right)^2}$$ (Bäcklund transformation)

where $X = \frac{2\sigma}{\sqrt{1-v^2}}$, $T = \frac{2v\tau}{\sqrt{1-v^2}}$, and *v* is the relative velocity of two solitons.

$$q = \frac{1}{\cosh T + v\cosh X}\begin{pmatrix} (v\mathrm{ch}T\mathrm{ch}\sigma + \mathrm{ch}X\mathrm{ch}\sigma - \gamma^{-1}\mathrm{sh}X\mathrm{sh}\sigma)\cos\tau + \gamma^{-1}\mathrm{sh}T\mathrm{ch}\sigma\sin\tau \\ -(v\mathrm{ch}T\mathrm{ch}\sigma + \mathrm{ch}X\mathrm{ch}\sigma - \gamma^{-1}\mathrm{sh}X\mathrm{sh}\sigma)\sin\tau + \gamma^{-1}\mathrm{sh}T\mathrm{ch}\sigma\cos\tau \\ (v\mathrm{ch}T\mathrm{sh}\sigma + \mathrm{ch}X\mathrm{sh}\sigma - \gamma^{-1}\mathrm{sh}X\mathrm{ch}\sigma)\cos\tau + \gamma^{-1}\mathrm{sh}T\mathrm{sh}\sigma\sin\tau \\ -(v\mathrm{ch}T\mathrm{sh}\sigma + \mathrm{ch}X\mathrm{sh}\sigma - \gamma^{-1}\mathrm{sh}X\mathrm{ch}\sigma)\sin\tau + \gamma^{-1}\mathrm{sh}T\mathrm{sh}\sigma\cos\tau \end{pmatrix}$$

$$E = \int d\sigma \mathcal{P}_t^\tau = \frac{1}{8v(\cosh T + v\cosh X)}\Big\{-4v^2\gamma^{-1}\sinh X - (2 - v^2 + 2\gamma^{-1})\sinh(2-2\gamma)\sigma$$
$$+ (2 - v^2 - 2\gamma^{-1})\sinh(2+2\gamma)\sigma + 2v\cosh T(2\sigma + \sinh 2\sigma) + 4v^2\sigma\cosh X\Big\}$$

$$S = \int d\sigma \mathcal{P}_\theta^\tau = \frac{1}{8v(\cosh T + v\cosh X)}\Big\{4v^2\gamma^{-1}\sinh X + (2 - v^2 + 2\gamma^{-1})\sinh(2-2\gamma)\sigma$$
$$+ (2 - v^2 - 2\gamma^{-1})\sinh(2+2\gamma)\sigma + 2v\cosh T(-2\sigma + \sinh 2\sigma) - 4v^2\sigma\cosh X\Big\}$$

$$\boxed{E - S = \int d\sigma(\mathcal{P}_t^\tau - \mathcal{P}_\theta^\tau) = \sigma - \frac{v\gamma^{-1}\sinh X}{\cosh T + v\cosh X}}$$

where $\gamma = (1 - v^2)^{-1/2}$.

Two-soliton solution

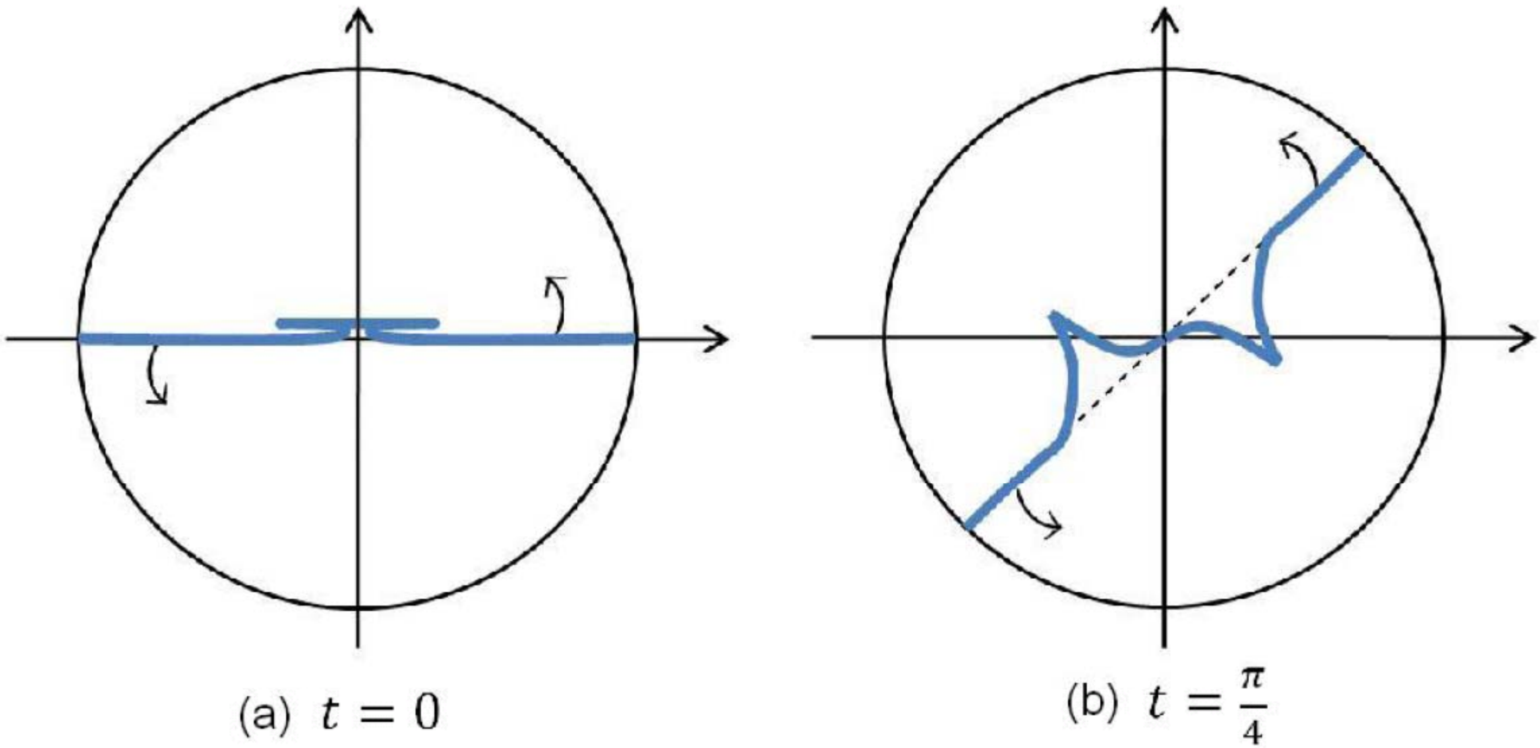


Figure 3: The Minkowskian two-soliton solution with $v = \frac{1}{\sqrt{5}}$ at different global time (a) $t = 0$, (b) $t = \pi/4$. The thick line denotes double-string.

Note: Solitons are localized near the center of AdS space.

Other two-soliton solutions

$$\alpha_{s\bar{s},\bar{s}s} = \ln 2 \pm \ln\Big[\frac{v\sinh X - \sinh T}{v\sinh X + \sinh T}\Big]^2$$

$$q = \frac{1}{\sinh T \pm v\sinh X}\begin{pmatrix} (v\mathrm{sh}T\mathrm{ch}\sigma \pm \mathrm{sh}X\mathrm{ch}\sigma \mp \gamma^{-1}\mathrm{ch}X\mathrm{sh}\sigma)\cos\tau + \gamma^{-1}\mathrm{ch}T\mathrm{ch}\sigma\sin\tau \\ -(v\mathrm{sh}T\mathrm{ch}\sigma \pm \mathrm{sh}X\mathrm{ch}\sigma \mp \gamma^{-1}\mathrm{ch}X\mathrm{sh}\sigma)\sin\tau + \gamma^{-1}\mathrm{ch}T\mathrm{ch}\sigma\cos\tau \\ (v\mathrm{sh}T\mathrm{sh}\sigma \pm \mathrm{sh}X\mathrm{sh}\sigma \mp \gamma^{-1}\mathrm{ch}X\mathrm{ch}\sigma)\cos\tau + \gamma^{-1}\mathrm{ch}T\mathrm{sh}\sigma\sin\tau \\ -(v\mathrm{sh}T\mathrm{sh}\sigma \pm \mathrm{sh}X\mathrm{sh}\sigma \mp \gamma^{-1}\mathrm{ch}X\mathrm{ch}\sigma)\sin\tau + \gamma^{-1}\mathrm{ch}T\mathrm{sh}\sigma\cos\tau \end{pmatrix}$$

Take v to be imaginary: $v = iw$,

$$\alpha_{B,\pm} = \ln 2 \pm \ln\Big[\frac{w\sinh X - \sin T}{w\sinh X + \sin T}\Big]^2$$

(breather solution)

$$q = \frac{1}{\sin T \pm w\sinh X}\begin{pmatrix} -(w\sin T\mathrm{sh}\sigma \mp \mathrm{sh}X\mathrm{sh}\sigma \pm \gamma_w^{-1}\mathrm{ch}X\mathrm{ch}\sigma)\cos\tau + \gamma_w^{-1}\cos T\mathrm{sh}\sigma\sin\tau \\ (w\sin T\mathrm{sh}\sigma \mp \mathrm{sh}X\mathrm{sh}\sigma \pm \gamma_w^{-1}\mathrm{ch}X\mathrm{ch}\sigma)\sin\tau + \gamma_w^{-1}\cos T\mathrm{sh}\sigma\cos\tau \\ -(w\sin T\mathrm{ch}\sigma \mp \mathrm{sh}X\mathrm{ch}\sigma \pm \gamma_w^{-1}\mathrm{ch}X\mathrm{sh}\sigma)\cos\tau + \gamma_w^{-1}\cos T\mathrm{ch}\sigma\sin\tau \\ (w\sin T\mathrm{ch}\sigma \mp \mathrm{sh}X\mathrm{ch}\sigma \pm \gamma_w^{-1}\mathrm{ch}X\mathrm{sh}\sigma)\sin\tau + \gamma_w^{-1}\cos T\mathrm{ch}\sigma\cos\tau \end{pmatrix}$$

Singular!

where $\gamma_w = (1+w^2)^{-1/2}$.

Properties of the soliton solutions

- Solitons (spikes) are located in the bulk of AdS
- Near the boundary the solution reduces to vacuum
- These solutions defined on an open line ($\omega=1$) are simple but not fully satisfactory :
  1) Energy is not conserved because there is momentum flow at the asymptotic ends of the string
  2) String is not closed
- To make the physical quantities conserved, and also to clarify the $\omega=1$ limit, we need to build string solutions on a closed circle.

6. Finite (closed) string solutions

$$\hat{\alpha}_{\xi\eta} - 2\sqrt{-uv}\sinh\hat{\alpha} = 0 \qquad u = 2, v = -2$$

$$\hat{\alpha}_1 = \ln[k\ \mathrm{sn}^2(\frac{\sigma}{\sqrt{k}}, k)] \Longrightarrow q_1 = \begin{pmatrix} \frac{1}{\sqrt{1-k^2}}\mathrm{dn}(\frac{\sigma}{\sqrt{k}}, k)\cos\sqrt{k}\tau \\ \frac{1}{\sqrt{1-k^2}}\mathrm{dn}(\frac{\sigma}{\sqrt{k}}, k)\sin\sqrt{k}\tau \\ \frac{k}{\sqrt{1-k^2}}\mathrm{cn}(\frac{\sigma}{\sqrt{k}}, k)\cos\frac{1}{\sqrt{k}}\tau \\ \frac{k}{\sqrt{1-k^2}}\mathrm{cn}(\frac{\sigma}{\sqrt{k}}, k)\sin\frac{1}{\sqrt{k}}\tau \end{pmatrix}$$

Periodicity : $L = 2\sqrt{k}K(k)$ ($0 < k < 1$)

k=1 limit :

$$\hat{\alpha}_{1,k=1} = \ln[\tanh^2\sigma] \Longrightarrow q_{1,k=1} = \frac{1}{2\sqrt{2}\cosh\sigma}\begin{pmatrix} 2\tau\cos\tau - \sin\tau(\cosh 2\sigma + 2) \\ 2\tau\sin\tau + \cos\tau(\cosh 2\sigma + 2) \\ -2\tau\cos\tau + \sin\tau\cosh 2\sigma \\ -2\tau\sin\tau - \cos\tau\cosh 2\sigma \end{pmatrix}$$

[5] A. Jevicki and K. Jin, *Solitons and AdS string solutions*, [arXiv: 0804.0412].

Second solution ?

$$\hat{\alpha}_2 = \ln[k \ \mathrm{cn}^2(\frac{\sigma}{\sqrt{k}}, k) \ \mathrm{nd}^2(\frac{\sigma}{\sqrt{k}}, k)] \Longrightarrow q_2 = \begin{pmatrix} \mathrm{nd}(\frac{\sigma}{\sqrt{k}}, k) \cos\sqrt{k}\tau \\ \mathrm{nd}(\frac{\sigma}{\sqrt{k}}, k) \sin\sqrt{k}\tau \\ k \ \mathrm{sd}(\frac{\sigma}{\sqrt{k}}, k) \cos\frac{1}{\sqrt{k}}\tau \\ k \ \mathrm{sd}(\frac{\sigma}{\sqrt{k}}, k) \sin\frac{1}{\sqrt{k}}\tau \end{pmatrix}$$

k=1 limit :

$$\hat{\alpha}_{2,k=1} = 0 \Longrightarrow q_{2,k=1} = \begin{pmatrix} \cosh\sigma\cos\tau \\ \cosh\sigma\sin\tau \\ \sinh\sigma\cos\tau \\ \sinh\sigma\sin\tau \end{pmatrix}$$

Relation of two solutions: σ translation

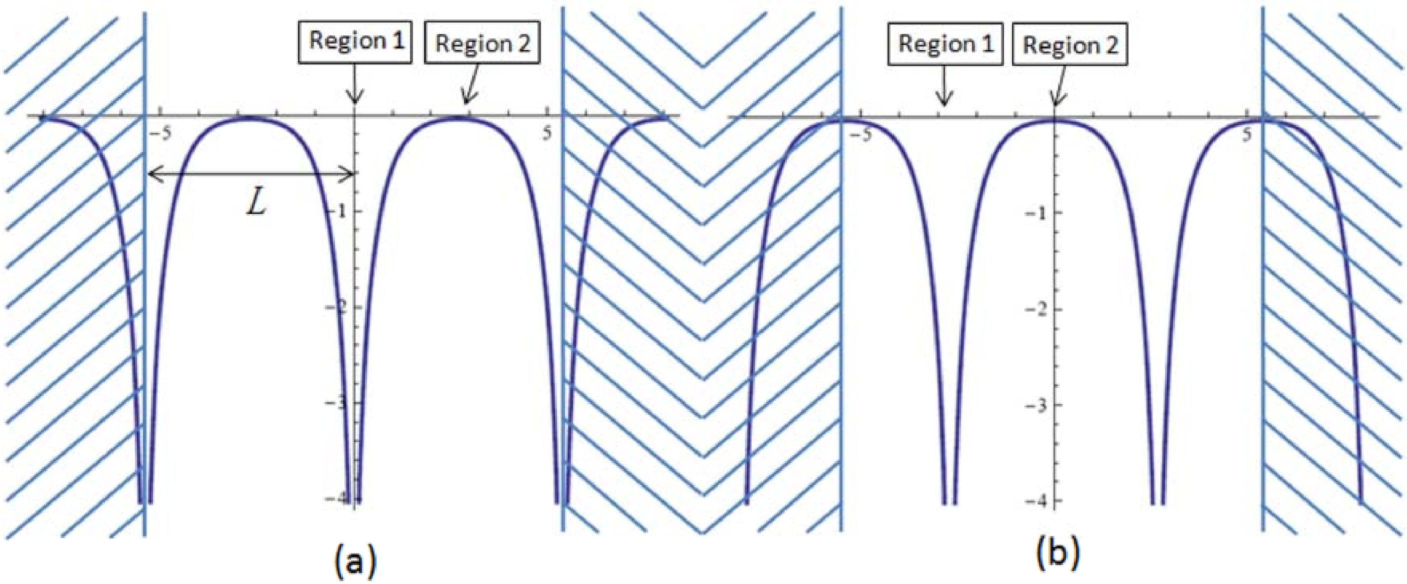


Fig. 1. (a) First periodic sinh-Gordon solution $\hat{\alpha}_1$ when $k = 0.964$; (b) Second periodic sinh-Gordon solution $\hat{\alpha}_2$ when $k = 0.964$. They are related by a translation of $\sigma \to \sigma + \sqrt{k}K(k)$.

$$\hat{\alpha}_1 = \ln[k\ \mathrm{sn}^2(\frac{\sigma}{\sqrt{k}}, k)] \qquad \hat{\alpha}_2 = \ln[k\ \mathrm{cn}^2(\frac{\sigma}{\sqrt{k}}, k)\ \mathrm{nd}^2(\frac{\sigma}{\sqrt{k}}, k)]$$

$$\sigma \to \sigma + \sqrt{k}K(k)$$

Reduction to the GKP solution

$$\begin{pmatrix} \mathrm{nd}(\frac{\sigma}{\sqrt{k}}, k) \cos\sqrt{k}\tau \\ \mathrm{nd}(\frac{\sigma}{\sqrt{k}}, k) \sin\sqrt{k}\tau \\ k\ \mathrm{sd}(\frac{\sigma}{\sqrt{k}}, k) \cos\frac{1}{\sqrt{k}}\tau \\ k\ \mathrm{sd}(\frac{\sigma}{\sqrt{k}}, k) \sin\frac{1}{\sqrt{k}}\tau \end{pmatrix}$$

Do the rescaling $\sqrt{k}\tau \to \tau, \sqrt{k}\sigma \to \sigma$ and write $k = 1/\omega$,

$$\Rightarrow \quad \boxed{\rho'^2(\sigma) = \cosh^2\rho - \omega^2 \sinh^2\rho}$$

This is exactly the GKP solution.

Folded rotating string along a straight line !

Therefore, the energy reads : $E - S = \frac{\sqrt{\lambda}}{\pi} \ln(\frac{S}{\sqrt{\lambda}}) + \cdots$

7. N-soliton (spike) construction

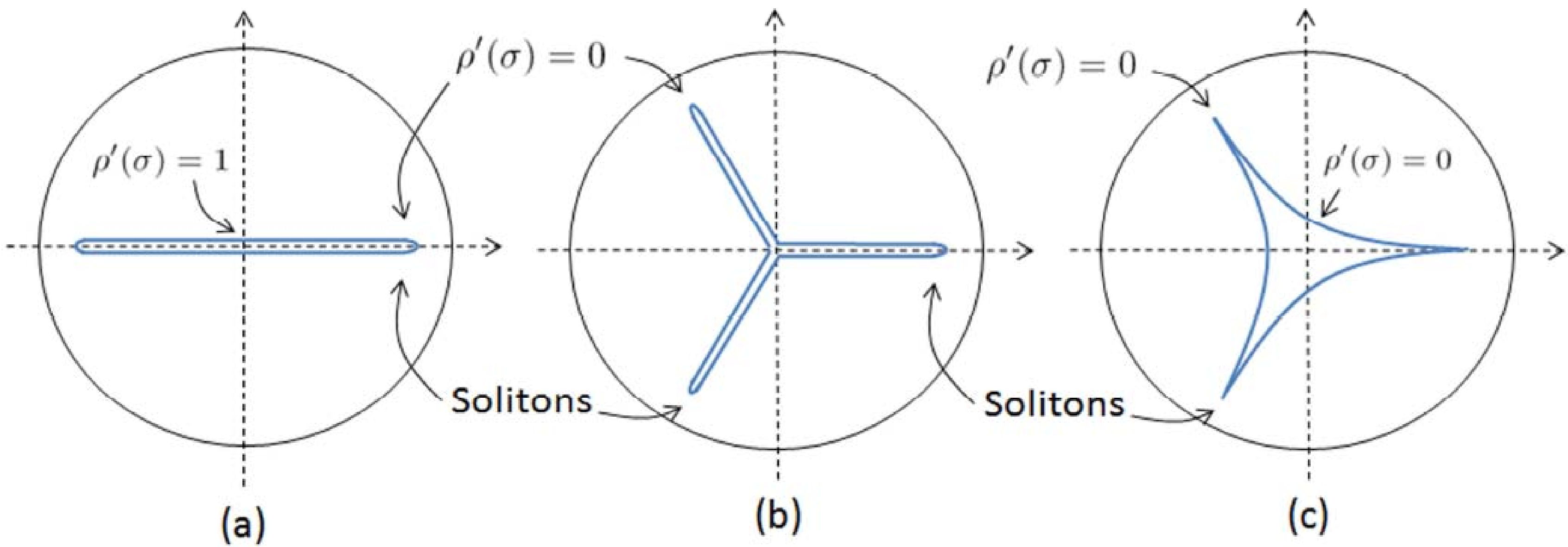


Fig. 2. (a) GKP two-soliton configuration plotted in the plane $x = \rho\cos\theta, y = \rho\sin\theta$ where ρ, θ are the global coordinates; (b) A attempt to construct the GKP type three-soliton solution; (c) Kruczenski's three-spike string solution.

ρ as a function of σ

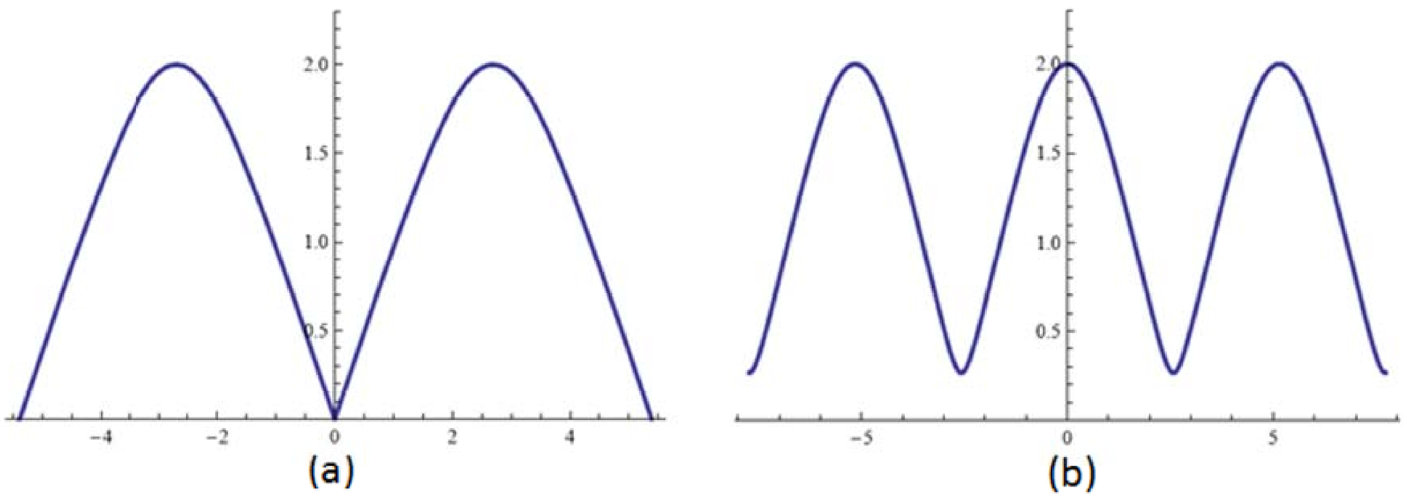


Fig. 3. (a) GKP ρ as a function of σ when $k = 0.964$; (b) Kruczenski ρ as a function of σ when $\rho_1 = 2, \rho_0 = 0.2688735$.

Kruczenski's solution in conformal gauge

ansatz :

$$\begin{aligned} t &= \tau + f(\sigma), \\ \theta &= \omega\tau + g(\sigma), \\ \rho &= \rho(\sigma). \end{aligned}$$

The equations of motion and the Virasoro constraints can be solved by :

$$f'(\sigma) = \frac{\omega \sinh 2\rho_0}{2\cosh^2\rho}, \qquad g'(\sigma) = \frac{\sinh 2\rho_0}{2\sinh^2\rho},$$

$$\rho'^2(\sigma) = \frac{(\cosh^2\rho - \omega^2\sinh^2\rho)(\sinh^2 2\rho - \sinh^2 2\rho_0)}{\sinh^2 2\rho}.$$

Near the spike, we have $\rho \sim \rho_1 \equiv \mathrm{arccoth}\omega$, further assume $\rho_1 \gg \rho_0$,

$$\rho'^2(\sigma) = \cosh^2\rho - \omega^2\sinh^2\rho \qquad \text{(GKP solution)}$$

Therefore, the n-spike configuration is a n-soliton solution in sinh-Gordon picture.

Exact solution

$$\rho = \frac{1}{2}\operatorname{arccosh}\left(\cosh 2\rho_1 \mathrm{cn}^2(\mathrm{u},\mathrm{k}) + \cosh 2\rho_0 \mathrm{sn}^2(\mathrm{u},\mathrm{k})\right)$$

where : $u \equiv \sqrt{\frac{\cosh 2\rho_1 + \cosh 2\rho_0}{\cosh 2\rho_1 - 1}}\sigma, \quad k \equiv \sqrt{\frac{\cosh 2\rho_1 - \cosh 2\rho_0}{\cosh 2\rho_1 + \cosh 2\rho_0}},$

The gauge transformation functions are :

$$f = \frac{\sqrt{2}\omega \sinh 2\rho_0 \sinh \rho_1}{(\cosh 2\rho_1 + 1)\sqrt{\cosh 2\rho_1 + \cosh 2\rho_0}}\Pi\left(\frac{\cosh 2\rho_1 - \cosh 2\rho_0}{\cosh 2\rho_1 + 1}, x, k\right)$$

$$g = \frac{\sqrt{2}\sinh 2\rho_0 \sinh \rho_1}{(\cosh 2\rho_1 - 1)\sqrt{\cosh 2\rho_1 + \cosh 2\rho_0}}\Pi\left(\frac{\cosh 2\rho_1 - \cosh 2\rho_0}{\cosh 2\rho_1 - 1}, x, k\right)$$

where : $x = \mathrm{am}(u, k)$

Exact energy

$$E = \frac{n\sqrt{\lambda}}{\pi}\sqrt{\frac{\mathrm{ch}2\rho_1 - 1}{\mathrm{ch}2\rho_1 + \mathrm{ch}2\rho_0}}\Big[\frac{1}{2}(\mathrm{ch}2\rho_1 + \mathrm{ch}2\rho_0)\mathrm{E}(k) - \mathrm{sh}^2\rho_0\mathrm{K}(k)\Big],$$

$$S = \frac{n\omega\sqrt{\lambda}}{\pi}\sqrt{\frac{\mathrm{ch}2\rho_1 - 1}{\mathrm{ch}2\rho_1 + \mathrm{ch}2\rho_0}}\Big[\frac{1}{2}(\mathrm{ch}2\rho_1 + \mathrm{ch}2\rho_0)\mathrm{E}(k) - \mathrm{ch}^2\rho_0\mathrm{K}(k)\Big],$$

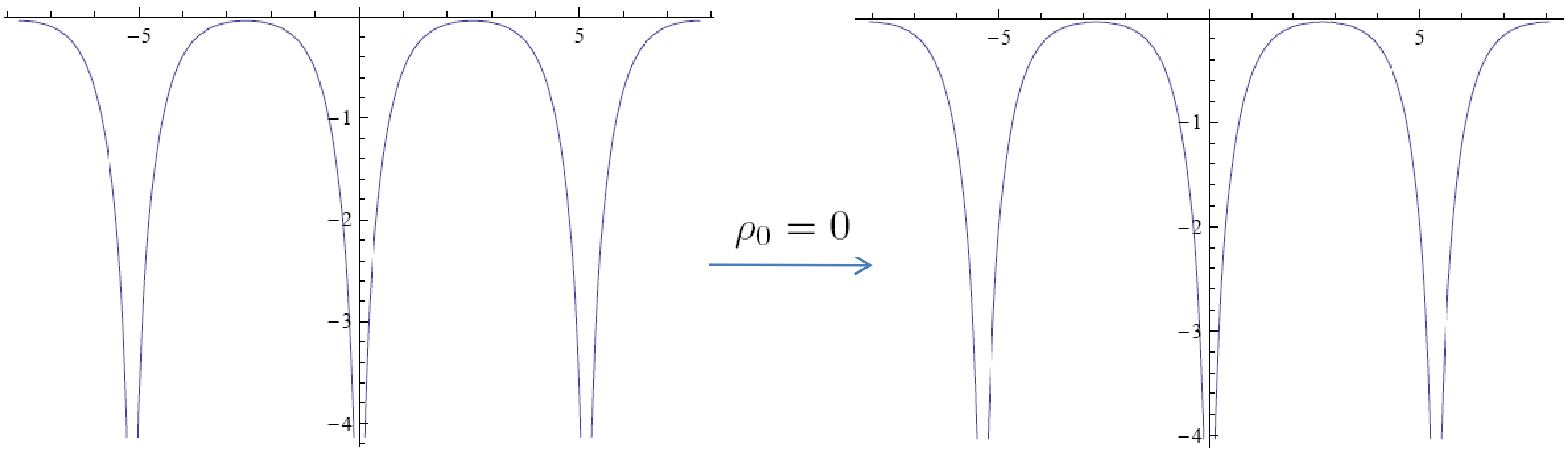
$$E - \omega S = \frac{n\sqrt{\lambda}}{\pi}\sqrt{\frac{\mathrm{ch}2\rho_1 + \mathrm{ch}2\rho_0}{\mathrm{ch}2\rho_1 - 1}}\Big[K(k) - E(k)\Big].$$

In the limit of $\omega \to 1$ and assume $\rho_1 \gg \rho_0$, we find

$$\boxed{E - S = n\frac{\sqrt{\lambda}}{2\pi}\ln S + \cdots}$$

Sinh-Gordon picture

$$\hat{\alpha} = \ln\Big[\frac{\cosh 2\rho_1 - \cosh 2\rho_0}{\sqrt{\sinh^2 2\rho_1 - \sinh^2 2\rho_0}}\mathrm{sn}^2(u,k)\Big] \xrightarrow{\rho_0 = 0} \hat{\alpha}_1 = \ln[k\ \mathrm{sn}^2(\frac{\sigma}{\sqrt{k}}, k)]$$



There is a tiny shift along y axis and a tiny expansion of the period.

Similarly, there is a σ shifted solution reducing to $\hat{\alpha}_2$ ı the limit of $\rho_0 = 0$

In this sense, we say Kruczenski's solution is a generalization of the GKP solution by lifting the minimum value of ρ.

8. Conclusion and Outlook

- ✓ Inverse scattering method is useful for finding the classical string solutions in AdS;
- ✓ Spikes in AdS are related to solitons in sinh-Gordon theory;
- ✓ The GKP solution is a two-soliton configuration with solitons localized at the boundary of AdS;
- ✓ We constructed new string solutions with spikes in the bulk of AdS corresponding to sinh-Gordon vacuum, one-soliton, two-soliton, and breathers (some are singular);
- ✓ N-spike solution (Kruczenski's solution) can be constructed from the GKP solution by lifting the minimum value of ρ;
- ✓ It is interesting to study the dynamics of the spikes.