AdS/CFT: an introduction to the Maldacena conjecture

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Suggested reading:
O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, hep-th/9905111
P. Di Vecchia, hep-th/9908148
E. D'Hoker and D.Z. Freedman, hep-th/0201253
J. Maldacena, hep-th/0309246

Outline

Gauge theories for large N: the t'Hooft limit

Pure SU(N) YM in 4D

$$\mathcal{L} = -\frac{1}{4g_{YM}^2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} \quad , \quad F_{\mu\nu} = F_{\mu\nu}^a T^a$$

Running coupling

$$\mu\frac{dg_{YM}}{d\mu} = -\frac{11}{3}N\frac{g_{YM}^3}{16\pi^2} + \mathcal{O}\left(g_{YM}^5\right)$$

't Hooft limit: $N \to \infty$, $\lambda = g_{YM}^2 N$ fix $\Rightarrow \Lambda_{QCD}$ fix.
More general theory

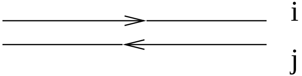
$$\mathcal{L} \sim \frac{1}{g_{YM}^2}\mathrm{Tr}\left(d\Phi_i d\Phi_i + c^{ijk}\Phi_i\Phi_j\Phi_k + d^{ijkl}\Phi_i\Phi_j\Phi_k\Phi_l\right) \ , \ \Phi_i = \Phi_i^a T^a$$

Vertices $\propto \frac{N}{\lambda}$, propagator $\propto \frac{\lambda}{N}$, loop $\propto N$.

Strings from large N

Field in $SU(N)$ adjoint : fundamental$\otimes$anti-fundamental - trace

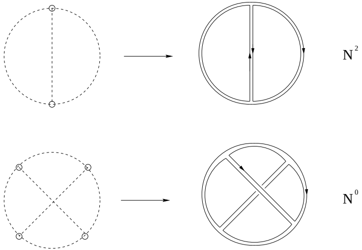
$$\Phi^a \equiv \Phi^i_j$$



$SU(N)$ propagator: $\left\langle \Phi^i_j \Phi^k_l \right\rangle \propto \left(\delta^i_l \delta^k_j - \frac{1}{N} \delta^i_j \delta^k_l \right)$
$\Rightarrow$ subleading term, absent in $U(N) \Rightarrow$ for simplicity switch to $U(N)$
Order of diagram:

$$N^{V-E+F} \lambda^{E-V} = N^{\chi} \lambda^{E-V}$$

E: edges (propagators), F: faces (loops), V : vertices
$\chi = V - E + F = 2 - 2g$: Euler characteristics with genus g



Perturbation series can be recast as

$$\sum_{g=0}^{\infty} N^{2-2g} \sum_{i=0}^{\infty} c_{g,i} \lambda^{i} = \sum_{g=0}^{\infty} N^{2-2g} f_{g}(\lambda)$$

$\Rightarrow$ string theory with

$$g_s \sim \frac{1}{N}$$

$N \to \infty$: planar diagrams. $SU(N)$: oriented diagrams. $SO(N)$: non-orientable surfaces (eg. Klein bottle).
Matter in fundamental: surface boundary (open strings).

Fundamental problem: what is the appropriate string theory?

Basic facts about superstrings

$$S_{ws} = \frac{1}{2\pi\alpha'} \times \text{Area (+susy)}$$

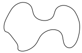
String scale: l_s String tension: $(2\pi\alpha')^{-1}$ with $\alpha' = l_s^2$

Open strings

$$S_{eff} = \int d^D x \left(-\frac{1}{4\,g^2} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i \not{D} \psi \right)$$

Oriented: $f \neq \bar{f}$
Consistent in flat spt:
$D = 10\,,\ G = SO(32)$
(non-oriented)

Closed strings

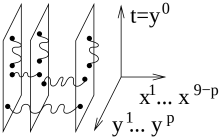


$$S_{eff} = \int d^D x \left(\frac{1}{2\kappa^2} R + \ldots \right)$$

$N = 2$ SUSY: Type IIA
(nonchiral) or Type IIB (chiral)
$N = 1$ SUSY: Heterotic (oriented)
or Type I (non-oriented)

$\kappa^2 \sim l_s^{D-2} g_s^2 \quad , \quad g^2 \sim l_s^{D-4} g_s \qquad (g_{closed} \sim g_{open}^2)$

Yet more basic facts, this time about D-branes



D-branes: surfaces for strings to end

generalized Chan-Paton factors

Degrees of freedom for N p-branes

1. Localized to brane: open strings $\Rightarrow$
dimensionally reduced $SU(N)$ SYM theory (valid for $g_s N \ll 1$)

$$
\begin{aligned}
SO(1,9) \quad &\rightarrow \quad SO(1,p) \times SO(9-p)_R \\
A^a_\mu(y^0,\dots,y^p) \quad &: \quad \begin{cases} \mu = 0,\dots,p & \text{gauge fields} \\ \mu = p+1,\dots 9 & \text{scalars } X_1,\dots,X_{9-p} \end{cases} \\
\psi^a_\alpha(y^0,\dots,y^p) \quad &: \quad \psi^a_{\xi i}(y^0,\dots,y^p) \quad i: \quad \text{R-symmetry index}
\end{aligned}
$$

Scalar potential: $V \sim \sum_{i,j} \mathrm{Tr}\,[X_i, X_j]^2 \Rightarrow$ moduli space:
$X_i = \mathrm{diag}(x_i^1,\dots,x_i^N)$ transverse positions ($i = 1,\dots,9-p$)

2. Bulk: closed strings

World-sheet CFT point of view: superconformal boundary condition

Flat brane: maximally supersymmetric (1/2-BPS).

D-branes as solitons

Low-energy effective action of type II strings

$$S = \frac{1}{(2\pi)^7 l_s^8} \int d^{10}x \sqrt{-g} \left\{ e^{-2\Phi} \left(R + 4(\nabla\Phi)^2 - \frac{2}{(8-p)!} F_{p+2}^2 \right) \right\}$$

$$F_{p+2} = dA_{p+1} \qquad (*F)_{8-p} : \quad *F_{\mu_1 \dots \mu_{8-p}} \propto \epsilon^{\mu_1 \dots \mu_{8-p} \nu_1 \dots \nu_{p+2}} F_{\nu_1 \dots \nu_{p+2}}$$

$$IIA :: \; p = 0, 2 \text{ (RR)} \qquad IIB \; : \; p = -1, 1, 3 \text{ (RR)}$$

p-brane: source for A_{p+1}

$$S_{\rm source} = e_p \int_{\rm brane} d^{p+1}x \, A_{p+1}$$

$$N = \int_{S^{8-p}} *F$$

6-p-brane: source for $\tilde{A}_{7-p}$ with $d\tilde{A}_{7-p} = (*F)_{8-p}$

$$S_{\rm source} = g_{6-p} \int_{\rm brane} d^{7-p}x \, \tilde{A}_{7-p}$$

$$\tilde{N} = \int_{S^{p+2}} F$$

electric-magnetic duality with Dirac quantization for charges

$$e_p g_{6-p} = \text{integer}$$

Black p-brane

Solution in string frame

$$
\begin{aligned}
ds^2 &= -\frac{f_+(\rho)}{\sqrt{f_-(\rho)}}dt^2 + \sqrt{f_-(\rho)}\sum_{i=1}^{p} dy^i dy^i + \frac{f_-(\rho)^{-\frac{1}{2}-\frac{5-p}{7-p}}}{f_+(\rho)}d\rho^2 \\
&\quad + r^2 f_-(\rho)^{-\frac{1}{2}-\frac{5-p}{7-p}} d\Omega_{8-p}^2 \\
\mathrm{e}^{-2\Phi} &= {g_s}^{-2} f_-(\rho)^{-\frac{p-3}{2}} \quad , \quad f_\pm(\rho) = 1 - \left(\frac{r_\pm}{\rho}\right)^{7-p} \\
M &= \frac{(8-p)r_+^{7-p} - r_-^{7-p}}{(7-p)(2\pi)^7 d_p l_P^8} \quad , \quad N = \frac{(r_+ r_-)^{\frac{7-p}{2}}}{d_p g_s l_s^{7-p}} \\
l_P &= g_s^{1/4} l_s \quad , \quad d_p = \frac{\Gamma\left(\frac{7-p}{2}\right)}{(4\pi)^{(p-5)/2}}
\end{aligned}
$$

Einstein frame: $g^E_{\mu\nu} = \sqrt{g_s \mathrm{e}^{-\Phi}}\, g_{\mu\nu}$: horizon at r_+, singularity at r_-

$$
r_+ > r_- \quad \Leftrightarrow \quad M \geq \frac{N}{(2\pi)^p g_s l_s^{p+1}}
$$

Extremal brane

Bound $r_+ > r_-$: also the BPS bound $\Rightarrow$ extremal keeps 1/2 SUSY.
Redefining $r^{7-p} = \rho^{7-p} - r_+^{7-p}$

$$ds^2 = \frac{1}{\sqrt{H(r)}}\left(-dt^2 + \sum_{i=1}^{p} dy^i dy^i\right) + \sqrt{H(r)}\sum_{a=1}^{9-p} dx^a dx^a$$

$$e^{\Phi} = g_s H(r)^{\frac{3-p}{4}} \quad , \quad H(r) = 1 + \frac{r_+^{7-p}}{r^{7-p}} \quad , \quad r_+^{7-p} = d_p g_s N l_s^{7-p}$$

BPS: no force $\Rightarrow$ multi-center solution

$$H(r) = 1 + \sum_{i=1}^{k} \frac{r_{(i)+}^{7-p}}{|\vec{r} - \vec{r}_i|^{7-p}} \quad , \quad r_{(i)+}^{7-p} = d_p g_s N_i l_s^{7-p}$$

$p = 3$: self-dual solution, constant dilaton.
SUGRA valid in all space-time if

$$l_P < l_s \ll r_+ \quad \text{i.e.} \quad 1 \ll g_s N < N$$

($g_s < 1$ can be assumed using S-duality)
$p \neq 3$: singularity at $r = 0 \Rightarrow$ SUGRA valid only in limited region.

Black holes, Hawking radiation and greybody factors

Extremal black p-brane soliton $\equiv$ D-brane

$\Rightarrow$ microscopic description of black holes

Systems considered: near-extremal $D3$ or $D5 + D1$ branes

Results:

1. Bekenstein-Hawking entropy is explained by open string excitations of D-branes

2. Hawking radiation is not exactly thermal: greybody factors σ_{absorb}

$$d\Gamma_{\mathrm{emit}} = \frac{v\sigma_{\mathrm{absorb}}}{\mathrm{e}^{\omega/T_H} \pm 1}$$

σ_{absorb} : computed for min. coupled scalar (dilaton) in SUGRA

But also in SYM: dilaton-brane coupling

$$S_{int} \propto \int d^{p+1}x \mathrm{e}^{-\Phi} \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} \sim \int d^{p+1}x \Phi \mathrm{Tr} F_{\mu\nu} F^{\mu\nu}$$

dilaton is absorbed by decay to SYM multiplet degrees of freedom.

The two results agree: first hint for gauge/gravity correspondence!

The Maldacena conjecture

Starting point: N coincident D3 branes in 10D IIB string theory

$$\mathcal{S} = \mathcal{S}_{bulk} + \mathcal{S}_{brane} + \mathcal{S}_{int}$$

Low-energy limit: $E \ll \sqrt{\alpha'^{-1}}$ or $\alpha' \rightarrow 0$.

$$\mathcal{S}_{bulk} \sim \frac{1}{2\kappa^2} \int \sqrt{-g} R = \int (\partial h)^2 + \kappa (\partial h)^2 h + \dots$$

$$g = \eta + \kappa h \quad , \quad \kappa \sim g_s \alpha'^2$$

$\Rightarrow$ free graviton supermultiplet.

$$\mathcal{S}_{int} \sim \mathcal{O}(\kappa)$$

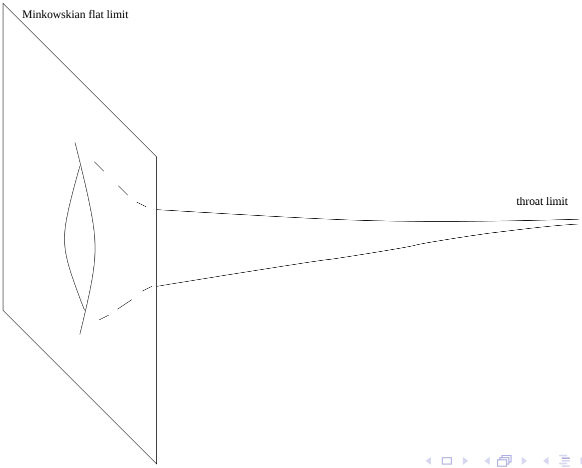
$$\mathcal{S}_{brane} = \mathcal{S}_{\mathcal{N}=4\, SYM} + \mathcal{O}(\kappa)$$

SUGRA description (Einstein frame)

$$ds^2 = f^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + f^{1/2}\left(dr^2 + r^2 d\Omega\right)$$

$$F_5 = (1 + *) dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge df^{-1}$$

$$f = 1 + \frac{R^4}{r^4} \quad , \quad R^4 \equiv 4\pi g_s \alpha'^2 N$$



Low-energy limit

1. Long wavelength excitations around $r = \infty$
   Scattering xsection: $\sigma \propto \omega^3 R^8 \Rightarrow$ decouple from horizon modes
2. Any excitation in near horizon region
   E_p : energy at $r \Rightarrow E = f^{-1/4} E_p$ energy for asymptotic observer, $f = 1 + \frac{R^4}{r^4}$

$\Rightarrow$ free gravity at infinity + IIB string theory around $r = 0$

$$r \to 0 : \quad ds^2 = \frac{r^2}{R^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2$$

$\Rightarrow$ near horizon region is an $AdS_5 \times S^5$ space-time!

The conjecture

$\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory in 3+1 is equivalent with type IIB superstrings on $AdS_5 \times S^5$.

Horizon limit in a more precise way

$\alpha' \to 0$: fix energy in string units so that any string excitation survives the limit, i.e. fix $\sqrt{\alpha'} E_p$

Asymptotically measured energy: $E \sim E_p r / \sqrt{\alpha'}$.
$\Rightarrow r \to 0 \quad , \quad U = r/\alpha' = fix$
with the metric

$$ds^2 = \alpha' \left[\frac{U^2}{\sqrt{4\pi g_s N}} \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \right) + \sqrt{4\pi g_s N} \frac{dU^2}{U^2} + \sqrt{4\pi g_s N} d\Omega_5^2 \right]$$

The Maldacena limit: a closer look

Decoupling in effective action

$$\begin{aligned}\mathcal{L} &= a_1\alpha'\mathcal{R} + a_2\left(\alpha'\mathcal{R}\right)^2 + a_3\left(\alpha'\mathcal{R}\right)^3 + \dots \\ &= b_1\alpha' y^{-2} + b_2\alpha'^2 y^{-4} + b_3\alpha'^3 y^{-6} + \dots \xrightarrow[\alpha'\to 0]{} 0 \\ &\quad (\mathcal{R}\sim 1/y^2 \text{ in as. flat region } y \gg R)\end{aligned}$$

$$\begin{aligned}ds^2 &= G_{MN}dx^M dx^N = R^2 \bar{G}_{MN}(x;R)dx^M dx^N \\ \bar{G}_{MN}(x;R)dx^M dx^N &= \left(1+\frac{R^4}{u^4}\right)^{1/2}\left(\frac{du^2}{u^2}+d\Omega_5^2\right) + \left(1+\frac{R^4}{u^4}\right)^{-1/2}\frac{1}{u^2}\eta_{ij}dx^i dx^j\end{aligned}$$

String σ model action

$$\begin{aligned}S &= \frac{1}{4\pi\alpha'}\int_\Sigma \sqrt{\gamma}\gamma^{mn}G_{MN}(x)\partial_m x^M\partial_n x^N \\ &\xrightarrow[\alpha'\to 0]{} \sqrt{\frac{\lambda}{4\pi}}\int_\Sigma \sqrt{\gamma}\gamma^{mn}\bar{G}_{MN}(x;R=0)\partial_m x^M\partial_n x^N\end{aligned}$$

$$\frac{R^2}{4\pi\alpha'} = \sqrt{\frac{\lambda}{4\pi}} \qquad \lambda = g_s N \sim g_{YM}^2 N$$

$\bar{G}_{MN}(x;R=0)$: metric of $AdS_5\times S^5$ with unit radius.

Approximations

Perturbative SYM

$$g_{YM}^2 N \sim g_s N \sim \frac{R^4}{l_s^4} \ll 1$$

Classical SUGRA

$$\frac{R^4}{l_s^4} \sim g_s N \sim g_{YM}^2 N \gg 1$$

Disjoint regions $\Rightarrow$ nontrivial duality!

Multi-center D3 branes

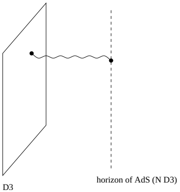
Multi-center solution

$$f \propto \frac{N}{r^4} \to \sum_{i=1}^{N} \frac{1}{\left|\vec{r} - \vec{r}_i\right|^4}$$

Case $N+1$: D3 brane action on $AdS_5 \times S^5$ in SUGRA approx:

$$S = -\frac{1}{(2\pi)^3 g_s \alpha'^2} \int d^4x\, f^{-1}$$
$$\times \left[\sqrt{-\det\left(\eta_{\alpha\beta} + f\,\partial_\alpha r \partial_\beta r + r^2 f\, g_{ij}\partial_\alpha\theta^i\partial_\beta\theta^j + 2\pi\alpha'\sqrt{f}F_{\alpha\beta}\right)} - 1\right]$$

$$f = \frac{4\pi g_s \alpha'^2 N}{r^4} \qquad \theta_i,\ g_{ij}:\ S^5$$



In SYM picture this corresponds to

$$U(N) \to U(N-1) \times U(1)$$

Integrating out the W-bosons results in the Born-Infeld action to $\mathcal{O}(\alpha^4)$.

Symmetries

AdS_5 isometries	$SO(4,2)$	SYM conformal group
S^5 isometries	$SO(6) \equiv SU(4)$	R-symmetry
32 supercharges		S , Q superconf generators
IIB S duality	$SL(2,\mathbb{Z})$	Montonen-Olive duality

Montonen-Olive duality

$$\tau \equiv \frac{4\pi i}{g_{YM}^2} + \frac{\theta}{2\pi} = \frac{i}{g_s} + \frac{\chi}{\pi}$$

$$\chi : \text{ IIB axion}$$

$$S : \tau \rightarrow \frac{a\tau + b}{c\tau + d}$$

$$a, b, c, d \in \mathbb{Z}, \; ad - bc = 1$$

Field-operator correspondence

SYM: conform field theory: $\beta(g_{YM}) \equiv 0$
$\Rightarrow$ no asymptotic states, no S matrix – basic objects are local operators.
Poincaré coordinates on AdS_5

$$ds^2 = R^2 \left(\frac{dz^2 - dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2}{z^2} \right)$$

Dilaton: string coupling $\Rightarrow$ change in boundary dilaton field: change in SYM coupling $\Rightarrow$ deformation with a local operator $\mathcal{O}(x)$.

$$\left\langle e^{\int d^4x \phi_0(x) \mathcal{O}(x)} \right\rangle_{CFT} = \mathcal{Z}_{string} \left[\phi(x,z)|_{z=0} = \phi_0(x) \right]$$

Free scalar field on Euclidean AdS_5

$$S = \int dx^5 \sqrt{g} \left[\frac{1}{2} (\partial\phi)^2 + \frac{1}{2} m^2 \phi^2 \right] \quad \Rightarrow \quad \left(\Box - m^2 \right) \phi = 0$$

Solution:

$$\phi = e^{ip\cdot x} Z(pz) \quad , \quad \left[u^3 \partial_u \frac{1}{u^3} \partial_u - u^2 - m^2 R^2 \right] Z(u) = 0$$

$$\Rightarrow \qquad Z_1(u) = u^2 I_{\Delta-2}(u) \quad , \quad Z_2(u) = u^2 K_{\Delta-2}(u)$$

$$\Delta = 2 + \sqrt{4 + m^2 R^2}$$

Finite action: $Z_2(u)$, with BC: $\phi(x, z = \epsilon) = \phi_0(x) = e^{ip\cdot x}$ $\Rightarrow$

$$\phi(x, z) = \frac{(pz)^2 K_{\Delta-2}(pz)}{(p\epsilon)^2 K_{\Delta-2}(p\epsilon)} e^{ip\cdot x}$$

$$\langle \mathcal{O}(x) \mathcal{O}(y) \rangle = \epsilon^{2\Delta-8} \frac{2\Delta - 4}{\Delta} \frac{\Gamma(\Delta + 1)}{\pi^2 \Gamma(\Delta - 2)} \frac{1}{|x - y|^{2\Delta}}$$

Δ: scaling dimension of $\mathcal{O}$. ϵ: UV cutoff.

Correct form of bulk-boundary coupling

$$\left\langle e^{\int d^4x \phi_0(x)\mathcal{O}(x)} \right\rangle_{CFT} = \mathcal{Z}_{string}\left[\phi(x,z)|_{z=\epsilon} = \phi_0(x)\right]$$

In AdS_{d+1}:

scalars:	$\Delta_\pm = \frac{1}{2}\left(d \pm \sqrt{d^2 + 4m^2R^2}\right)$
spinors:	$\Delta = \frac{1}{2}\left(d + 2\lvert mR\rvert\right)$
vectors:	$\Delta_\pm = \frac{1}{2}\left(d \pm \sqrt{(d-2)^2 + 4m^2R^2}\right)$
p-forms:	$\Delta_\pm = \frac{1}{2}\left(d \pm \sqrt{(d-2p)^2 + 4m^2R^2}\right)$
spin-3/2:	$\Delta = \frac{1}{2}\left(d + 2\lvert mR\rvert\right)$
massless spin-2:	$\Delta = d \Rightarrow T_{\mu\nu}$ on the boundary

Choice of solution: unitarity bound on Δ (in some cases further conditions necessary).

Conformal group

Conformal group: diffeomorphisms

$$g_{\mu\nu}(x) \rightarrow w(x) g_{\mu\nu}(x)$$

3+1 Minkowski : P_μ , $M_{\mu\nu}$, K_μ , D

$$[D, K_\mu] = iK_\mu \,, \quad [D, P_\mu] = -iP_\mu \,, \quad [P_\mu, K_\nu] = 2iM_{\mu\nu} - 2i\eta_{\mu\nu} D$$

$$SO(d+2,2) : \; J_{ab} \,, \; a, b = 0, \ldots, d+1 \,, \; (- + + \cdots + -)$$

$$J_{\mu\nu} = M_{\mu\nu} \,, \; J_{\mu d} = \frac{1}{2}(K_\mu - P_\mu) \,, \; J_{\mu d+1} = \frac{1}{2}(K_\mu + P_\mu) \,, \; J_{(d+1)d} = \ldots$$

Scaling operator: $[D, \mathcal{O}(0)] = -i\Delta \mathcal{O}(0)$. Primary operator: $[K_\mu, \mathcal{O}(0)] = 0 \Rightarrow$

$$\begin{aligned}
[P_\mu, \mathcal{O}(x)] &= i\partial_\mu \mathcal{O}(x) \\
[M_{\mu\nu}, \mathcal{O}(x)] &= [i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu}] \mathcal{O}(x) \\
[K_\mu, \mathcal{O}(x)] &= \left[i\left(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu + 2x_\mu \Delta\right) - 2x^\nu \Sigma_{\mu\nu}\right] \mathcal{O}(x) \\
[D, \mathcal{O}(x)] &= i(-\Delta + x^\mu \partial_\mu) \mathcal{O}(x)
\end{aligned}$$

Spectrum of chiral primary operators

Superconformal algebra: Q and S

$$[D,Q]=-\frac{i}{2}Q\,,\ [D,S]=\frac{i}{2}S\,,\ [K,Q]\simeq S\,,\ [P,S]\simeq Q$$
$$\{Q,Q\}\simeq P\,,\ \{S,S\}\simeq K\,,\ \{Q,S\}\simeq M+D+R$$

Superprimary operator: $[S,\mathcal{O}(0)]_{\pm}=0 \Rightarrow$ multiplet is built with Q
Chiral primary: annihilated by a combination of Qs $\Rightarrow$ short multiplet $\Rightarrow$

$$\Delta=\Delta(\Sigma,R)$$

no quantum corrections!
Unitarity bound:

$$\Delta\geq\frac{d-2}{2}$$

(equality only for free scalar field).

SYM chiral primaries

$$\mathcal{O}^{I_1\dots I_n} = \mathrm{Tr}\left[\phi^{I_1}\dots\phi^{I_n}\right]$$

I_k : $SU(4)_R$ 6 in dices
Symmetric traceless component is a chiral primary:
$SU(4)_R$ $(0, n, 0)$ weight representation and

$$\Delta\left(\mathcal{O}^{I_1\dots I_n}\right) = n$$

with quadratic Casimir of conformal group: $C_2 = \Delta(\Delta - 4)$.
This multiplet contains a complex scalar field at level $n + 2$in
$SU(4)_R$ $(0, n-2, 0)$:

$$Q^4\mathcal{O}_n = \mathrm{Tr}\left[F_{\mu\nu}^2\phi^{I_1}\dots\phi^{I_{n-4}}\right] + \dots$$

SUGRA multiplet

IIB SUGRA on $AdS_5 \times S^5$

AdS_5 SUSY algebra $\equiv \mathcal{N} = 4$ superconformal.

Quadratic Casimir for scalar field with mass m: $C_2 = m^2 R^2$.

Dilaton Kaluza-Klein spectrum

$$\tau(x) = \sum_k \tau^k(x) Y^k(y) \ , \ x \in AdS_5 \ , \ y \in S^5$$

S^5 spherical functions, symmetric and traceless in $SO(6) \simeq SU(4)$ indices:

$$Y^k(y) \sim y^{I_1} \dots y^{I_k} \ , \ I_l = 1 \dots 6 \ , \ \sum_{I=1}^{6} \left(y^I \right)^2 = 1$$

Mass of τ^kin AdS_5: $k(k+4)/R^2 \Rightarrow Q^4 \mathcal{O}_n \ , \ n = k + 4$.
$(\tau^0 \ \rightarrow \ \mathrm{Tr}\, F^2)$

Result: $\mathcal{O}_n$ ***SYM multiplets*** $\equiv$ ***KK modes of IIB graviton supermultiplet***

Anomalies

$\mathcal{N}=4$ SYM $SU(4)_R$ anomaly (with background gauge field)

$$(\mathcal{D}^\mu J_\mu)^a = \frac{N^2-1}{384\pi^2} i d^{abc} \epsilon^{\mu\nu\rho\sigma} F^b_{\mu\nu} F^c_{\rho\sigma}$$

IIB SUGRA on $AdS_5 \times S^5$: Chern-Simons term

$$\mathcal{S}_{CS} = \frac{iN^2}{96\pi^2} \int_{AdS_5} d^5x \left(d^{abc} \epsilon^{\mu\nu\rho\sigma\lambda} A^a_\mu \partial_\nu A^b_\rho \partial_\sigma A^c_\lambda \right)$$

$$A^a_\mu \;\rightarrow\; A^a_\mu + (\mathcal{D}_\mu \Lambda)^a$$

$$\delta\mathcal{S}_{CS} = -\frac{iN^2}{384\pi^2} \int_{\partial AdS_5} d^4x \, d^{abc} \Lambda^a \epsilon^{\mu\nu\rho\sigma} F^b_{\mu\nu} F^c_{\rho\sigma}$$

Boundary coupling: $\int d^4x A^\mu_a J^a_\mu \Rightarrow$

$$\delta \int d^4x A^\mu_a J^a_\mu = \int d^4x \, (\mathcal{D}^\mu \Lambda)_a \, J^a_\mu = \int d^4x \Lambda_a \, (\mathcal{D}^\mu J_\mu)^a$$

Similar agreement for conformal (Weyl) anomaly.

Correlation functions and Witten diagrams

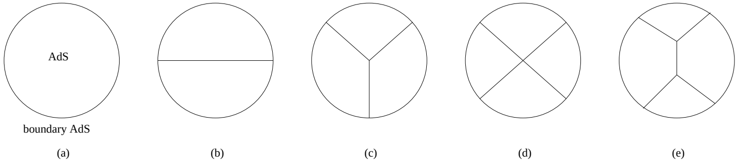


Diagram rules:

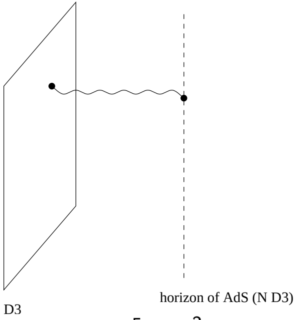
1. Boundary point: source $\phi_\Delta(x)$ in boundary point x.
2. "Bulk-to-boundary" propagators link boundary points to interior.
3. Internal vertices: SUGRA action on AdS_5.
4. Internal lines: SUGRA "bulk" propagators.
5. Internal vertices must be integrated over.

1, 2 and 3-point functions

For superconformal primaries:

$$
\begin{aligned}
\langle \mathcal{O}_\Delta(x) \rangle &= \delta_{\Delta,0} \\
\langle \mathcal{O}_{\Delta_1}(x_1)\, \mathcal{O}_{\Delta_2}(x_2) \rangle &= \frac{\delta_{\Delta_1 \Delta_2}}{|x_1 - x_2|^{2\Delta_1}} \\
\langle \mathcal{O}_{\Delta_1}(x_1)\, \mathcal{O}_{\Delta_2}(x_2)\, \mathcal{O}_{\Delta_3}(x_3) \rangle &= \frac{c_{\Delta_1 \Delta_2 \Delta_3}(g_s, N)}{|x_{12}|^{\Delta - 2\Delta_3}\, |x_{13}|^{\Delta - 2\Delta_2}\, |x_{23}|^{\Delta - 2\Delta_1}} \\
\Delta &= \Delta_1 + \Delta_2 + \Delta_3
\end{aligned}
$$

Wilson loops



$U(N+1) \rightarrow U(N) \times U(1)$

D3-horizon strings: $U(N)$ fundamental rep

$\Rightarrow$ quarks with $m = U/2\pi$

Static quark: $U \rightarrow \infty \Rightarrow$ string goes all the way to AdS_5 boundary.

$$ds^2 = \alpha' \left[\frac{U^2}{\sqrt{4\pi g_s N}} \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \sqrt{4\pi g_s N} \frac{dU^2}{U^2} + \sqrt{4\pi g_s N} d\Omega_5^2 \right]$$

SUSY $\Rightarrow$ Wilson loop couples to full SYM multiplet; omitting gauginos

$$\mathcal{W}(\mathcal{C}) = \mathrm{Tr}\left[P \exp\left(\oint \left(i A_\mu \dot{x}^\mu + \theta^I \phi^I \sqrt{\dot{x}^2} \right) d\tau \right) \right]$$

$$\mathcal{C} : x^\mu = x^\mu(\tau) \quad , \quad \sum_{I=1}^{6} \left(\theta^I\right)^2 = 1 \, .$$

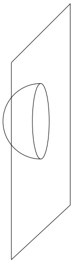
Wilson loop in SUGRA approximation

Minimal area problem

SYM: $\langle \mathcal{W}(\mathcal{C}) \rangle$ perturbatively finite
SUGRA: area diverges.

Using a cutoff $U < r$ the divergent term is

$$r\,|\mathcal{C}|$$



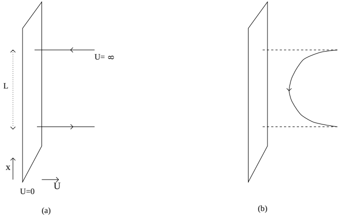
Solution: action gives string BC Neumann

$\Rightarrow$ to obtain Dirichlet, need to take Legendre transform of string action in $\theta^I$and U
$\Rightarrow$ exactly cancels divergent part.

Quark: $x = L/2$, antiquark: $x = -L/2$, action

$$
\begin{aligned}
S &= \frac{1}{2\pi\alpha'}\int d\tau d\sigma \sqrt{\det\left(G_{MN}\partial_\alpha X^M \partial_\beta X^N\right)} \\
&= \frac{TR^2}{2\pi}\int_{-L/2}^{L/2} dx \frac{\sqrt{(\partial_x z)^2+1}}{z^2}
\end{aligned}
$$

$$\text{static configuration}: \ \tau = t,\ \sigma = x$$



Result: the quark-antiquark potential is

$$E = V(L) = -\frac{4\pi^2\left(2g_{YM}^2 N\right)^{1/2}}{\Gamma\left(\frac{1}{4}\right)^4 L}$$

1. Conformally invariant
2. No confinement
3. D-string gives the same for two magnetic monopoles, but $g_{YM} \to 4\pi/g_{YM}$

Barions and instantons

D5-brane on S^5 $\Rightarrow$ particle in AdS_5

N units of F_5 flux + Wess-Zumino term (anomaly cancellation)

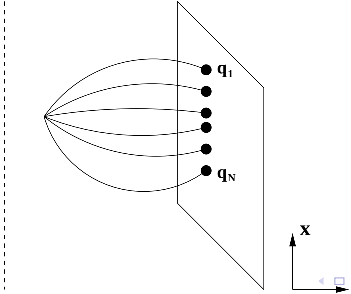
$$\frac{1}{2\pi}\int d^6x\, a \wedge F_5$$

$\Rightarrow$ N unit of $U(1)$ charge induced on brane.

Compact manifold: $\sum Q = 0 \Rightarrow$ need $N \times (-1)$ charge $\Rightarrow$ N strings.

Ground state: antisymmetric, fermionic.

∂AdS_5: N copies of $SU(N)$ fundamental $\Rightarrow$ barion operator.



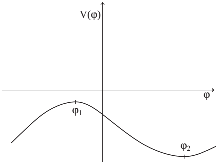
Holographic renormalization group

Perturbed superconformal QFT: relevant deformations

$$\mathcal{L} = \mathcal{L}_{\mathcal{N}=4} + \frac{1}{2} m^2_{ij} \mathrm{Tr}\phi^i \phi^j + \frac{1}{2} M_{ab} \mathrm{Tr}\psi^a \psi^b + b_{ijk} \mathrm{Tr}\phi^i \phi^j \phi^k$$

Holographic RG flows: toy model

$$S = \frac{1}{4\pi G} \int d^{d+1}x \sqrt{g} \left(-\frac{1}{4} R + \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + V(\varphi) \right)$$



Critical solution

$$\phi = \phi_i \quad , \quad \text{geometry}: \quad AdS_{d+1}(R_i) \quad , \quad 4V(\varphi_i) = -\frac{d(d-1)}{R_i^2}$$

Sign convention: $r \to \infty$ boundary, $r \to -\infty$ AdS interior

Domain wall solution

$$
\begin{aligned}
ds^2 &= e^{2A(r)}\delta_{ij}dx^i dx^j + dr^2 \\
\varphi &= \varphi(r) \\
&\Downarrow \\
A'^2 &= \frac{2}{d(d-1)}\left(\varphi'^2 - 2V(\varphi)\right) \\
\varphi'' + d\,A'\,\phi' &= \frac{dV(\varphi)}{d\varphi} \\
r \to \pm\infty &: \text{critical}
\end{aligned}
$$

Result: $r \to \infty$ V maximal, $r \to -\infty$ V minimal, $A'' < 0$
$\Rightarrow$ irreversible RG flow (c-theorem).
Holographic correspondence: $\phi \to \mathcal{O}_\Delta$

$$
S = S_{UV\,CFT} + \int d^d x\, \mathcal{O}_\Delta
$$

$$
r \to \infty : \text{UV}\,, r \to -\infty : IR
$$

$$
UV\ CFT \to IR\ CFT
$$

Towards QCD: AdS/CFT at finite T

Non-extremal black D3 brane with N units of RR charge

$$
\begin{aligned}
ds^2 &= -\frac{f_+}{f_-}dt^2 + \sqrt{f_-}\sum_{i=1}^{3} dx^i dx^i + \frac{1}{f_+ f_-}d\rho^2 + \rho^2 d\Omega_5^2 \\
e^{-2\phi} &= g_s^{-2} \quad , \quad f_\pm = 1 - \left(\frac{r_\pm}{\rho}\right)^2 \\
M &= \frac{1}{4(2\pi)^7 d_3 g_s^2 \alpha'^4}\left(5r_+^4 - r_-^4\right) \quad , \quad N = \frac{(r_+ r_-)^2}{d_3 g_s \alpha'^2} \quad , \quad d_3 = 4\pi
\end{aligned}
$$

Horizon limit, $M - M_{extr}$ fixed: X_1 space-time

$$
ds^2 = R^2\left[u^2\left(-hdt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{du^2}{hu^2} + d\Omega_5^2\right]
$$

$$
h = 1 - \frac{u_0^4}{u^4} \quad , \quad u_0 = \pi T \, .
$$

$t_E = it \Rightarrow$ compact Euclidean time, SUGRA- action:

$$
I = -\frac{1}{16\pi G_5}\int d^5x\sqrt{g}\left(\mathcal{R} + \frac{12}{R^2}\right)
$$

Boundary : $\partial X_1 = S^3 \times S^1 \Rightarrow$ finite-T QFT on S^3

AdS black hole

Another solution: X_2 AdS black hole

$$
\begin{aligned}
ds^2 &= f dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega_3^2 \\
f &= 1 + \frac{r^2}{R^2} - \frac{\mu}{r^2} \quad , \quad r \geq r_+ \quad , \quad f(r_+) = 0
\end{aligned}
$$

Conical singularity at $r = r_+$ is avoided by

$$
t \sim t + \beta \quad , \quad \beta = \frac{2\pi R^2 r_+}{2r_+^2 + R^2}
$$

Topology

$$
X_2 \sim S^3 \times B^2 \quad , \quad \partial X_2 = S^3 \times S^1
$$

X_1's relation to X_2:

$$
X_1 \simeq X_2(\mu = 0) \sim B^4 \times S^1 \quad , \quad \partial X_1 = S^3 \times S^1
$$

Finite T: fermions antiperiodic on S^1 $\Rightarrow$ breaks SUSY

$$I(X_2) - I(X_1) = \frac{\pi^2 r_+^3 \left(R^2 - r_+^2\right)}{4G_5 \left(2r_+^2 + R^2\right)}$$

Dominant configuration in SUGRA path integral:

$$\left.\begin{array}{c} R > r_+(\text{low } T) : \; X_1 \\ R < r_+(\text{high } T) : \; X_2 \end{array}\right\} \Rightarrow \text{phase transition}$$

Polyakov loop

$$\langle \mathcal{W}(\mathcal{C}) \rangle \sim \exp\left(-F_q(T)/T\right) \quad , \quad \mathcal{C} = S^1$$

Low T: $X_1 \sim B^4 \times S^1$, $\partial X_1 \sim S^3 \times S^1$ $\Rightarrow$ $\mathcal{C} \neq \partial \mathcal{S}$ $\Rightarrow$

$$\langle \mathcal{W}(\mathcal{C}) \rangle = 0 \quad , \quad F_q(T) = \infty$$

$\Rightarrow$ confinement.

High T: $X_2 \sim S^3 \times B^2$, $\partial X_2 \sim S^3 \times S^1$ $\Rightarrow$ $\mathcal{C} = \partial B^2$ $\Rightarrow$

$$\langle \mathcal{W}(\mathcal{C}) \rangle \neq 0 \quad , \quad F_q(T) < \infty$$

$\Rightarrow$ deconfinement?

Not so simple: X_1 is not a unique solution

$$\int_{B^2} B_{NS} = \psi \mod 2\pi \ , \ dB = 0$$

String action:

$$\mathcal{S}(\psi) = \mathcal{S}(\psi = 0) + i\psi$$

$$\langle \mathcal{W}(\mathcal{C}) \rangle = \int d\psi \exp(-\mathcal{S}(\psi)) = 0$$

But: in SYM $\psi = const$ $\Rightarrow$ no integration $\Rightarrow$ $\langle \mathcal{W}(\mathcal{C}) \rangle \neq 0$.

$$\psi \in U(1) = \lim_{N \to \infty} \mathbb{Z}_N$$

$\Rightarrow$ SSB of center symmetry.

QCD_3

Finite T: fermions massive, scalars get mass at one-loop, SUSY broken.

$T \rightarrow \infty$: pure QCD_3, dominated by AdS black hole X_2

$$ds^2 = \alpha' \sqrt{4\pi g_s N} \left[u^2 \left(h(u) d\tau^2 + \sum_{i=1}^{3} dx_i^2 \right) + h(u)^{-1} \frac{du^2}{u^2} + d\Omega_5^2 \right]$$

$$h(u) = 1 - \frac{u_0^4}{u^4} \quad , \quad \tau \sim \tau + 2\pi R_0 \quad , \quad u_0 = \frac{1}{2R_0}$$

$1/R_0$: UV cutoff. QCD_3:
$g_4^2 N \rightarrow 0\,,\ R_0 \rightarrow 0\,,\ g_3^2 N = g_4^2 N / 2\pi R_0 = \mathrm{fix}.$
Scale-dependent effective coupling

$$\lambda_{\mathrm{eff}}\,(R_0) = g_s N \quad , \quad g_s = \frac{g_4^2}{4\pi}$$

Validity of SUGRA: $g_s N \gg 1$. $\Rightarrow$ QCD_3 mass scale $g_3^2 N \gg 1/R_0$ (UV cutoff)$\Rightarrow$ large N QCD_3 with fix cutoff in strong coupled limit ($\sim$ lattice strong coupling, but here: full 3d Poincaré invariance).
$R_0 \rightarrow 0$: singular geometry, SUGRA invalid
$\Rightarrow$ full string theory needed.

Confinement

Area law

$$\langle \mathcal{W}(\mathcal{C}) \rangle \simeq \exp\left(-\sigma A(\mathcal{C})\right)$$

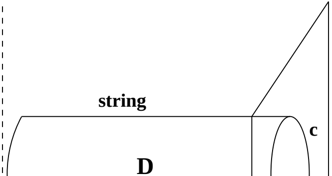
On the other hand

$$\langle \mathcal{W}(\mathcal{C}) \rangle = \int \exp\left(-\mu(D)\right) \quad , \quad \partial D = \mathcal{C}$$

μ: regularized string action. In SUGRA limit

$$\langle \mathcal{W}(\mathcal{C}) \rangle = \exp\left(-\mu(D_0)\right)$$

D_0: minimal area surface.
$\mathcal{N} = 4$ SYM: $\mathcal{C} \rightarrow \alpha\mathcal{C}$, conformal invariance $\Rightarrow$ $D_0 \rightarrow \alpha D_0$, but $\mu(D_0) = \mu(\alpha D_0)$.



However: now we have BH horizon at $u = u_0$!

$g_{ii}\big|_{u=u_0}$ $\sqrt{g_s N}$

Mass spectrum

$J^{PC} = 0^{++}$ glueballs $\Rightarrow \mathcal{O} = \mathrm{Tr}\, F^2$

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \simeq \sum c_i \exp\left(-M_i\,|x-y|\right)$$

IIB strings: dilaton field Φ.

$$\begin{aligned}
\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\Phi\right) &= 0 \\
\Downarrow &\quad \Phi = f(u)e^{ikx} \\
\partial_u\left[u\left(u^4-u_0^4\right)\partial_u f(u)\right] + M^2 u f(u) &= 0 \qquad , \qquad M^2 = -k^2
\end{aligned}$$

WKB solution (almost exact):

$$M^2_{0^{++}} \approx \frac{1.44 n(n+1)}{R_0^2} \quad , \quad n = 1, 2, 3, \ldots$$

Mass ratios are very close to lattice results.

$J^{PC} = 0^{--}$: $B_{NS} \Rightarrow$

$$\left(\frac{M_{0^{--}}}{M_{0^{++}}}\right)_{\mathrm{SUGRA}} = 1.50 \qquad \left(\frac{M_{0^{--}}}{M_{0^{++}}}\right)_{\mathrm{lattice}} = 1.45 \pm 0.08$$

Caveats and QCD_4

<u>Validity range of lattice and SUGRA is very different!</u>

$\Rightarrow$ agreement is puzzling.

Lattice strong coupling expansion $\equiv$ string α' expansion. First correction:

$$M^2 = \frac{c_0 + c_1 \alpha'^3 / R_0^6}{R_0^2}$$

c_1 can be computed from $\mathcal{L}_{\mathrm{SUGRA}}$ α'^3 corrections: mass changes significantly, but not the ratios.

QCD_4

N D4 branes (IIA), $T = 1/2\pi R_0$, horizon limit

$$ds^2 = \frac{2\pi\lambda}{3u_0}\left(4u^2 \sum_{i=1}^{4} dx_i^2 + \frac{4}{9u_0^2} u^2 \left(1 - \frac{u_0^6}{u^6}\right) d\tau^2 + 4\frac{du^2}{u^2\left(1 - \frac{u_0^6}{u^6}\right)} + d$$

$$e^{2\phi} = \frac{8\pi\lambda^3 u^3}{27 u_0^3 N^2} \quad , \quad u_0 \leq u \leq \infty \quad , \quad \tau \sim \tau + 2\pi \quad , \quad u_0 = \frac{1}{3R_0}$$

Needs strongly coupled IIA: M-theory (low-energy limit: 11D SUGRA)

Successes in understanding QCD

1. confinement, dual Meissner effect, θ-vacua are correctly described
2. glueball spectrum agrees with lattice
3. confinement-deconfinement transition at finite T
4. barions described
5. topological susceptibility, gluon condensate, string tension agrees in magnitude with lattice

Outlook 1: strings on pp waves

Plane wave limit of $AdS_5 \times S_5$

$$
\begin{aligned}
ds^2 &= R^2(-dt^2\cosh^2\rho + d\rho^2 + \sinh^2\rho d\Omega_5 + d\psi^2\cos^2\theta + d\theta^2 + \sin^2\theta d\Omega_3'^2) \\
\downarrow & \quad x^- = R^2(t-\psi)\ ,\ x^+ = t\ ,\ r = R\rho\ ,\ y = R\theta\ ,\ R\to\infty \\
ds^2 &= -2dx^+dx^- - A_{ij}(x^+)y^iy^j(dx^+)^2 + d\vec{y}^2 + d\vec{r}^2 \\
& \quad A_{ij}(x^+)y^iy^j = \vec{r}^2 + \vec{y}^2 \qquad \vec{r},\vec{y}\in\mathbb{R}^4
\end{aligned}
$$

String propagation can be solved exactly (e.g. in light cone gauge).
In gauge theory:

$$
\begin{aligned}
-p_+ &= i\partial_{x^+} = i(\partial_t + \partial_\psi) = \Delta - J \\
-p_- &= \frac{1}{R^2}i(-\partial_\psi) = \frac{J}{R^2}
\end{aligned}
$$

i.e. $J\to\infty$ with $\Delta - J$ fixed

gauge theory $\Delta \leftrightarrow$ string state energy E

Outlook 2: integrability in AdS/CFT

More generally: string propagation on AdS/CFT is integrable
$\rightarrow$ world-sheet action: coset σ model

This corresponds to $g_s = 1/N \rightarrow 0$: large N gauge theory

but: 't Hooft coupling $\lambda = g_s N = g_{YM}^2 N$ fixed.

Classical string solutions: highly excited string states carrying large charges (e.g. J) $\sim$ plane wave limit

But integrability allows solution for all states $\Rightarrow$

$$N = \infty \text{ SYM theory is integrable}$$

Outlook 3: AdS/CFT in other settings

1. Other branes $\rightarrow$ more AdS/CFT dualities

AdS_3/CFT_2: for pure AdS_3 gravity with negative cosmological constant

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{g} \left(\mathcal{R} + \frac{2}{\ell^2} \right)$$

CFT_2 is extremal and self-dual CFT (Witten) with

$c = 3\ell/2G = 24k$

(self-dual: only vacuum character; extremal: only Virasoro descendants of vacuum up to level $k+1$)

$k = 1$: Monster theory

Perturbative gravity excitations: Virasoro descendants

Black holes: Virasoro primaries

Gaberdiel: such beast does not exist for $k \neq 1, 2, 3, 4, 5, 7, 8, 11, 13$!

$\Rightarrow$additional states: is there a minimal version?

2. Other "physics" from AdS:

AdS/DIS(deep inelastic scattering); AdS/HD(hydrodynamics – heavy ions); AdS/CM(condensed matter) ...