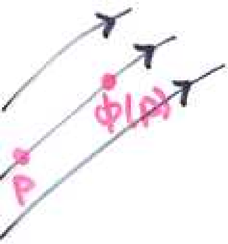
GR and the geometry of AdS

- Maximally symmetric geometries
  - $R > 0$ de Sitter
  - $R = 0$ Minkowski
  - $R < 0$ anti de Sitter
- Global properties of AdS
- AdS and Kottler solutions of Einstein's equations

Geometric Symmetries

- M – connected smooth m-manifold
  g_{ab} – metric with signature (p,q), $p+q=m$

- Symmetries: $\phi : M \to M$ – diffeomorph. preserving some geom. object:



$\phi_* g_{ab} = g_{ab}$ – isometry (Killing, rigid mot

$\phi_* g_{ab} = e^{st} g_{ab}$ – homothetic m. (self-similari

$\phi_* g_{ab} = \omega^2 g_{ab}$ – conformal m. (preserv. ang

- Infinitesimal symmetries: $\phi_t : M \to M$ – diffeo gener. by some vector field K^a

$$\mathcal{L}_K g_{ab} = \nabla_a K_b + \nabla_b K_a = 0$$

$$\mathcal{L}_K g_{ab} = \frac{2}{m} (\nabla_e K^e) g_{ab}, \quad \nabla_e K^e = \text{const.}$$

$$\mathcal{L}_K g_{ab} = \frac{2}{m} (\nabla_e K^e) g_{ab}$$

Proposition: The set of infinitesimal symm. form a Lie algebra w.r.t. Lie bracket.

Eg.: $\mathcal{L}_{[K,\bar{K}]} g_{ab} = \mathcal{L}_K (\mathcal{L}_{\bar{K}} g_{ab}) - \mathcal{L}_{\bar{K}} (\mathcal{L}_K g_{ab}) =$

$= \mathcal{L}_K (\bar{f} g_{ab}) - \mathcal{L}_{\bar{K}} (f g_{ab}) \sim g_{ab}$

- Maximal number of isometries $= ?$
  Max. symmetric geometries $= ?$

COMPLETELY INTEGRABLE P.D.E.

- First order system of p.d.e.:

$$(*) \qquad \frac{\partial y^A}{\partial x^\alpha} = f_\alpha^A(y^B, x^\beta) \qquad \begin{matrix} A = 1, \dots, M \\ \alpha = 1, \dots, m \end{matrix}$$

given functions: $f_\alpha : \mathcal{U} \times \mathcal{V} \subset \mathbb{R}^M \times \mathbb{R}^m \to \mathbb{R}^M$

The condition of integrability:

$$0 = \frac{\partial^2 y^A}{\partial x^\beta x^\alpha} - \frac{\partial^2 y^A}{\partial x^\alpha \partial y^\beta} =$$

$$= \frac{\partial f_\alpha^A}{\partial x^\beta} + \frac{\partial f_\alpha^A}{\partial y^B} f_\beta^B - \frac{\partial f_\beta^A}{\partial x^\alpha} - \frac{\partial f_\beta^A}{\partial y^B} f_\alpha^B =: F_{\alpha\beta}^A(y, x)$$

- Important special case (L.P. Eisenhart):

  The system $(*)$ is called completely integrable (or complete) on $\mathcal{U}_0 \times \mathcal{V}_0 \subset \mathcal{U} \times \mathcal{V}$ if $F_{\alpha\beta}^A(y,x) = 0 \quad \forall (y,x) \in \mathcal{U}_0 \times \mathcal{V}_0$.

Theorem: (Darboux)

If $(*)$ is completely integrable on $\mathcal{U}_0 \times \mathcal{V}_0$ then for any $x_0^\alpha \in \mathcal{V}_0$ and $y_0^A \in \mathcal{U}_0$ there is a uniquely determined solution $y^A = y^A(x^\alpha$ of $(*)$ on $\mathcal{V}_0$ such that

$$y^A(x_0^\alpha) = y_0^A .$$

– the solutions are parameterized by y_0^A, they form an M parameter family.

MAXIMALLY SYMMETRIC GEOMETRIES

- Killing equations:

(K): $\nabla_a K_b + \nabla_b K_a = \mathcal{L}_K g_{ab} = 0$ or $\partial_a K_b + \partial_b K_a = 2\Gamma^e_{ab} K_e$

– its form is different from that of (*) !

Idea: $K_{ab} := \nabla_{[a} K_{b]}$ – extra variable

$$\left.\begin{aligned} \nabla_a K_b &= K_{ab} \\ \nabla_a K_{bc} &= K_e R^e{}_{abc} \end{aligned}\right\}$$

– the Killing eq.

– its integrability cond

– the new system is equivalent to (K)

Integrability cond.:

$0 = K_e (R^e{}_{abc} + R^e{}_{bca} + R^e{}_{cab})$ – identically satisf.

$0 = K_{e[a} R^e{}_{b]cd} + K_{e[c} R^e{}_{d]ab} - K_e \nabla_{[a} R^e{}_{b]cd}$

- For completely integrable system:

i. The number of Killing fields:

Free data: K_a, K_{ab} at any fixed $p \in M$

$m + \frac{1}{2} m(m-1) = \frac{1}{2} m(m+1)$.

ii. Geometries with max. symmetries:

Integr. conds. $\rightarrow$

$\delta^{[e}_{[a} R^{f]}{}_{b]cd} + \delta^{[e}_{[c} R^{f]}{}_{d]ab} = 0$, $\nabla_{[a} R^e{}_{b]cd} = 0$

$\Downarrow$

- $R_{abcd} = \dfrac{R}{m(m-1)} (g_{ac} g_{bd} - g_{ad} g_{bc})$

- $\nabla_a R = 0$ – constant curvature

Explicitly:

(M, g_{ab}) admits the max. number of infinit. isometries precisely when:

1. If $R =: \frac{m(m-1)}{\alpha^2} > 0$ then (M, g_{ab}) is locally isometric to

$$S^{+1}_\alpha := \left\{ x^A \in \mathbb{R}^{m+1} \,\middle|\, (x^0)^2 + (x^1)^2 + \dots + (x^p)^2 - (x^{p+1})^2 - \dots - (x^m)^2 = \alpha^2 \right\} \subset \mathbb{E}^{p+1,q}$$

and $\mathcal{K} = so(p+1, q)$;

<u>NB</u>: $(x^0)^2 + \dots + (x^p)^2 = \alpha^2 + (x^{p+1})^2 + \dots + (x^m)^2$
– topologically $\mathbb{S}^p \times \mathbb{R}^q$ (disconnect. p=0)

or 2. If $R = 0$ then (M, g_{ab}) is locally flat, $\mathbb{E}^{p,q}$ and $\mathcal{K} = so(p,q) \oplus \mathbb{R}^m$;

or 3. If $R =: -\frac{m(m-1)}{\alpha^2} < 0$ then (M, g_{ab}) is locally isometric to

$$S^{-1}_\alpha := \left\{ x^A \in \mathbb{R}^{m+1} \,\middle|\, (x^1)^2 + \dots + (x^p)^2 - (x^{p+1})^2 - \dots - (x^m)^2 - (x^{m+1})^2 = -\alpha^2 \right\} \subset \mathbb{E}^{p,q+1}$$

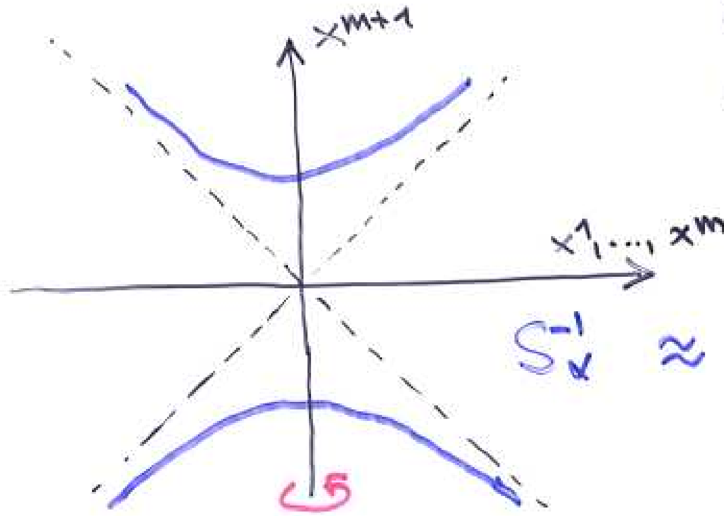
and $\mathcal{K} = so(p, q+1)$.

<u>NB</u>: $(x^1)^2 + \dots + (x^p)^2 + \alpha^2 = (x^{p+1})^2 + \dots + (x^{m+1})^2$
– topologically $\mathbb{R}^p \times \mathbb{S}^q$, disconnected for $q=0$

"hyperspheres with radius α and index ± 1 in $\mathbb{E}^{p+1,q}$, $\mathbb{E}^{p,q+1}$, resp."

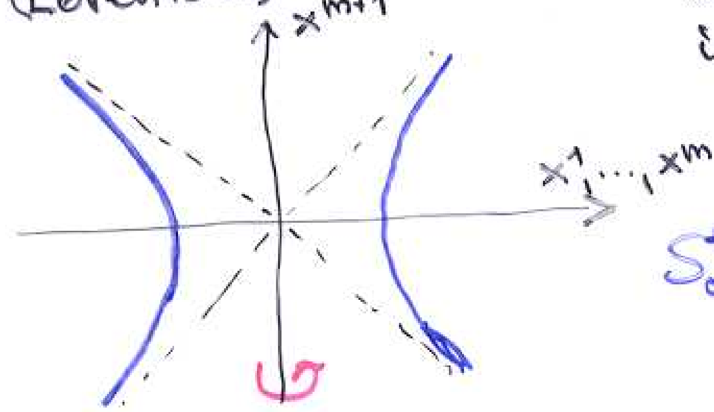
Examples (with $R \neq 0$)

- $(m,0)$ (Riemannian)
  - $R > 0$ — S^m embedded in $\mathbb{E}^{m+1}$
  - $R < 0$ — Bolyai-Lobachevsky hyperboloidal m-space, as a spacelike hypersurf. in $\mathbb{E}^{m,1}$ (Minkowski)



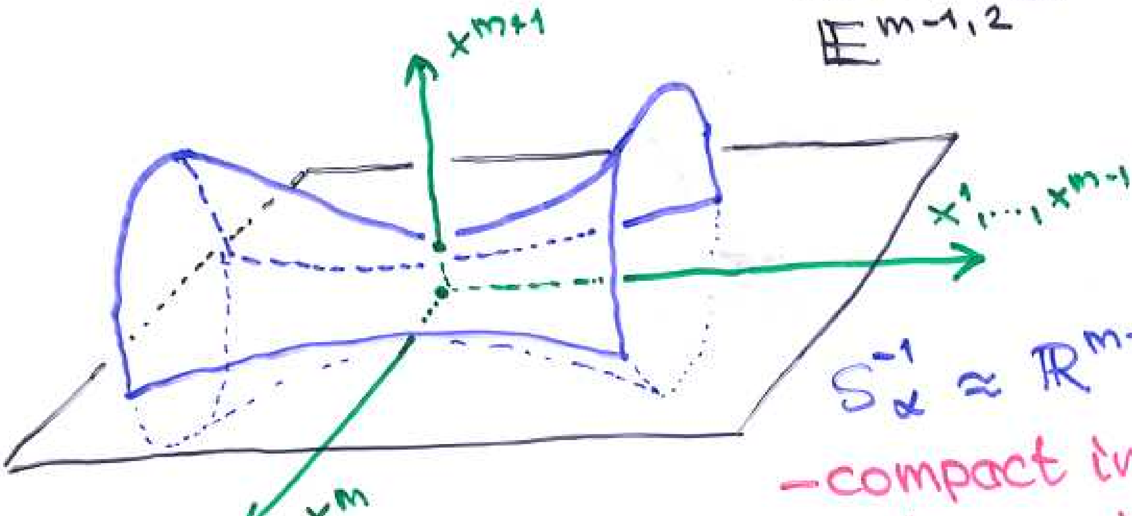
$S^{-1}_{\alpha} \approx \mathbb{R}^m \times S^0$ — disconnected

- $(m-1,1)$ (Lorentzian)
  - $R > 0$ — de Sitter spacetime as a timelike hypersurf. in $\mathbb{E}^{m,1}$ (Minkowski)



$S^{+1}_{\alpha} \approx S^{m-1} \times \mathbb{R}$

  - $R < 0$ — anti de Sitter spacetime as a hypersurface in $\mathbb{E}^{m-1,2}$



$S^{-1}_{\alpha} \approx \mathbb{R}^{m-1} \times S^1$

— compact in the time direction!

Spaces of Const. Curvature as Homogeneous Spaces

<u>Def</u>: M is called homogeneous if $\exists\, C^\infty$ transitive left Lie group action $\tau: M \times G \to M$ on M. (recall: transitive means: $\forall p, q \in M\ \exists g \in G: q = \tau(p,g)$; i.e M is an orbit of G)

Spaces of const curvature ar homogeneous:

- $\mathbb{E}^{p+1,q} \times SO_0(p+1,q) \to \mathbb{E}^{p+1,q}$ yields
  $S_\alpha^{+1} \times SO_0(p+1,q) \to S_\alpha^{+1}$ – transitive
- $\mathbb{E}^{p,q} \times \left(SO_0(p,q) \ltimes \mathbb{R}^m\right) \to \mathbb{E}^{p,q}$ – clearly transitive
- $\mathbb{E}^{p,q+1} \times SO_0(p,q+1) \to \mathbb{E}^{p,q+1}$ yields
  $S_\alpha^{-1} \times SO_0(p,q+1) \to S_\alpha^{-1}$ – transitive

<u>Theorem</u> (Warner, Helgason)

If (M, G, τ) is a homogeneous space and $H \subset G$ is the isotropy group of a point $p \in M$ (i.e. $H = \{g \in G \mid \tau(p,g) = p\}$), then there is a natural diffeomorphism $\phi: M \to G/H$; ie every homogeneous space is coset.

- $S_\alpha^{+1} \approx SO_0(p+1,q)/SO_0(p,q)$
- $\mathbb{E}^{p,q} \approx \left(SO_0(p,q) \ltimes \mathbb{R}^m\right)/SO_0(p,q) \approx \mathbb{R}^m$
- $S_\alpha^{-1} \approx SO_0(p,q+1)/SO_0(p,q)$.

GLOBAL PROPERTIES OF AdS SPACES

- m-dim, Lorentzian $(+\cdots+-)$ geom. with const. curvature

$$R_{abcd} = \frac{R}{m(m-1)}(g_{ac}g_{bd} - g_{ad}g_{bc}) \quad , \quad R < 0$$

— loc. conformally flat

Global topology of S^{-1}_α: $\mathbb{R}^{m-1} \times S^1$

— a priori closed timelike curves !

From now on: to rule out these

AdS := universial covering space of S^{-1}_α with topol $\mathbb{R}^m$ + same metric

- Global coordinates: by the def. of S^{-1}_α

$$(x^1)^2 + \cdots + (x^{m-1})^2 + \alpha^2 = (x^m)^2 + (x^{m+1})^2$$

new coordinates on $\mathbb{E}^{m-1,2}$ adapted to S^{-1}_α:

$$x^1 =: \alpha \operatorname{sh}\chi \sin\phi_1 \sin\phi_2 \cdots \sin\phi_{m-3} \sin\phi_{m-2}$$
$$x^2 =: \alpha \operatorname{sh}\chi \sin\phi_1 \sin\phi_2 \cdots \sin\phi_{m-3} \cos\phi_{m-2}$$
$$x^3 =: \alpha \operatorname{sh}\chi \sin\phi_1 \sin\phi_2 \cdots \cos\phi_{m-3}$$
$$\vdots$$
$$x^{m-1} =: \alpha \operatorname{sh}\chi \cos\phi_1$$
$$x^m =: \alpha \operatorname{ch}\chi \sin\psi$$
$$x^{m+1} =: \alpha \operatorname{ch}\chi \cos\psi$$

$\alpha > 0$, $\chi \geq 0$, $(\phi_1, \ldots, \phi_{m-2}) \in S^{m-2}$, $\psi \in S^1$

$\Downarrow$

$(\alpha, \psi, \chi, \phi_1, \ldots, \phi_{m-2})$ coord. system on $\mathbb{E}^{m-1,2}$ outside its "null cone".

The line element of $\mathbb{E}^{m-1,2}$:

$$dl^2 = -d\alpha^2 - \alpha^2 \operatorname{ch}^2\chi \, d\psi^2 + \alpha^2 d\chi^2 + \alpha^2 \operatorname{sh}^2\chi \underbrace{\left(d\phi_1^2 + \sum_{k=2}^{m-2} \sin^2\phi_1 \cdots \sin^2\phi_{k-1} \, d\phi_k^2 \right)}_{d\omega^2}$$

$d\omega^2$ – line element on the unit $m-2$ -sphere

⇓

line element of the induced metric on S_α^{-1} :

$$ds^2 = \alpha^2\left(-\operatorname{ch}^2\chi \, d\psi^2 + d\chi^2 + \operatorname{sh}^2\chi \, d\omega^2\right)$$

on AdS : $[0, 2\pi) \ni \psi \rightsquigarrow t \in \mathbb{R}$

– extension to the covering space

– Conformal boundary, PENROSE diagram

$$ds^2 = \Omega^{-2} \, d\bar{s}^2 = \alpha^2 \operatorname{ch}^2\chi \, d\bar{s}^2$$

– conformal to

$$d\bar{s}^2 = -dt^2 + d\bar{\chi}^2 + \sin^2\bar{\chi} \, d\omega^2$$

where

$$\bar{\chi} := 2 \operatorname{arctg} \exp(\chi) - \tfrac{1}{2}\pi \in [0, \tfrac{\pi}{2})$$

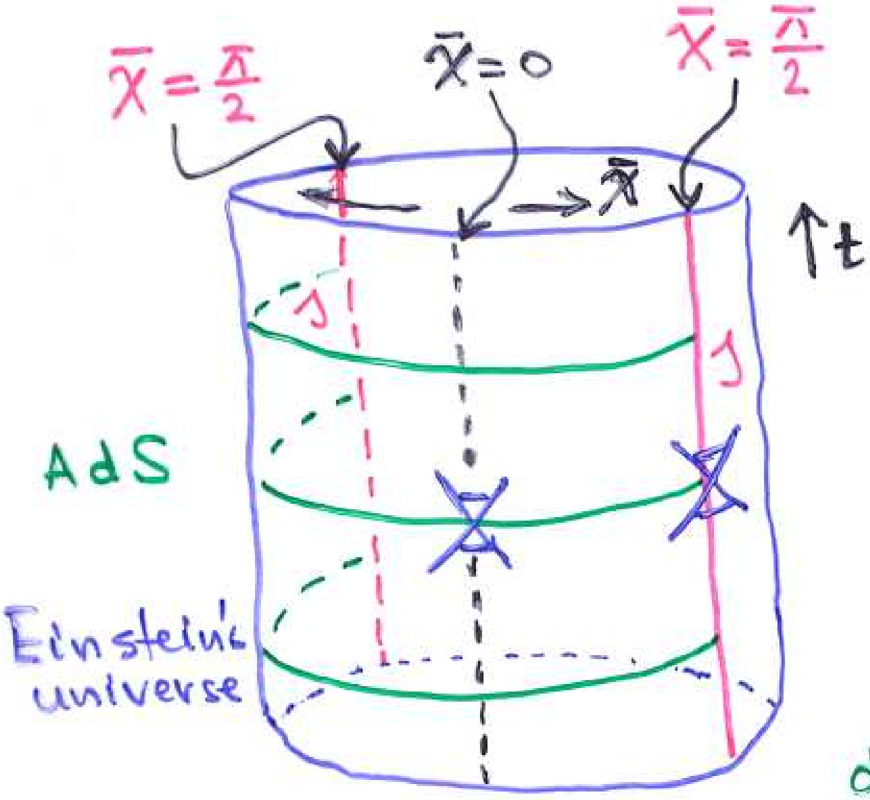
– half of Einstein's static universe (for which $\bar{\chi} \in [0, \pi)$ and $\approx \mathbb{R} \times \mathbb{S}^{m-1}$)

i.e.

AdS is conformal to a proper subset of Einstein's cylinder

⇓

from the ~~an~~ embedding we obtain the conformal boundary of AdS :



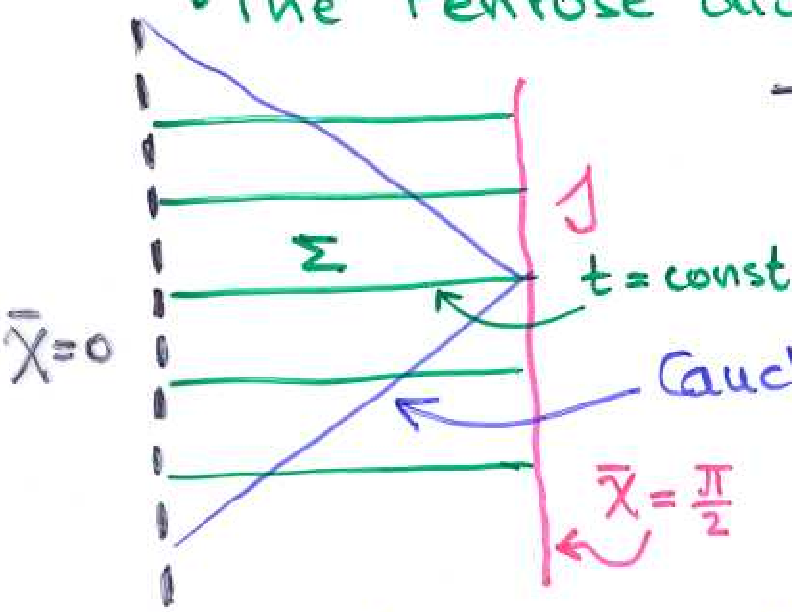
Conformal bound.:

$\mathcal{J} := \{\Omega = 0\} =$
$= \{\bar{\chi} = \frac{\pi}{2}\} \approx \mathbb{R}^1 \times S^{m-2}$

Coordinates on $\mathcal{J}$:
$(t, \phi_1, \dots, \phi_{m-2})$

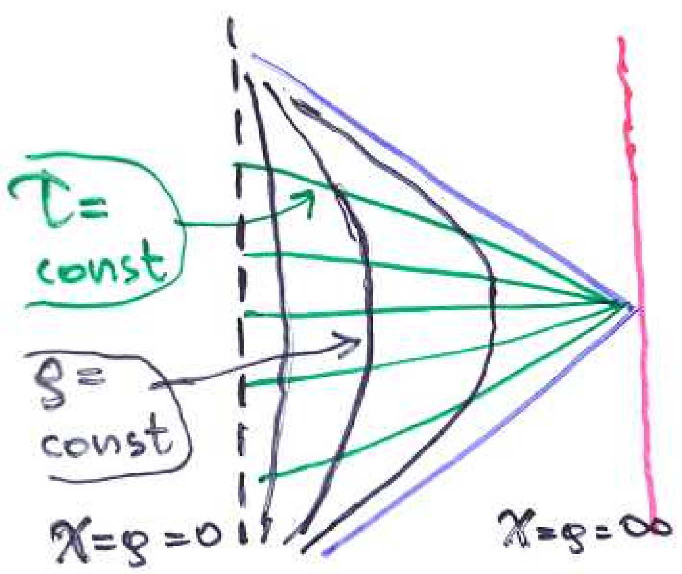
Conformal metric:
$db^2 = -dt^2 + d\omega^2$
– Lorentzian !

• The Penrose diagram:



– the normals of $t = \text{const}$ are NOT geodesic

– Another (geodesic, not global) coordinate sys.:



$$t - t_0 = -\operatorname{arctg}(\operatorname{ctg}\tau \operatorname{ch}\varrho)$$
$$\chi = \operatorname{ar\,sh}(\cos\tau \operatorname{sh}\varrho)$$

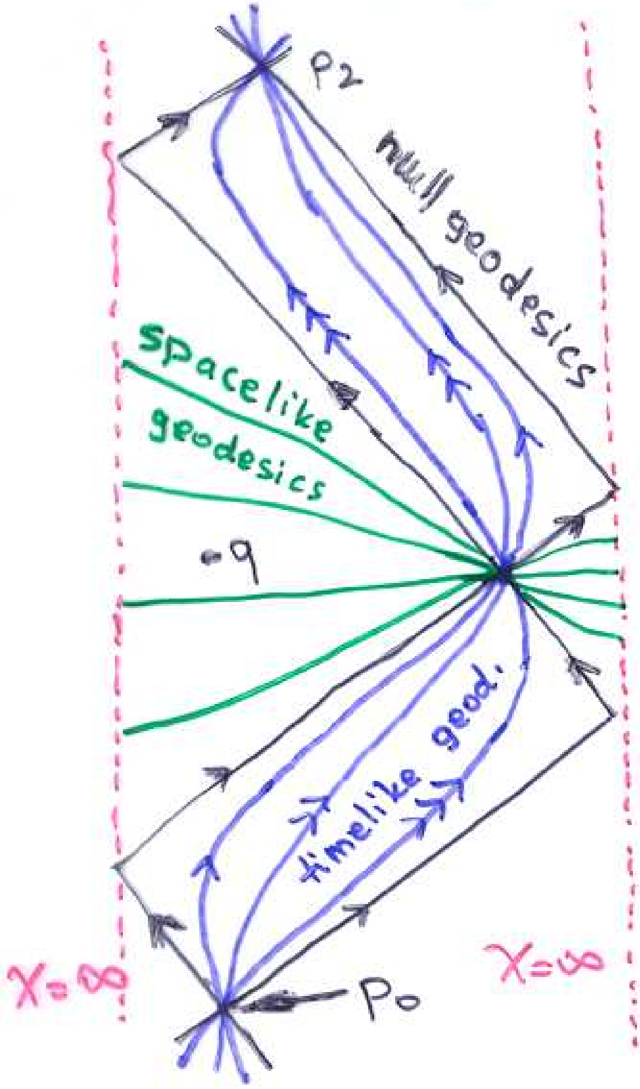
$\downarrow$

$$ds^2 = -\alpha^2 d\tau^2 + \alpha^2 \cos^2\tau \left(d\varrho^2 + \operatorname{sh}^2\varrho \, d\omega^2\right)$$

"time depend."

– the normals to $\tau = \text{const}$ are focussing

– The geodesic structure of AdS:



Solving the geodesic eq. or the deviation equation

$$\ddot{Z}^a + R^a{}_{bcd} K^b K^d Z^c = 0$$

in a p.p. frame along K^a:

- Null geod.:

$$Z^a(u) = \dot{Z}^a(0)\, u$$

  – like in Minkowski

- Timelike geod.:

$$Z^a(u) = \sqrt{\frac{m(m-1)}{|R|}}\, \dot{Z}^a(0) \sin\left(\sqrt{\frac{|R|}{m(m-1)}}\, u\right)$$

  – isotropic focussing with period $\Delta u = \pi \sqrt{\frac{m(m-1)}{|R|}}$!

- Spacelike geod.:

$$Z^a(u) = \sqrt{\frac{m(m-1)}{|R|}}\, \dot{Z}^a(0)\, \mathrm{sh}\left(\sqrt{\frac{|R|}{m(m-1)}}\, u\right)$$

  – separation grows exponentially!

<u>NB</u>: NO geodesic between p_0 and q

– in contrast to (complete) Riemannian geometries

AdS AS A SPACETIME

- Einstein's equations: $G_{ab} + \Lambda g_{ab} = k T_{ab}$

But: $G_{ab} := R_{ab} - \frac{1}{2} R g_{ab} = -\frac{m-2}{m} R g_{ab}$

$\Downarrow$

i. AdS is a solution with $T_{ab} = 0$ if

$$\Lambda = \frac{R}{2m}(m-2) = -\frac{1}{2}(m-1)(m-2)\frac{1}{\alpha^2}$$

or

ii. AdS is a solution with $\Lambda = 0$ if

$$k\mu := k T_{ab} t^a t^b = \frac{m-2}{m} R < 0 \quad \text{... strange!}$$

$$k p := k T_{ab} v^a v^b = -\frac{m-2}{2} R > 0$$

- Schwarzschild - AdS (Kottler) solution:

$$ds^2 = -f(r)\, d\tilde{t}^2 + \frac{1}{f(r)} dr^2 + r^2 d\omega^2$$

line element on the unit $(m-2)$-sphere

where

$$f(r) = 1 - c_m \frac{2M}{r^{m-3}} + \frac{r^2}{\alpha^2} \qquad (c_4 = 1)^*$$

- static, spherically symmetric, asymptotic. AdS vacuum solution

Event horizon: $f(r_H) = 0$ - null hyper surface

$$2 c_m M = (r_H)^{m-3} + \frac{1}{\alpha^2}(r_H)^{m-1}$$

and: Killing horizon: $K^a = \left(\frac{\partial}{\partial t}\right)^a$, $K_a K^a \big|_{r_H} = 0$

- changes its causal character

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* Fixing the const c_m: From $M = E_{ADM} = H$ (on shell)

$$c_m = \frac{8\pi}{m-2}\left[\mathrm{vol}(S_1^{m-1})\right]^{-1}$$

The surface gravity and the Hawking temperature:

$$\kappa^2 := -\frac{1}{2}(\nabla_a K_b)(\nabla^a K^b) = \left(c_m (m-3)\frac{M}{r^{m-2}} + \frac{r}{\alpha^2}\right)^2$$

– a measure of the strength of K^a

$\downarrow$

at the event horizon:

$$T_H := \frac{1}{2\pi}\kappa\Big|_{r_H} = \frac{m-3}{4\pi}\left(\frac{1}{r_H} + \frac{m-1}{m-3}\,\frac{r_H}{\alpha^2}\right)$$

– a measure of the strength of the grav. field on the event horizon

– the physical temperature in the black body radiation (Hawking)

– the acceleration of the Killing field

$$|a| = \frac{\kappa}{\sqrt{F}}$$ – surface grav / local red-shift fact