Uniform LC gauge and Giant Magnons

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- Polyakov string

- Uniform LC gauge: flat space + $\mathrm{AdS}_5 \otimes \mathrm{S}_5$

- GM solution: mapping to SG solitons

- GM scattering: time delay and phase shift

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NAMBU-GOTO STRING ACTION

closed string embedding

$$X^A(\tau,\sigma) \qquad\qquad X^A(\tau,r) = X^A(\tau,-r)$$

target space metric

$$\mathrm{d}X^2 = G_{AB}(X)\,\mathrm{d}X^A\,\mathrm{d}X^B$$

$$\Downarrow$$

WS (induced) metric

$$h_{\mu\nu} = G_{AB}(X)\,\partial_\mu X^A\,\partial_\nu X^B$$

NG string action: "area"

$$S_{\rm NG} = \frac{1}{\alpha'} \int_{-r}^{r} {\rm d}\sigma \int {\rm d}\tau \sqrt{-h}$$

EOM

$$\partial_\mu \left(\sqrt{-h} G_{AB} h^{\mu\nu} \partial_\nu X^B \right) = \frac{1}{2} \sqrt{-h} h^{\mu\nu} \frac{\partial G_{BC}}{\partial X^A} \partial_\mu X^B \partial_\nu X^C$$

"mini"-GR on WS: reparametrization invariance

$$\begin{aligned} \tau \to \tau' &= f_1(\tau, \sigma) \\ \sigma \to \sigma' &= f_2(\tau, \sigma) \end{aligned}$$

conformal gauge:

$$h_{\mu\nu} = \mathrm{e}^{\psi}\, \eta_{\mu\nu} \qquad \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Downarrow$$

WS "Poincaré-invariant"

$$\partial_\mu \left(G_{AB} \partial^\mu X^B \right) = \tfrac{1}{2} \tfrac{\partial G_{BC}}{\partial X^A} \partial_\mu X^B \partial^\mu X^C$$

conformal gauge constraints: $h_{01} = 0, \quad h_{00} + h_{11} = 0$

$$\begin{aligned} G_{AB} \dot{X}^A X'^B &= 0 \\ G_{AB} \left(\dot{X}^A \dot{X}^B + X'^A X'^B \right) &= 0 \end{aligned}$$

LC coordinates $\quad (\partial_\pm = \partial_0 \pm \partial_1)$

$$\partial_+ \left(G_{AB}\partial_- X^B\right) + \partial_- \left(G_{AB}\partial_+ X^B\right) = \frac{\partial G_{BC}}{\partial X^A}\partial_+ X^B \partial_- X^C$$

$$G_{AB}\partial_\pm X^A \partial_\pm X^B = 0$$

Residual gauge invariance

$$\tau \pm \sigma \to f_\pm(\tau \pm \sigma) \qquad\qquad \eta_{\mu\nu} \to (\qquad)\, \eta_{\mu\nu}$$

flat space:

$$G_{AB} = \eta_{AB} \qquad \partial_+\partial_- X^A = 0 \qquad \partial_+ X^A \partial_- X_A = 0$$

POLYAKOV ACTION

Rescaling: $X^A = R x^A$

$$\frac{\sqrt{\lambda}}{4\pi} = \frac{R^2}{\alpha'} \sim M^2_{\mathrm{Planck}} R^2 \qquad\qquad g = \frac{\sqrt{\lambda}}{2\pi}$$

alternative action

$$S_{Pol} = \frac{g}{2} \int_{-r}^{r} \mathrm{d}\sigma \int \mathrm{d}\tau \gamma^{\alpha\beta} G_{AB}(x)\, \partial_\alpha x^A\, \partial_\beta x^B$$

independent WS metric

$$\gamma^{\alpha\beta} \sim \sqrt{-h} h^{\alpha\beta} \qquad\qquad \det(\gamma) = -1$$

parametrization

$$\gamma^{00} = \mathrm{e}^{u}\cos\phi \qquad \gamma^{11} = -\mathrm{e}^{-u}\cos\phi \qquad \gamma^{01} = \sin\phi$$

x^A EOM

$$\partial_\alpha\left(G_{AB}\gamma^{\alpha\beta}\partial_\beta x^B\right) = \frac{1}{2}\gamma^{\alpha\beta}\frac{\partial G_{BC}}{\partial x^A}\partial_\alpha x^B \partial_\beta x^C$$

$\gamma^{\alpha\beta}$ EOM

$$\gamma^{\alpha\beta} = \sqrt{-h}h^{\alpha\beta}$$

Hamiltonian analysis

canonical momenta

$$p_A = \frac{\partial \mathcal{L}_{\mathrm{Pol}}}{\partial \dot{x}^A} = g\, G_{AB}\left(\gamma^{00}\dot{x}^B + \gamma^{01}x'^B\right)$$

Hamiltonian

$$\mathcal{H} = \frac{1}{2g\gamma^{00}} G^{AB} p_A p_B - \frac{\gamma^{01}}{\gamma^{00}} p_A x'^A + \frac{g}{2\gamma^{00}} G_{AB} x'^A x'^B + P_u \omega_u + P_\phi \omega_\phi$$

constraints: $P_u \approx P_\phi \approx 0$

consistency:

$$\mathcal{K} = p_A x'^A \approx 0 \qquad \mathcal{M} = \frac{1}{g} G^{AB} p_A p_B + g\, G_{AB} x'^A x'^B \approx 0$$

conformal generators $\quad \mathcal{L}_\pm = \mathcal{K} \pm \frac{1}{2}\mathcal{M}$

classical Virasoro algebra

$$\{\mathcal{L}_\pm(\sigma_1), \mathcal{L}_\pm(\sigma_2)\} = 2\mathcal{L}_\pm(\sigma_2)\delta'(\sigma_1 - \sigma_2) - \mathcal{L}'_\pm(\sigma_2)\delta(\sigma_1 - \sigma_2)$$

$$\{\mathcal{L}_+(\sigma_1), \mathcal{L}_-(\sigma_2)\} = 0$$

Fourier components

$$\mathcal{L}^{(\pm)}_m = \tfrac{r}{\pi} \int_{-r}^{r} \mathrm{d}\sigma \mathrm{e}^{-im\frac{\sigma\pi}{r}} \mathcal{L}_\pm(\sigma)$$

$$\left\{\mathcal{L}^{(\pm)}_m, \mathcal{L}^{(\pm)}_n\right\} = i(m-n)\,\mathcal{L}^{(\pm)}_{m+n}$$

$u \approx \phi \approx 0$ gauge fixing:

$$\mathcal{H} = \tfrac{1}{2g} G^{AB}\, p_A p_B + \tfrac{g}{2} G_{AB}\, x'^A x'^B$$

$$\text{constraints}: \qquad \mathcal{L}_\pm \approx 0 \qquad\qquad \mathcal{H} = \mathcal{L}_+ - \mathcal{L}_-$$

Hamiltonian reduction problem:

$\Rightarrow$ covariant quantization

$\Rightarrow$ LC quantization

Dirac procedure

$$H\,,\quad \phi_a \approx 0 \qquad a = 1, 2, \ldots \qquad (\dot{\phi}_a \approx 0)$$

$$\{\phi_a, \phi_b\} \approx 0$$ first class (gauge)

gauge fixing: $\chi_a \approx 0$ a=1,2,. . . $(\dot{\chi}_a \approx 0)$

$\{\phi_a, \chi_b\}$ invertible matrix: C_{ab}

$\{\chi_a, \chi_b\} = 0$ (for simplicity)

Dirac bracket

$$\{\mathcal{A}, \mathcal{B}\}^* = \{\mathcal{A}, \mathcal{B}\} + \{\mathcal{A}, \phi_a\}\, C^{-1}_{ba}\, \{\chi_b, \mathcal{B}\} - \{\mathcal{B}, \phi_a\}\, C^{-1}_{ba}\, \{\chi_b, \mathcal{A}\}$$

$$\{\mathcal{A}, \phi_a\} \approx \{\mathcal{A}, \chi_a\} \approx 0$$

$$\dot{\mathcal{A}} = \{\mathcal{A}, H\} \approx \{\mathcal{A}, H\}^*$$

works if: $\{H, \phi_a\} \approx \{H, \chi_a\} \approx 0$

time dependent case:

$$\dot{\mathcal{O}} = \frac{\partial \mathcal{O}}{\partial \tau} + \{\mathcal{O}, H\}$$

Time-dependent gauge fixing: $\{\ ,\ \}^*$ still works

modified Hamiltonian: $H \Rightarrow H^*$

Uniform LC gauge

Symmetries (isometries): $\delta x^A = \varepsilon_a \xi_a^A$

$$\delta \mathcal{L} = 0 \;\; \Leftrightarrow \;\; \frac{\partial G_{BC}}{\partial x^A}\, \xi_a^A + G_{AB}\, \frac{\partial \xi_a^A}{\partial x^C} + G_{AC}\, \frac{\partial \xi_a^A}{\partial x^B} = 0$$

Noether procedure:

$$\delta \mathcal{L} = J_a^\beta\, \partial_\beta \varepsilon_a \qquad\qquad J_a^\beta = g\, \gamma^{\alpha\beta}\, G_{AB}\, \partial_\alpha\, x^A\, \xi_a^B$$

conserved charges

$$\partial_\beta\, J_a^\beta = 0 \qquad\qquad \dot{Q}_a = 0 \qquad\qquad Q_a = \int_{-r}^{r} \mathrm{d}\sigma\, J_a^0(\tau, \sigma)$$

symmetry generators

$$J_a^0 = p_A \, \xi_a^A \qquad\qquad \{x^A, Q_a\} = \xi_a^A(x)$$

Flat space

$$G_{AB} = \eta_{AB} \qquad \mathcal{L} = \frac{g}{2}\,\gamma^{\alpha\beta}\,\partial_\alpha\, x^A \partial_\beta\, x_A \qquad A = 1, 2, \ldots, D$$

D-dimensional Poincaré group

$\mathrm{AdS}_5 \otimes \mathrm{S}_5$

S_5: $\quad b_1^2 + b_2^2 + \cdots + b_6^2 = 1$

AdS_5: $\quad -d_{-1}^2 - d_0^2 + d_1^2 + \cdots + d_4^2 = -1$

$$\mathcal{L} = \frac{g}{2}\gamma^{\alpha\beta}\,\left(\partial_\alpha b^a \partial_\beta b^a + \partial_\alpha d^M \partial_\beta d_M\right)$$

$$\mathrm{SO}(6) \quad \otimes \quad \mathrm{SO}(4,2)$$

unconstrained coordinates

b_1, b_2, b_3, b_4, ϕ $\qquad b_5 + ib_6 = \mathrm{e}^{i\phi}\sqrt{1-b^2}$ $\qquad \phi$ angle

d_1, d_2, d_3, d_4, t $\qquad d_{-1} + id_0 = \mathrm{e}^{it}\sqrt{1+d^2}$ $\qquad t$ NOT angle!

(covering space)

Light-cone coordinates

$$\mathrm{d}x^2 = -G_{tt}(\xi)\mathrm{d}t^2 + G_{\phi\phi}(\xi)\mathrm{d}\phi^2 + G_{ab}(\xi)\mathrm{d}\xi^a\,\mathrm{d}\xi^b$$

$\delta t = \varepsilon_t$ $\qquad J^0_t = p_t$

$\delta\phi = \varepsilon_\phi$ $\qquad J^0_\phi = p_\phi$

$$x_- = \phi - t \qquad\qquad p_- = p_\phi + p_t$$

$$x_+ = (1-a)t + a\phi \qquad\qquad p_+ = (1-a)p_\phi - a p_t$$

$$\{x_-(\sigma_1), p_+(\sigma_2)\} = \{x_+(\sigma_1), p_-(\sigma_2)\} = \delta(\sigma_1 - \sigma_2)$$

uniform LC gauge: $x_+ = \tau \quad p_+ = 1$

$$\mathcal{K} = p_t\, t' + p_\phi\, \phi' + \mathcal{K}_T(p_\xi, \xi)$$

$$\mathcal{M} = -\tfrac{1}{g\, G_{tt}}\, p_t^2 + \tfrac{1}{g\, G_{\phi\phi}}\, p_\phi^2 - g\, G_{tt}\, (t')^2 + g\, G_{\phi\phi}\, (\phi')^2 + \mathcal{M}_T(p_\xi, \xi)$$

Modified Hamiltonian

$$H^* = H - P_- \qquad P_- = \int_{-r}^{r} \mathrm{d}\sigma\, p_-$$

$$H \approx 0 \qquad H^* = -P_-$$

flat space: $E = -\int_{-r}^{r} \mathrm{d}\sigma\, p_t \qquad P = \int_{-r}^{r} \mathrm{d}\sigma\, p_\phi \qquad H^* = E - P$

$\mathrm{AdS}_5 \otimes \mathrm{S}_5$: $\Delta = -\int_{-r}^{r} \mathrm{d}\sigma\, p_t \qquad J = \int_{-r}^{r} \mathrm{d}\sigma\, p_\phi \qquad H^* = \Delta - J$

solve $\mathcal{K}$ constraint: $x'_- = -p_a\, \xi'^a$

level-matching condition:

$$\Delta x_- = x_-(r) - x_-(-r) = -\int_{-r}^{r} \mathrm{d}\sigma\, p_a\, \xi'^a = 0$$

$\Delta x_- = p_{\mathrm{ws}}$ WS momentum:

$$\{\Delta x_-, \xi^a(\sigma)\} = \xi'^a(\sigma) \qquad \{\Delta x_-, p_a(\sigma)\} = p'_a(\sigma)$$

$$P_+ = (1-a)J + a\Delta = 2r$$

Decompactification limit

$r \to \infty$: $J \to \infty$, $\Delta \to \infty$, $H^* = \Delta - J$ finite

level-matching relaxed

$\mathcal{M}$ constraint:

$$-\frac{1}{g\,G_{tt}}\left[(1-a)p_- - 1\right]^2 + \frac{1}{g\,G_{\phi\phi}}(1+ap_-)^2$$

$$-g\,G_{tt}(ax'_-)^2 + g\,G_{\phi\phi}\left[(1-a)x'_-\right]^2 + \mathcal{M}_T = 0$$

$$\mathcal{M}_T = \frac{1}{g}\,G^{ab}(\xi)\,p_a p_b + g\,G_{ab}(\xi)\,\xi'^a\xi'^b$$

flat space: $G_{tt} = G_{\phi\phi} = 1 \quad G_{ab} = \delta_{ab} \qquad a = 1/2$

$$H^* = -P_- = \tfrac{1}{2} \int \mathrm{d}\sigma \sum_a \left[p_a^2 + g^2 \left(\xi'^a\right)^2 \right]$$

manifest symmetry: $\mathrm{SO}(D-2)$

$\mathrm{AdS}_5 \otimes \mathrm{S}_5$: complicated p_-, $4+4\ \xi^a$

manifest symmetry: $\mathrm{SO}(4) \otimes \mathrm{SO}(4) = [\mathrm{SU}(2)]^4$

Giant Magnon on $\mathrm{R} \otimes \mathrm{S}_2$

$$\mathcal{L} = \frac{g}{2}\gamma^{\alpha\beta}\left(-\partial_\alpha t \partial_\beta t + \partial_\alpha \mathbf{n} \cdot \partial_\beta \mathbf{n}\right)$$

polar coordinates:

$$n^1 = \sin\theta\ \cos\phi \qquad n^2 = \sin\theta\ \sin\phi \qquad n^3 = \cos\theta$$

Uniform LC gauge ($a = 0$):

$$H^* = \Delta - J \qquad p_{\mathrm{ws}} = \Delta\phi \qquad t = \tau \qquad p_\phi = 1$$

Conformal-temporal gauge:

$$\gamma^{\alpha\beta} = \eta^{\alpha\beta} \qquad t = \tau \qquad p_t = -g$$

- time (velocities) same
- Δ, J, $\Delta\phi$ physical (gauge invariant)

$$\Downarrow$$

- $H^*(p_{\rm ws})$ dispersion relation
- Δt time delay (classical scattering)

Pohlmeyer reduction

$$\dot{\mathbf{n}}^2 + \mathbf{n}'^2 = 1 \qquad \dot{\mathbf{n}} \cdot \mathbf{n}' = 0 \qquad \Rightarrow \qquad \ddot{\mathbf{n}} - \mathbf{n}'' + \lambda \mathbf{n} = 0$$

$$\lambda = \dot{\mathbf{n}}^2 - \mathbf{n}'^2 = \cos\alpha$$

Sine-Gordon equation:

$$\partial_+ \partial_- \alpha = -\sin\alpha$$

Classically only!

$$\mathrm{R} \otimes \mathrm{S}_2 \quad \text{string problem} \qquad \Longleftrightarrow \qquad \text{SG model}$$

$$\cos\alpha = \partial_+\theta\partial_-\theta + \sin^2\theta\,\partial_+\phi\partial_-\phi$$

Giant Magnon Ansatz:

$$\theta = b(\sigma - v\tau) \qquad \phi = \tau + f(\sigma - v\tau)$$

$$\cos b = \frac{1}{\gamma\cosh(\gamma\sigma)} \qquad \tan f = \frac{1}{\gamma v}\tanh(\gamma\sigma)$$

$$\gamma = \frac{1}{\sqrt{1-v^2}}$$

SG Lorentz – invariant!

SG-soliton solution:

$$\tan\frac{\alpha}{4} = \mathrm{e}^{-\gamma(\sigma - v\tau)}$$

Magnon dispersion relation

$$p_{\rm ws} = \Delta\phi = f(\infty) - f(-\infty) = 2\arctan\frac{1}{\gamma v} = p$$

parametrization:

$$v = \cos\frac{p}{2} \qquad\qquad \gamma = \frac{1}{\sin\frac{p}{2}}$$

Noether current:

$$J_\mu = g(n^1\partial_\mu n^2 - n^2\partial_\mu n^1) = g\sin^2\theta\partial_\mu\phi \qquad J_0 = g\tanh^2\left[\gamma(\sigma - v\tau)\right]$$

dispersion relation:

$$H^* = \Delta - J = g\int_{-\infty}^{\infty} \mathrm{d}\sigma \left\{1 - \tanh^2\left[\gamma\sigma\right]\right\} = \frac{2g}{\gamma} = 2g\sin\frac{p}{2} = \frac{\sqrt{\lambda}}{\pi}\sin\frac{p}{2}$$

Semi-classical phase shift

Quantum Mechanics:

$$S(p_1, p_2) = \mathrm{e}^{i\delta(p_1,p_2)} \qquad \delta(p_1, p_2) = \tfrac{1}{\hbar}\,\delta_{\mathrm{sc}}(p_1, p_2) + \mathrm{O}(1)$$

classical time delay:

$$v_1\,\Delta t_1(p_1, p_2) = \frac{\partial \delta_{\mathrm{sc}}(p_1,p_2)}{\partial p_1}$$

AdS: $\Delta t_1 \quad \Rightarrow \quad \delta_{\mathrm{sc}}$ effortlessly!

$$\delta(p_1, p_2) = \tfrac{\sqrt{\lambda}}{2\pi}\,\delta_{\mathrm{sc}}(p_1, p_2) + \mathrm{O}(1)$$

SG-scattering (center-of-mass frame):

$$\tan\frac{\alpha}{4} = \frac{\sinh(\gamma v \tau)}{v\cosh(\gamma\sigma)}$$

$\tau \to \pm\infty$ limit:

$$\alpha \approx \alpha_s^{(v)}\left(\sigma, \tau \pm \tfrac{\Delta T}{2}\right) + \alpha_{\bar{s}}^{(-v)}\left(\sigma, \tau \pm \tfrac{\Delta T}{2}\right)$$

COM time delay:

$$\Delta T = \frac{2}{v\gamma}\ln v$$

WS Lorentz transformation:

$$\Delta t_1 = \frac{2\ln v}{v_1\gamma_1} \qquad \gamma_1 = \frac{1}{\sqrt{1-v_1^2}}$$

$$v = \frac{\sin\frac{p_1-p_2}{2}}{\cos\frac{p_1+p_2}{2}} \qquad v_{1,2} = \cos\frac{p_{1,2}}{2}$$

Large coupling result

semi-classical phase shift:

$$\delta_{\rm sc}(p_1,p_2) = 2\left(\cos\tfrac{p_2}{2} - \cos\tfrac{p_1}{2}\right)\ln\frac{\sin^2\left(\frac{p_1-p_2}{2}\right)}{\cos^2\left(\frac{p_1+p_2}{2}\right)} \quad + \quad 2p_1\,\sin\tfrac{p_2}{2}$$

note ambiguity

$$\Delta\,\delta_{\rm sc}(p_1,p_2) = p_1\,f(p_2)$$