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Scattering amplitudes in $\mathcal{N} = 4$ SYM

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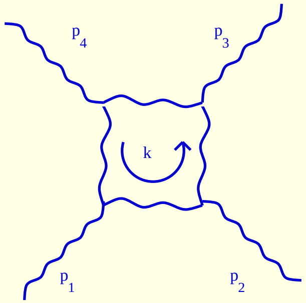


AdS/CFT correspondence (Maldacena 1997)

II_B superstring on $AdS_5 \times S^5$ $\frac{R^2}{\alpha'}\int \frac{d\tau d\sigma}{4\pi}\left(\partial_a X^M \partial^a X_M + \partial_a Y^M \partial^a Y_M\right) + \dots$	$\equiv$	$\mathcal{N}=4$ D=4 $SU(N)$ SYM $\frac{2}{g_{YM}}\int d^4x \mathrm{Tr}\left[-\frac{1}{4}F^2 - \frac{1}{2}(D\Phi)^2 + i\bar{\Psi}\not{D}\Psi + V\right]$

Dictionary

Couplings: $\sqrt{\lambda} = \frac{R^2}{\alpha'}$, $g_s = \frac{\lambda}{N}$ String spectrum $E(\lambda)$ Minimal surface g_{ab}	strong$\leftrightarrow$weak	$\lambda = g_{YM}^2 N$, N Anomalous dim $\Delta(\lambda)$ Scattering amplitudes=Wilson loops $\langle T_{\mu\nu}\rangle$ =hydro

Plan of the school

AdS : Szabados string: Balog super: Cynolter	$\rightarrow$	Sinkovics: AdS/CFT

Jevicki: classical strings Balog: giant magnon Palla: symmetries	anomalous dimension planar: integrability Ahn: S-matrix $\longrightarrow$ Bethe Ansatz	Hegedűs: gauge theory, magnon $\nearrow$

Jevicki: minimal surfaces	scattering amplitudes	Bajnok: 4 gluon, Wilson loops

RHIC, LHC: Csörgő, Barnaföldi, Regős: Hydro	AdShydro $\langle T_{\mu\nu}\rangle$	Bajnok: g_{ab}

Kormos	appl. to cond mat:	

$\mathcal{N}=4$ super Yang-Mills in 4d

$\mathcal{N}=4$ D=4 $SU(N)$ SYM

$$A_\mu, \Psi_i, \overline{\Psi}_i, \Phi_a$$

$$\frac{2}{g_{YM}} \int d^4x \mathrm{Tr}\left[-\frac{1}{4}F^2 - \frac{1}{2}(D\Phi)^2 + i\overline{\Psi}\not{D}\Psi + V\right]$$

$$V(\Phi,\Psi) = \frac{1}{4}[\Phi,\Phi]^2 + \overline{\Psi}[\Phi,\Psi]$$

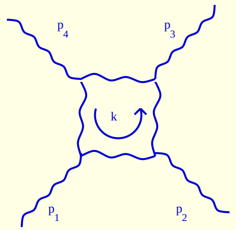
$\beta = 0$: superconformal $\frac{PSU(2,2|4)}{SO(5)\times SO(1,4)}$

$\lambda = g_{YM}^2 N$, $N \to \infty$ planar

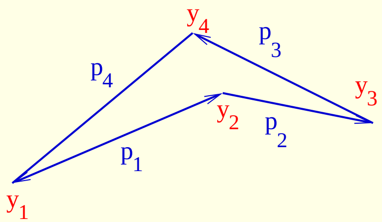
Alternative descriptions:

QFT: Smatrix

asym. states: (gluon+...)$|h, p_\mu, a\rangle$



Wilson loops



CFT: 2pt, 3pt

2pt: $\langle \mathcal{O}_n(x)\mathcal{O}_m(0)\rangle = \frac{\delta_{nm}}{|x|^{2\Delta_n(\lambda)}}$

Anomalous dim $\Delta(\lambda)$

3pt: $\langle \mathcal{O}_n(\infty)\mathcal{O}_m(1)\mathcal{O}_k(0)\rangle = C_{nmk}$

Gluon scattering amplitudes: summary

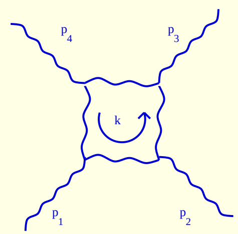
Literature: Alday: 0804.0951, Alday-Roiban 0807.1889, Henn arXiv:0903.0522, +100 papers,

Motivation: tree level=QCD, higher levels: nice iterative structure+ helps in QCD, $f(\lambda)$

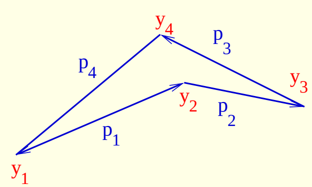
$YM^{L=1} = (\mathcal{N}=4)^{L=1} - 4(\mathcal{N}=1)^{L=1} + (boson\,loop)^{L=1}$ relation to $\mathcal{N}=8$ SUGRA

Plan:

1. Four leg amplitude: $\mathcal{A}_4^{(L)}$



2. Light-like Wilson loops



3. BDS conjecture: Bern-Dixon-Smirnov hep-th0505205

$A_4^L(p_i,\epsilon) = A_4^{L=0}(p_i,\epsilon)\mathcal{M}_4^{(L)}(\epsilon)$ where

$\mathcal{M}_4^{(L)}(\epsilon) = \exp_s\left[div.part\right]\exp_t\left[div.part\right]\exp\left[\frac{f(\lambda)}{8}\log^2\frac{s}{t}\right]$ cusp anomalous dimension $f(\lambda)$

Gluon scattering amplitudes: definition

Asymptotic states: $|h, p_\mu, a\rangle$

$h = \pm$ helicity
p_μ light-like momentum $p_\mu p^\mu = 0$
a color index, in the adjoint of $SU(n)$

N-leg amplitude: "S-matrix"

Color structure: $A_n^L = g^{n-2}(g^2N)^L \sum_{S_{n-1}} \mathrm{Tr}(T^{a_{\rho_1}} \dots T^{a_{\rho_n}}) \mathcal{A}_n^L(\rho_1, \dots \rho_n) +$ multiple traces

't Hooft (planar) limit ($N \to \infty$):

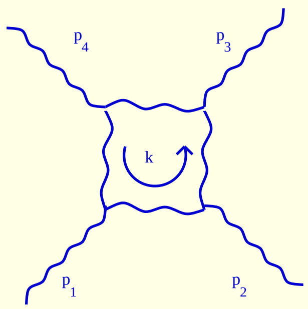
single traces are dominant,
$\mathcal{A}$ does not depend on ρ

still non-trivial Lorentz and helicity structure

Helicity: $| + + \dots + \rangle = | - + \dots + \rangle = 0$ (SuSy on $|\psi gg...gg\rangle$)

Simplest: Maximal Helicity Violating: $| - - + \dots + \rangle$ MHV (4pt and 5pt the only ones).

4 gluon scattering amplitude



Tree level: $\mathcal{A}_4^{L=0}(s,t)$ where $s=(p_1+p_2)^2$, $t=(p_2+p_3)^2$

One loop: $\mathcal{A}_4^{L=1}=\mathcal{A}_4^{L=0}\left[1-\frac{a}{2}st\,I_4+O(a^2)\right]$; $a=\frac{g^2N}{8\pi^2}$

$$I_4=C\int d^4k\frac{1}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2}$$

momentum conservation for onshell states $p_1+p_2+p_3+p_4=0$, $p_\mu p^\mu=0$

Divergences!	Soft $k_\mu=0$ collinear $k_\mu\propto p_\mu$

Dimensional regularization $4\to D=4-2\epsilon$ singularities $\mathcal{A}_n\propto\frac{1}{\epsilon^{2L}}+\ldots$

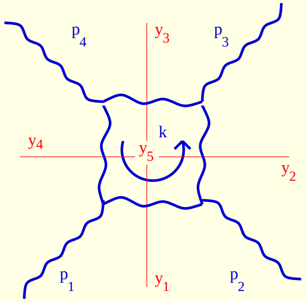
dimension $C=\mu^{2\epsilon}e^{-\epsilon\gamma_E}(4\pi)^{2-\epsilon}$

Compute infrared safe quantities

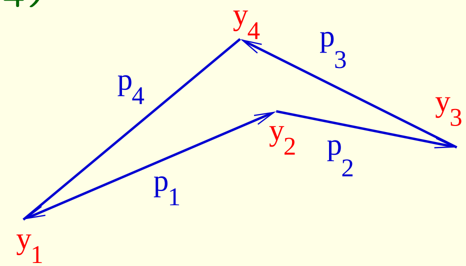
Break conformal symmetry in a specific way

general form (valid for any MHV) $\mathcal{A}_4^L(h_i,\{p_i\})=\mathcal{A}_4^{L=0}(h_i,\{p_i\})\mathcal{M}_4^L(\epsilon,\{p_i\})$

Dual superconformal symmetry



1-loop 4 leg amplitude: Tree level: $\mathcal{M}_4 \propto \int d^D k \frac{1}{k^2 (k-p_1)^2 (k-p_1-p_2)^2 (k+p_4)^2}$



Dual coordinates (in momentum space)

$p_1^\mu = y_1^\mu - y_2^\mu = y_{12}^\mu, \ldots, p_4^\mu = y_4^\mu - y_1^\mu = y_{41}^\mu$ and $k^\mu = y_1^\mu - y_5^\mu$

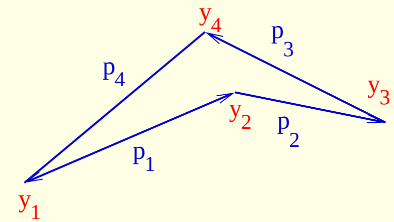
amplitude: $\mathcal{M}_4 \propto \int d^D y_5 \frac{1}{y_{15}^2 y_{25}^2 y_{35}^2 y_{45}^2}$ is conformal modulo Λ^{D-4}.

and looks like in coordinate space$\langle \Phi(x_1) \Phi(x_2) \rangle \propto \frac{1}{x_{12}^2}$

$$\mathcal{M}_4 \propto \int d^D y_5 \frac{1}{y_{15}^2 y_{25}^2 y_{35}^2 y_{45}^2} \propto \left[\frac{1}{\epsilon^2} (\frac{\mu^2}{-s})^\epsilon + \frac{1}{\epsilon^2} (\frac{\mu^2}{-t})^\epsilon + \frac{1}{2} \log^2 \frac{s}{t} + 4\zeta_2 + \ldots \right]$$

Old wisdom: it is the same what appears in the Wilson loops

Wilson loop



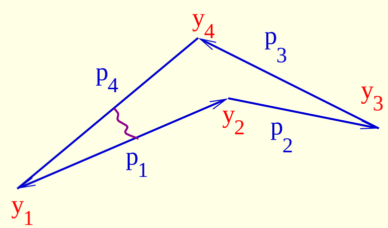
on light-like polygon
$p_1^\mu = y_1^\mu - y_2^\mu = y_{12}^\mu, \ldots, p_4^\mu = y_4^\mu - y_1^\mu = y_{41}^\mu$ and $k^\mu = y_1^\mu - y_5^\mu$

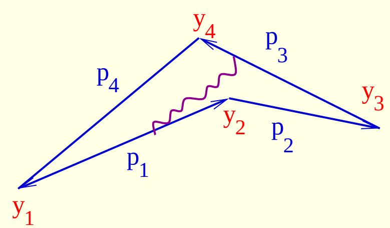
$$\boxed{W(C) = \frac{1}{N}\langle 0|\mathrm{Tr}(\mathcal{P}\exp\{ig\oint dx^\mu A_\mu\})|0\rangle}$$

Weak coupling expansion: Tree $W(C) = 1 + \frac{(ig)^2}{2}\frac{N^2-1}{2N}\oint_x dx^\mu \oint_y dy^\nu \langle A_\mu^a(x) A_\nu^b(y)\rangle$

where $A_\mu = A_\mu^a t_a$ and $\langle A_\mu^a(x) A_\nu^b(y)\rangle = D_{\mu\nu}(x-y)\delta^{ab}$

Feynman gauge: $D_{\mu\nu}(x) = \eta_{\mu\nu}\left[-\frac{\Gamma(1-\epsilon)}{4\pi^2}(-x^2+i0)^{-1+\epsilon}(\mu^2 e^{-\gamma_E})^\epsilon\right]$





loop contributions

UV divergence, singular part: $W(C) = 1 + \frac{(ig)^2}{2}\frac{N^2-1}{2N}\left\{-\frac{1}{2\epsilon^2}(-y_{24}^2\mu^2)^\epsilon\right\}$

Regular part $\frac{(ig)^2}{2}\frac{N^2-1}{2N}\left[\log^2\left(\frac{y_{13}^2}{y_{24}^2}\right) + \pi^2\right]$ agrees if $\epsilon_{UV} = \epsilon_{IR}$ and $s = y_{13}^2$

BDS Ansatz

One loop 4 leg amplitude factorizes:

$$\mathcal{A}_4^{L=1} = \mathcal{A}_4^{L=0}\mathcal{M}_4^{L=1} = \mathcal{A}_4^{L=0}\left[1 - \frac{a}{\epsilon^2}(\frac{\mu^2}{-s})^{\epsilon}\right]\left[1 - \frac{a}{\epsilon^2}(\frac{\mu^2}{-s})\right]\left[1 + a\left(\frac{1}{2}\log^2\frac{s}{t} + 4\zeta_2\right)\right] + O(a^2)$$

two loop result:

$$\mathcal{M}_4^{L=2}(\epsilon) = \frac{1}{2}(\mathcal{M}_4^{L=1}(\epsilon))^2 + f^{(2)}(\epsilon)\mathcal{M}_4^{L=1}(2\epsilon) + C(\epsilon)$$

Conjecture: $\mathcal{M}_4^{(L)}(\epsilon) = \exp_s[div.part]\exp_t[div.part]\exp\left[\frac{f(\lambda)}{8}\log^2\frac{s}{t}\right]$where

$div.part = \frac{-1}{8\epsilon^2}f^{(-2)}(a(\frac{\mu^2}{s})^{\epsilon}) - \frac{1}{4\epsilon}g^{(-1)}(a(\frac{\mu^2}{s})^{\epsilon})$ where $(x\partial_x)^2 f^{(-2)}(x) = f(x)$

This can be derived from:

Dual conformal anomaly: $K^{\mu}W(C) = \frac{1}{2}f(\lambda)\sum y^{\mu}_{i,i+1}\log(\frac{y^2_{i,i+2}}{y_{i-1,i+1}}) \neq 0$ fixes the 4 and 5 point functions.

In general BDS: $\mathcal{M}_n^{(L)}(\epsilon) = \exp\left[\sum_L a^L f^{(L)}(\epsilon)\mathcal{M}_n^{(1)}(L\epsilon) + const\right]$

where $f^{(L)}(\epsilon) = f_0^{(L)} + \epsilon f_1^{(L)} + \epsilon^2 f_2^{(L)}$

Conclusion

We calculated: One loop 4 leg amplitude using dual conformal symmetry

It agreed perturbatively with the VEV of a light-like Wilson loop (origin of the symmetry)

Dual conformal anomaly can be used to fix the 4leg and 5leg amplitudes and BDS works

On tree level original+dual conformal symmetry form a Yangian

No physical justification for the amplitude$\leftrightarrow$Wilson loop correspondence

BDS ansatz can be extended to higher MHV amplitudes but fails over 6legs,

still amplitude$\leftrightarrow$Wilson loop correspondence remains valid