Exact solutions of Navier-Stokes equations
and effects on slopes, elliptic flow and HBT radii

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Introduction:
Zimányi-Bondorf-Garpman solution of perfect fluid hydrodynamics
„There blows the gluon wind”, introduction of inflation to hydrodynamics
2005 AIP top physics story, 2006 “silver medal” nucl-ex paper

Indication of hydro in RHIC/SPS data: hydrodynamical scaling behavior
Appear in beautiful, exact family of solutions of fireball hydro
non-relativistic, perfect and dissipative exact solutions
relativistic, perfect, accelerating solutions -> M. Csanád’s talk

Their application to data analysis at RHIC energies -> Buda-Lund

Exact results: tell us what can (and what cannot) be learned from data
#1: Zimányi – Bondorf – Garpman solution


Old idea: perfect fluid of nucleons
started by looking at experimental data

More recent: perfect liquid of quarks
Buda-Lund hydro model
started by looking at HBT data at SPS
was based on the ZBG flow profile (Jozsó’s advice)
many families of exact solutions
describes spectra, v2, and HBT data at RHIC
successfull predictions: v2 & HBT scaling at RHIC
#2: Our last completed work: inflation at RHIC

Proc. Budapest’02
Quark & Hadron

Inflation of the HBT

T. Csörgő

\[ n(t, r') = n_0 \frac{V_0}{V} \exp \left( \frac{X}{X} r'_x \right) \]

\[ v'(t, r') = \left( \frac{X}{X} r'_x, 1 \right) \]

\[ T(t) = T_i(t) \left( \frac{V_0}{V} \right) \]

\[ T_i(t) = \kappa T_0 \left[ 1 + j_e \right] \]

Workshop on Dynamics

Homogeneity

\[ = 0, \]

\[ = - \frac{(\nabla (p + p_G))}{(mn)}, \]

\[ = j_G, \]

\[ = j_v(t) n T, \]

\[ = j_E(t) \kappa n T, \]
Milestone #3: Top Physics Story 2005

http://arxiv.org/abs/nucl-ex/0410003

Inverse slopes $T$ of single particle $p_t$ distribution increase \~ linearly with mass:

$$T = T_0 + m \langle u_t \rangle^2$$

Increase is stronger in more head-on collisions.
Suggests collective radial flow, local thermalization and hydrodynamics

Nu Xu, NA44 collaboration, Pb+Pb @ CERN SPS
T. Cs. and B. Lörstad, hep-ph/9509213
Successfully predicted by the Buda-Lund hydro model (T. Cs et al, hep-ph/0108067)
Notation for fluid dynamics

- **nonrelativistic hydro:**
  - \( t \): time,
  - \( r \): coordinate 3-vector, \( r = (r_x, r_y, r_z) \),
  - \( m \): mass,

- **field i.e. \((t,r)\) dependent variables:**
  - \( n \): number density,
  - \( \sigma \): entropy density,
  - \( p \): pressure,
  - \( \varepsilon \): energy density,
  - \( T \): temperature,
  - \( v \): velocity 3-vector, \( v = (v_x, v_y, v_z) \)
Nonrelativistic perfect fluid dynamics

Equations of nonrelativistic hydro:
local conservation of
charge: continuity
momentum: Euler
energy

\[
\begin{align*}
\partial_t n + \nabla(n \mathbf{v}) &= 0, \\
mn \left[ \partial_t \mathbf{v} + (\mathbf{v} \nabla) \mathbf{v} \right] &= 0, \\
\partial_t \epsilon + \nabla(\epsilon \mathbf{v}) + p \nabla \mathbf{v} &= 0.
\end{align*}
\]

EoS needed:

\[ p = nT, \quad \epsilon = \kappa(T)nT, \]

Perfect fluid: 2 equivalent definitions, term used by PDG

#1: no bulk and shear viscosities, and no heat conduction.
#2: \( T^\mu = \text{diag}(e,-p,-p,-p) \) in the local rest frame.

Ideal fluid: ambiguously defined term, discouraged

#1: keeps its volume, but conforms to the outline of its container
#2: an inviscid fluid
Dissipative, non-relativistic fluid dynamics

Navier-Stokes equations: dissipative, nonrelativistic hydro:

\[ \partial_t n + \nabla (n \mathbf{v}) = 0, \]
\[ m n [\partial_t \mathbf{v} + (\mathbf{v} \nabla) \mathbf{v}] = -\nabla p + \eta \left[ \Delta \mathbf{v} + \frac{1}{3} \nabla (\nabla \mathbf{v}) \right] + \zeta \nabla (\nabla \mathbf{v}), \]
\[ \partial_t \epsilon + \nabla (\epsilon \mathbf{v}) + p \nabla \mathbf{v} = \nabla (\lambda \nabla T) + \zeta (\nabla \mathbf{v})^2 + 2\eta \left[ T r D^2 - \frac{1}{3} (\nabla \mathbf{v})^2 \right], \]

EoS needed:

\[ p = nT, \]
\[ \epsilon = \frac{1}{c_s^2(T)} p \equiv \kappa p, \]

Shear and bulk viscosity, heat conduction effects:

\[ \eta_s \quad \zeta \quad \lambda \]
Old idea: Quark Gluon Plasma
More recent: Liquid of quarks

$T_c = 176 \pm 3 \text{ MeV} \ (\sim 2 \text{ terakelvin})$

(hep-ph/0511166)
at $\mu = 0$, a cross-over

Aoki, Endrődi, Fodor, Katz, Szabó

hep-lat/0611014

LQCD input for hydro: $p(\mu T)$
LQCD for RHIC region: $p \sim p(T)$,

$c_s^2 = \frac{\delta p}{\delta \varepsilon} = c_s^2(T) = 1/\kappa(T)$

It’s in the family exact hydro solutions!
New exact, parametric hydro solutions

Ansatz: the density $n$ (and $T$ and $\varepsilon$) depend on coordinates only through a scale parameter $s$


$$n = f(t)g(s).$$

$$\partial_t n = f'(t)g(s) + f(t)g'(s)\partial_t s,$$

$$\nabla (vn) = f(t)g(s)\nabla v + f(t)g'(s)v\nabla s.$$

Principal axis of ellipsoid:

$(X,Y,Z) = (X(t), Y(t), Z(t))$

$$f(t) = \frac{X_0Y_0Z_0}{XYZ}$$

$$s = \frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2} + \frac{r_z^2}{Z^2}$$

Density=const on ellipsoids.
g(s): arbitrary scaling function.

$$f'(t) \frac{f(t)}{f(t)} = -\nabla v,$$

$$\partial_t s + v\nabla s = 0$$

$$v = \left( \frac{\dot{X}}{X}, \frac{\dot{Y}}{Y}, \frac{\dot{Z}}{Z}, r_x, r_y, r_z \right)$$

Directional Hubble flow.

Notation: $n \sim \nu(s)$, $T \sim \tau(s)$ etc.
Perfect, ellipsoidal hydro solutions

A new family of PARAMETRIC, exact, scale-invariant solutions


Volume is introduced as $V = XYZ$

\[
n(t, r) = n_0 \frac{V_0}{V} \nu(s) \\
\mathbf{v}(t, r) = \left( \frac{\dot{X}}{X}, \frac{\dot{Y}}{Y}, \frac{\dot{Z}}{Z}, \frac{\dot{r}_x}{r_x}, \frac{\dot{r}_y}{r_y}, \frac{\dot{r}_z}{r_z} \right) \\
T(t, r) = T_0 \left( \frac{V_0}{V} \right)^{1/\kappa} T(s) \\
\nu(s) = \frac{1}{T(s)} \exp \left( - \frac{T_i}{2T_0} \int_0^s \frac{du}{T(u)} \right)
\]

\[
s = \frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2} + \frac{r_z^2}{Z^2}
\]

For $\kappa = \kappa(T)$ exact solutions, see
T. Cs, S.V. Akkelin, Y. Hama,
B. Lukács, Yu. Sinyukov,
hep-ph/0108067,
or see the solutions of Navier-Stokes later on.

The dynamics is reduced to coupled, nonlinear but ordinary differential equations for the scales $X, Y, Z$

\[
X \ddot{X} = Y \ddot{Y} = Z \ddot{Z} = \frac{T_i}{m} \left( \frac{V_0}{V} \right)^{1/\kappa}
\]

Many hydro problems (initial conditions, role of EoS, freeze-out conditions) can be easily illustrated and understood on the equivalent problem:
a classical potential motion of a mass-point in a conservative potential (a shot)!

Note: temperature scaling function $\tau(s)$ remains arbitrary! $\nu(s)$ depends on $\tau(s).$ -> FAMILY of solutions.
Dynamics of principal axis:

The role of initial boundary conditions, EoS and freeze-out in hydro can be understood from potential motion!
From the new family of exact solutions, the initial conditions:

**Initial coordinates:**
(nuclear geometry +
time of thermalization)

\[(X_0, Y_0, Z_0)\]

**Initial velocities:**
(pre-equilibrium + time of thermalization)

\[(\dot{X}_0, \dot{Y}_0, \dot{Z}_0)\]

**Initial temperature:**

\[T_0\]

**Initial density:**

\[n_0\]

**Initial profile function:**
(energy deposition
and pre-equilibrium process)

\[\tau(s)\]
Role of initial temperature profile

- Initial temperature profile = arbitrary positive function
- Infinitely rich class of solutions
- Matching initial conditions for the density profile

\[ \nu(s) = \frac{1}{\mathcal{T}(s)} \exp \left( -\frac{T_i}{2T_0} \int_0^s \frac{du}{\mathcal{T}(u)} \right) \]

- Homogeneous temperature ⇒ Gaussian density

\[ \nu(s) = \exp(-s/2), \quad \mathcal{T}(s) = 1. \]

\[ s = \frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2} + \frac{r_z^2}{Z^2} \]

- Buda-Lund profile:

\[ \mathcal{T}(s) = \frac{1}{1 + bs} \]
\[ \nu(s) = (1 + bs) \exp \left[ -\frac{T_i}{2T_0} (s + bs^2/2) \right] \]

- Zimányi-Bondorf-Garpman profile:

\[ \mathcal{T}(s) = (1 - s) \Theta(1 - s) \]
\[ \nu(s) = (1 - s)^\alpha \Theta(1 - s) \]
Illustrated initial T-> density profiles

- Illustrations of density profiles:
  - Fireball
  - Ring of fire
  - Embedded shells of fire

Exact integrals of hydrodynamics:
Scales expand in time.

Time evolution of the scales (X,Y,Z) follows a classic potential motion.
Scales at freeze out -> observables.
Info on history LOST!
No go theorem - constraints on initial conditions (penetrating probes) indispensable.
Illustrations of exact hydro results

- Propagate the hydro solution in time numerically:

\[ R_x(t), R_y(t), R_z(t) \]

Correlation radii

\[ t \]

\[ 0 \rightarrow 70 \]
Final (freeze-out) boundary conditions

From the new exact hydro solutions, the conditions to stop the evolution:

**Freeze-out temperature:**

Final coordinates:
(cancel from measurables, diverge)

Final velocities:
(determine observables, tend to constants)

Final density:
(cancels from measurables, tends to 0)

Final profile function:
(= initial profile function! from solution)
Role of the Equation of States:

The potential depends on $\kappa = \frac{\delta}{\delta p}$:

$$T_0 \left( \frac{V_0}{V} \right)^{1/\kappa}$$

Time evolution of the scales ($X,Y,Z$) follows a classic potential motion. Scales at freeze out determine the observables. Info on history LOST! No go theorem - constraints on initial conditions (information on spectra, elliptic flow of penetrating probes) indispensable.

The arrow hits the target, but can one determine $g$ from this information??
Initial conditions <-> Freeze-out conditions:

Different initial conditions
but
same freeze-out conditions
ambiguity!
Penetrating probes radiate through the time evolution!
Solution of the “HBT puzzle”

Geometrical sizes keep on increasing. Expansion velocities tend to constants. HBT radii $R_x$, $R_y$, $R_z$ approach a direction independent constant.

Slope parameters tend to direction dependent constants.

General property, independent of initial conditions - a beautiful exact result.
Dissipative, ellipsoidal hydro solutions

A new family of dissipative, exact, scale-invariant solutions

T. Cs. and Y. Hama, in preparation ...

Volume is $V = XYZ$

$$n(t, r) = n_0 \frac{V_0}{V} \nu(s)$$

$$\mathbf{v}(t, r) = \left( \frac{\dot{X}}{X} r_x, \frac{\dot{Y}}{Y} r_y, \frac{\dot{Z}}{Z} r_z \right)$$

$$T(t, r) = T_0 f(t) \bar{T}(s),$$

$$\nu(s) = \frac{1}{\bar{T}(s)} \exp \left( -\frac{T_i}{2T_0} \int_0^s \frac{du}{\bar{T}(u)} \right)$$

The dynamics is reduced to coupled, nonlinear but ordinary differential equations for the scales $X,Y,Z$

$$X\dddot{X} = Y\dddot{Y} = Z\dddot{Z} = \frac{T_i f(t)}{m}$$

$$s = \frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2} + \frac{r_z^2}{Z^2}$$

Even VISCOUS hydro problems (initial conditions, role of EoS, freeze-out conditions, DISSIPATION) can be easily illustrated and understood on the equivalent problem:

a classical potential motion of a mass-point in a conservative potential (a shot)!

Note: temperature scaling function $\tau(s)$ remains arbitrary! $\nu(s)$ depends on $\tau(s)$. -> FAMILY of solutions.
Dissipative, ellipsoidal hydro solutions

A new family of PARAMETRIC, exact, scale-invariant solutions

T. Cs. and Y. Hama, in preparation

Introduction of kinematic bulk and shear viscosity coefficients:

\[
\nu_S = \frac{\eta}{mn} = c_1 \\
\nu_B = \frac{\zeta}{mn} = c_2
\]

Note that the Navier-Stokes (gen. Euler) is automatically solved by the directional Hubble ansatz, as the 2nd gradients of the velocity profile vanish!

Only non-trivial contribution from the energy equation:

\[
\dot{T} - \dot{T} \frac{d \ln c_s^2(T)}{dT} = -c_s^2(T)T \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right) + m\nu_B \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right)^2 + \\
+ 2m\nu_S \left[ \left( \frac{\dot{X}}{X} \right)^2 + \left( \frac{\dot{Y}}{Y} \right)^2 + \left( \frac{\dot{Z}}{Z} \right)^2 - \frac{1}{3} \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right)^2 \right]
\]

Asymptotics: T -> 0 for large times, hence X ~ t, Y ~ t, Z ~ t, and asymptotic analysis possible!

- EOS: drives dynamics, asymptotically dominant term: perfect fluid!!
- Shear: asymptotically sub-subleading correction, ~ 1/t^3
- bulk: asymptotically sub-leading correction, ~ 1/t^2
Dissipative, heat conductive hydro solutions

A new family of PARAMETRIC, exact, scale-invariant solutions

T. Cs. and Y. Hama, in preparation

Introduction of ‘kinematic’ heat conductivity:

The Navier-Stokes (gen. Euler) is again automatically solved by the directional Hubble ansatz!

Only non-trivial contribution from the energy equation:

\[ \dot{T} - \dot{T} \frac{d \ln c_s^2(T)}{d \ln T} \approx -c_s^2(T)T \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right) + m\nu_B \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right)^2 + \\
+ 2m\nu_s \left[ \left( \frac{\dot{X}}{X} \right)^2 + \left( \frac{\dot{Y}}{Y} \right)^2 + \left( \frac{\dot{Z}}{Z} \right)^2 - \frac{1}{3} \left( \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} \right)^2 \right] + \\
+ m \left[ \nu_Q T_i T'(0) \left( \frac{1}{X^2} + \frac{1}{Y^2} + \frac{1}{Z^2} \right) \right] \]

Role of heat conduction can be followed asymptotically
- same order of magnitude \((1/t^2)\) as bulk viscosity effects
- valid only for nearly constant densities,
- destroys self-similarity of the solution if there are strong irregularities in temperature

\[ \nabla \nu(s) = 0 \]

\[ \Delta T \approx -T_i \left( \frac{1}{X^2} + \frac{1}{Y^2} + \frac{1}{Z^2} \right) \]
Scaling predictions, for (viscous) fluid dynamics

\[ T_x' = T_f + m \dot{X}_f^2 , \]
\[ T_y' = T_f + m \dot{Y}_f^2 , \]
\[ T_z' = T_f + m \dot{Z}_f^2 . \]

- Slope parameters increase linearly with mass
- Elliptic flow is a universal function its variable w is proportional to transverse kinetic energy and depends on slope differences.

\[ \nu_2 = \frac{I_1(w)}{I_0(w)} \]
\[ w = \frac{k_t^2}{4m} \left( \frac{1}{T_y'} - \frac{1}{T_x} \right) , \]
\[ w = \frac{E_K}{2T_*} \varepsilon \]

Inverse of the HBT radii increase linearly with mass analysis shows that they are asymptotically the same

Relativistic correction: \( m \rightarrow m_t \)

hep-ph/0108067, nucl-th/0206051

\[ R_x'^{-2} = X_f^{-2} \left( 1 + \frac{m}{T_f} \dot{X}_f^2 \right) , \]
\[ R_y'^{-2} = Y_f^{-2} \left( 1 + \frac{m}{T_f} \dot{Y}_f^2 \right) , \]
\[ R_z'^{-2} = Z_f^{-2} \left( 1 + \frac{m}{T_f} \dot{Z}_f^2 \right) . \]
Understanding hydro results

Hydro problem equivalent to potential motion (a shot)!

**Hydro:**
- Description of data
- Initial conditions (IC)
  - Initial position and velocity
- Equations of state
  - Strength of the potential
- Freeze-out (FC)
  - Position of the target
- Data constrain EOS

**Shot of an arrow:**
- Hitting the target
- Different IC yields same FC
- Different archers can hit the target
- EOS and IC can co-vary
- Universal scaling of $v^2/F/ma = 1$

**Viscosity effects:**
- Drag force of air
- Numerical hydro fails (HBT)

**Arrow misses the target (!)**
Buda-Lund and exact hydro sols

- Perfect non-relativistic solutions
- Dissipative non-relativistic solutions
- Hwa Bjorken Hubble
- Exact relativistic solutions w/o acceleration
- Exact relativistic solutions w/ acceleration
- Data

Buda-Lund
Femtoscopy signal of sudden hadronization

Buda-Lund hydro fit indicates hydro predicted (1994-96) scaling of HBT radii

T. Cs, L.P. Csernai
hep-ph/9406365
T. Cs, B. Lörstad
hep-ph/9509213

Hadrons with $T > T_c$:
a hint for cross-over
M. Csanád, T. Cs, B. Lörstad and A. Ster,
nucl-th/0403074
Universal hydro scaling of $v_2$

Black line: Theoretically predicted, universal scaling function from analytic works on perfect fluid hydrodynamics:

$$v_2 = \frac{I_1(w)}{I_0(w)}$$

hep-ph/0108067, nucl-th/0310040, nucl-th/0512078
Illustration, (in)dependence on EOS

$m=940$ MeV, $T_0 = 180$ MeV
Same with bulk and shear viscosity

\[ m = 940 \text{ MeV}, \quad T_0 = 180 \text{ MeV}, \quad \nu(B) = \nu(S) = 0.1 \]
Summary

Au+Au elliptic flow data at RHIC satisfy the UNIVERSAL scaling laws, predicted (2001, 2003) by the (Buda-Lund) hydro model, based on exact solutions of PERFECT FLUID hydrodynamics:

quantitative evidence for a perfect fluid in Au+Au at RHIC

New, rich families of exact hydrodynamical solutions discovered when searching for dynamics in Buda-Lund

- non-relativistic perfect fluids
- non-relativistic, Navier-Stokes
- scaling predictions of hydro DO NOT depend on viscosity (!)

Pros: late time perfect fluid  Contras: initially viscous

- relativistic perfect fluids -> see M. Csanád’s talk
Backup slides from now on
Illustration, (in)dependence on EOS

![Graph showing the dependence of Xdot, Ydot, and Zdot on time (t) with different values of $c_s^2$.]
"In general we look for a new law by the following process. First we guess it. Then we compare the consequences of the guess to see what would be implied if this law that we guessed is right. Then we compare the result of the computation to nature, with experiment or experience, compare it directly with observation, to see if it works. If it disagrees with experiment it is wrong.

In that simple statement is the key to science. It does not make any difference how beautiful your guess is. It does not make any difference how smart you are, who made the guess, or what his name is — if it disagrees with experiment it is wrong.”

/R.P. Feynman/
Principles for Buda-Lund hydro model

- Analytic expressions for all the observables
- 3d expansion, local thermal equilibrium, symmetry
- Goes back to known exact hydro solutions:
  - nonrel, Bjorken, and Hubble limits, 1+3 d ellipsoids
  - but phenomenology, extrapolation for unsolved cases
- Separation of the Core and the Halo
  - Core: perfect fluid dynamical evolution
  - Halo: decay products of long-lived resonances
- Missing links: phenomenology needed
  - search for accelerating ellipsoidal rel. solutions
  - first accelerating rel. solution: nucl-th/0605070
Hydro scaling of Bose-Einstein/HBT radii

\[ \frac{1}{R_{\text{eff}}^2} = \frac{1}{R_{\text{geom}}^2} + \frac{1}{R_{\text{thrm}}^2} \]
and \[ \frac{1}{R_{\text{thrm}}^2} \sim m_t \]

intercept is nearly 0, indicating \[ \frac{1}{R_G^2} \sim 0, \]
thus \( \mu(x)/T(x) = \text{const}! \)

reason for success of thermal models @ RHIC!
A useful analogy

Fireball at RHIC ↔ our Sun

- Core ↔ Sun
- Halo ↔ Solar wind
- $T_{0,\text{RHIC}} \sim 210 \text{ MeV} \leftrightarrow T_{0,\text{SUN}} \sim 16 \text{ million K}$
- $T_{\text{surface,RHIC}} \sim 100 \text{ MeV} \leftrightarrow T_{\text{surface,SUN}} \sim 6000 \text{ K}$
Buda-Lund hydro model

The general form of the emission function:

\[
S_c(x, p) d^4x = \frac{g}{(2\pi)^3} \frac{p^\mu d^4\Sigma_\mu(x)}{\exp \left( \frac{p^\nu u_\nu(x)}{T(x)} - \frac{\mu(x)}{T(x)} \right)} + s_q
\]

Calculation of observables with core-halo correction:

\[
N_1(p) = \frac{1}{\sqrt{\lambda_\star}} \int d^4x S_c(p, x)
\]

\[
C(Q, p) = 1 + \left| \frac{\tilde{S}(Q, p)}{\tilde{S}(0, p)} \right|^2 = 1 + \lambda_\star \left| \frac{\tilde{S}_c(Q, p)}{\tilde{S}_c(0, p)} \right|^2
\]

Assuming profiles for flux, temperature, chemical potential and flow.
The generalized Buda-Lund model

The original model was for axial symmetry only, central coll.
In its general hydrodynamical form:

Based on 3d relativistic and non-rel solutions of perfect fluid dynamics:

\[ S_c(x, \mu) d^4 x = \frac{g}{(2\pi)^3} \exp \left( \frac{p^\nu u_\nu(x)}{T(x)} - \frac{\mu(x)}{T(x)} \right) + s_q \]

Have to assume special shapes:

Generalized Cooper-Frye prefactor:

\[ p^\mu d^4 \Sigma_\mu(x) = p^\mu u_\mu(x) H(\tau) d^4 x \]

Four-velocity distribution:

\[ u^\mu = (\gamma, \sinh \eta_x, \sinh \eta_y, \sinh \eta_z) \]

Temperature:

\[ \frac{1}{T(x)} = \frac{1}{T_0} \left(1 + \frac{T_0 - T_s}{T_s} s\right) \left(1 + \frac{T_0 - T_e}{T_e} \frac{r^2}{2 \Delta \tau^2}\right) \]

\[ \frac{\mu(x)}{T(x)} = \frac{\mu_0}{T_0} - s \]

Fugacity:

\[ s = \frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2} + \frac{r_z^2}{Z^2} \]
Buda-Lund model is based on *fluid dynamics*

First formulation: parameterization based on the flow profiles of
- Zimanyi-Bondorf-Garpman non-rel. exact sol.
- Bjorken rel. exact sol.
- Hubble rel. exact sol.

Remarkably successful in describing h+p and A+A collisions at CERN SPS and at RHIC

led to the discovery of *an incredibly rich family* of parametric, *exact solutions* of
- non-relativistic, perfect hydrodynamics
- imperfect hydro with bulk + shear viscosity + heat conductivity
- relativistic hydrodynamics, finite dn/dη and initial acceleration
- all cases: with temperature profile!

Further research: relativistic ellipsoidal exact solutions with acceleration and dissipative terms
Scaling predictions: Buda-Lund hydro

\[ T_x = T_0 + \overline{m}_t \frac{\dot{X}^2}{T_0 + \overline{m}_t a^2}, \]

\[ \overline{m}_t = m_t \cosh(\eta_s - y). \]

\[ \nu_2 = \frac{I_1(w)}{I_0(w)} \]

\[ w = \frac{E_K}{2T_*} \varepsilon \]

\[ E_K = \frac{p_t^2}{2\overline{m}_t} \]

Inverse of the HBT radii increase linearly with mass analysis shows that they are asymptotically the same

Relativistic correction: \( m \to m_t \)

hep-ph/0108067, nucl-th/0206051

- Slope parameters increase linearly with transverse mass
- Elliptic flow is same universal function.
- Scaling variable \( w \) is prop. to generalized transv. kinetic energy and depends on effective slope diffs.

\[ \varepsilon = \frac{T_x - T_y}{T_x + T_y}. \]

\[ \frac{1}{T_*} = \frac{1}{2} \left( \frac{1}{T_x} + \frac{1}{T_y} \right). \]

\[ \frac{1}{R_{i,i}^2} = \frac{B(x_s, p)}{B(x_s, p) + s_q} \left( \frac{1}{X_i^2} + \frac{1}{R_T^2} \right) \]

\[ \frac{1}{R_{T,i}^2} = \frac{m_t}{T_0} \left( \frac{a^2}{X_i^2} + \frac{\dot{X}_i^2}{X_i^2} \right) \]
Hydro scaling of slope parameters

Buda-Lund hydro prediction:  

\[ T_{*,i} = T_0 + m_t \dot{X}_i^2 \frac{T_0}{T_0 + m_t a^2} \]

Exact non-rel. hydro solution:

\[ T'_x = T_f + m\dot{X}_f^2, \]
\[ T'_y = T_f + m\dot{Y}_f^2, \]
\[ T'_z = T_f + m\dot{Z}_f^2. \]

PHENIX data:
Buda-Lund hydro and Au+Au@RHIC

BudaLund v1.5 hydro fits to 200 AGeV Au+Au

Spectra

\[ \chi^2/NDF = 126/208 \]

(stat + syst errors added in quadrature)

nucl-th/0311102, nucl-th/0207016, nucl-th/0403074

T. Csörgő @ Zimányi'75, Budapest, 2007/7/2
Confirmation

see nucl-th/0310040 and nucl-th/0403074,
R. Lacey@QM2005/ISMD 2005
A. Ster @ QM2005.

Universal scaling
PHOBOS $v_2(\eta \leq w)$
Scaling and scaling violations

Universal hydro scaling breaks where scaling with number of VALENCE QUARKS sets in, \( p_t \sim 1-2 \) GeV

Fluid of QUARKS!!

R. Lacey and M. Oldenburg, proc. QM’05
A. Taranenko et al, PHENIX: nucl-ex/0608033
Geometrical & thermal & HBT radii

3d analytic hydro: exact time evolution

- Geometrical size (fugacity ~ const)
- Thermal sizes (velocity ~ const)
- HBT sizes (phase-space density ~ const)

HBT dominated by the smaller of the geometrical and thermal scales

- nucl-th/9408022, hep-ph/9409327

HBT radii approach a constant of time
HBT volume becomes spherical
HBT radii -> thermal ~ constant sizes

- hep-ph/0108067, nucl-th/0206051

Animation by Máté Csanád
Exact scaling laws of non-rel hydro

\[ T'_x = T_f + m \dot{X}_f^2, \]
\[ T'_y = T_f + m \dot{Y}_f^2, \]
\[ T'_z = T_f + m \dot{Z}_f^2. \]

- Slope parameters increase linearly with mass
- Elliptic flow is a universal function and variable \( w \) is proportional to transverse kinetic energy and depends on slope differences.

\[ \nu_2 = \frac{I_1(w)}{I_0(w)} \]
\[ w = \frac{k_t^2}{4m} \left( \frac{1}{T'_y} - \frac{1}{T_x} \right), \]
\[ w = \frac{E_K}{2T*} \varepsilon \]

Inverse of the HBT radii increase linearly with mass analysis shows that they are asymptotically the same

Relativistic correction: \( m \to m_t \)

hep-ph/0108067, nucl-th/0206051

\[ R'_{x}^{-2} = X_f^{-2} \left( 1 + \frac{m}{T_f} \dot{X}_f^2 \right), \]
\[ R'_{y}^{-2} = Y_f^{-2} \left( 1 + \frac{m}{T_f} \dot{Y}_f^2 \right), \]
\[ R'_{z}^{-2} = Z_f^{-2} \left( 1 + \frac{m}{T_f} \dot{Z}_f^2 \right). \]
Some analytic Buda-Lund results

HBT radii widths:

\[
\frac{1}{R_{i,i}^2} = \frac{B(x_s, p)}{B(x_s, p) + s_q} \left( \frac{1}{X_i^2} + \frac{1}{R_{T,i}^2} \right)\]

\[
\frac{1}{R_{T,i}^2} = \frac{m_t}{T_0} \left( \frac{a^2}{X_i^2} + \frac{\dot{X}_i^2}{X_i^2} \right)\]

Slopes, effective temperatures:

\[
\frac{1}{T_*} = \frac{1}{2} \left( \frac{1}{T_x} + \frac{1}{T_y} \right).
\]

Flow coefficients are universal:

\[
v_{2n} = \frac{I_n(w)}{I_0(w)}
\]

\[
v_{2n+1} = 0
\]

\[
\varepsilon = \frac{T_x - T_y}{T_x + T_y}.
\]

\[
w = \frac{E_K}{2T_*} \varepsilon
\]

\[
E_K = \frac{p_t^2}{2m_t}
\]
Hydro scaling of elliptic flow

Extended longitudinal scaling: $v_2$

A surprising scaling!

Not an initial state effect

Scaling reproduced by the Buda-Lund parametrization of the emitting source.

G. Veres, PHOBOS data, proc QM2005
Hydro scaling of $v_2$ and $\sqrt{s}$ dependence

**PHOBOS, nucl-ex/0406021**
Universal scaling and $v_2$(centrality, $\eta$)

**PHOBOS, nucl-ex/0407012**
Universal v2 scaling and PID dependence

PHENIX, nucl-ex/0305013
Universal scaling and fine structure of $v_2$

STAR, nucl-ex/0409033
Geometrical sizes keep on increasing. Expansion velocities tend to constants. HBT radii $R_x$, $R_y$, $R_z$ approach a direction independent constant.

Slope parameters tend to direction dependent constants.

General property, independent of initial conditions - a beautiful exact result.