Self-Similar Solution of the three dimensional compressible Navier-Stokes Equations

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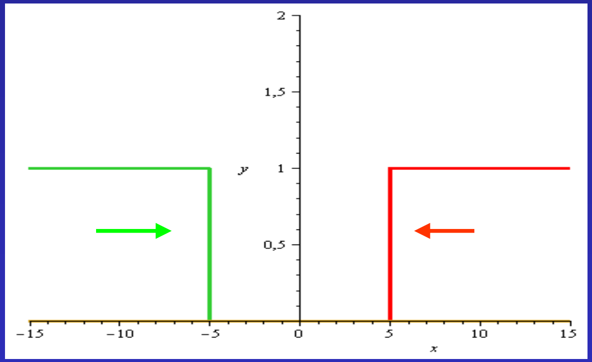
Outline

- **Solutions of PDEs** *self-similar, traveling wave*
- *non-compressible Navier-Stokes equation with my 3D Ansatz & geometry* *my solution + other solutions, replay from last year*
- *compressible Navier-Stokes equation* *with the same Ansatz, some part of the solutions, traveling wave analysis*
- *Summary & Outlook* *more EOS & viscosity functions*

Physically important solutions of PDEs

- *Travelling waves: arbitrary wave fronts*

  u(x,t) ~ g(x-ct), g(x+ct)



- *Self-similar*

$$u(x,t) = t^{-\alpha} f(x/t^{\beta})$$

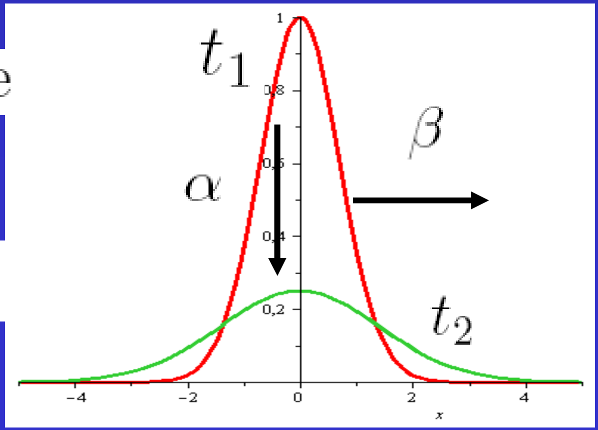
Sedov, Barenblatt, Zeldovich

α and β are of primary physical importance

α represents the rate of decay

β is the rate of spread (or contraction if $\beta < 0$)

$t_1 < t_2$ *in Fourier heat-conduction*



The non-compressible Navier-Stokes equation

$$\nabla \mathbf{v} = 0,$$

$$\mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v} = \nu \triangle \mathbf{v} - \frac{\nabla p}{\rho} + a$$

3 dimensional cartesian coordinates,
Euler description
v velocity field, p pressure, a external field
ν kinematic viscosity, ρ constant density
Newtonian fluid

$\mathbf{v}(x,y,z,t) = u(x,y,z,t), v(x,y,z,t), w(x,y,z,t)$ $p(x,y,z,t)$

Consider the most general case

$$u_x + v_y + w_z = 0$$

$$u_t + uu_x + vu_y + wu_z = \nu(u_{xx} + u_{yy} + u_{zz}) - \frac{p_x}{\rho}$$

$$v_t + uv_x + vv_y + wv_z = \nu(v_{xx} + v_{yy} + v_{zz}) - \frac{p_y}{\rho}$$

$$w_t + uw_x + vw_y + ww_z = \nu(w_{xx} + w_{yy} + w_{zz}) - \frac{p_z}{\rho} + a.$$

just to write out all the coordinates

My 3 dimensional Ansatz

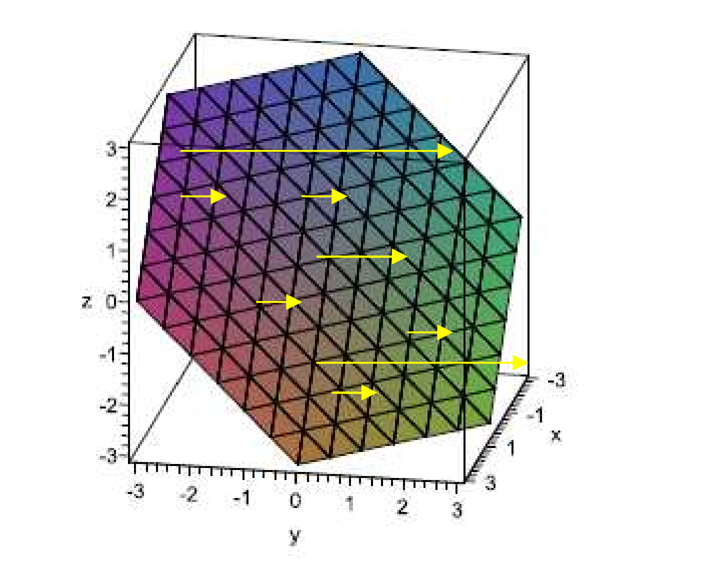
$$u(x,t) = t^{-\alpha} f(x/t^{\beta})$$

$$u(x,y,z,t) = t^{-\alpha} f\left(\frac{F(x,y,z)}{t^{\beta}}\right) := t^{-\alpha} f\left(\frac{x+y+z}{t^{\beta}}\right) := t^{-\alpha} f(\omega)$$

$$F(x,y,z) = x+y+z = 0$$

A more general function does not work for N-S

$$u(x,y,z,t) = t^{-\alpha} f\left(\frac{\sqrt{x^2+y^2+z^2-a}}{t^{\beta}}\right)$$



The graph of the $x+y+z=0$ plane.

Geometrical meaning:
all v components with coordinate constrain x+y+z=0 lie in a plane = equivalent

The final applied forms

$$u(x,y,z,t) = t^{-\alpha} f\left(\frac{x+y+z}{t^{\beta}}\right), \quad v(x,y,z,t) = t^{-\gamma} g\left(\frac{x+y+z}{t^{\delta}}\right)$$

$$w(x,y,z,t) = t^{-\epsilon} h\left(\frac{x+y+z}{t^{\zeta}}\right), \quad p(x,y,z,t) = t^{-\eta} l\left(\frac{x+y+z}{t^{\theta}}\right)$$

The obtained ODE system

$$
\begin{aligned}
f'(\omega)+g'(\omega)+h'(\omega) &= 0\\
-\frac{1}{2}f(\omega)-\frac{1}{2}\omega f'(\omega)+[f(\omega)+g(\omega)+h(\omega)]f'(\omega) &= 3\nu f''(\omega)-\frac{l'(\omega)}{\rho}\\
-\frac{1}{2}g(\omega)-\frac{1}{2}\omega g'(\omega)+[f(\omega)+g(\omega)+h(\omega)]g'(\omega) &= 3\nu g''(\omega)-\frac{l'(\omega)}{\rho}\\
-\frac{1}{2}h(\omega)-\frac{1}{2}\omega h'(\omega)+[f(\omega)+g(\omega)+h(\omega)]h'(\omega) &= 3\nu h''(\omega)-\frac{l'(\omega)}{\rho}+a.
\end{aligned}
$$

as constraints we got for the exponents:

$\alpha,\beta,\gamma,\delta,\epsilon,\zeta,\theta=1/2, \qquad \eta=1$ **universality relations**

$$
u(x,y,z,t)=t^{-1/2}f\left(\frac{x+y+z}{t^{1/2}}\right)=t^{-1/2}f(\omega),\quad v(x,y,z,t)=t^{-1/2}g(\omega),
$$
$$
w(x,y,z,t)=t^{-1/2}h(\omega),\quad p(x,y,z,t)=t^{-1}l(\omega),
$$

Continuity eq. helps us to get an additional constraint:

$$
f(\omega)+g(\omega)+h(\omega)=c,\quad \mathit{and}\quad f''(\omega)+g''(\omega)+h''(\omega)=0
$$

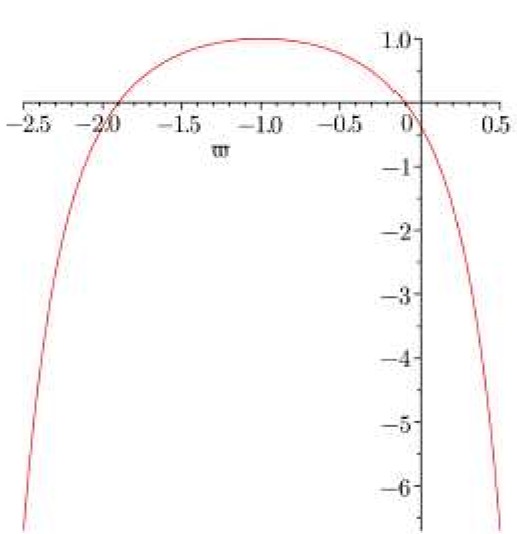
c is prop. to mass flow rate

Solutions of the ODE

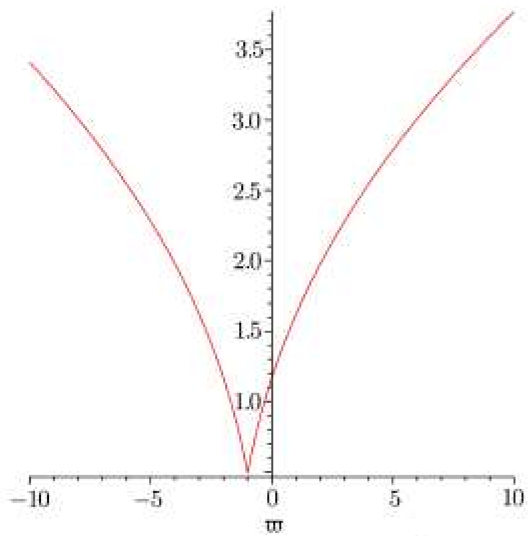
a single Eq. remains

$$9\nu f''(\omega) - 3(\omega + c) f'(\omega) + \frac{3}{2} f(\omega) - \frac{c}{2} + a = 0.$$

$$f(\omega) = c_1 \cdot KummerU\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\omega + c)^2}{6\nu}\right) + c_2 \cdot KummerM\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\omega + c)^2}{6\nu}\right) + \frac{c}{3} - \frac{2a}{3}$$



The KummerM$(-1/4, 1/2, (\omega + c)^2/6\nu)$ function for $c = 1$ and $\nu = 0.1$.



The KummerU$(-1/4, 1/2, (\omega + c)^2/6\nu)$ function for $c = 1$ and $\nu = 0.1$.

Kummer is spec.

$(a)_n$ is the Pochhammer symbol

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), (a)_0 = 1$$

$$M(a, b, z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \cdots + \frac{(a)_n z^n}{(b)_n n!},$$

$$U(a, b, z) = \frac{\pi}{\sin(\pi b)}\left[\frac{M(a, b, z)}{\Gamma(1 + a - b)\Gamma(b)} - z^{1-b}\frac{M(1 + a - b, 2 - b, z)}{\Gamma(a)\Gamma(2 - b)}\right]$$

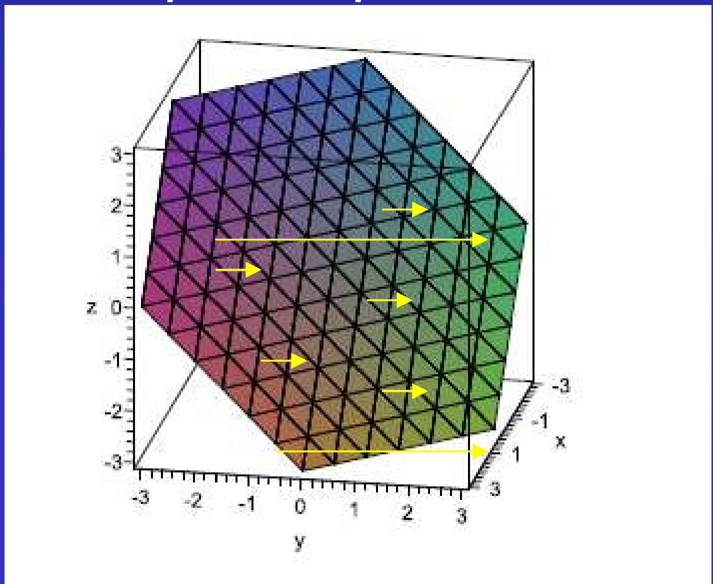
Solutions of N-S

$$u(x,y,z,t) = t^{-1/2} f(\omega) = t^{-1/2}\left[c_1 \cdot \mathrm{KummerU}\left(\frac{-1}{4}, \frac{1}{2}, \frac{((x+y+z)/t^{1/2}+c)^2}{6\nu}\right)\right] + t^{-1/2}\left[c_2 \cdot \mathrm{KummerM}\left(-\frac{1}{4}, \frac{1}{2}, \frac{((x+y+z)/t^{1/2}+c)^2}{6\nu}\right) + \frac{c}{3} - \frac{2a}{3}\right]$$

analytic only for one velocity component ☹

Geometrical explanation:

all v components with coordinate constrain x+y+z=0 lie in a plane = equivalent



Naver-Stokes makes a dynamics of this plane

getting a multi-valued surface

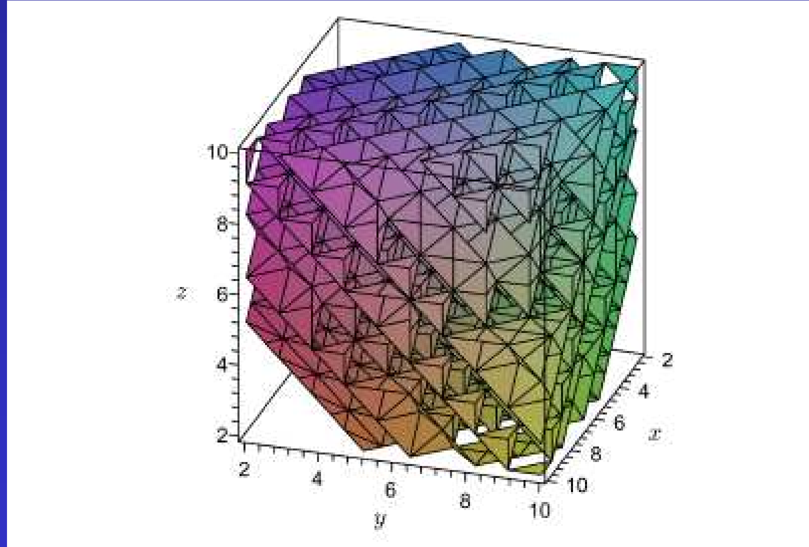


Fig. 5 The implicit plot of the self-similar solution Eq. (17). Only the KummerU function is presented for $t = 1$, $c_1 = 1$, $c_2 = 0$, $a = 0$, $c = 1$, and $\nu = 0.1$.

I.F. Barna http://arxiv.org/abs/1102.5504
Commun. Theor. Phys. 56 (2011) 745-750

for fixed space it decays in time
t^-1/2 KummerT or U(1/t) ☺

Other analytic solutions

Without completeness, usually from Lie algebra studies
all are for non-compressible N-S

W. I. Fushchich, W. M. Shtelen and S. L. Slavutsky J. Phys. A: Math. Gen. 24 (1990) 971.

$\omega = z/\sqrt{t}$

Presented 19 various solutions one of them is:

$$u(z,t) = \frac{f(\omega)}{\sqrt{t}}, \quad v(y,z) = \frac{g(\omega)}{\sqrt{t}} + \frac{y}{t}, \quad w(z,t) = \frac{h(\omega)}{\sqrt{t}}, \quad p(t,z) = \frac{l(\omega)}{\sqrt{t}}$$

V. Grassi, R.A. Leo, G. Soliani and P. Tempesta, Physica **286** (2000) 79

Ansatz: $U_1 = Y(y)T(z)\Phi(t).$

Solutions are Kummer functions as well

JOURNAL OF MATHEMATICAL PHYSICS **50**, 083101 (2009)

Analytical solutions to the Navier–Stokes equations with density-dependent viscosity and with pressure

Ling Hei Yeung[1,a)] and Yuen Manwai[2,b)]

"Only" Radial solution for 2 or 3 D
Ansatz:

$$\rho(t,r) = \frac{f(r/a(t))}{a(t)^N}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)}r,$$

Nonlinear Instability of the Solutions of the Navier–Stokes Equations: Formulas for Constructing Exact Solutions

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Received May 5, 2009

Ansatz:

$$V_n = f_n(z,t)x + g_n(z,t)y, \quad n = 1, 2; \quad V_3 = F(z,t).$$

Ukrainian Mathematical Journal, Vol. 49, No. 9, 1997

ON NAVIER–STOKES FIELDS WITH LINEAR VORTICITY

G. V. Popovich and R. O. Popovich

$$\text{rot}\,\vec{u} = H(t)\vec{x} + \vec{k}(t).$$

$$\vec{u} = \nabla\varphi + \frac{1}{3}(H\vec{x})\times\vec{x} + \frac{1}{2}\vec{k}\times\vec{x},$$

Sedov, stationary N-S, only the angular part

$$v_r = \frac{\nu}{r}f(\theta), \quad v_\theta = \frac{\nu}{r}\varphi(\theta), \quad v_\lambda = \frac{\nu}{r}\psi(\theta), \quad U - \frac{p}{\rho} = \frac{\nu^2}{r^2}F(\theta).$$

The compressible Navier-Stokes eq.

$$\rho_t + div(\rho \mathbf{v}) = 0$$

$$\rho[\mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v}] = \nu_1 \triangle \mathbf{v} + \frac{\nu_2}{3} grad\, div\, \mathbf{v} - \nabla p + a; \qquad EOS\; p = \kappa \rho^n$$

3 dimensional cartesian coordinates,
Euler description, Newtonian fluid, politropic EOS (these can be changed later)
v velocity field, p pressure, a external field
ν_1, ν_2 **viscosities,** ρ **density** *(No temperature at this point)*

Consider the most general case: $\mathbf{v}(x,y,z,t) = u(x,y,z,t), v(x,y,z,t), w(x,y,z,t)$ $\rho(x,y,z,t)$

just write out all the coordinates:

$$\rho_t + \rho_x u + \rho_y v + \rho_z w + \rho(u_x + v_y + w_z) = 0$$

$$\rho[u_t + uu_x + vu_y + wu_z] - \nu_1(u_{xx} + u_{yy} + u_{zz}) - \frac{\nu_2}{3}(u_{xx} + v_{xy} + w_{xz}) + \kappa n \rho^{n-1}\rho_x = 0$$

$$\rho[v_t + uv_x + vv_y + wv_z] - \nu_1(v_{xx} + v_{yy} + v_{zz}) - \frac{\nu_2}{3}(u_{xy} + v_{yy} + w_{yz}) + \kappa n \rho^{n-1}\rho_y = 0$$

$$\rho[w_t + uw_x + vw_y + ww_z] - \nu_1(w_{xx} + w_{yy} + w_{zz}) - \frac{\nu_2}{3}(u_{xz} + v_{yz} + w_{zz}) + \kappa n \rho^{n-1}\rho_z = 0.$$

The applied Ansatz & Universality Relations

$$\rho(x,y,z,t) = t^{-\alpha} f\left(\frac{x+y+z}{t^{\beta}}\right) = t^{-\alpha} f(\eta) \quad u(x,y,z,t) = t^{-\delta} g(\eta),$$

$$v(x,y,z,t) = t^{-\epsilon} h(\eta), \quad w(x,y,z,t) = t^{-\omega} i(\eta).$$

Where all the exponents $\alpha, \beta, \delta, \epsilon, \omega$ are real numbers.

as constraints we got for the exponents:

universality relations

$$\alpha = \beta = \frac{2}{n+1} \qquad \& \qquad \delta = \epsilon = \omega = 2 - \frac{4}{n+1}.$$

Note, that n remains free, presenting some physics in the system, polytropic EOS

$$p = \kappa \rho^{n}$$

The obtained ODE system

The most general case, n is free

Continuity can be integrated

$$\alpha[f + f'\eta] = f'[g + h + i] + f[g' + h' + i'] \rightarrow \alpha f\eta = f[g + h + i] + c_0$$

$$f[-\delta g - \alpha\eta g' + gg' + hg' + ig'] = -\kappa n f^{n-1} f' + 3\nu_1 g'' + \frac{\nu_2}{3}[g'' + h'' + i'']$$

$$f[-\delta h - \alpha\eta h' + gh' + hh' + ih'] = -\kappa n f^{n-1} f' + 3\nu_1 h'' + \frac{\nu_2}{3}[g'' + h'' + i'']$$

$$f[-\delta i - \alpha\eta i' + gi' + hi' + ii'] = -\kappa n f^{n-1} f' + 3\nu_1 i'' + \frac{\nu_2}{3}[g'' + h'' + i'']$$

if $\delta = 0$ N-S can be intergated once, after some algebra getting an ODE of:

$$4\nu c_0 f' + 3\kappa f^3 + f^2[-c_0\eta - 4\nu + c_4] + c_0^2 f = 0$$

This is for the density

where $\nu = \nu_1 = \nu_2$ and c_4 is a new constant

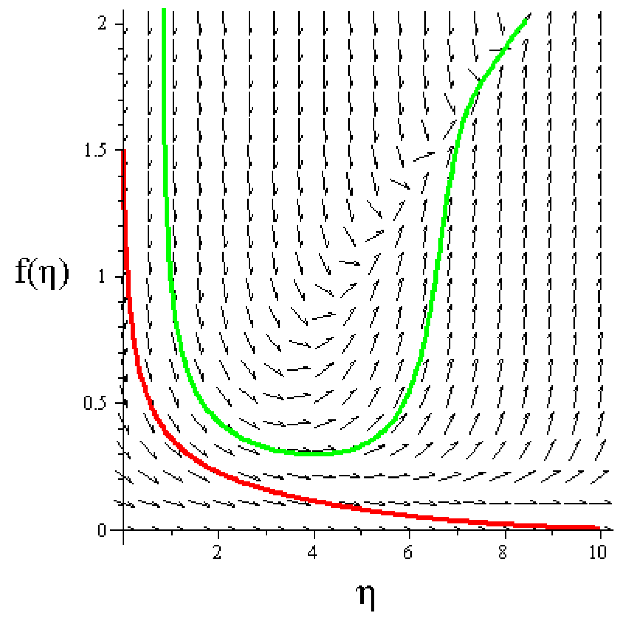
No analytic solutions exist , but the direction field can be investigated for reasonable parameters

The properties of the solutions

From the universality relations the global properties of the solutions are known

$$\delta = 0 \rightarrow \alpha = \beta = 1,\ n = 1,\ \rho = t^{-1} f\left(\frac{x+y+z}{t}\right),\ u = g\left(\frac{x+y+z}{t}\right),\ p = \kappa\rho$$

has decay & spreading — just spreading in time for fixed space



The small unit vectors represent the direction tangent of the numerical solution, the line is a numerical solution itself

$\delta \neq 0$ is under recent investigation

```
v := 0.4; c0 := 1; c4 := 1; k := 1.4;
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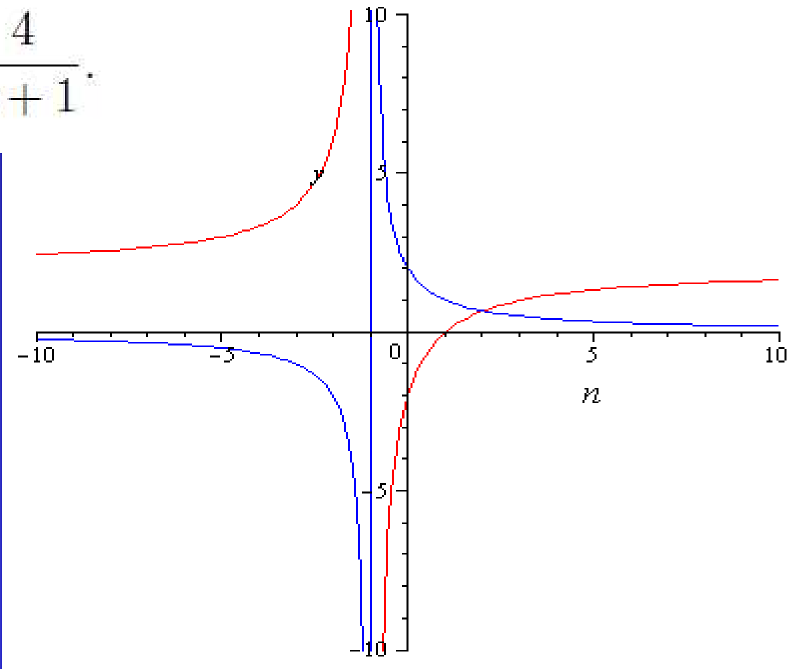
General properties of the solutions for other exponents

$$\rho(x,y,z,t) = t^{-\alpha} f\left(\frac{x+y+z}{t^{\beta}}\right) = t^{-\alpha} f(\eta) \qquad u(x,y,z,t) = t^{-\delta} g(\eta).$$

$$\alpha = \beta = \frac{2}{n+1} \qquad \& \qquad \delta = \epsilon = \omega = 2 - \frac{4}{n+1}.$$



There are different regimes for different ns

n > 1 all exponents are positive decaying, spreading solutions for speed and density
n = 1 see above
-1 ≤ n ≤ +1 decaying and spreading density & enhancing velocity in time
n ≠ -1
***n ≤ -1* sharpening and enhancing density & decaying and sharpening velocity**

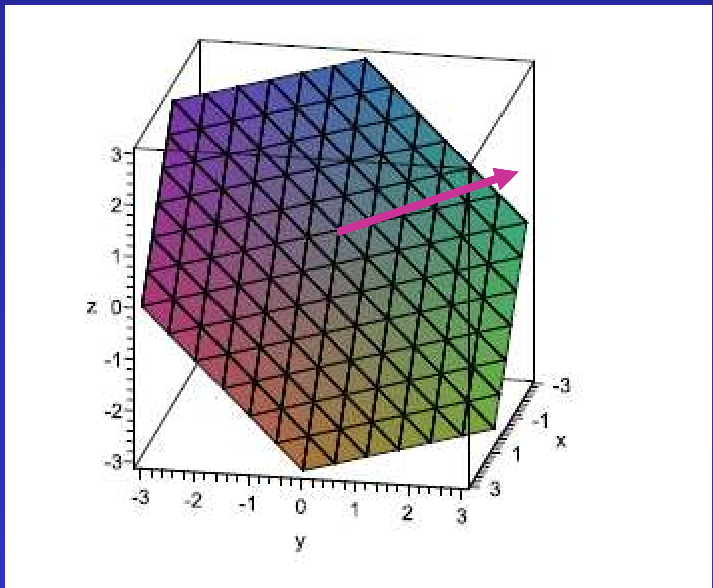
Relevant physics is for n >1 the analysis is in progress to see the shape functions

Traveling wave solutions

As a second Ansatz we may try to find traveling wave solutions of the NS system.

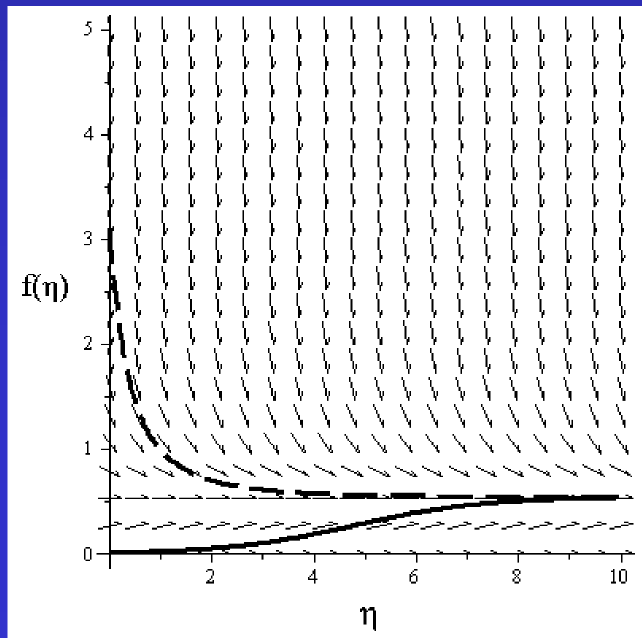
$$\rho(x,y,z,t) = f(x+y+z-Ct) = f(\eta),\ u = g(\eta),\ v = h(\eta),\ w = i(\eta).$$

where C is the wave velocity



After some algebra the next ODE can be obtained: (for n = 1)

$$-4\nu c_0 f' + 3\kappa f^3 + f^2[c_4 - c_0 c] + c_0^2 f = 0$$



Detailed analysis is in progress

Summary & Outlook

- *The self-similar Ansatz is presented as a tool for non-linear PDA*
- *The non-compressible & compressible N-S eq. is investigated and the results are discussed*

- *An in-depth analysis is in progress for further EOS, more general viscosity functions could be analysed like the Ostwald-de Waele power law*

$$\tau_{yx} = K\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}$$

K: *consistency index* for non-Newtonian viscosity

$$\left(\gamma = \frac{K}{\rho}\right)$$

$n > 0$: *power-law index* $\quad 0 < n < 1$ *pseudoplastic fluids* or shear-thinning fluids

$n = 1$ *Newtonian fluid*

$n > 1$ *dilatant* or shear-thickening fluids-

- *To investigate some relativistic cases, which may attracts the interest of the recent community*

Thank you for your attention!

Questions, Remarks, Comments?…