Self-similar solutions for classical heat-conduction mechanisms

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Outline

- **Motivation** *(infinite propagation speed with the diffusion/heat equation)*
- **A way-out** *(Cattaneo equ. OR using a hyperbolic first order PDE system*
- **Derivation of a self-similar telegraph-type equation & analysing the properties**
- **Non-continuous solutions for the hyperbolic system for heat propagation**
- **Summary**

Ordinary diffusion/heat conduction equation

$$\mathbf{q} = -k\nabla U(x,t), \quad \nabla \mathbf{q} = -\gamma \frac{\partial U(x,t)}{\partial t}$$

U(x,t) temperature distribution
Fourier law + conservation law

- $$\begin{cases} u_t(x,t) - k u_{xx}(x,t) = 0 & -\infty < x < \infty, \quad 0 < t < \infty \\ u(x,t=0) = \delta(x) \end{cases}$$ *parabolic PDA, no time-reversal sym.*

- *strong maximum principle ~ solution is smeared out in time*
- *the fundamental solution:* $$\Phi(x,t) = \int \frac{1}{\sqrt{4\pi kt}} exp\left(-\frac{x^2}{4kt}\right)$$
- *general solution is:*

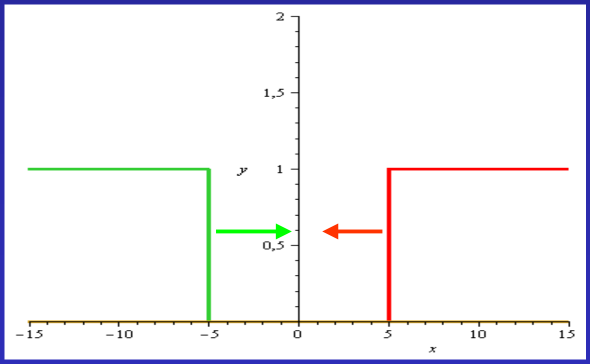
$$u(x,t) = \int \Phi(x-y,t) g(y) dy$$ $$u(x,0) = g(x) \;\; for -\infty < x < \infty \;\; and \;\; 0 < t < \infty$$

- *kernel is non compact = inf. prop. speed*
- *Problem from a long time ☹*
- *But have self-similar solution ☺* $$u(x,t) = t^{-\alpha} f(x/t^{\beta})$$

Important kind of PDA solutions

- Travelling waves: arbitrary wave fronts

  u(x,t) ~ g(x-ct), g(x+ct)



- Self-similar solutions

$$u(x,t) = t^{-\alpha} f(x/t^{\beta})$$

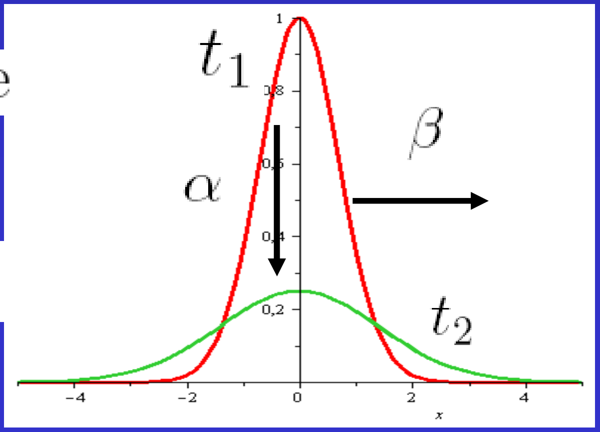
Sedov, Barenblatt, Zeldovich

α and β are of primary physical importance

α represents the rate of decay

β is the rate of spread (or contraction if $\beta < 0$)

$t_1 < t_2$



Cattaneo heat conduction equ.

$$\tau\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -k\nabla T(x,t)$$

Cattaneo heat conduction law, new term $\tau\frac{\partial \mathbf{q}}{\partial t}$

$$\nabla \mathbf{q} = -\gamma\frac{\partial T(x,t)}{\partial t}$$

Energy conservation law

$$\frac{\partial^2 T(x,t)}{\partial t^2} + \frac{1}{\tau}\frac{\partial T(x,t)}{\partial t} = c^2\nabla^2 T(x,t)$$

T(x,t) temperature distribution
q heat flux
k effective heat conductivity
γ heat capacity
τ relaxation time
Telegraph equation(exists in Edyn., Hydrodyn.)

$c = \sqrt{k/\tau\gamma}$ is the sound of the transmitted heat wave.

General properties of the telegraph eq. solution

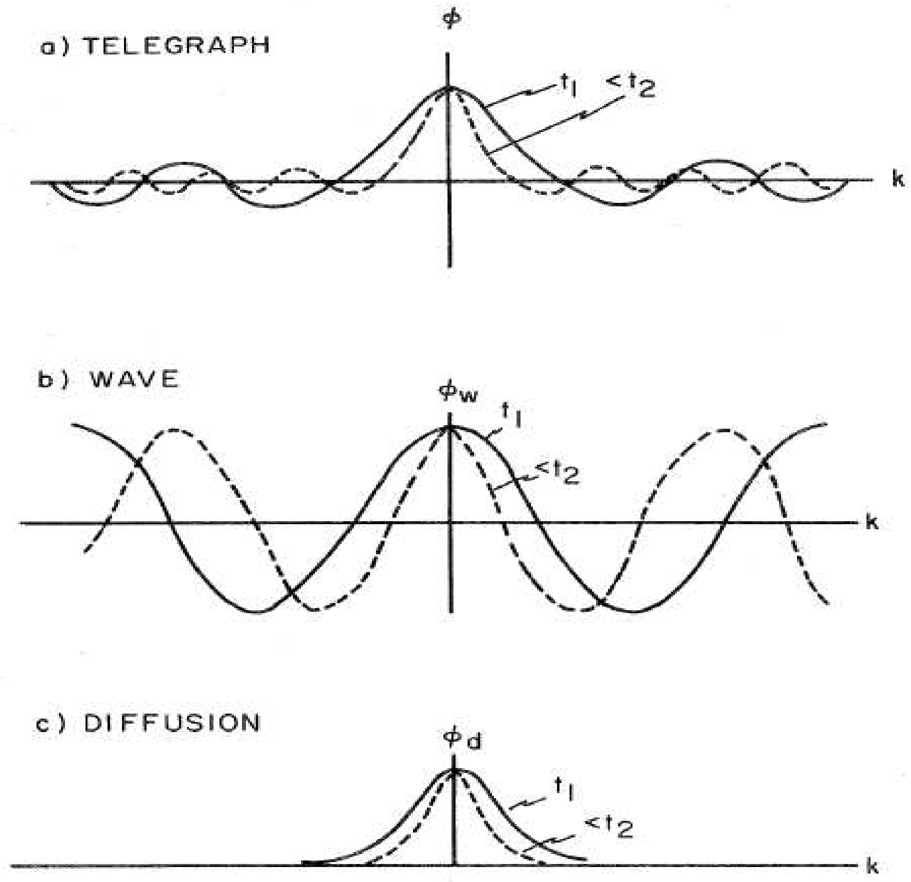


Figure 1. Behaviors in wavenumber space
a) Telegraph equation
b) Wave equation
c) Diffusion equation

decaying travelling waves

$$T(x,t) \propto e^{-\lambda t} f(x - ct)$$

$$T(x,t) = e^{-\lambda t} I_0\left(\frac{\lambda}{2c}\sqrt{(c^2t^2 - x^2)}\right)$$

Bessel function

Problem:

1) no self-similar diffusive solutions $T(x,t) = t^{-\alpha} f(\eta)\ \eta = \frac{x}{t^\beta}$

2) oscillations, T<0 ?
maybe not the best eq.

Our alternatives

- Way 1
- Def. new kind of Cattaneo law (with physical background)

  ⟶ new telegraph-type equation

  ⟶ with self-similar and compact solutions ☺

- Way 2

  instead of a 2^{nd} order parabolic(?) PDA

  use a first order hyperbolic PDA system with 2 Eqs.

  these are not equivalent!!!

  ⟶ non-continuous solutions and also self-similar

General derivation for heat conduction law (Way 1)

$$\tau\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -k\nabla T(x,t)$$

Cattaneo heat conduction law, there is a general way to derive

$$q = -\int_{-\infty}^{t} Q(t-t')\frac{\partial T(x,t')}{\partial x}\,\mathrm{d}t'$$

T(x,t) temperature distribution
q heat flux

Joseph D D and Preziosi L 1989 *Rev. Mod. Phys.* **61** 41
Joseph D D and Preziosi L 1990 *Rev. Mod. Phys.* **62** 375

$$Q(t-t') = \frac{k\tau^l}{(t-t'+\omega)^l}$$

the kernel can have microscopic interpretation

$$\epsilon\frac{\partial^2 T(x,t)}{\partial t^2} + \frac{a}{t}\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2}$$

telegraph-type time dependent eq. with self sim. solution

$$T(x,t) = t^{-\alpha} f(\eta) \quad with \quad \eta = \frac{x}{t^{\beta}}$$

Solutions

There are differential eqs. for $f(\eta)$ only for $\alpha = \beta = +1$ or for $\alpha = -2 \text{ and } \beta = +1$

$\alpha = \beta = +1$ a total difference = conserved quantity

$$\epsilon(\eta^2 f(\eta))'' - \alpha(\eta f(\eta))' = f''(\eta)$$

$$\epsilon(\eta^2 f(\eta))' - \alpha\eta f(\eta) = f'(\eta) + c_1$$

There are two different solutions:

$c_1 = 0$ physically relevant solution, compact support with vanishing derivatives at the boarders

I.F. Barna and R. Kersner, http://arxiv.org/abs/1002.099

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$c_1 \neq 0.$ Not so nice ☹

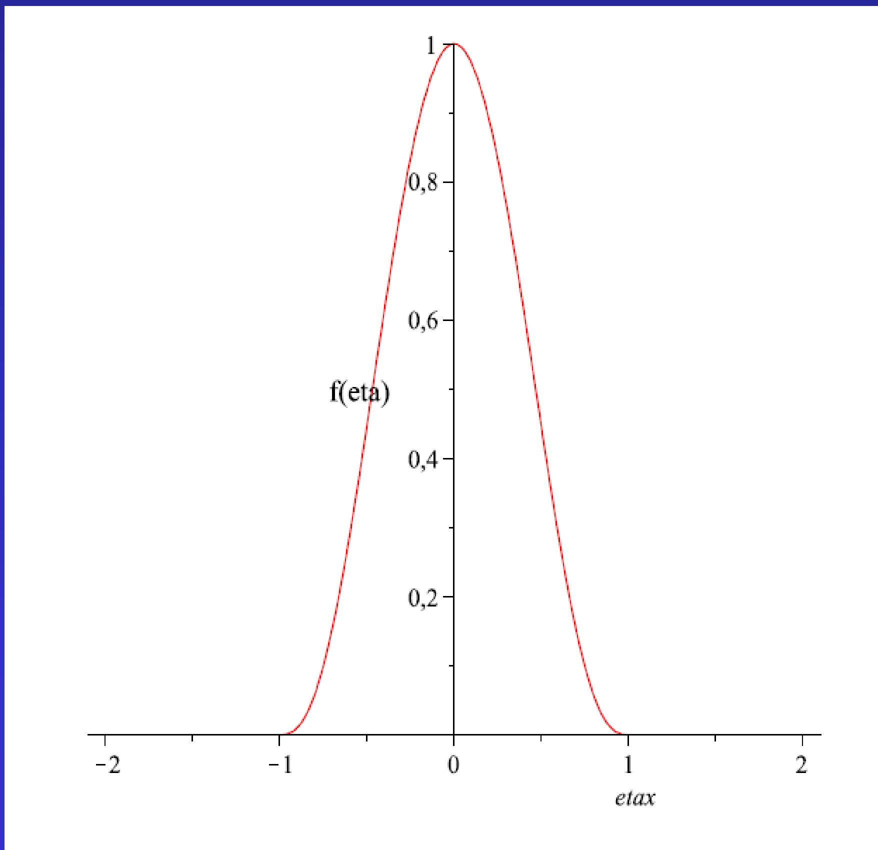
Solutions

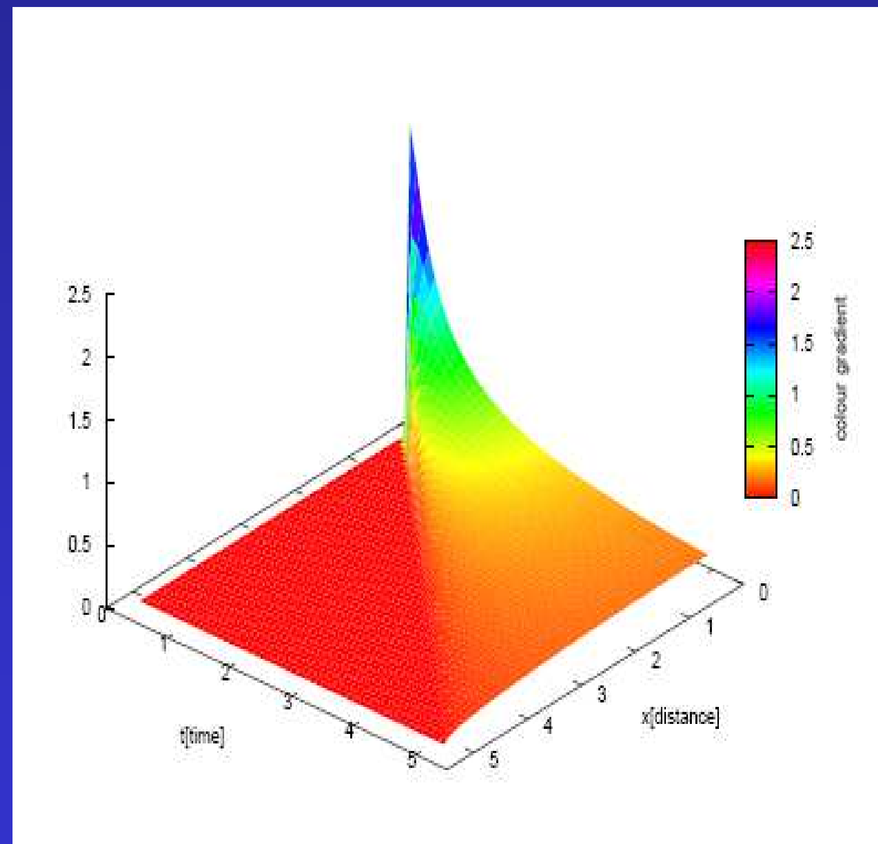
$c_1 = 0$

$a = 4.1, \epsilon = 1$

$$f(\eta) = \left(1 - \epsilon\eta^2\right)^{\frac{a}{2\epsilon} - 1}$$

$$T(x,t) = \frac{1}{t}\left(1 - \epsilon\frac{x^2}{t^2}\right)^{\frac{a}{2\epsilon} - 1}$$





2 Important new feature: the solution is a product of 2 travelling wavefronts

$if\ a > 4\epsilon,\ f'(\eta) = 0$ no flux conservation problem

$T(x,t) \sim U(x - ct)U(x + ct)$

Solutions

$c_1 \neq 0.$

$$f(\eta) = \quad (\epsilon\eta^2 - 1)^{\frac{a}{2\epsilon}-1}\left[c_1\{signum(\epsilon\eta^2 - 1)\}^{\frac{a}{2\epsilon}-1}\{-signum(\epsilon\eta^2 - 1)\}^{\frac{a}{2\epsilon}-1} \eta F(1/2, a/2/\epsilon; 3/2; \epsilon\eta^2) + c_2\right]$$

- *where F(a,b;c;z) is the hypergeometric function*

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \qquad a,b,c,z \;\; \epsilon\, C$$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \qquad (a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1) \quad n = 1,2,3,\ldots$$

- *some elementary functions can be expressed via F*
- *In our case if* $\frac{a}{2\epsilon}$ *is Integer or Half-Integer are important the 4 basic cases:*

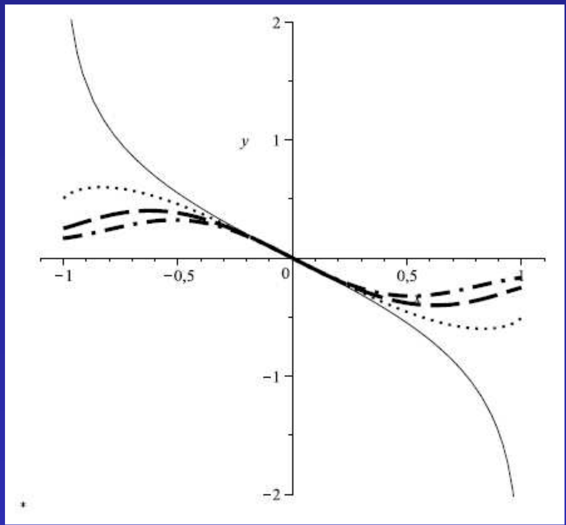
$$\frac{a}{2\epsilon} = 0, \quad F\left(0, \frac{1}{2}; \frac{3}{2}; \epsilon\eta^2\right) = 1$$

$$\frac{a}{2\epsilon} = 1, \quad F\left(1, \frac{1}{2}; \frac{3}{2}; \epsilon\eta^2\right) = \frac{1}{2\sqrt{\epsilon}\eta} \ln\left(\frac{1+\sqrt{\epsilon}\eta}{1-\sqrt{\epsilon}\eta}\right)$$

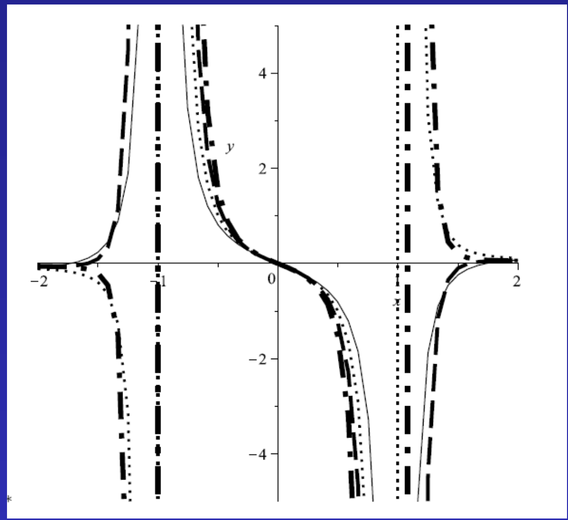
$$\frac{a}{2\epsilon} = \frac{1}{2}, \quad F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \epsilon\eta^2\right) = \frac{arccos(\sqrt{\epsilon}\eta)}{\sqrt{\epsilon}\eta}$$

$$\frac{a}{2\epsilon} = \frac{3}{2}, \quad F\left(\frac{3}{2}, \frac{1}{2}; \frac{3}{2}; \epsilon\eta^2\right) = \frac{1}{(1-\epsilon\eta^2)}$$

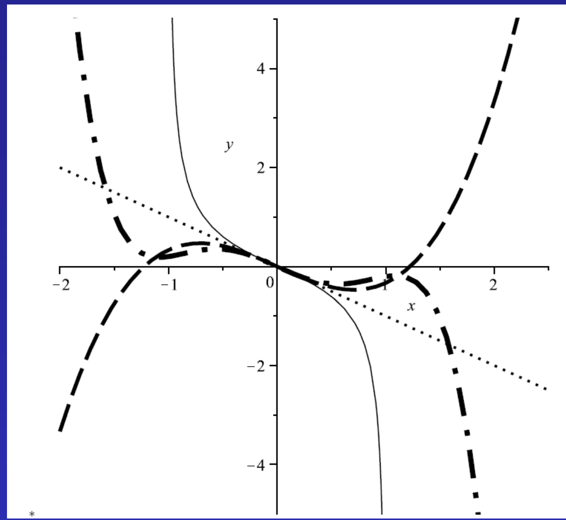
Solutions



negative integer $a/2\epsilon$ values.



positive integer $a/2\epsilon$ values.



positive half-integer $a/2\epsilon$ values.

not-so-nice solutions, non-compact no-finite derivatives
just have a rich mathematical structure

I.F. Barna and R. Kersner http://arxiv.org/abs/1009.6085
Adv. Studies Theor. Phys. 5, (2011) 193

Self-similar, non-continous shock-wave behaviour for heat-propagation (Way 2)

$$\frac{\partial q(r,t)}{\partial t} = -\frac{q}{\tau} - \frac{\kappa}{\tau}\frac{\partial T(r,t)}{\partial r}$$
$$c_0 \frac{\partial T(r,t)}{\partial t} = -\frac{\partial q(r,t)}{\partial r} - \frac{q(r,t)}{r}$$

general Cattaneo heat conduction law,
+ cylindrically symmetric conservation law

$\kappa = \kappa(T) = \kappa_0 T^{\omega}$ heat conduction coefficient temperature dependent (e.q. plasmas)

$\tau = \tau(T) = \tau_0 T^{\epsilon}$ relaxation time also temperature dependent (e.q. plasmas)

using the first oder PDA system (not second order) looking for self-similar solutions in the form

$$T(r,t) = t^{-\alpha} f\left(\frac{r}{t^{\beta}}\right), \quad q(r,t) = t^{-\delta} g\left(\frac{r}{t^{\gamma}}\right)$$

$$\delta g(\eta) + \beta \eta g'(\eta) = f^{-\epsilon}(\eta) g(\eta) + f^{\omega-\epsilon}(\eta) f'(\eta)$$
$$c_0[\alpha f(\eta) + \beta \eta f'(\eta)] = g'(\eta) + g(\eta)/\eta$$

Parameters are fixed,
coupled sytem of ODE
but Eq. 2. can be integrated
→ only one ODE

Properties of the model

originaly there are 5 independent parameters, exponents α, β, δ ϵ, ω
only one remained independent , we fixed omega

$$\alpha = \frac{1}{\omega - 1}, \quad \beta = \frac{1}{2(\omega - 1)} \quad \delta = \frac{\omega - 1/2}{\omega - 1}, \quad \epsilon = 1 - \omega.$$

the parameter dependence of the solutions is now dictated

$$T(x,t) = t^{-\alpha} f\left(\frac{r}{t^{\beta}}\right) = t^{\frac{-1}{\omega-1}} f\left(\frac{r}{t^{\frac{1}{2(\omega-1)}}}\right), \qquad q(x,t) = t^{-\delta} g\left(\frac{r}{t^{\beta}}\right) = t^{\frac{\omega-1/2}{\omega-1}} g\left(\frac{r}{t^{\frac{1}{2(\omega-1)}}}\right)$$

heat conduction and relaxation time terms are also known

$$\kappa = \kappa_0 \cdot T(x,t)^{\omega} = \kappa_0 t^{\frac{-\omega}{\omega-1}} f^{\omega}\left(\frac{r}{t^{\frac{1}{2(\omega-1)}}}\right), \qquad \tau = \tau_0 \cdot T(x,t)^{\epsilon} = \kappa_0 t^{\frac{1-\omega}{\omega-1}} f^{1-\omega}\left(\frac{r}{t^{\frac{1}{2(\omega-1)}}}\right)$$

Properties of the solution

$$f'\left[\frac{1}{2(\omega-1)}\eta^2 - 2a(\omega-1)f^{2\omega-1}\right] + \eta f\left[\frac{2\omega}{2(\omega-1)} - \frac{f^{\omega-1}}{\tau_0}\right] = 0$$

first order non-linear ODE (no analytic solution) BUT
-Variable transformations, $y = \eta^2$ and $x(y) = f(\eta)$ and considering the inverse of the first derivative → linear inhomogeneous ODE

$$\frac{dy}{dx} = \frac{y}{x[bx^{\omega-1} - \omega]} - \frac{4a(\omega-1)^2 x^{2\omega-2}}{bx^{\omega-1} - \omega}$$

$a = \frac{\kappa_0}{\tau_0 c_0}$

$b = (2\omega - 1)/\tau_0$

→ can be integrated

$$y = 4a(\omega-1)^2\left[x^{-\frac{1}{\omega-1}}(-2\omega x^{\omega} + x^{\omega} + x\omega)^{\frac{1}{\omega(\omega-1)}}\right]\left(\int x^{\frac{2\omega^2-3\omega+2}{\omega-1}}(-2\omega x^{\omega} + x^{\omega+x\omega})^{\frac{-\omega^2+\omega-1}{\omega(\omega-1)}}dx + c\right)$$

general solution of the homogeneous equation times the particular solution of the inhomogeneous one, there is only one parameter dependence

Properties of the inverse solution

it is not singular for $2(\omega-1)x^{\omega-1}-\omega \neq 0$ so for $f(\omega)=\left(\frac{\omega}{2\omega-1}\right)^{\frac{1}{\omega-1}}$ $0<\omega\leq 1/2$
different ω means different kind of solutions

$1/2<\omega$ blow up solution at $x=\left(\frac{\omega}{2\omega-1}\right)^{\frac{1}{\omega-1}}$ •

$\omega\neq 1$ fordbidden value

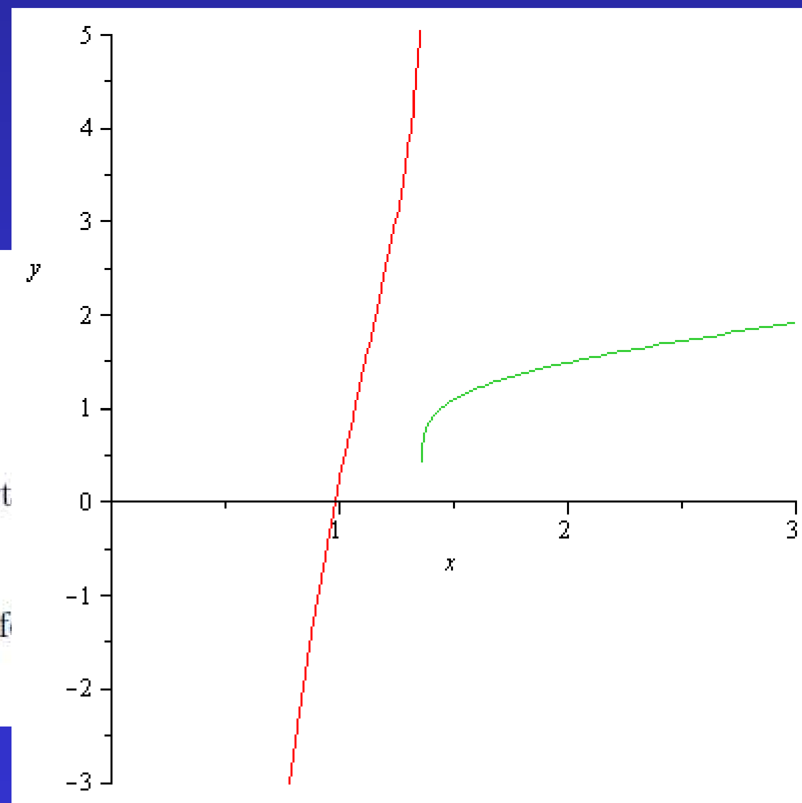
$\omega=1/2$ solution: $y(x)=1+c/x^2$

$0<\omega<1/2$ non-compact solution with a maxima, can be invert

$\omega=0$ solution:$(-4Ei(1,-x)+x)e^{-x}$ no positive domain

$-1<\omega<0$ singular in origin, have a max and can be inverted f

$\omega\leq -1$ two distinct solutions with a cut in between •



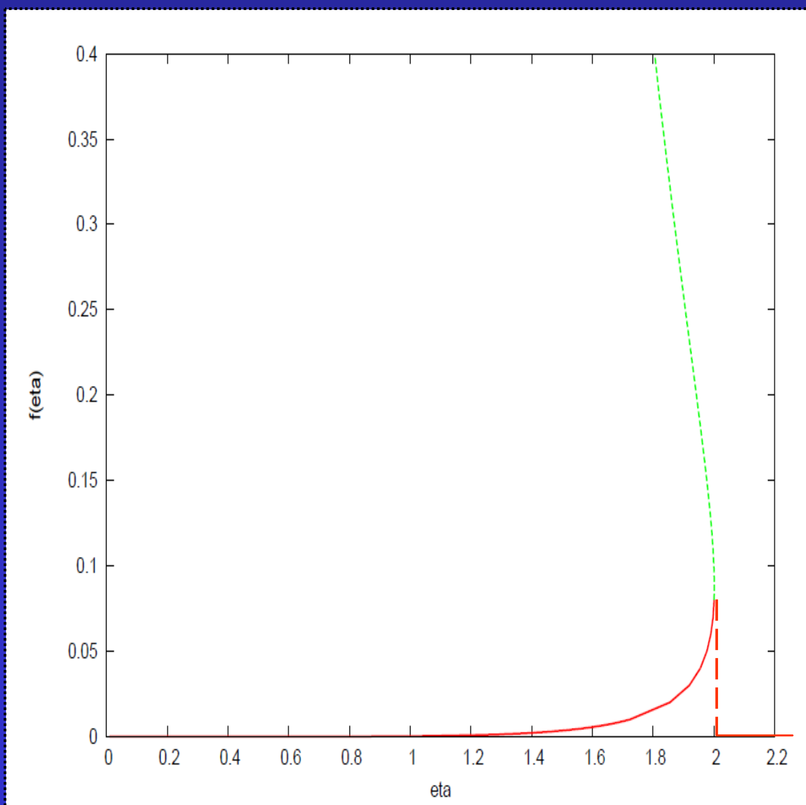
Non-continous solutions *of* PDE

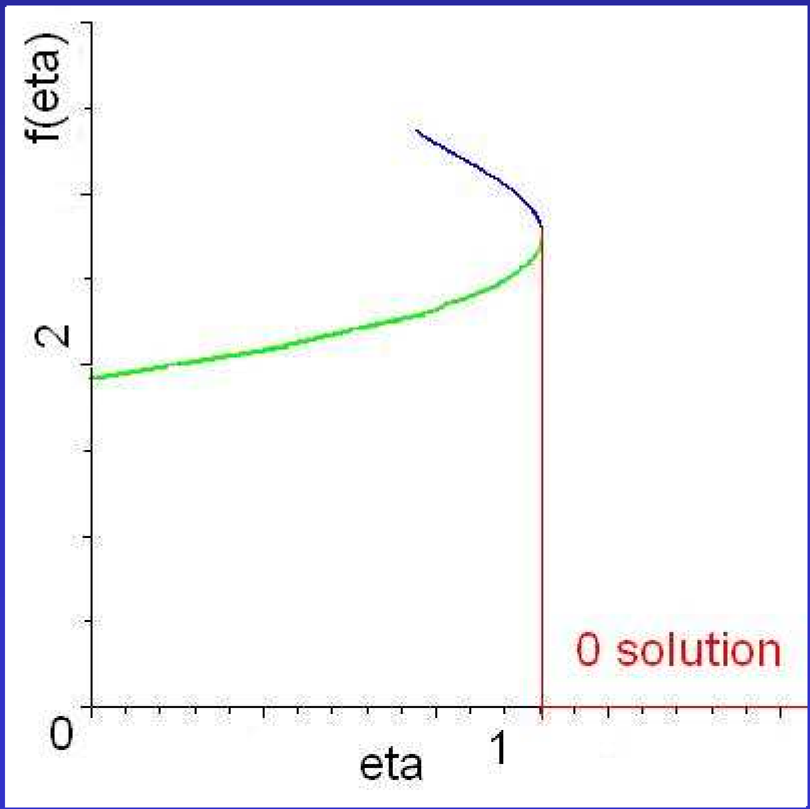
applying the back-transformation (inversion + square root)

$0 < \omega < 1/2$ non-compact solution with a maxima, can be inverted $[0..\infty]$

$-1 < \omega < 0$ singular in origin, have a maxima and can be inverted in positive x domain

principal value have to be fixed!!





we may define zero solutions outside this eta domain

Summary and Outlook

we presented the problem of the heat conduction eq.
defined two possible way-outs

As a new feature we presented a new telegraph-type equation with self-similar solutions
It has both parabolic and hyperbolic properties

As a second point we use a hyperbolic system to investigate heat conduction, can have non-continuous solutions

Thank you for

your attention!

Questions, Remarks, Comments?...