Self-Similar Solutions of the non-linear Maxwell equations

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Outline

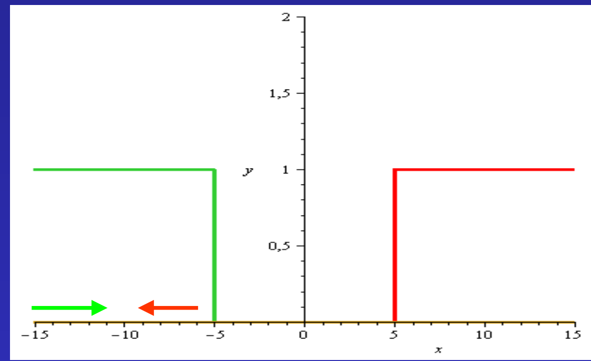
- ***Important solutions of PDEs*** *self-similar, travelling-waves,*
  examples Fourier heat conduction, Cattaneo equ.
- ***The non-linear Maxwell equation*** *and constitutive relations*
- ***My Ansatz & Solution***
- ***Summary***

Physically important solutions of PDEs

- *Travelling waves:*

  arbitrary wave fronts

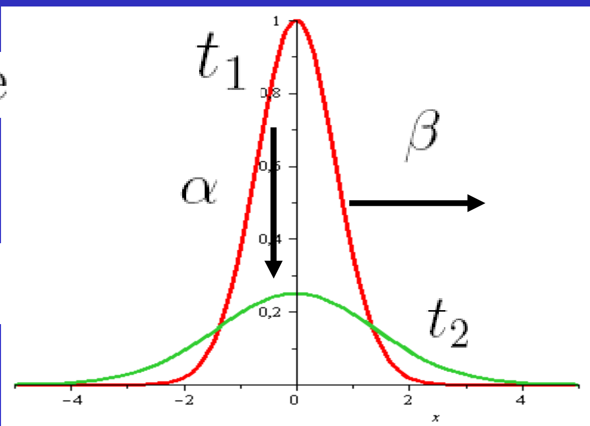
  u(x,t) ~ g(x-ct), g(x+ct)



- *Self-similar*

$$u(x,t) = t^{-\alpha} f(x/t^{\beta})$$

Sedov, Barenblatt, Zeldovich



α and β are of primary physical importance

α represents the rate of decay

β is the rate of spread (or contraction if $\beta < 0$)

$t_1 < t_2$

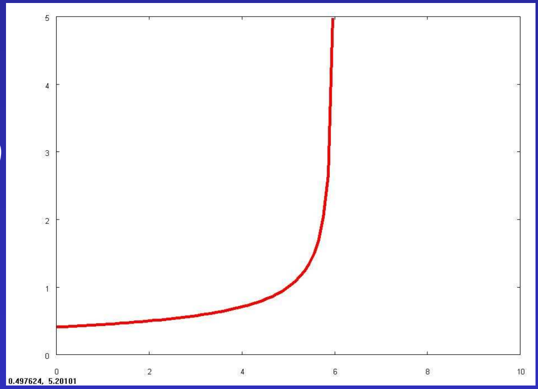
Physically important solutions of PDEs II

- blow-up solution

$$u(x,t) = \frac{1}{\sqrt{T-t}} f\left(\frac{x}{\sqrt{T-t}}\right)$$

goes to infinity in finite time

other forms are available too



additive separable

multiplicative separable

generalised separable eg.

$$u(x,t) = \varphi(t) + \phi(x)$$

$$u(x,t) = \varphi(t)\phi(x)$$

$$u(x,t) = \frac{x + C_1}{at + C_2} + \frac{2ab}{(at + C_1)^2}$$

Examples for Self-similar Solutions – Fourier heat conduction

$$\mathbf{q} = -k\nabla U(x,t), \quad \nabla\mathbf{q} = -\gamma\frac{\partial U(x,t)}{\partial t}$$

U(x,t) temperature distribution
Fourier law + conservation law

- $$\begin{cases} u_t(x,t) - k u_{xx}(x,t) = 0 & -\infty < x < \infty, \quad 0 < t < \infty \\ u(x,t=0) = \delta(x) \end{cases}$$ *parabolic PDA, no time-reversal sym.*

- *strong maximum principle ~ solution is smeared out in time*
- *the fundamental solution:* $$\Phi(x,t) = \int \frac{1}{\sqrt{4\pi kt}} exp\left(-\frac{x^2}{4kt}\right)$$
- *general solution is:*

$$u(x,t) = \int \Phi(x-y,t)g(y)dy \qquad u(x,0) = g(x) \;\; for -\infty < x < \infty \;\; and \;\; 0 < t < \infty$$

- *kernel is non compact = inf. prop. speed paradox of heat cond.*
- *Problem from a long time ☹*
- *But it has a self-similar solution ☺* $$u(x,t) = t^{-\alpha} f(x/t^{\beta})$$

Additional example

first order eq. for *large* disturbances in media (eq.shock-waves in gases)
second order eq. *small* disturbances in media (eq. sound waves in gases)

the Cattaneo heat cond. law with power law temperature dependent heat conduction coefficient and relaxation time, considering the hyperbolic and not the parabolic theory

$$q_t = -\frac{q}{\tau} - \frac{\kappa}{\tau}T_r, \qquad c_0 T_t = -q_r - \frac{q}{r}.$$

$\kappa = \kappa_0 T^{\omega},$ $\tau = \tau_0 T^{-\epsilon}$

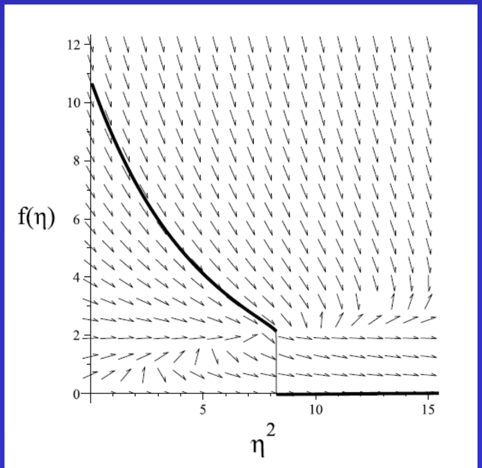
Ansatz: $T = t^{-\alpha} f(\eta),$ $q = t^{-\delta} g(\eta).$

leads to a final ODE:

$$\frac{df}{d\eta^2}\left(\beta^2\eta^2 - f^{2\omega+1}\right) = \frac{\beta f}{2}[f^{\omega+1} - (2\beta+1)].$$

Which has non-continuous and continuous solutions as well

$\omega = 0$



$\omega = -1/2,$

$$T = \frac{9t}{(r^{3/2} + t^{3/2})^2},$$

I.F. Barna and R. Kersner
http://arxiv.org/abs/1204.4386
will be published

The Maxwell equations & non-linearity

The field equations of Maxwell

$$\nabla \cdot \mathbf{D} = \rho, \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}, \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

The constitutive Eqs + differential Ohm's Law

$$\mathbf{D} = \epsilon \mathbf{E} \qquad \mathbf{B} = \mu \mathbf{H} \qquad \mathbf{J} = \sigma \mathbf{E}$$

Our consideration, power law:

$$\mu(\mathbf{H}) = a\mathbf{H}^q \qquad \epsilon(\mathbf{E}) = b\mathbf{E}^r \qquad \sigma(\mathbf{E}) = h\mathbf{E}^p \quad a, b, h, p, r, q \; are \; \mathrm{Real}$$

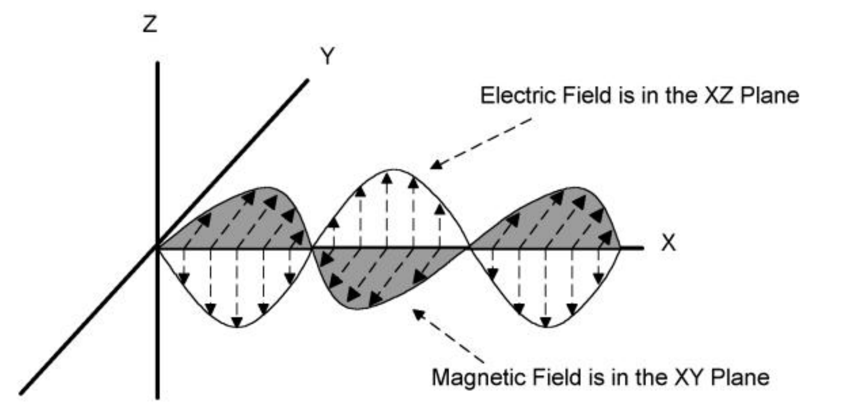
$$\text{constrain } c^2 = \frac{1}{\mu\epsilon}$$

$$\mu(\mathbf{H}) = a\mathbf{H}^q \qquad \epsilon(\mathbf{H}) = \frac{1}{c^2 a^1 H^q}$$

Geometry & Applied Ansatz

For the sake of simplicity we consider the following one dimensional problem

$$\mathbf{E} = (0, E_y(x,t), 0) \qquad \mathbf{H} = (0, 0, H_z(x,t))$$



$$\nabla \times \mathrm{E} = -\frac{\partial \mathrm{H}}{\partial t}, \qquad \nabla \times \mathrm{H} = \frac{\partial \mathrm{D}}{\partial t} + \mathrm{J}$$

$$\frac{\partial E_y}{\partial x} = -\frac{\partial H_z}{\partial t}, \qquad -\frac{\partial H_z}{\partial x} = \frac{\partial D_y}{\partial t} + J_y$$

$$\begin{aligned} E_y(x,t) &= t^{-\alpha} f\left(\frac{x}{t^{\beta}}\right) = t^{-\alpha} f(\eta) \\ H_z(x,t) &= t^{-\delta} g\left(\frac{x}{t^{\beta}}\right) = t^{-\delta} g(\eta) \end{aligned}$$

where α, β, δ are **Real**

no second order wave equ.
but the first order Maxw. equ

$$\begin{aligned} \frac{\partial}{\partial x}[t^{-\alpha} f(\eta)] &= -\frac{\partial}{\partial t}[a t^{-\delta(q+1)} g^{q+1}(\eta)] \\ -\frac{\partial}{\partial x}[t^{-\delta} g(\eta)] &= \frac{\partial}{\partial t}[c^{-2} a^{-1} t^{\delta q - \alpha} g^{-q}(\eta) f(\eta)] + h t^{-\alpha(p+1)} f^{p+1}(\eta), \quad \eta = x/t^{\beta} \end{aligned}$$

The final ordinary differential equations

$$f' = a(q+1)[\delta g^{q+1} - g^q g' \eta \beta]$$
$$-g' = \frac{1}{ac^2}[(q+1)g^q f + q(q+1)g^{q-1}g' f\eta + (q+1)g^q f'\eta]$$

' means derivation with respect to eta

Universality relations among the parameters: $\beta = -q-1$ $\alpha = -1$ $\delta = 1$ $p = 1$

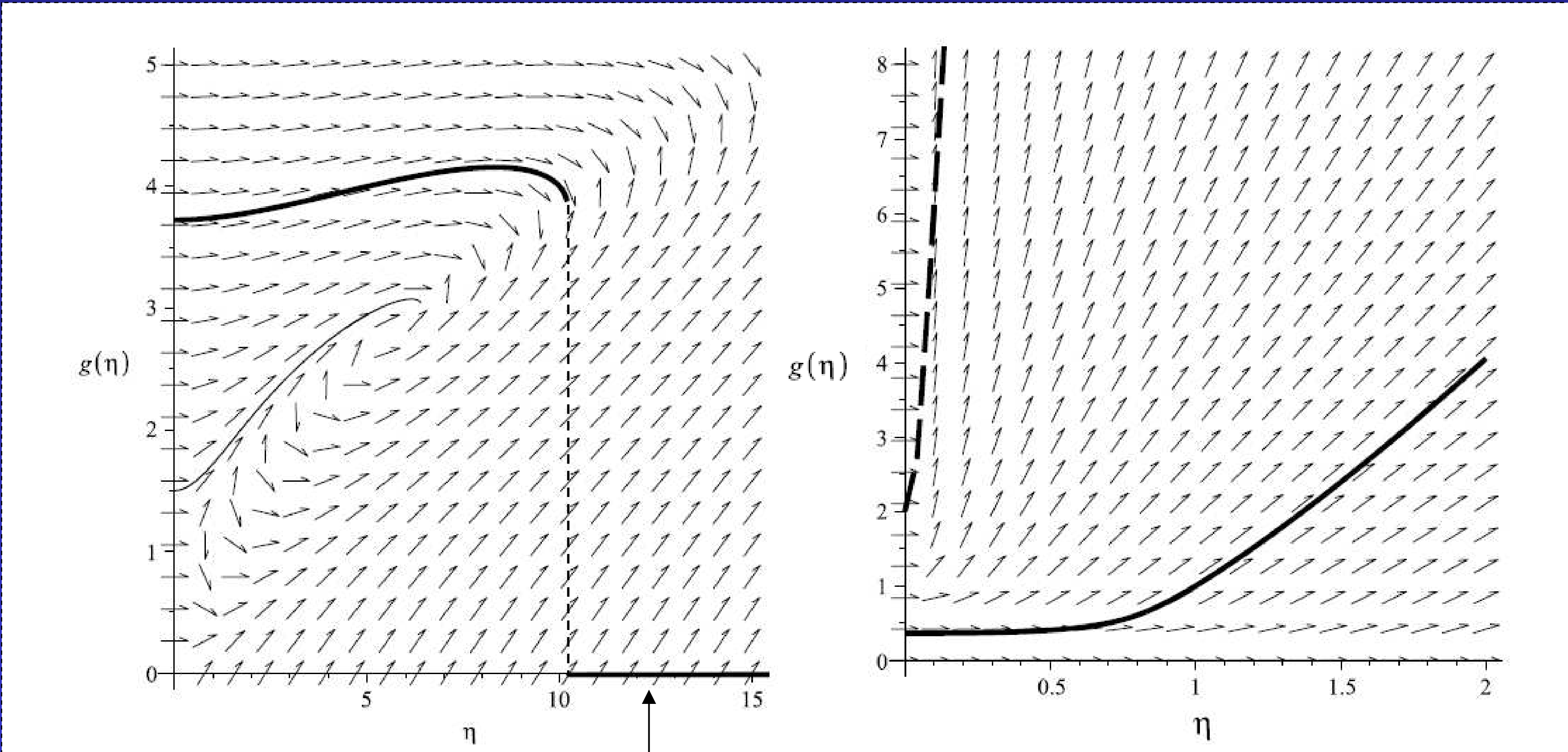
The first equ. is a total difference so can be integrated : $f = a(q+1)\eta g^{q+1}$

Remaining a simple ODE with one parameter q: $g' = \frac{2(q+1)^2 \eta g^{2q+1} + (q+1)^3 \eta^2 g^{2q} + h}{1 + q(q+1)^2 \eta^2 g^{2q}}$

There is no theory for the existence and uniqueness of such ODE (no fix-point theory, nothing works)

Different kind of solutions

No closed-forms are available so the direction fields are presented



Compact solutions for q < -1
shock-waves

Extended 0 solution the for PDE!

Non-compact solutions for q > -1

Physically relevant solutions

We consider the Poynting vector which gives us the energy flux (in W/m^2) in the field

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = t^{-\alpha-\delta} f g = a(q+1)\eta g^{q+2}$$

Note that for $q < -2$ the $\int_0^{cut} S d\eta$ is finite which is a good reason.

Unfortunatelly, there are two contraversial definition of the Poynting vector in media, unsolved problem R.N. Pfeifer et al. Rev. Mod. Phys. 79 (2007) 1197.

Summary

we presented physically important,
self-similar solutions for various PDEs

as a new feature we defined non-linear,
field-dependent magnetic permeability for
the Maxwell equations

found and presented non-continous shock-wave
solutions with compact support
further investigation is in progress to clear up all
dark points