Self-Similar Solution of the three dimensional Navier-Stokes Equation

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Outline

- ***Solutions of PDEs*** *self-similar, various heat conduction examples*
- ***Navier-Stokes equation***
- ***My 3D Ansatz & geometry*** *my solution + other solutions*
- ***Summary & Outlook*** *additional new systems to study*

The basic idea

Initially: a (nonlinear) Partial Differential Equation (system)

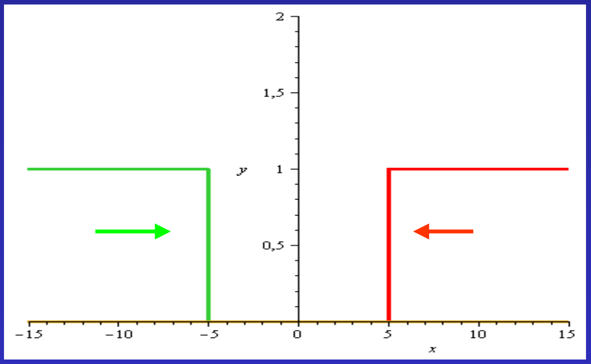
Ansatz combination from x,t to a new variable eta(x,t)

Result: a (nonlinear) Ordinary Differential Equation (system)

+ with some tricks analytic solutions can be found which depend on some free physical parameter(s)

Physically important solutions of PDEs

- *Travelling waves:*
  arbitrary wave fronts
  $u(x,t) \sim g(x-ct), g(x+ct)$



- *Self-similar*

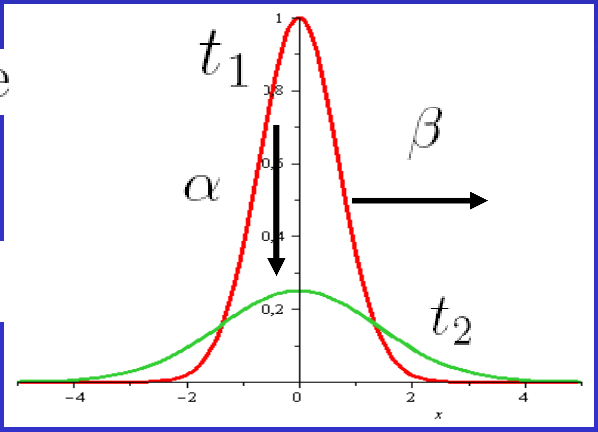
$$u(x,t) = t^{-\alpha} f(x/t^{\beta})$$

Sedov, Barenblatt, Zeldovich

α and β are of primary physical importance

α represents the rate of decay

β is the rate of spread (or contraction if $\beta < 0$)



$t_1 < t_2$ *in Fourier heat-conduction*

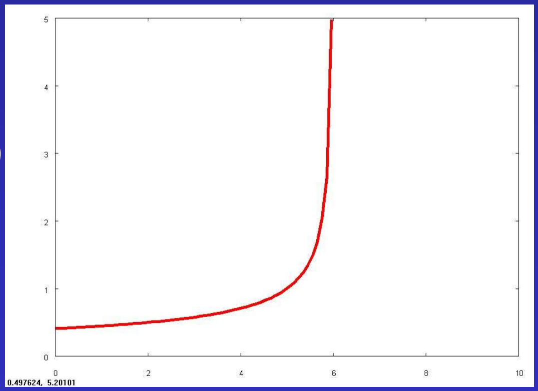
Physically important solutions of PDEs II

- *blow-up solution*

$$u(x,t) = \frac{1}{\sqrt{T-t}} f\left(\frac{x}{\sqrt{T-t}}\right)$$

goes to infinity in finite time

other forms are available too



additive separable

$$u(x,t) = \varphi(t) + \phi(x)$$

multiplicative separable

$$u(x,t) = \varphi(t)\phi(x)$$

generalised separable eg.

$$u(x,t) = \frac{x + C_1}{at + C_2} + \frac{2ab}{(at + C_1)^2}$$

Ordinary diffusion/heat conduction equation

$$\mathbf{q} = -k\nabla U(x,t), \quad \nabla \mathbf{q} = -\gamma \frac{\partial U(x,t)}{\partial t}$$

U(x,t) temperature distribution
Fourier law + conservation law

- $$\begin{cases} u_t(x,t) - k u_{xx}(x,t) = 0 & -\infty < x < \infty, \quad 0 < t < \infty \\ u(x,t=0) = \delta(x) \end{cases}$$

parabolic PDA, no time-reversal sym.

- ***strong maximum principle ~ solution is smeared out in time***
- ***the fundamental solution:***
- ***general solution is:***

$$\Phi(x,t) = \int \frac{1}{\sqrt{4\pi k t}} exp\left(-\frac{x^2}{4kt}\right)$$

$$u(x,t) = \int \Phi(x-y,t) g(y) dy$$

$$u(x,0) = g(x) \ \ for -\infty < x < \infty \ \ and \ \ 0 < t < \infty$$

- ***kernel is non compact = inf. prop. Speed paradox of heat cond.***
- ***Problem from a long time ☹***
- ***But have self-similar solution ☺***

$$u(x,t) = t^{-\alpha} f(x/t^{\beta})$$

General derivation for heat conduction law

$$\tau \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -k \nabla T(x,t)$$

Cattaneo heat conduction law, there is a general way to derive

$$q = -\int_{-\infty}^{t} Q(t-t') \frac{\partial T(x,t')}{\partial x} \, \mathrm{d}t'$$

T(x,t) temperature distribution
q heat flux

Joseph D D and Preziosi L 1989 *Rev. Mod. Phys.* **61** 41
Joseph D D and Preziosi L 1990 *Rev. Mod. Phys.* **62** 375

$$Q(t-t') = \frac{k\tau^{l}}{(t-t'+\omega)^{l}}$$

the kernel can have microscopic interpretation

$$\epsilon \frac{\partial^2 T(x,t)}{\partial t^2} + \frac{a}{t}\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2}$$

telegraph-type time dependent eq. with self sim. solution

$$T(x,t) = t^{-\alpha} f(\eta) \;\; with \;\; \eta = \frac{x}{t^{\beta}}$$

Solutions

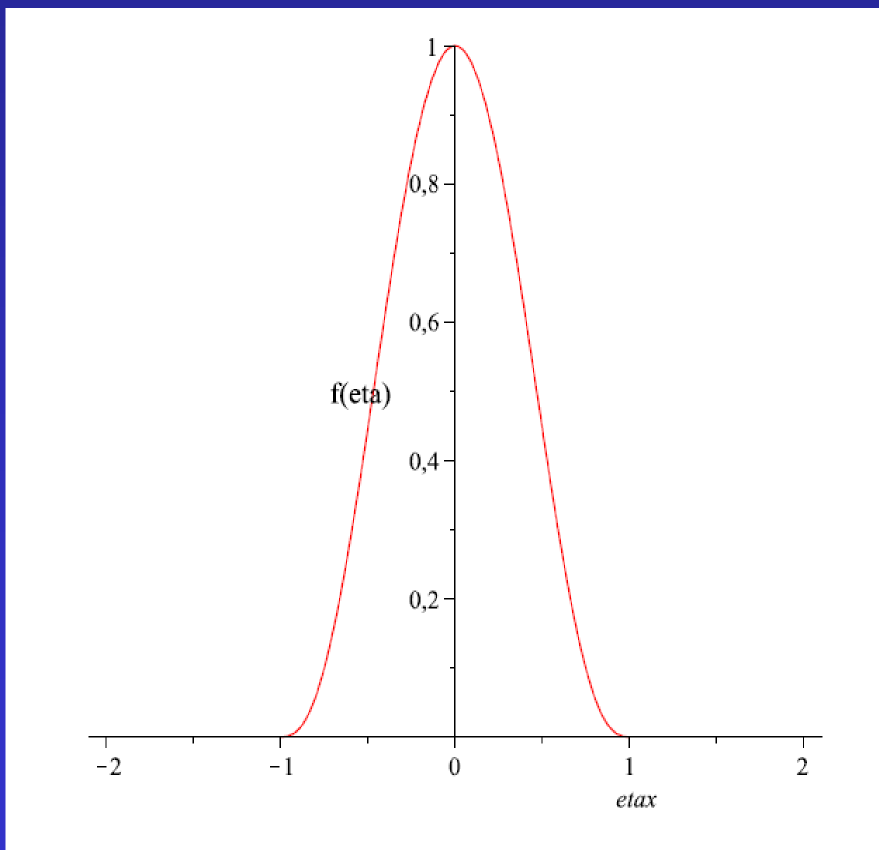
J. Phys. A: Math. Theor. 43 (2010) 375210

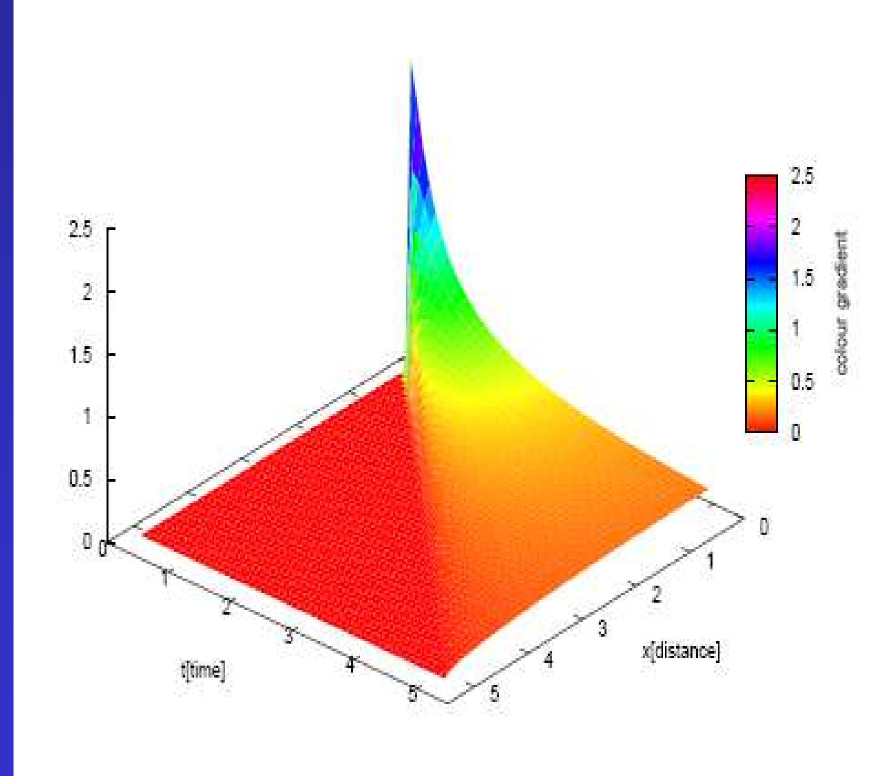
$c_1 = 0$

$a = 4.1, \epsilon = 1$

$$f(\eta) = \left(1 - \epsilon\eta^2\right)^{\frac{a}{2\epsilon} - 1}$$

$$T(x,t) = \frac{1}{t}\left(1 - \epsilon\frac{x^2}{t^2}\right)^{\frac{a}{2\epsilon} - 1}$$





2 Important new feature: the solution is a product of 2 travelling wavefronts

$if\ a > 4\epsilon,\ f'(\eta) = 0$ no flux conservation problem

$$T(x,t) \sim U(x - ct)U(x + ct)$$

The Navier-Stokes equation

$$\nabla \mathbf{v} = 0,$$
$$\mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v} = \nu \triangle \mathbf{v} - \frac{\nabla p}{\rho} + a$$

3 dimensional cartesian coordinates,
Euler description
v velocity field, p pressure, a external field
ν kinematic viscosity, ρ constant density
Newtonian fluid

$\mathbf{v}(x,y,z,t) = u(x,y,z,t), v(x,y,z,t), w(x,y,z,t)$ $p(x,y,z,t)$

Consider the most general case

$$u_x + v_y + w_z = 0$$
$$u_t + uu_x + vu_y + wu_z = \nu(u_{xx} + u_{yy} + u_{zz}) - \frac{p_x}{\rho}$$
$$v_t + uv_x + vv_y + wv_z = \nu(v_{xx} + v_{yy} + v_{zz}) - \frac{p_y}{\rho}$$
$$w_t + uw_x + vw_y + ww_z = \nu(w_{xx} + w_{yy} + w_{zz}) - \frac{p_z}{\rho} + a.$$

just to write out all the coordinates

My 3 dimensional Ansatz

$$u(x,t) = t^{-\alpha} f(x/t^{\beta})$$

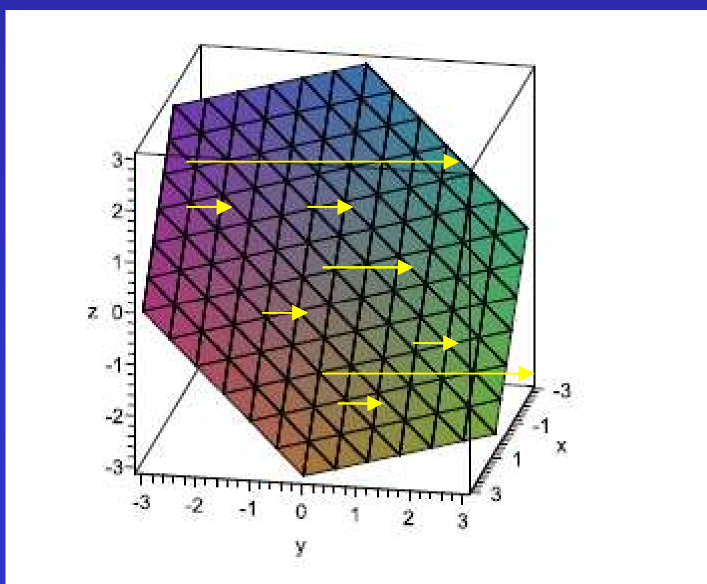
→

$$u(x,y,z,t) = t^{-\alpha} f\left(\frac{F(x,y,z)}{t^{\beta}}\right) := t^{-\alpha} f\left(\frac{x+y+z}{t^{\beta}}\right) := t^{-\alpha} f(\omega)$$

$$F(x,y,z) = x+y+z = 0$$

A more general function does not work for N-S

$$u(x,y,z,t) = t^{-\alpha} f\left(\frac{\sqrt{x^2+y^2+z^2}-a}{t^{\beta}}\right)$$



The graph of the $x+y+z=0$ plane.

Geometrical meaning:
all v components with
coordinate constrain x+y+z=0
lie in a plane = equivalent

The final applied forms

$$u(x,y,z,t) = t^{-\alpha} f\left(\frac{x+y+z}{t^{\beta}}\right), \quad v(x,y,z,t) = t^{-\gamma} g\left(\frac{x+y+z}{t^{\delta}}\right)$$
$$w(x,y,z,t) = t^{-\epsilon} h\left(\frac{x+y+z}{t^{\zeta}}\right), \quad p(x,y,z,t) = t^{-\eta} l\left(\frac{x+y+z}{t^{\theta}}\right)$$

Just a technical detail

Just building the time and space derivatives of

$$u(x,y,z,t) = t^{-\alpha} f\left(\frac{x+y+z}{t^{\beta}}\right), \quad v(x,y,z,t) = t^{-\gamma} g\left(\frac{x+y+z}{t^{\delta}}\right)$$

$$w(x,y,z,t) = t^{-\epsilon} h\left(\frac{x+y+z}{t^{\zeta}}\right), \quad p(x,y,z,t) = t^{-\eta} l\left(\frac{x+y+z}{t^{\theta}}\right)$$

and pluging back to

$$u_t + uu_x + vu_y + wu_z = \nu(u_{xx} + u_{yy} + u_{zz}) - \frac{p_x}{\rho}$$

we arrive at:

$$-\alpha t^{-\alpha-1} f(\omega) - \beta t^{-\alpha-1} f'(\omega)\omega + t^{-2\alpha-\beta} f(\omega) f'(\omega) + t^{-\gamma-\alpha-\beta} g(\omega) f'(\omega) + t^{-\epsilon-\alpha-\beta} h(\omega) f'(\omega) = \nu 3 t^{-\alpha-2\beta} f''(\omega) - \frac{t^{-\mu-\beta} l'(\omega)}{\rho}.$$

which must not depend on time

The obtained ODE system

$$\begin{aligned}
f'(\omega)+g'(\omega)+h'(\omega) &= 0\\
-\frac{1}{2}f(\omega)-\frac{1}{2}\omega f'(\omega)+[f(\omega)+g(\omega)+h(\omega)]f'(\omega) &= 3\nu f''(\omega)-\frac{l'(\omega)}{\rho}\\
-\frac{1}{2}g(\omega)-\frac{1}{2}\omega g'(\omega)+[f(\omega)+g(\omega)+h(\omega)]g'(\omega) &= 3\nu g''(\omega)-\frac{l'(\omega)}{\rho}\\
-\frac{1}{2}h(\omega)-\frac{1}{2}\omega h'(\omega)+[f(\omega)+g(\omega)+h(\omega)]h'(\omega) &= 3\nu h''(\omega)-\frac{l'(\omega)}{\rho}+a.
\end{aligned}$$

as constraints we got for the exponents:

$\alpha,\beta,\gamma,\delta,\epsilon,\zeta,\theta=1/2,\qquad \eta=1$ **universality relations**

$$u(x,y,z,t)=t^{-1/2}f\left(\frac{x+y+z}{t^{1/2}}\right)=t^{-1/2}f(\omega),\quad v(x,y,z,t)=t^{-1/2}g(\omega),$$
$$w(x,y,z,t)=t^{-1/2}h(\omega),\quad p(x,y,z,t)=t^{-1}l(\omega),$$

Continuity eq. helps us to get an additional constraint:

$f(\omega)+g(\omega)+h(\omega)=c,\quad and\quad f''(\omega)+g''(\omega)+h''(\omega)=0$ **c is prop. to mass flow rate**

Solutions of the ODE

a single Eq. remains

$$9\nu f''(\omega) - 3(\omega + c)f'(\omega) + \frac{3}{2}f(\omega) - \frac{c}{2} + a = 0.$$

$$f(\omega) = c_1 \cdot KummerU\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\omega+c)^2}{6\nu}\right) + c_2 \cdot KummerM\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\omega+c)^2}{6\nu}\right) + \frac{c}{3} - \frac{2a}{3}$$

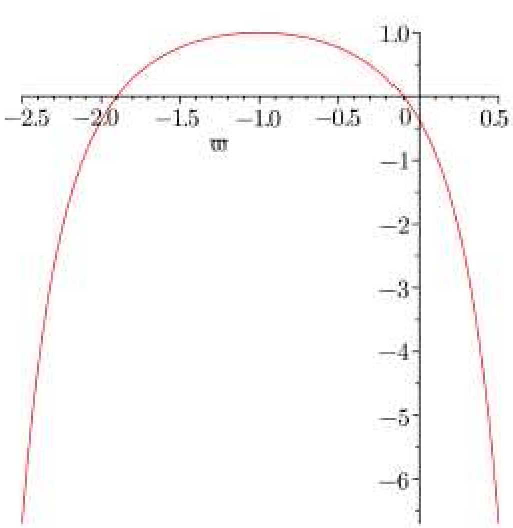


Fig. 3 The KummerM$(-1/4, 1/2, (\omega+c)^2/6\nu)$ function for $c = 1$ and $\nu = 0.1$.

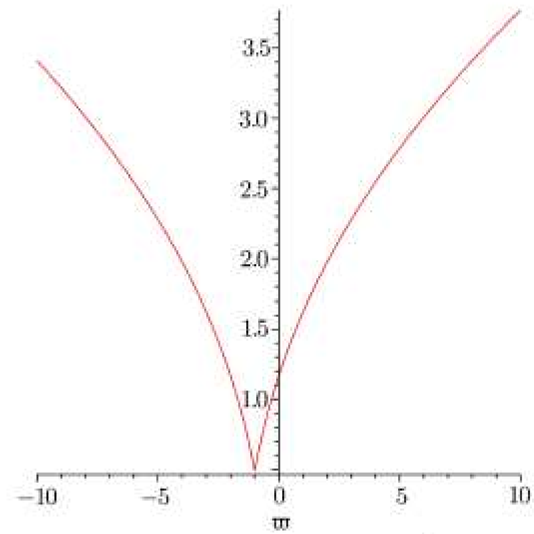


Fig. 4 The KummerU$(-1/4, 1/2, (\omega+c)^2/6\nu)$ function for $c = 1$ and $\nu = 0.1$.

Kummer is spec.

$(a)_n$ is the Pochhammer symbol

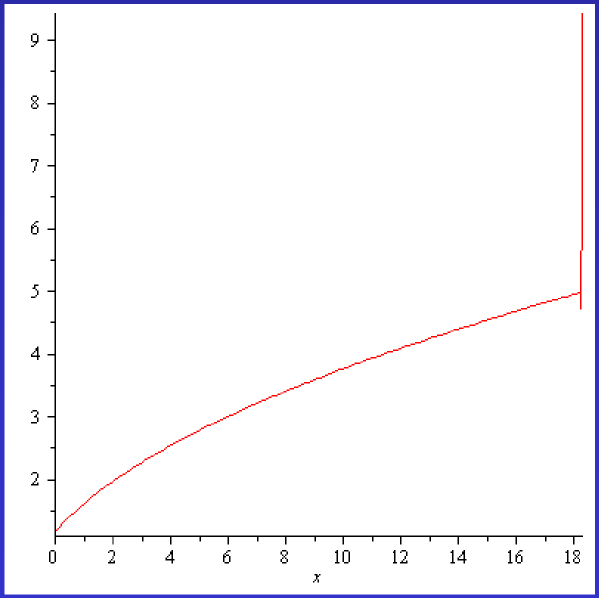
$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), (a)_0 = 1$$

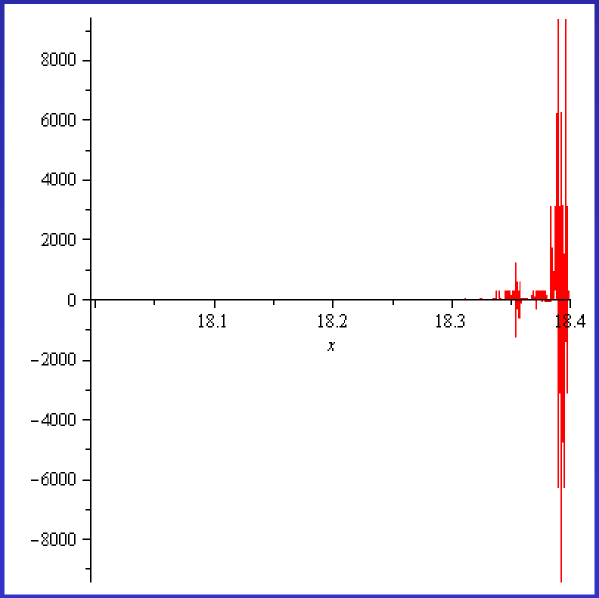
$$M(a, b, z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \cdots + \frac{(a)_n z^n}{(b)_n n!},$$

$$U(a, b, z) = \frac{\pi}{\sin(\pi b)}\left[\frac{M(a, b, z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(1+a-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)}\right]$$

Numerical properies of the solution

- *Convergence problem at evaluation of KummerU, personal experience*
- *Using Maple 12 with 50 digits accuracy*





An example:
for given c and nu above a defined x value the result is not convergent any more

Of course, this should be checked with great care but compare with the von Neumann Stability Analysis for numerical schemes for N-S Eq.

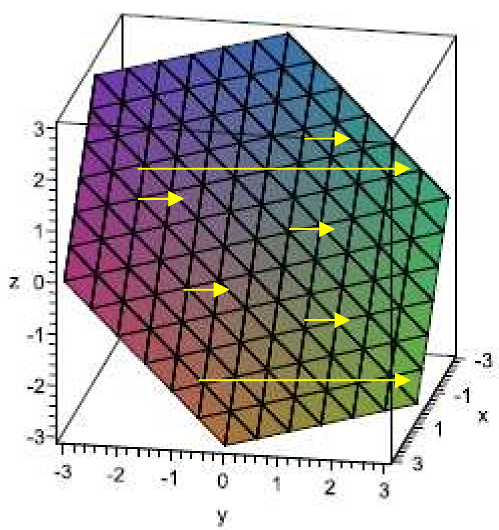
Solutions of N-S

$$u(x,y,z,t) = t^{-1/2} f(\omega) = t^{-1/2}\Big[c_1 \cdot \mathrm{KummerU}\Big(\frac{-1}{4}, \frac{1}{2}, \frac{((x+y+z)/t^{1/2}+c)^2}{6\nu}\Big)\Big] + t^{-1/2}\Big[c_2 \cdot \mathrm{KummerM}\Big(-\frac{1}{4}, \frac{1}{2}, \frac{((x+y+z)/t^{1/2}+c)^2}{6\nu}\Big) + \frac{c}{3} - \frac{2a}{3}\Big]$$

only for one velocity component ☹

Geometrical explanation:

all v components with coordinate constrain x+y+z=0 lie in a plane = equivalent



Naver-Stokes makes a dynamics of this plane

→

getting a multi-valued surface

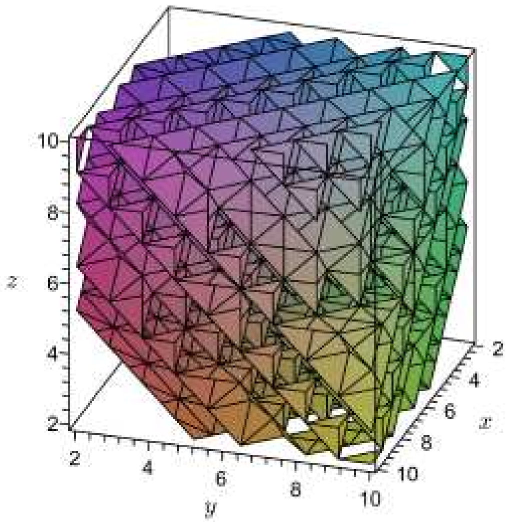


Fig. 5 The implicit plot of the self-similar solution Eq. (17). Only the KummerU function is presented for $t = 1$, $c_1 = 1$, $c_2 = 0$, $a = 0$, $c = 1$, and $\nu = 0.1$.

I.F. Barna http://arxiv.org/abs/1102.5504
Commun. Theor. Phys. 56 (2011) 745-750

Solution for the other two velocity components

$$v(x, y, z, t) = t^{-1/2} g(\omega)$$

$$-3\nu g''(\omega) + g'(\omega)\Big(-\frac{\omega}{2} + c\Big) - \frac{g(\omega)}{2} + F(f''(\omega), f'(\omega), f(\omega)) = 0\,,$$

The general form for v:

$$g(\omega) = \Big[c_2 + \int \Big\{\frac{-c_1 + \int F(f''(\omega), f'(\omega), f(\omega))\mathrm{d}\omega \cdot \exp((-\omega^2/4 + c\omega)/-3\nu)}{3\nu}\Big\}\mathrm{d}\omega\Big] \exp\Big(\frac{-\omega^2/4 + c\omega}{3\nu}\Big)\,.$$

The 1st and 2nd derivatives of the Kummer functions are needed

I have no closed formula.

Other analytic solutions I

Without completeness, usually from Lie algebra studies

W. I. Fushchich, W. M. Shtelen and S. L. Slavutsky J. Phys. A: Math. Gen. 24 (1990) 971.

$$\omega = z/\sqrt{t}$$

Presented 19 various solutions one of them is:

$$u(z,t) = \frac{f(\omega)}{\sqrt{t}}, \quad v(y,z) = \frac{g(\omega)}{\sqrt{t}} + \frac{y}{t}, \quad w(z,t) = \frac{h(\omega)}{\sqrt{t}}, \quad p(t,z) = \frac{l(\omega)}{\sqrt{t}}$$

the obtained ODE system

$$\begin{aligned} h'(\omega) + 1 &= 0 \\ -\frac{1}{2}(f(\omega) + \omega f'(\omega)) + h(\omega) f'(\omega) &= f''(\omega), \\ \frac{1}{2}(g(\omega) + \omega g'(\omega)) + h(\omega) g'(\omega) &= g''(\omega), \\ -\frac{1}{2}(h(\omega) + \omega h'(\omega)) + h(\omega) h'(\omega) + l'(\omega) &= f''(\omega). \end{aligned}$$

the solution:

$$\begin{aligned} f(\omega) &= (\frac{3}{2}\omega - c)^{-1/2} exp\left[-\frac{1}{6}(\frac{3}{2}\omega - c)^2\right] w\left[-\frac{1}{12}, \frac{1}{4}, \frac{1}{3}(\frac{3}{2}\omega - c)^2\right] \\ g(\omega) &= (\frac{3}{2}\omega - c)^{-1/2} exp\left[-\frac{1}{6}(\frac{3}{2}\omega - c)^2\right] w\left[-\frac{5}{12}, \frac{1}{4}, \frac{1}{3}(\frac{3}{2}\omega - c)^2\right] \\ h(\omega) &= -\omega + c \\ l(\omega) &= \frac{3}{2}c\omega - \omega^2 + c_1 \end{aligned}$$

with:

$$w(\kappa, \mu, z) = e^{-1/2z} z^{1/2+\mu} KummerM(1/2 + \mu - \kappa, 1 + 2\mu, z).$$

Other analytic solutions II

V. Grassi, R.A. Leo, G. Soliani and P. Tempesta, Physica **286** (2000) 79

The initial Navier-Stokes

velocity components $U_i(y,z,t)$ and π is the pressure.

$$\begin{aligned} U_{1t} + cU_1 + U_2U_{1y} + U_3U_{1z} - \nu(U_{1yy} + U_{1zz}) &= 0, \\ U_{2t} + U_2U_{2y} + U_3U_{2z} + \pi_y - \nu(U_{2yy} + U_{2zz}) &= 0, \\ U_{3t} + U_2U_{3y} + U_3U_{3z} + \pi_z - \nu(U_{3yy} + U_{3zz}) &= 0, \\ U_{2y} + U_{3z} + c &= 0 \end{aligned}$$

After some transformation got a PDE:

$$U_{1t} + k_1yU_{1y} + (\sigma - k_1z)U_{1z} - \nu(U_{1yy} + U_{1zz}) = 0$$

applied Ansatz:

$$U_1 = Y(y)T(z)\Phi(t).$$

Solutions:
where M is a Kummer function

$$\begin{aligned} \Phi &= c_1exp(c_2)t \\ Y &= c_3M\left(-c_4, \frac{1}{2}, \frac{y^2}{2\nu}\right) + yc_5M\left(\frac{1}{2} - c_4, \frac{3}{2}, \frac{y^2}{2\nu}\right) \\ T &\approx M\left(c_6, \frac{1}{2}, \frac{z^2}{2\nu}\right) + zM\left(\frac{1}{2} - c_6, \frac{3}{2}, \frac{z^2}{2\nu}\right) \end{aligned}$$

Other analytic solutions III

JOURNAL OF MATHEMATICAL PHYSICS **49**, 113102 (2008)

Analytical solutions to the Navier–Stokes equations

Yuen Manwai[a)]

JOURNAL OF MATHEMATICAL PHYSICS **50**, 083101 (2009)

Analytical solutions to the Navier–Stokes equations with density-dependent viscosity and with pressure

Ling Hei Yeung[1,a)] and Yuen Manwai[2,b)]

"Only" Radial solution for 2 or 3 D

$$\rho_t + u\rho_r + \rho u_r + \frac{N-1}{r}\rho u = 0,$$

$$\rho(u_t + uu_r) + K\rho_r = \nu\left(u_{rr} + \frac{N-1}{r}u_r - \frac{N-1}{r^2}u\right).$$

Ansatz:

$$\rho(t,r) = \frac{f(r/a(t))}{a(t)^N}, \quad u(t,r) = \frac{\dot{a}(t)}{a(t)}r,$$

Various viscosity terms are investigated, only the final ODEs were given

Other Analytic solutions IV

Nonlinear Instability of the Solutions of the Navier–Stokes Equations: Formulas for Constructing Exact Solutions

A. D. Polyanin
Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, Russia
e-mail: polyanin@ipmnet.ru
Received May 5, 2009

Investigate the whole system with the Ansatz:

$$V_n = f_n(z, t)x + g_n(z, t)y, \quad n = 1, 2; \quad V_3 = F(z, t).$$

$$V_1 = x\left(-\frac{1}{2}\frac{\partial F}{\partial z} + w\right) + yv, \quad V_2 = xu - y\left(\frac{1}{2}\frac{\partial F}{\partial z} + w\right),$$
$$V_3 = F,$$
$$\frac{P}{\rho} = p_0 - \frac{1}{2}\alpha x^2 - \frac{1}{2}\beta y^2 - \gamma xy - \frac{1}{2}F^2 + \nu\frac{\partial F}{\partial z} - \int\frac{\partial F}{\partial t}dz, \quad (5)$$

Polyanin has a book on analitic solutions of PDEs

Other Analytic solutions V

Ukrainian Mathematical Journal, Vol. 49, No. 9, 1997

ON NAVIER–STOKES FIELDS WITH LINEAR VORTICITY

G. V. Popovich and R. O. Popovich

$$\mathrm{rot}\,\vec{u} = H(t)\vec{x} + \vec{k}(t).$$

$$\vec{u} = \nabla\varphi + \frac{1}{3}(H\vec{x})\times\vec{x} + \frac{1}{2}\vec{k}\times\vec{x},$$

(i) $H \neq 0$:

(a)

$$u^1 = (\zeta^1 + \beta^1)x_1 + \left(\beta^2 - \frac{1}{2}\kappa\right)x_2,$$

$$u^2 = \left(\beta^2 + \frac{1}{2}\kappa\right)x_1 + (\zeta^1 - \beta^1)x_2,$$

$$u^3 = -\frac{1}{2}\lambda(x_1^2 + x_2^2) + \frac{1}{2}(x_1^2 - x_2^2)\mu\sin\theta + x_1x_2\mu\cos\theta - 2\zeta^1 x_3,$$

$$p = -\frac{1}{2}\left(\frac{1}{4}\mu^{-2}(\lambda_t)^2 + (\zeta^1)^2 + (\zeta^2)^2 - \frac{1}{4}\kappa^2 + \zeta_t^1\right)(x_1^2 + x_2^2)$$

$$-\frac{1}{2}(\beta_t^1 + 2\zeta^1\beta^1)(x_1^2 - x_2^2) - (\beta_t^2 + 2\zeta^1\beta^2)x_1x_2 - \left(\zeta_t^1 - 2(\zeta^1)^2\right)x_3^2 - 2\lambda x_3,$$

where $\kappa, \lambda, \mu, \beta^1, \beta^2, \zeta^1, \zeta^2$, *and* θ *are smooth functions of* t *such that the relations*

Other Analytic solutions VI

SIMILARITY
and
DIMENSIONAL METHODS
in
MECHANICS
10th Edition

L. I. Sedov
Russian Academy of Sciences
Moscow, Russia

Translated by
A. G. Volkovets

CRC Press
Boca Raton Ann Arbor London Tokyo

On Page 120

The spherical problem, with the asimuthal angle dependence:

$$v_r = \frac{\nu}{r} f(\theta), \qquad v_\theta = \frac{\nu}{r}\varphi(\theta), \qquad v_\lambda = \frac{\nu}{r}\psi(\theta), \qquad U - \frac{p}{\rho} = \frac{\nu^2}{r^2}F(\theta).$$

The obtained ODE system

$$f'' + f'(\cot\theta - \varphi) + f^2 + \varphi^2 + \psi^2 - 2F = 0,$$

$$\varphi\varphi' - \psi^2 \cot\theta - f' - F' = 0,$$

$$\psi'' - \varphi\psi' - \varphi\psi\cot\theta + \psi'\cot\theta - \frac{\psi}{\sin^2\theta} = 0,$$

$$f + \varphi' + \varphi\cot\theta = 0.$$

The solution:

$$\varphi = \frac{2\sin\theta}{A + \cos\theta},$$

$$f = -2 + \frac{2(A^2 - 1)}{(A + \cos\theta)^2},$$

$$F(\theta). = \left(\cos\frac{\theta}{2}\right)^{\gamma}\left(\sin\frac{\theta}{2}\right)^{1+\alpha+\beta-\gamma}\left\{PF\left(\alpha,\beta,\gamma,\cos^2\frac{\theta}{2}\right) + Q\left(\cos^2\frac{\theta}{2}\right)^{1-\gamma} F\left(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma,\cos^2\frac{\theta}{2}\right)\right\},$$

F on right side hypergeometric function

Other Analytic solutions VII

- *the Russians and the NASA have probably even more…*
- *Sometimes I find scientific reports – for flows - which were confidental 40 years ago*
- *“The results are sometimes difficult to find because the studies were conducted in the last century and often published in reports of limited circulation”*

Summary

we presented the self-similar Ansatz as a tool for non-linear PDEs with two examples for heat conduction

as a new feature we presented a 3d self-similar Ansatz for the fully 3d N-S system with explanation and results

further work is in progress to clear out all dark points and give a bit more generality (ax – by + z = 0)

Numerical investigation of the surface of

$$u(x,y,z,t) = t^{-1/2} f(\omega) = t^{-1/2}\Big[c_1 \cdot \mathrm{KummerU}\Big(\frac{-1}{4}, \frac{1}{2}, \frac{((x+y+z)/t^{1/2}+c)^2}{6\nu}\Big)\Big]$$
$$+ t^{-1/2}\Big[c_2 \cdot \mathrm{KummerM}\Big(-\frac{1}{4}, \frac{1}{2}, \frac{((x+y+z)/t^{1/2}+c)^2}{6\nu}\Big) + \frac{c}{3} - \frac{2a}{3}\Big]$$

Outlook I

systems to investigate/or in progress

$$\rho(x,t)_t + [\rho(x,t)v(x,t)]_x = 0,$$
$$v(x,t)_x + v(x,t)v(x,t)_x = -\frac{1}{\rho(x,t)}p(x,t)_x,$$
$$T(x,t)_t + v(x,t)T(x,t)_x = \lambda T(x,t)_{xx},$$

1 dim compressible flow + heat conduction with various Equation of states *virial*

$$p(x,t) = a\rho(x,t)^n$$

$$p(x,t) = AT(x,t)\rho(x,t)$$

Looks good ☺

The same system for a spherical bubble: *looks good ☺*

Uniform homogeneous Two-phase flow, 3 equation model

$p \propto \rho_G^k$. ρ_L constant

$$\frac{\partial}{\partial t}(1-\alpha) + \frac{\partial}{\partial s}[(1-\alpha)u] = 0$$

$$\frac{\partial}{\partial t}(\rho_G\alpha) + \frac{\partial}{\partial s}(\rho_G\alpha u) = 0$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial s}\right) = -\frac{\partial p}{\partial s}$$

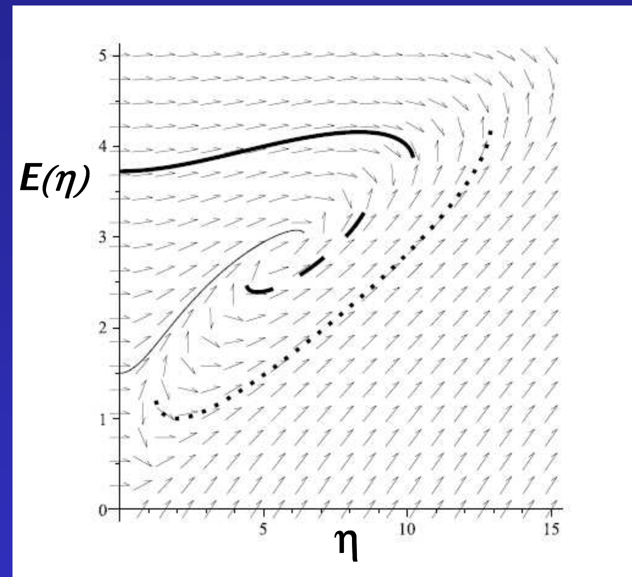
looks bad ☹

Outlook II

Self-similar model is used for additional systems:

non-linear Maxwell equ. to find shock-wave/compact solutions
Idea: power-law field dependent materials $\mu(\mathrm{H}) = a\mathrm{H}^q \qquad \epsilon(\mathrm{E}) = b\mathrm{E}^r$

first results: if q < -1 →



Generalized Cattaneo eq. for heat conduction In solids

$$q_t = -T^\epsilon q - T^{\epsilon+\omega} T_r,$$
$$T_t = -q_r - \frac{q}{r}.$$

almost ready ☺

$$\kappa = \kappa_0 T^\omega, \qquad \tau = \tau_0 T^{-\epsilon}.$$

Thank you for your attention!

Questions, Remarks, Comments?...