Self-similar shock wave solutions for heat conductions in solids and for non-linear Maxwell equations

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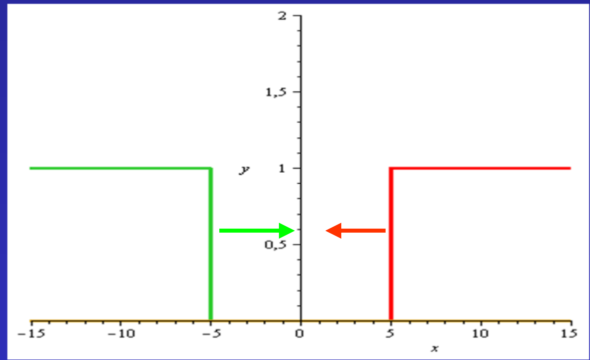
Outline

- **Introduction** (physically relevant solutions for PDEs)
- **Motivation** (infinite propagation speed with the diffusion/heat equation)
- **A way-out** (Cattaneo equ. OR using a hyperbolic first order PDE system
- **Heat conduction model with a non-linear Cattaneo-Vernot law**
- **The non-linear Maxwell equation with self-similar shock wave solutions**
- **Summary**

Important kind of solutions for non-linear PDEs

- **Travelling waves: arbitrary wave fronts**

$u(x,t) \sim g(x-ct), g(x+ct)$



- **Self-similar solutions**

$$u(x,t) = t^{-\alpha} f(x/t^{\beta})$$

Sedov, Barenblatt, Zeldovich

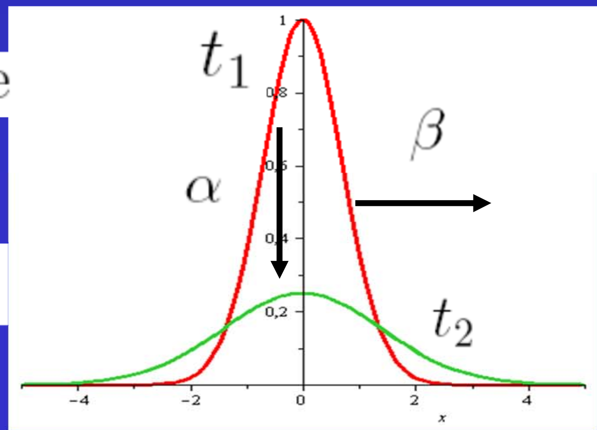
α and β are of primary physical importance

α represents the rate of decay

β is the rate of spread (or contraction if $\beta < 0$)

gives back the Gaussian for heat conduction

$t_1 < t_2$



Ordinary diffusion/heat conduction equation

$$\mathbf{q} = -k\nabla U(x,t), \quad \nabla \mathbf{q} = -\gamma \frac{\partial U(x,t)}{\partial t}$$

U(x,t) temperature distribution
Fourier law + conservation law

- $$\begin{cases} u_t(x,t) - k u_{xx}(x,t) = 0 & -\infty < x < \infty, \quad 0 < t < \infty \\ u(x,t=0) = \delta(x) \end{cases}$$ *parabolic PDA, no time-reversal sym.*

- *strong maximum principle ~ solution is smeared out in time*
- *the fundamental solution:* $$\Phi(x,t) = \int \frac{1}{\sqrt{4\pi k t}} exp\left(-\frac{x^2}{4kt}\right)$$
- *general solution is:*

$$u(x,t) = \int \Phi(x-y,t) g(y) dy \qquad u(x,0) = g(x) \;\; for -\infty < x < \infty \;\; and \;\; 0 < t < \infty$$

- *kernel is non compact = inf. prop. speed*
- *Problem from a long time ☹*
- *But have self-similar solution ☺* $$u(x,t) = t^{-\alpha} f(x/t^{\beta})$$

Our alternatives

- Way 1
- Def. new kind of time-dependent Cattaneo law (with physical background)

→ new telegraph-type equation

→ with self-similar and compact solutions ☺

I.F. Barna and R. Kersner, http://arxiv.org/abs/1002.099

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- Way 2

instead of a 2nd order parabolic(?) PDA

use a first order hyperbolic PDA system with 2 Eqs.

these are not equivalent!!!

→ non-continuous solutions shock waves

Cattaneo heat conduction equ.

$$\tau\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -k\nabla T(x,t)$$

Cattaneo heat conduction law, new term $\tau\frac{\partial \mathbf{q}}{\partial t}$

$$\nabla \mathbf{q} = -\gamma\frac{\partial T(x,t)}{\partial t}$$

Energy conservation law

$$\frac{\partial^2 T(x,t)}{\partial t^2} + \frac{1}{\tau}\frac{\partial T(x,t)}{\partial t} = c^2\nabla^2 T(x,t)$$

T(x,t) temperature distribution
q heat flux
k effective heat conductivity
γ **heat capacity**
τ **relaxation time**
Telegraph equation(exists in Edyn., Hydrodyn.)

$c = \sqrt{k/\tau\gamma}$ is the sound of the transmitted heat wave.

General properties of the telegraph eq. solution

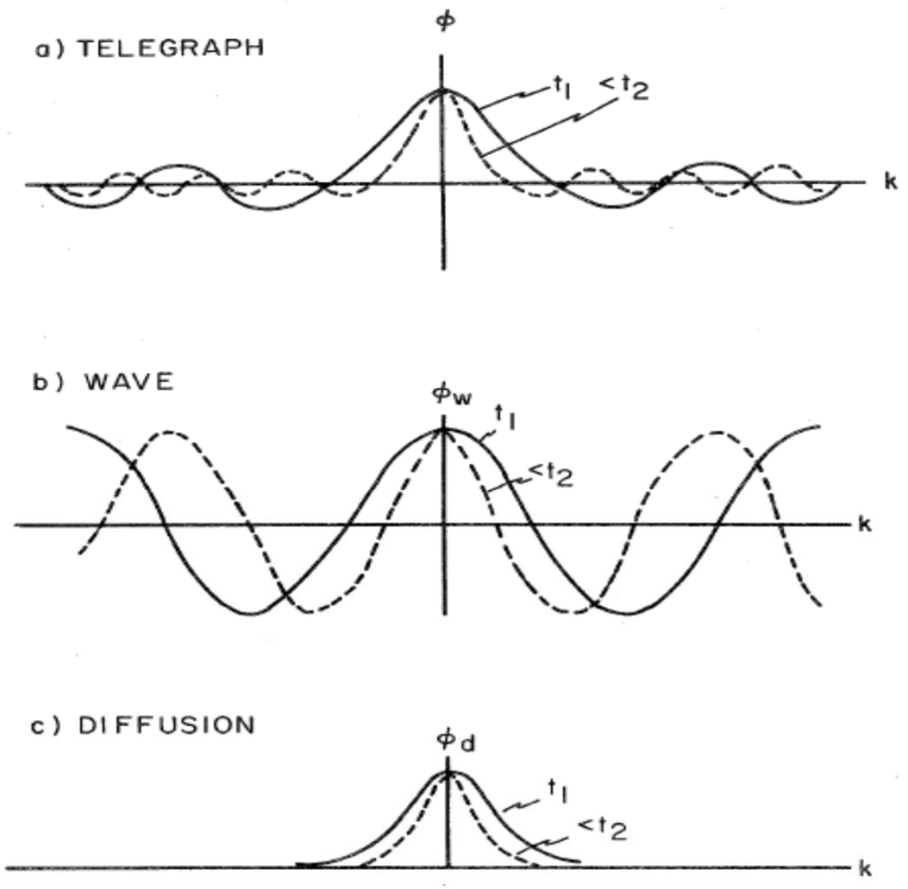


Figure 1. Behaviors in wavenumber space
a) Telegraph equation
b) Wave equation
c) Diffusion equation

decaying travelling waves

$$T(x,t) \propto e^{-\lambda t} f(x-ct)$$

$$T(x,t) = e^{-\lambda t} I_0\left(\frac{\lambda}{2c}\sqrt{(c^2t^2 - x^2)}\right)$$

Bessel function

Problem:

1) no self-similar diffusive solutions $T(x,t) = t^{-\alpha} f(\eta)\ \eta = \frac{x}{t^{\beta}}$.

2) oscillations, T<0 ?
maybe not the best eq.

Self-similar, non-continous shock wave behaviour for heat-propagation (Way 2)

$$\frac{\partial q(r,t)}{\partial t} = -\frac{q}{\tau} - \frac{\kappa}{\tau}\frac{\partial T(r,t)}{\partial r}$$
$$c_0\frac{\partial T(r,t)}{\partial t} = -\frac{\partial q(r,t)}{\partial r} - \frac{q(r,t)}{r}$$

general Cattaneo heat conduction law, + cylindrically symmetric conservation law

heat conduction coefficient (temperature dependent e.q. plasmas) relaxation time also temperature dependent (e.q. plasma phys.)

$$\kappa = \kappa(T) = \kappa_0 T^{\omega}$$

$$\tau = \tau(T) = \tau_0 T^{\epsilon}.$$

using the first oder PDA system (not second order) looking for self-similar solutions in the form

$$T(r,t) = t^{-\alpha} f\left(\frac{r}{t^{\beta}}\right),$$

$$q(r,t) = t^{-\delta} g\left(\frac{r}{t^{\gamma}}\right)$$

Way to the solutions

The following universality relations are hold:

$$\alpha = 2\beta$$

$$\alpha = \frac{1}{\omega+1}, \quad \beta = \frac{1}{2(\omega+1)}, \quad \delta = \frac{2\omega+3}{2(\omega+1)}, \quad \epsilon = \omega + 1.$$

Form of the solutions:
Note the ω dependence representing different physics

$$T = t^{\frac{-1}{\omega+1}} f\left(\frac{r}{t^{\frac{1}{2(\omega+1)}}}\right) \qquad q = t^{\frac{2\omega+3}{2(\omega+1)}} g\left(\frac{r}{t^{\frac{1}{2(\omega+1)}}}\right)$$

$$\kappa = \kappa_0 t^{\frac{-\omega}{\omega+1}} f^{\omega}\left(\frac{r}{t^{\frac{1}{2(\omega+1)}}}\right) \qquad \tau = \kappa_0 t^{-1} f^{\omega+1}\left(\frac{r}{t^{\frac{1}{2(\omega+1)}}}\right)$$

The ODE system for the shape functions:

$$\begin{aligned} \delta g + \beta\eta g' &= g f^{\omega+1} + f^{2\omega+1} f' \\ (\eta g)' &= \beta(\eta^2 f)' \end{aligned}$$

The second eq. can be integrated getting the relation:

$$g = \beta\eta f$$

The solutions

The final ODE reads:

$$\frac{df}{d\eta^2}\left(\beta^2\eta^2 - f^{2\omega+1}\right) = \frac{\beta f}{2}[f^{\omega+1} - (2\beta+1)].$$

Just putting:

$y = \eta^2$ and $x = f$.

Getting:
which is linear in y

$$\frac{dy}{dx} = \frac{y(x) - 4(\omega+1)^2 x^{2\omega+1}}{x[(\omega+1)x^{\omega+1} - \omega - 2]}$$

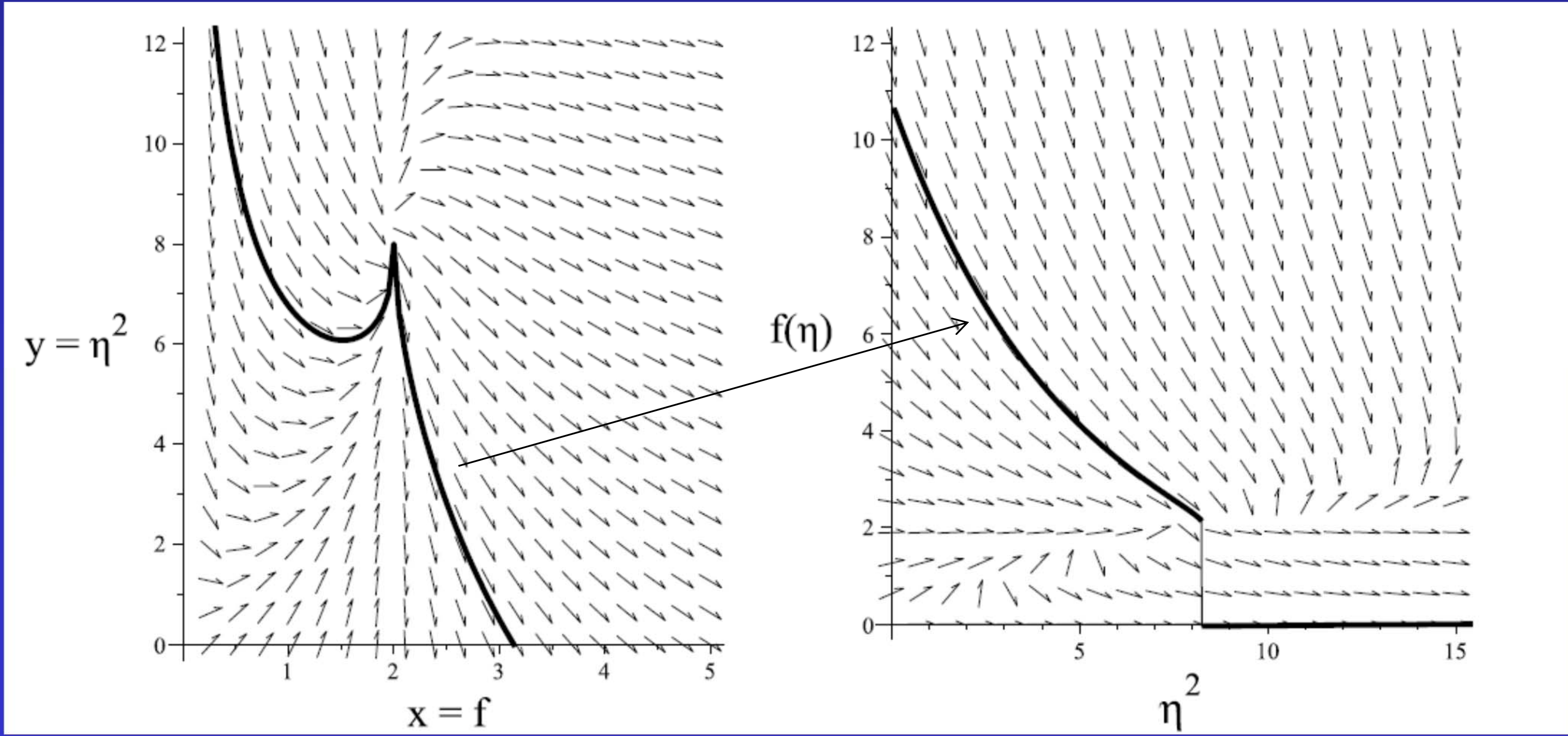
The first case is for $\omega = 0, (\alpha = 1, \beta = 1/2, \delta = 3/2, \epsilon = 1)$

$$y' = (y - 4x)/x(x-2)$$

$$y = 8 + [(x-2)/x]^{1/2}[c_1 - 8ln(\sqrt{x} + \sqrt{x-2})]$$

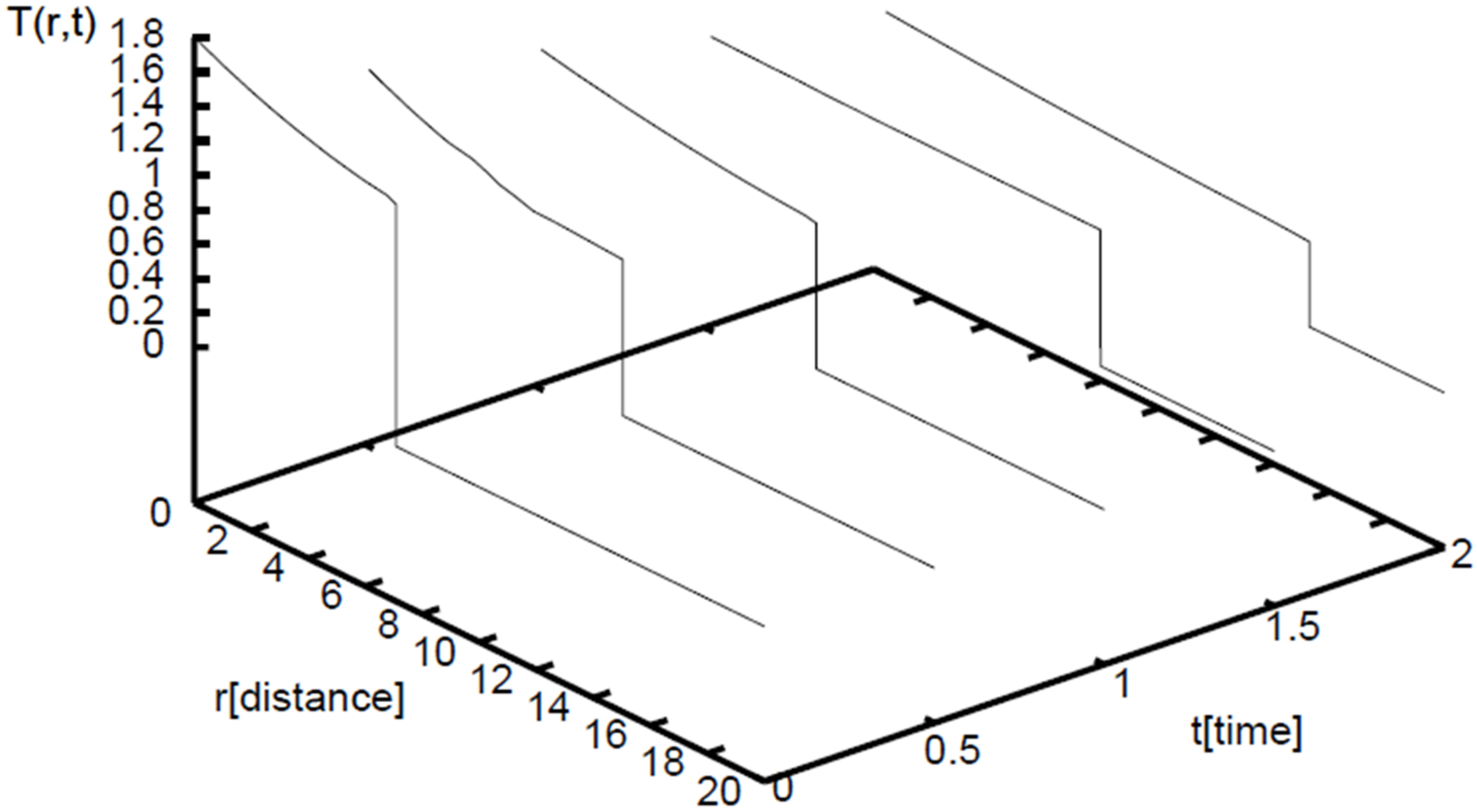
The solutions

The direction field of the solutions for y and
for the inverse function for eta



Arrow shows how the inverse function were defined CUT and 0 solution

How the shock propagates in time



The solutions

The second case is for $\omega = -1/2, (\alpha = 2, \beta = 1, \delta = 2, \epsilon = 1/2)$

The corresponding ODE : $$\frac{dy}{dx} = 2(y-1)/[x(\sqrt{x}-3)]$$

solution for the shape function: $$y = c_2 x^{-2/3}(x^{1/2}-3)^{4/3}$$

Returning to original variables: $$f = 9/[(\eta^2)^{3/4}+1]^2$$

And the final analytic and continous solutions are:

$$T = \frac{9t}{(r^{3/2}+t^{3/2})^2}$$

$$q = \frac{9r}{(r^{3/2}+t^{3/2})^2}$$

Summary of the solutions

$\omega < -1$ unphysical regime (negative flux etc.)

$\omega \neq -1$ forbidden value

$-1 < \omega \leq -1/2$ continuous solutions

$\omega = -1/2$ critical exponent

$-1/2 < \omega$ shocks always appear

The second example

Idea is similar:
instead of the second order wave equation
we investigate the two coupled first-order Maxwell PDEs in one dimension
with a non-linear power-law
material or constitutive equation

depending on the exponent
(meaning different material and physics)

Having continous or shock wave solutions

The Maxwell equations & non-linearity

The field equations of Maxwell

$$\nabla \cdot \mathbf{D} = \rho, \qquad \nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = \frac{\partial \mathbf{H}}{\partial t}, \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

The constitutive Eqs + differential Ohm's Law

$$\mathbf{D} = \epsilon \mathbf{E} \qquad \mathbf{B} = \mu \mathbf{H} \qquad \mathbf{J} = \sigma \mathbf{E}$$

Our consideration, power law:

$$\mu(\mathbf{H}) = a\mathbf{H}^q \qquad \epsilon(\mathbf{E}) = b\mathbf{E}^r \qquad \sigma(\mathbf{E}) = h\mathbf{E}^p \quad a, b, h, p, r, q \; are \; \mathrm{Real}$$

But:

$$\text{constrain } c^2 = \frac{1}{\mu\epsilon}$$

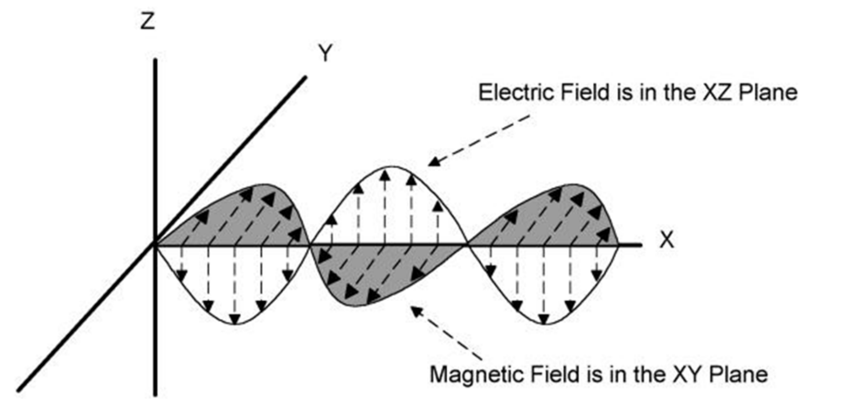
→

$$\mu(\mathbf{H}) = a\mathbf{H}^q \qquad \epsilon(\mathbf{H}) = \frac{1}{c^2 a^1 H^q}$$

Geometry & Applied Ansatz

For the sake of simplicity we consider the following one dimensional problem

$$\mathbf{E} = (0, E_y(x,t), 0) \qquad \mathbf{H} = (0, 0, H_z(x,t))$$



$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}, \qquad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

$$\frac{\partial E_y}{\partial x} = -\frac{\partial H_z}{\partial t}, \qquad -\frac{\partial H_z}{\partial x} = \frac{\partial D_y}{\partial t} + J_y$$

$$\begin{aligned} E_y(x,t) &= t^{-\alpha} f\left(\frac{x}{t^\beta}\right) = t^{-\alpha} f(\eta) \\ H_z(x,t) &= t^{-\delta} g\left(\frac{x}{t^\beta}\right) = t^{-\delta} g(\eta) \end{aligned}$$

where α, β, δ are **Real**

$$\begin{aligned} \frac{\partial}{\partial x}[t^{-\alpha} f(\eta)] &= -\frac{\partial}{\partial t}[a t^{-\delta(q+1)} g^{q+1}(\eta)] \\ -\frac{\partial}{\partial x}[t^{-\delta} g(\eta)] &= \frac{\partial}{\partial t}[c^{-2} a^{-1} t^{\delta q - \alpha} g^{-q}(\eta) f(\eta)] + h t^{-\alpha(p+1)} f^{p+1}(\eta), \quad \eta = x/t^\beta \end{aligned}$$

The final ordinary differential equations

$$f' = a(q+1)[\delta g^{q+1} - g^q g' \eta \beta]$$
$$-g' = \frac{1}{ac^2}[(q+1)g^q f + q(q+1)g^{q-1}g' f\eta + (q+1)g^q f'\eta]$$

' means derivation with respect to eta

Universality relations among the parameters:

$\beta = -q - 1$ $\alpha = -1$ $\delta = 1$ $p = 1$

The first equ. is a total difference so can be integrated :

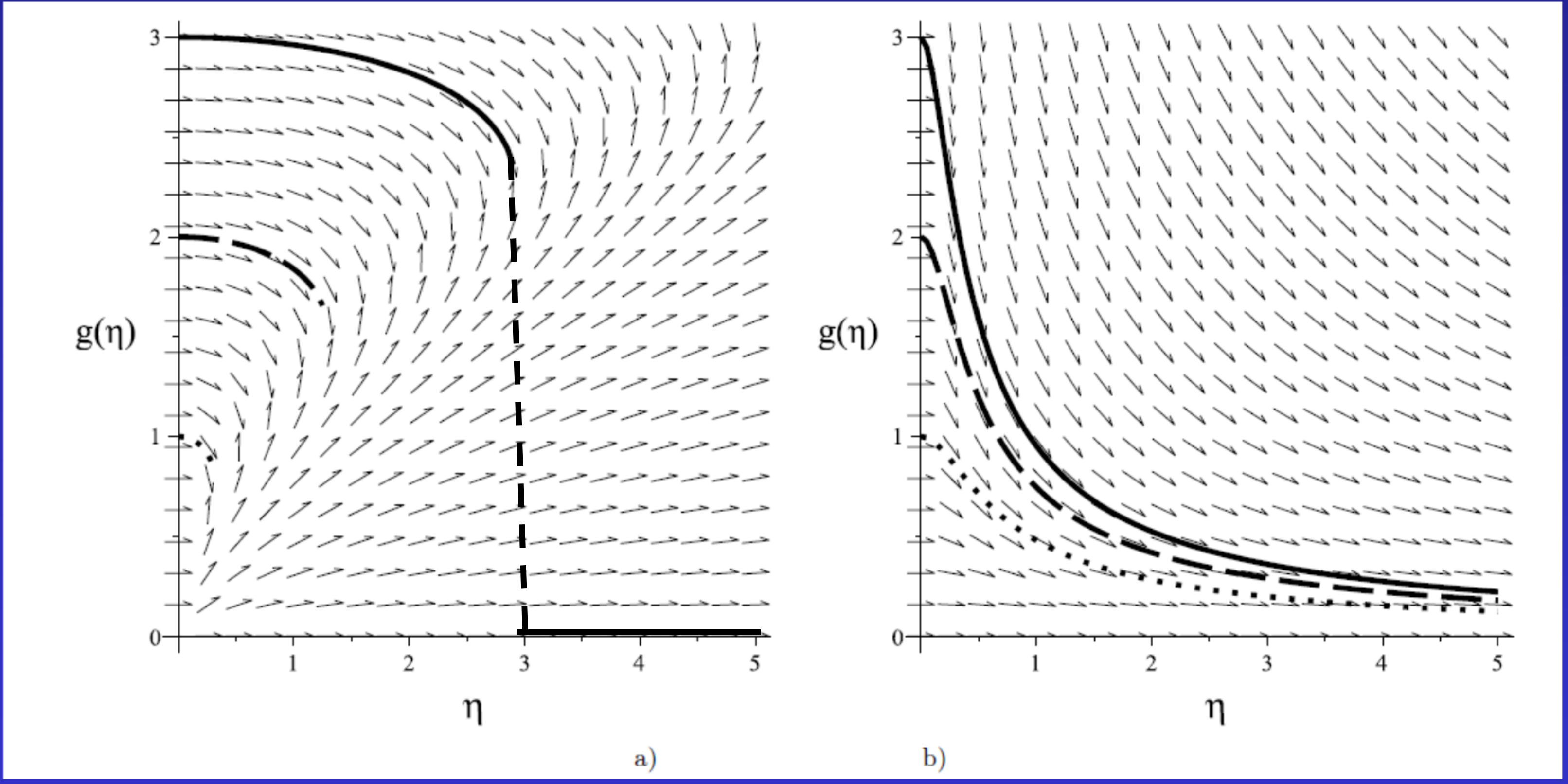
$$f = a(q+1)\eta g^{q+1}$$

Remaining a simple ODE with one parameter q (material exponent):

$$-g' = \frac{2(q+1)^2 \eta g^{2q+1} + h}{1 + (2q+1)(q+1)^2 \eta^2 g^{2q}}$$

Different kind of solutions

No closed-forms are available so the direction fields are presented



Compact solutions for $q < -1$
shock-waves

Non-compact solutions for $q > -1$

Physically relevant solutions

We consider the Poynting vector which gives us the energy flux (in W/m^2) in the field

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = t^{-\alpha-\delta} f g = a(q+1)\eta g^{q+2}$$

Note that for $q < -2$ the $\int_0^{cut} S d\eta$ is finite which is a good reason.

Unfortunatelly, there are two contraversial definition of the Poynting vector in media, unsolved problem R.N. Pfeifer et al. Rev. Mod. Phys. 79 (2007) 1197.

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I.F. Barna Laser Phys. 24 (2014) 086002 &
arXiv : http://arxiv.org/abs/1303.7084

Summary

we presented physically important,
self-similar solutions for various PDEs
which can describe shock waves

as a first example we investigated a genaralized one dimensional Cattaneo-Vernot heat conduction problem

as a second example we investigated the 1 dim. Maxwell equations (instead fo the wave equation) closing with non-linear (power law) constitutive equations

In both cases we found shock-wave solutions for different material constants

Thank you for your attention!

Questions, Remarks, Comments?...