Analytic solutions for the three dimensional compressible Navier-Stokes equation

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Abstract

We investigate the three dimensional compressible Navier-Stokes and the continuity equations in Cartesian coordinates for Newtonian fluids. The problem has an importance in different fields of science and engineering like fluid, aerospace dynamics or transfer processes. Finding an analytic solution may bring a considerable progress in understanding the transport phenomena and in the design of different equipments where the Navier Sotkes equation is applicable. For solving the equation the polytropic equation of state is used as closing condition. The key idea is the three-dimensional generalization of the well-known self-similar Ansatz which was already used for non-compressible viscous flow in our former study. The geometrical interpretations of the trial function is also discussed. We compared our recent results to the former non-compressible ones.

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I. INTRODUCTION

In the process of understanding the environment there is a need of a knowledge of their features. The surrounding world of beings or even certain life mechanisms of them is strongly related to the phenomena of gaseous or liquid phase. The notion of fluid tries to connect relevant properties of these two phases. The study of fluid means finding a basic equation of it based on mechanical and conservation laws. The fundamental equation is the Navier-Stokes (NS) equation, which in a quite wide range of velocity describes the behavior of the fluid. By this the equation becomes applicable in biophysics, meteorology, and aerospace dynamics. Once the equation is settled appears the need of solving it. The NS system is sufficiently complex to give certain difficulties in the attempts which go for a solution. Certain steps has been made for a better understanding of the NS equation by finding quasi-conservation laws [1]. Practical aspects regarding chemical reactions in a Navier-Stokes flow can be found in [2]. Applications and transfer phenomena [3] may require solutions for NS equation as general as possible. With the help of certain numerical methods one tries to shed lights on the evolution of flow fields determined by the equation [4, 5]. An interesting review has been realized in Ref. [6], where one may find discussions about possible ways, which may lead to a solution of the three dimensional incompressible Navier Stokes equation. Regarding the present work we focus on the *compressible* NS equation and we will try to find a solution of it. The study involves mathematical techniques, which for certain type of systems of partial differential equations have been successfully applied.

To study the dynamics of viscous compressible fluids the compressible Navier-Stokes (NS) partial differential equation (PDE) together with the continuity equation have to be investigated. In Eulerian description in Cartesian coordinates these equations are the following:

$$\begin{aligned} \rho_t + div[\rho \mathbf{v}] &= 0 \\ \rho[\mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v}] &= \nu_1 \triangle \mathbf{v} + \frac{\nu_2}{3} grad\, div\, \mathbf{v} - \nabla p + a \end{aligned} \tag{1}$$

where $\mathbf{v}, \rho, p, \nu_{1,2}$ and a denote respectively the three-dimensional velocity field, density, pressure, kinematic viscosities and an external force (like gravitation) of the investigated fluid. To avoid further misunderstanding we use a for external field instead of the letter g which is reserved for a self-similar solution. In the later we consider no external force, so $a = 0$. For physical completeness we need an equation of state (EOS) to close the equations. We start with the polytropic EOS $p = \kappa\rho^n$, where κ is a constant of proportionality to fix the dimension and n is a free real parameter. (n is usually less than 2) In astrophysics, the Lane −Emden equation is a dimensionless form of the Poisson's equation for the gravitational potential of a Newtonian self-gravitating, spherically symmetric, polytropic fluid. It's solution is the polytropic EOS which we use in the following. The question of more complex EOSs will be concerned shortly later on. Now $\nu_{1,2}, a, \kappa, n$ are parameters of the flow. For a better transparency we use the coordinate notation $\mathbf{v}(x,y,z,t) = (u(x,y,z,t), v(x,y,z,t), w(x,y,z,t))$ and for the scalar density variable $\rho(x,y,z,t)$ from now on. Having in mind the correct forms of the mentioned complicated vector operations, the PDE system reads the following:

$$
\begin{aligned}
\rho_t + \rho_x u + \rho_y v + \rho_z w + \rho[u_x + v_y + w_z] &= 0 \\
\rho[u_t + uu_x + vu_y + wu_z] - \nu_1[u_{xx} + u_{yy} + u_{zz}] - \frac{\nu_2}{3}[u_{xx} + v_{xy} + w_{xz}] + \kappa n \rho^{n-1}\rho_x &= 0 \\
\rho[v_t + uv_x + vv_y + wv_z] - \nu_1[v_{xx} + v_{yy} + v_{zz}] - \frac{\nu_2}{3}[u_{xy} + v_{yy} + w_{yz}] + \kappa n \rho^{n-1}\rho_y &= 0 \\
\rho[w_t + uw_x + vw_y + ww_z] - \nu_1[w_{xx} + w_{yy} + w_{zz}] - \frac{\nu_2}{3}[u_{xz} + v_{yz} + w_{zz}] + \kappa n \rho^{n-1}\rho_z &= 0. \quad (2)
\end{aligned}
$$

The subscripts mean partial derivations. Note, that the formula for EOS is already applied.

There is no final and clear-cut existence and uniqueness theorem for the most general non-compressible NS equation. However, large number of studies deal with the question of existence and uniqueness theorem related to various viscous flow problems. Without completeness we mention two works which (together with the references) give a transparent overview about this field [7, 8].

According to our best knowledge there are no analytic solutions for the most general three dimensional NS system even for non-compressible Newtonian fluids.

However, there are various examination techniques available in the literature with analytic solutions for the restricted problem in one or two dimensions. Manwai [9] studied the N-dimensional ($N \geq 1$) radial Navier-Stokes equation with different kind of viscosity and pressure dependences and presented analytical blow up solutions. His works are still 1+1 dimensional (one spatial and one time dimension) investigations. Another well established and popular investigation method is based on Lie algebra. There are other numerous studies available. Some of them are for the three dimensional case, for more see [10]. Unfortunately, no explicit solutions are shown and analyzed there. Fushchich *et al.* [11] construct a complete set of $\tilde{G}(1,3)$-inequivalent Ansätze of co-dimension one for the NS system, they present 19 different analytical solutions for one or two space dimensions.

Recently, Hu *et al.* [13] presents a study where symmetry reductions and exact solutions of the (2+1)-dimensional NS are calculated. Aristov and Polyanin [14] use various methods like Crocco transformation, generalized separation of variables or the method of functional separation of variables for the NS and present large number of new classes of exact solutions. Sedov in his classical work [15] derive analytic solutions for the tree dimensional spherical NS equation where all three velocity components and the pressure have only polar angle dependence (θ). Even this kind of restricted symmetry leads to a non-linear coupled ordinary differential equation system with a very rich mathematical structure. Additional similarity reduction studies are available from various authors as well [16–18]. A full three dimensional Lie group analysis is available for the three dimensional Euler equation of gas dynamics, with polytropic EOS [19] unfortunately without any kind of viscosity. Analytical solutions of the Navier-Stokes equations for non-Newtonian fluid is presented for one radial and one time dimension by [20].

In our study we apply the physically relevant self-similar Ansatz to system (2).

The form of the one-dimensional self-similar Ansatz is well-known [15, 21, 22]

$$T(x,t) = t^{-\alpha} f\left(\frac{x}{t^{\beta}}\right) := t^{-\alpha} f(\eta), \tag{3}$$

where $T(x,t)$ can be an arbitrary variable of a PDE and t means time and x means spatial dependence. The similarity exponents α and β are of primary physical importance since α represents the rate of decay of the magnitude $T(x,t)$, while β is the rate of spread (or contraction if $\beta < 0$) of the space distribution $fort > 0$. The most powerful result of this Ansatz is the fundamental or Gaussian solution of the Fourier heat conduction equation (or for Fick's diffusion equation) with $\alpha = \beta = 1/2$. These solutions are exhibited on Figure 1. for time-points $t_1 < t_2$. This transformation is based on the assumption that a self-similar solution exists, i.e., every physical parameter preserves its shape during the expansion. Self-similar solutions usually describe the asymptotic behavior of an unbounded or a far-field problem; the time t and the space coordinate x appear only in the combination of $f(x/t^{\beta})$. It means that the existence of self-similar variables implies the lack of characteristic lengths and times. These solutions are usually not unique and do not take into account the initial stage of the physical expansion process.

There is a reasonable generalization of (3) in the form of $T(x,t) = h(t) \cdot f[x/g(t)]$, where $h(t), g(t)$ are continuous functions. The choice of $h(t) = g(t) = \sqrt{T-t}$ is called the blow-up

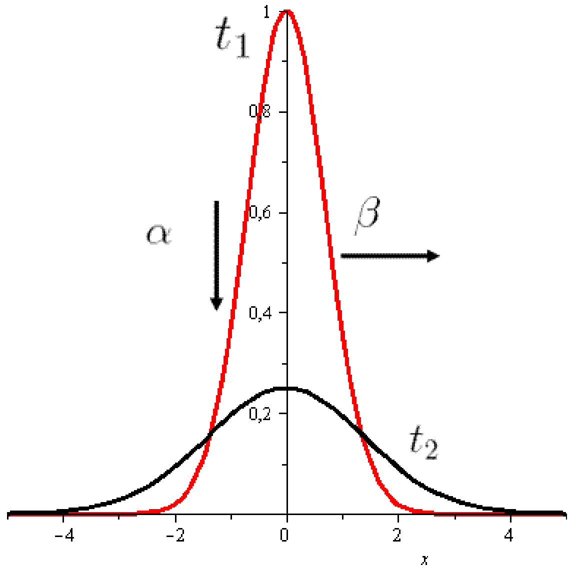

FIG. 1: A self-similar solution of Eq. (3) for $t_1 < t_2$. The presented curves are Gaussians for regular heat conduction.

solution, which means that the solution goes to infinity during a finite time duration.

Leray [23] in his pioneering work in 1934 at the end of the manuscript put the question whether it is possible to construct self-similar solutions to the NS system in $\mathbf{R}^3$ in the form of $p(x,t) = \frac{1}{T-1}P(x/\sqrt{T-t})$ and $\mathbf{v}(x,t) = \frac{1}{\sqrt{T-t}}\mathbf{V}(x/\sqrt{T-t})$. In 2001 Miller *et al.* [24] proved the non-existence of singular pseudo-self-similar solutions of the NS system in the above form. Okamato has given an exact backward finite-time blow-up self-similar solution via Leray's scheme.[25]

Unfortunately, there is no any direct analytic calculation for the three dimensional self-similar generalization of this Ansatz in the literature.

The applicability of Eq. (3) is quite wide and comes up in various transport systems [15, 21, 22, 26–28]. This Ansatz can be generalized for two or three dimensions in various ways. One is the following

$$u(x,y,z,t) = t^{-\delta} g\left(\frac{F(x,y,z)}{t^{\beta}}\right) = t^{-\delta} g(\eta), \tag{4}$$

where $F(x,y,z)$ can be understood as an implicit parametrization of a two dimensional surface. One of the most simple function (which we use in the following) is $F(x,y,z) = x+y+z = 0$ which represents a plane passing through the origin. It can be easily shown than even a more general plane, like $ax + by + dz + 1 = 0$ makes the remaining ODE system much more complicated. (The second term in the NS equation on the right hand side (*grad div* $\mathbf{v}$ term) creates distinct a^2, b^2, c^2 terms which cannot be transformed out, and a coupled system of three equations remain.)

At this point we can give a geometrical interpretation of the Ansatz. Note that the dimension of $F(x,y,z)$ still has to be a spatial coordinate. With this Ansatz we consider the velocity field ($\mathbf{v_x} = u$) - where the sum of the spatial coordinates lies on a plane - as a new entity. We are not considering all the $\mathbf{R}^3$ velocity fields but a plane of the $\mathbf{v}_x$ coordinate as an independent variable. This is the trick of the Ansatz. The NS equation which is responsible for the dynamics maps this kind of velocities which are on this plane surface to another more complex geometry. In this sense we can investigate the dynamical properties of the NS equation in details.

II. SELF-SIMILAR SOLUTION

Now, we concentrate on the first Ansatz Eq. (4) and search the solution of the Navier-Stokes PDE system in the following form:

$$\rho(x,y,z,t) = t^{-\alpha} f\left(\frac{x+y+z}{t^{\beta}}\right) = t^{-\alpha} f(\eta), \quad u(\eta) = t^{-\delta} g(\eta),$$
$$v(\eta) = t^{-\epsilon} h(\eta), \quad w(\eta) = t^{-\omega} i(\eta), \tag{5}$$

where all the exponents $\alpha, \beta, \delta, \epsilon, \omega$ are real numbers. (Solutions with integer exponents are called self-similar solutions of the first kind and can be obtained from dimensional argumentation as well.) According to Eq. (2), we need to calculate all the first time derivatives of the velocity field, all the first and second spatial derivatives of the velocity field and the first spatial derivatives of the pressure. All these derivatives are not presented in details. More technical details of such a derivation is presented and explained in our former study [29]. To get a final ODE system which depends only on the variable η, the following universality relations have to be hold

$$\alpha = \beta = \frac{2}{n+1} \qquad \& \qquad \delta = \epsilon = \omega = \frac{2n-2}{n+1}, \tag{6}$$

where $n \neq -1$. Note, that the self-similarity exponents are not fixed values thanks to the existence of the polytropic EOS exponent n. (In other systems e.g. heat conduction or non-compressible NS system, all the exponents have a fixed value, usually $+1/2$.) This means that our self-similar Ansatz is valid for different kind of materials with different kind of EOS. Different exponents represent different materials with different physical properties which results different final ODEs with diverse mathematical properties.

At this point we have to mention that a NS equation even with a more complicated EOS like $p \sim f(\rho^l v^m)$ can have a self-similar solution. The investigation of such problems will be performed in the near future but not in the recent study.

Our goal is to analyze the asymptotic properties of Eq. (5) with the help of Eq. (6). According to Eq. (3) the signs of the exponents automatically dictates the asymptotic behavior of the solution at sufficiently large time. All physical velocity components should decay at large times for a viscous fluid without external energy source term. The role of α and β was explained after Eq. (3). Figure 2 shows the $\alpha(n)$ and $\delta(n)$ functions. There are five different regimes:

- $n > 1$ all exponents are positive - physically fully meaningful scenario - spreading and decaying density and all speed components for large time - will be analyzed in details for general n
- $n = 1$ spreading and decaying density in time and spreading but non-decaying velocity field in time - not completely physical but the simplest mathematical case
- $-1 \leq n \leq 1$ spreading and decaying density in time and and spreading and enhancing velocity in time - not a physical scenario
- $n \neq -1$ not allowed case
- $n \leq -1$ sharpening and enhancing density and sharpening and decaying velocity in time, we consider it an non-physical scene and neglect further analyzis.

The corresponding coupled ODE system is:

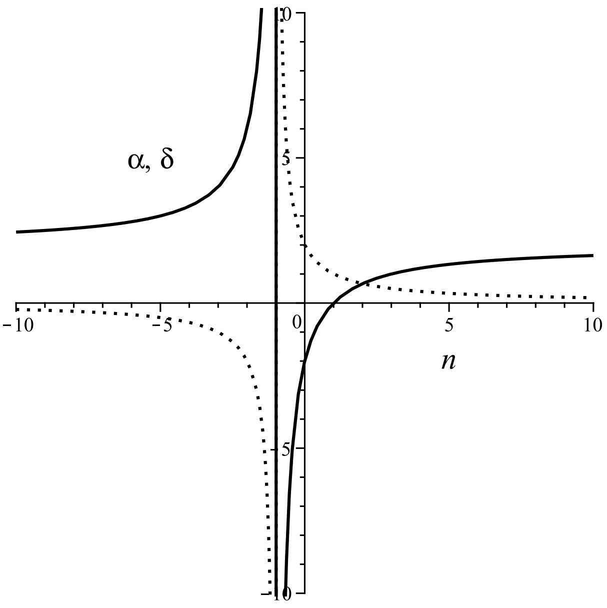

FIG. 2: Eq. (6) dotted line is $\alpha(n) = 2/(n+1)$ and solid line is $\delta(n) = 2 - 4/(n+1)$.

$$
\begin{aligned}
\alpha[f + f'\eta] &= f'[g+h+i] + f[g'+h'+i'], \\
f[-\delta g - \alpha\eta g' + gg' + hg' + ig'] &= -\kappa n f^{n-1} f' + 3\nu_1 g'' + \frac{\nu_2}{3}[g''+h''+i''], \\
f[-\delta h - \alpha\eta h' + gh' + hh' + ih'] &= -\kappa n f^{n-1} f' + 3\nu_1 h'' + \frac{\nu_2}{3}[g''+h''+i''], \\
f[-\delta i - \alpha\eta i' + gi' + hi' + ii'] &= -\kappa n f^{n-1} f' + 3\nu_1 i'' + \frac{\nu_2}{3}[g''+h''+i''],
\end{aligned} \tag{7}
$$

where prime means derivation with respect to η. The first (continuity) equation is a total derivative (if $\alpha = \beta$) so we can integrate automatically getting $\alpha f \eta = f[g+h+i] + c_0$, where c_0 is proportional to the mass flow rate. Now, we simplify the NS equation with introducing only a single viscosity $\nu = \nu_1 = \nu_2$. There are still too many free parameters remain for the general investigation.

We consider the $c_0 = 0$ from now on. Having in mind that the density of a fluid should be positive so $f \neq 0$, we get $\alpha\eta = g+h+i$. With the help of the first and second derivatives of this formula Eq. (7) can be reduced to the next non-linear first order ODE

$$
-3\kappa n f^{n-1} f' + \left(\frac{4n-4}{(n+1)^2} \right) \eta f = 0. \tag{8}
$$

Note, that it is a first order equation, so there is a conserved quantity which should be a kind of general impulse in the parameter space η. We can also see that this equation has no contribution from the viscous terms with ν just from the pressure and from the convective term. The general solution reads

$$f(\eta) = 3^{\frac{-1}{n-1}} \left(\frac{2\eta^2[n-1]}{\kappa n[n+1]} + 3c_1 \right)^{\frac{1}{n-1}}. \tag{9}$$

Note that for $\{n; n \in \mathbf{Z}\backslash\{-1\}\}$ exists n different solutions for $n > 0$ (one of them is the $f(\eta) = 0$) and n-1 different solutions for $n < 0$ these are the n or (n-1)th roots of the expression. For $\{n : n \in \mathbf{R}\backslash\{-1\}\}$ there is one real solution. Note, that for fixed $\kappa, c_1 > 0$ when n and η tend to infinity, the limit of Eq. (9) tends to zero. This meets our physical intuition for a viscous flow, we get back solutions which have an asymptotic decay. (In the limiting case $n = 1$ (which means the $\delta = 0$) we get back the trivial result $f = const$ which has no relevance.) For the $n = 2$, the least radical case $f(\eta) = \eta^2/(27\kappa) + c_1$ which is a quadratic function in η however, the density function $\rho = t^{-2/3}[(x+y+z)^2/t^{4/3}] = (x+y+z)^2/t^2$ has a proper time decay for large times.

All the three velocity field components can be derived independently from the last three Eqs. (7). For the $v = t^{-\delta}g(\eta)$ the ODE reads:

$$-3\nu g'' + \left(\frac{2n-2}{n+1} \right) gf - \kappa n f^{n-1} f' = 0. \tag{10}$$

Unfortunately, there is no solution for general n in a closed form. However, for $n = 2$ the solutions can be given inserting $f(\eta) = \eta^2/(27\kappa)$ into Eq. (10). These are the Whittaker W and Whittaker M functions [30]

$$g = \frac{\tilde{c}_1}{\sqrt{\eta}} \mathrm{M}_{-\frac{c_1\sqrt{2\kappa}}{4\sqrt{\nu}}, \frac{1}{4}} \left(\frac{\sqrt{2}\eta^2}{9\sqrt{\nu\kappa}} \right) + \frac{\tilde{c}_2}{\sqrt{\eta}} \mathrm{W}_{-\frac{c_1\sqrt{2\kappa}}{4\sqrt{\nu}}, \frac{1}{4}} \left(\frac{\sqrt{2}\eta^2}{9\sqrt{\nu\kappa}} \right) + \frac{2}{3}\eta, \tag{11}$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are integration constants. The M is the irregular and the W is the regular Whittaker function, respectively. These functions can be expressed via the Kummer's confluent hypergeometric functions M and U in general (for details see [30])

$$\begin{aligned} M_{\lambda,\mu}(z) &= e^{-z/2} z^{\mu+1/2} M(\mu - \lambda + 1/2, 1 + 2\mu; z); \\ W_{\lambda,\mu}(z) &= e^{-z/2} z^{\mu+1/2} U(\mu - \lambda + 1/2, 1 + 2\mu; z). \end{aligned} \tag{12}$$

Is some special cases when $\kappa = \nu/2$ the Whittaker functions can formally be expressed with other functions (e.g. Bessel, Err) when $\{c_1 : c_1 \in \mathbf{N}\backslash\{-2, -4\}\}$. It can be shown with the help of asymptotic forms that the velocity field $u \sim t^{-1/3}[M \mathrm{or} W(\cdot, \cdot; t^{-4/3})]$ decays for sufficiently large time which is a physical property of a viscous fluid. (Just to mention, we found additional closed solutions only for $n = 1/2$ and for $n = 3/2$ from Eq. (9-10) for the density and velocity field which contain the HeunT functions, in a confusingly complicated expression.)

At this point we compare our recent results to the former non-compressible ones. In the non-compressible case of the three dimensional NS equation, all the exponents have the 1/2 value - like in the regular diffusion equation - except the decaying exponent of the pressure field which is 1. For non-compressible fluids the x component of the velocity field is described with the help of the Kummer functions

$$\tilde{g}(\eta) = c_1 U\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\eta+c)^2}{6\nu}\right) + c_2 M\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\eta+c)^2}{6\nu}\right) + \frac{c}{3} - \frac{2a}{3}, \tag{13}$$

where c_1 and c_2 are the usual integration constants. The viscosity is ν the external field is a and the c is the non-zero integration constant from the continuity equation - which can be set to zero. We showed in our former study than only one of the velocity components can be written in the following form, the other two component do not have a closed form. Additional properties of this formula was analyzed in our last study[29] in depth.

Figure 3 compares the regular parts of the solutions of Eq. (11) and Eq. (13) with the same viscosity value $\nu = 0.1$ and for $\tilde{c}_1 = 1, \tilde{c}_2 = 1, c_1 = 0$. The compressible parameters are $\kappa = 1$ and $n = 2$. Note that the shape function of velocity of the compressible flow has a maximum and a quick decay, the incompressible velocity shape function has no decay. However, these are the reduced one dimensional shape functions, and the total thee dimensional velocity fields have proper time decay for large time as it should be. The c_1 in the Whittaker function cannot be negative because it comes from the density equation. If it is zero or any other positive number plays no difference in the form of the shape function.

Figure 4 presents how the regular part of the solution Eq. (11) depends on the compressibility for a well defined viscosity. Note, the higher the compressibility the lower the maximum of the top speed of the system.

As a second case study Figure 5 presents how the regular part of the solution Eq. (11) depends on the viscosity for a well defined compressibility. Higher the viscosity the higher the maximal reached speed and the range of the system. In our investigation the role of the two viscosities cannot be separated from each other therefore this effect cannot be seen more clearly.

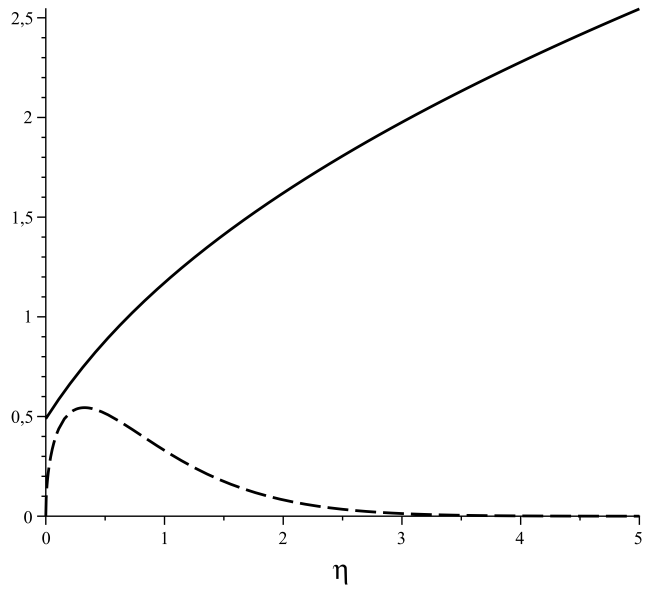

FIG. 3: Comparison of the regular solutions for the non-compressible (solid line) Eq. (13) and the compressible case(dashed line) Eq. (11). The viscosities have the same numerical value $\mu = 0.1$.

Note, that Eq. (11) is not a direct limit of Eq. (13) just a very similar one. Therefore this is the maximum what we could learn from the system.

Our long range aim is to study the properties of non-Newtonian fluids or fluids with various heat conduction mechanisms.

III. SUMMARY

In our study we investigated the compressible three dimensional Navier-Stokes equation with the self-similar Ansatz. The existence of the polytropic EOS for the compressibility makes the calculations more complicated than for the non-compressible case. There is no general closed form self-similar solution available for the density and the velocity fields for any kind of material. There are different scenarios available, some materials $n < 1$ dictate non-physical exploding solutions for $n > 1$ both density and velocity fields show a decay property for large times which is the only reasonable solutions for dissipative systems. There is a special case for $n = 2$ where both fields can be expressed with the help of closed formulas which is our mayor result. We could compare the velocity fields of the compressible and non-compressible fluids. The formulas show some kind of similarity which was analyzed in details. In the future we plan to analyse non-Newtonian fluids with the same kind of

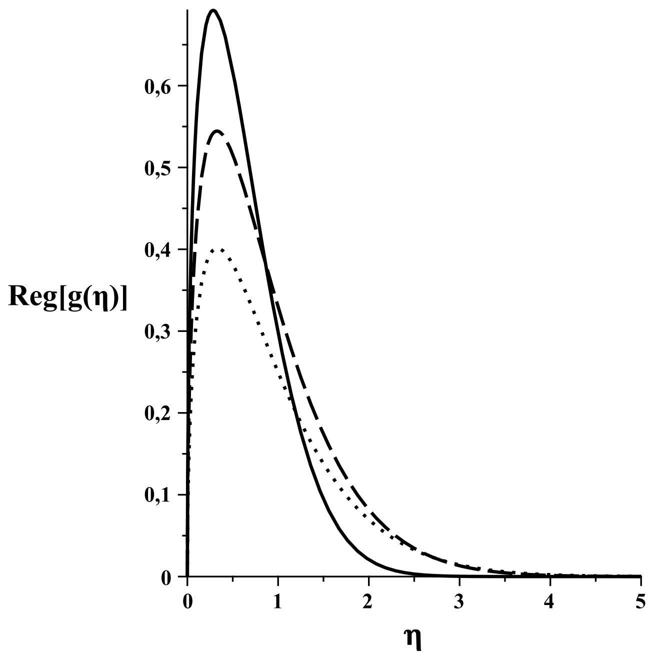

FIG. 4: The compressibility dependence of the regular solution of Eq. (11) for $n = 2$ and for $\nu = 0.1$ viscosity. The solid line is for $\kappa = 0.1$ the dotted line is for $\kappa = 1$ and the dashed line is for $\kappa = 2$.

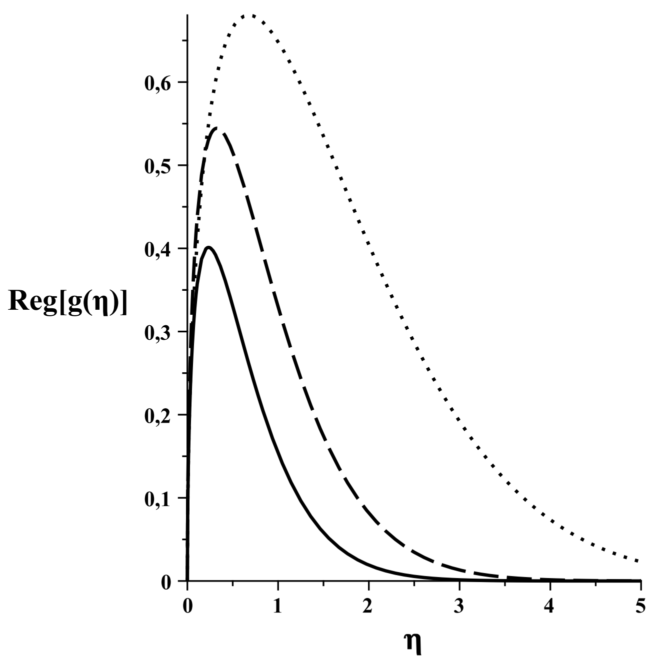

FIG. 5: The viscosity dependence of the regular solution of Eq. (11) for $n = 2$ and $\kappa = 1$. The solid line is for $\nu = 0.05$ the dotted line is for $\nu = 0.1$ and the dashed line is for $\nu = 0.5$.

method.

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