Open Problems of Liouville Field Theory

Harald Dorn and George Jorjadze

“O’Raifeartaigh Memorial Symposium”

Budapest, June 2006

Content:

- Introduction
- The classical Liouville theory
- The quantum Liouville theory
- Conclusions

Liouville equation

$$\partial^2_{x\bar{x}}\,\varphi(x,\bar{x}) + m^2\, e^{2\varphi(x,\bar{x})} = 0$$

$x = \tau + \sigma$ and $\bar{x} = \tau - \sigma$ are chiral coordinates; τ is time, σ is the spatial coordinate (Minkowskian space-time) and $m > 0$ is a constant.

The general solution

$$\varphi(x,\bar{x}) = \frac{1}{2}\,\log \frac{A'(x)\,\bar{A}'(\bar{x})}{[1 + m^2\,A(x)\bar{A}(\bar{x})]^2}$$

(Liouville 1853)

Conformal symmetry

2d conformal transformations:

$$x \quad \mapsto \quad \eta(x), \qquad\qquad \bar{x} \quad \mapsto \quad \bar{\eta}(\bar{x})$$

Liouville equation is invariant under

$$\varphi(x, \bar{x}) \quad \mapsto \quad \varphi(\eta(x), \bar{\eta}(\bar{x})) + \frac{1}{2} \log \eta'(x) \bar{\eta}'(\bar{x})$$

Cylindrical space-time $\tau \in \mathbb{R}^1$, $\sigma \in S^1$

$$\eta(x+2\pi) = \eta(x) + 2\pi, \qquad \eta'(x) > 0.$$

$$\bar{\eta}(\bar{x}+2\pi) = \bar{\eta}(\bar{x}) + 2\pi, \qquad \bar{\eta}'(\bar{x}) > 0.$$

Periodic Liouville field

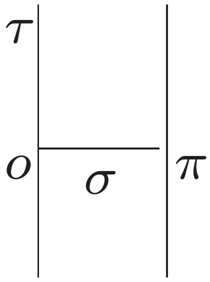
$$\varphi(\tau, \sigma + 2\pi) = \varphi(\tau, \sigma)$$

.

Introduction

Strip $\tau \in \mathbb{R}$, $\sigma \in (0, \pi)$; $\bar{\eta} = \eta$,

$$\eta(x + 2\pi) = \eta(x) + 2\pi, \qquad \eta'(x) > 0 \ .$$



2d conformal group $\sim$ $\mathrm{Diff}_+ S^1$

Invariant boundary conditions for $V \equiv e^{-\varphi}$

$a.$ $\partial_\sigma V|_{\sigma=0} = 2\mu_1 \quad \partial_\sigma V|_{\sigma=\pi} = 2\mu_2$ (FZZT)

$b.$ $V|_{\sigma=0} = 0 = V|_{\sigma=\pi}$ (ZZ)

(Gervais and Neveu 1983)

Energy-momentum tensor

$$T = \partial_x \varphi^2 - \partial^2_{xx}\varphi \qquad T = \partial_{\bar{x}} \varphi^2 - \partial^2_{\bar{x}\bar{x}}\varphi$$

$$\partial_{\bar{x}} T = 0 \qquad \partial_x \bar{T} = 0$$

T is periodic on the cylinder

$$T(x + 2\pi) = T(x) \qquad \bar{T}(\bar{x} + 2\pi) = \bar{T}(\bar{x})$$

T is periodic on the strip as well and

$$T(\tau) = \bar{T}(\tau)$$

Since $V = e^{-\varphi}$ satisfies

$$\partial^2_{xx} V = T\, V \qquad \partial^2_{\bar{x}\bar{x}} V = \bar{T}\, V$$

one can use the Hill equation

$$\psi''(x) = T(x)\psi(x)$$

to classify the Liouville fields by

coadjoint orbits of $T(x)$.

(Balog, Feher, Palla 1997)

Orbits of constant $T(x) = s$, with

$$s \geq -\frac{1}{4}$$

For other orbits:

a. The energy functional

$$E = \int_0^{2\pi} T(x)dx$$

is not bounded below.

b. Liouville fields are not regular.

Liouville fields are parameterized by s and $\eta(x) \in \mathrm{Diff}S^1$.

The periodic case:

regularity requires $\bar{s} = s > 0$

and V is given by

$$V = \frac{e^{-\sqrt{s}[\eta(x)+\bar{\eta}(\bar{x})]}}{\sqrt{\eta'(x)\bar{\eta}'(\bar{x})}} + m^2 \, \frac{e^{\sqrt{s}[\eta(x)+\bar{\eta}(\bar{x})]}}{\sqrt{\eta'(x)\bar{\eta}'(\bar{x})}}$$

Transformation to free-field variables

$$\phi(x) = \frac{p}{4\pi}\eta(x) + \frac{1}{2}\log\eta'(x)$$

where $p \equiv 4\pi\sqrt{s}$ is the zero momentum mode

$$\phi(x) = q + \frac{p}{4\pi}x + \frac{i}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{a_n}{n}e^{-inx}$$

Free-field parameterization

Asymptotic fields

$$e^{-\varphi(x,\bar{x})} = e^{-\varphi_{in}(x,\bar{x})} + e^{-\varphi_{out}(x,\bar{x})}$$

The relation between in-out fields

$$e^{-\varphi_{out}(x,\bar{x})} = m^2 \, e^{-\varphi_{in}(x,\bar{x})} \, S(x,\bar{x}),$$

where $S(x,\bar{x})$ is 'screening charge'

$$S(x,\bar{x}) = \frac{1}{(e^p - 1)^2} \int_0^{2\pi} dz \int_0^{2\pi} \, d\bar{z} \; e^{2\varphi_{in}(x+z,\bar{x}+\bar{z})}$$

Canonical transformations

a. From in-field to the Liouville field.

b. From in-field to out-field.

(Ford, Sachs 1997), (G.J. 1983, 2003)

The generating functional

$$F = \int_0^{2\pi} 2\lambda(x) + {\phi_{in}}'(x) \log \frac{\lambda(x) - {\phi_{in}}'(x)}{\lambda(x) + {\phi_{in}}'(x)}$$

where

$$\lambda(x) = \sqrt{{\phi'}^2_{in}(x) + 4e^{\phi_{out}(x)}}$$

The classical S-matrix.

Strip: the parameters of the theory

$$\rho_1 = \frac{\mu_1}{m} \qquad\qquad \rho_2 = \frac{\mu_2}{m}$$

Liouville fields are regular if and only if

$$\rho_1 \leq 1 \qquad \text{and} \qquad \rho_2 \geq -1$$

$$V = \frac{m\left(R_p \cosh p\eta_\tau + \rho_2 \cosh p\eta_\sigma - \rho_1 \cosh p(\eta_\sigma - \pi)\right)}{p \sinh 2\pi p \sqrt{\eta'(x)\,\eta'(\bar{x})}}$$

where $p = \sqrt{s}$,

$$\eta_\tau = \eta(x) + \eta(\bar{x}),$$

$$\eta_\sigma = \eta(x) - \eta(\bar{x}),$$

$$R_p = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cosh 2\pi p + \sinh^2 2\pi p}$$

Symplectic form

$$\omega = \delta s \wedge \int_0^{2\pi} \eta'(x)\delta\eta(x) + s\int_0^{2\pi} \delta\eta'(x) \wedge \delta\eta(x)$$

$$+\frac{1}{4}\int_0^{2\pi} \frac{\delta\eta''(x) \wedge \delta\eta'(x)}{\eta'(x)^2}$$

PB

$$\{s, \eta(x)\} = 1$$

$$\eta(x) = x + \eta_0 + \sum_{n\neq 0} \eta_n e^{-inx}$$

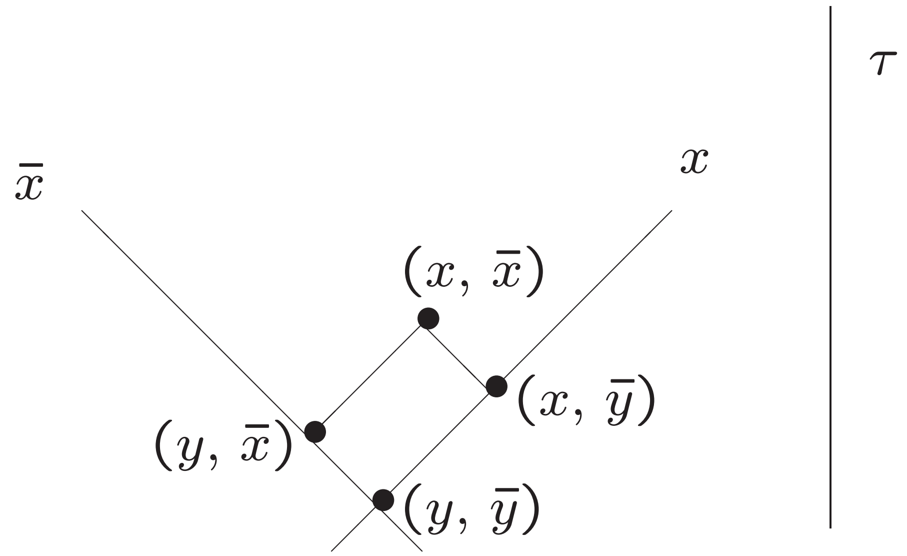
η_0 is a canonical conjugated to s.

Quantization of s for $s < 0$.

Non-equal time PB

$$\{V(x,\bar{x}),V(y,\bar{y})\} = \frac{1}{4}\Big(\text{sign}(x-y)+\text{sign}(\bar{x}-\bar{y})\Big)$$

$$\times\ \ (V(x,\bar{x})V(y,\bar{y}) - 2V(x,\bar{y})V(y,\bar{x}))$$



(G.J., G. Weigt 2001)

Fundamental domain (the periodic case):

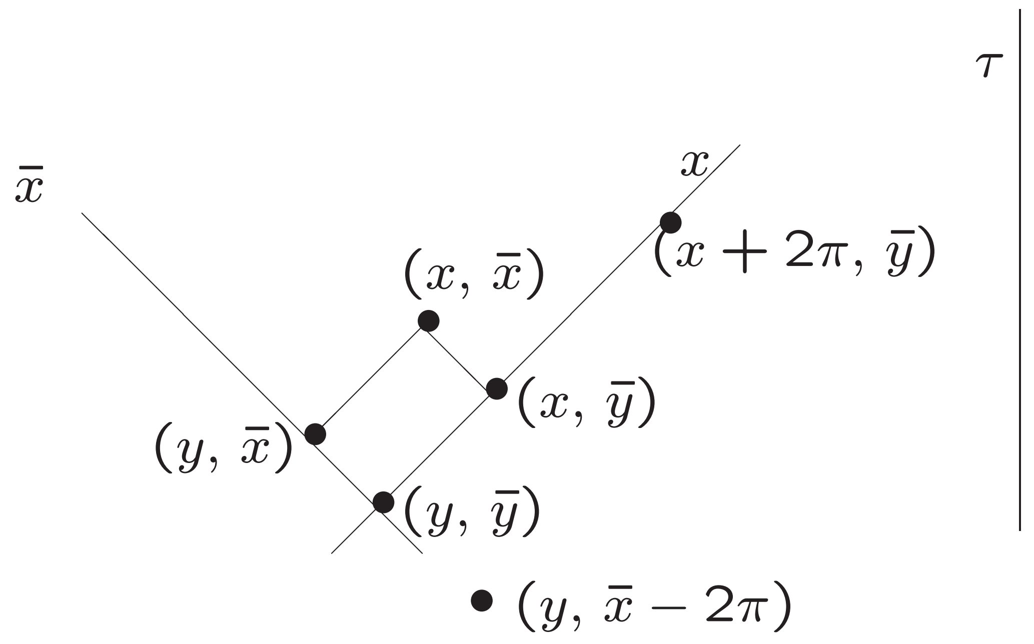
$$x-y \in (-2\pi, 2\pi), \qquad \bar{x}-\bar{y} \in (-2\pi, 2\pi)$$

The general non-equal time Poisson bracket

$$\{V(x,\bar{x}),V(y,\bar{y})\} =$$

$$\frac{1}{4}\sum_{n=-\infty}^{\infty}\Big(\mathrm{sign}(x-y+2\pi n)+\mathrm{sign}(\bar{x}-\bar{y}-2\pi n)\Big)\times$$

$$\Big(V(x,\bar{x})V(y,\bar{y})-2V(y,\bar{x}-2\pi n)V(x+2\pi n,\bar{y})\Big)$$



(G.J, C.Ford 2005)

The simplest vertex operator

$$V(x,\bar{x}) = E(x,\bar{x}) + F(x,\bar{x})$$

The *in*-field operator

$$E(x,\bar{x}) =: e^{-\varphi_{in}(x,\bar{x})} :$$

The *out*-field operator

$$F(x,\bar{x}) = \frac{\mu_\alpha^2}{2\sinh\frac{1}{2}p} E(x,\bar{x})\, S(x,\bar{x})\, \frac{1}{2\sinh\frac{1}{2}p}$$

The screening charge operator

$$S(x,\bar{x}) = \int_0^{2\pi} dz \int_0^{2\pi} d\bar{z} : e^{2\varphi_{in}(x+z,\bar{x}+\bar{z})-p} :$$

$\mu_a lpha$ is a renormalized mass.

The causal commutator of the Liouville theory

$$[V(x,\bar{x}),V(y,\bar{y})] = i\ \sin\frac{\hbar}{4}\Big(\text{sign}(x-y)+\text{sign}(\bar{x}-\bar{y})\Big)\times$$

$$\left(\frac{V(x,\bar{x})V(y,\bar{y})+V(y,\bar{y})V(x,\bar{x})}{2\cos\frac{1}{4}\hbar} - V(x,\bar{y})V(y,\bar{x}) - V(y,\bar{x})V(x,\bar{y})\right)$$

The corresponding PB

$$\{V(x,\bar{x}),V(y,\bar{y})\} = \frac{1}{4}\Big(\text{sign}(x-y)+\text{sign}(\bar{x}-\bar{y})\Big)$$

$$\times\ [V(x,\bar{x})V(y,\bar{y}) - 2V(x,\bar{y})V(y,\bar{x})]$$

The general causal commutator

$$\frac{V(x,\bar{x})V(y,\bar{y})}{e^{\frac{1}{4}i\hbar\Theta(x-y,\bar{x}-\bar{y})}} - \frac{V(y,\bar{y})V(x,\bar{x})}{e^{-\frac{1}{4}i\hbar\Theta(x-y,\bar{x}-\bar{y})}} =$$

$$-\frac{i}{2}\sin\frac{\hbar}{2}\sum_{n=-\infty}^{\infty}\Big(\text{sign}(x-y+2\pi n)+\text{sign}(\bar{x}-\bar{y}-2\pi n)\Big)\times$$

$$\left(\frac{V(y,\bar{x}-2\pi n)V(x+2\pi n,\bar{y})}{e^{-\frac{1}{4}i\hbar\Theta(x-y+2\pi n,\bar{y}-\bar{x}+2\pi n)}} + \frac{V(x+2\pi n,\bar{y})V(y,\bar{x}-2\pi n)}{e^{\frac{1}{4}i\hbar\Theta(x-y+2\pi n,\bar{y}-\bar{x}+2\pi n)}}\right)$$

with

$$\Theta(x,\bar{x}) = \frac{\epsilon(x)+\epsilon(\bar{x})}{2}$$

Correlation functions

Liouville vertex operators

$$V_\beta(x,\bar{x}) \sim e^{\beta\varphi(x,\bar{x})}$$

3-point function

$$\langle p|\, V_\beta(0,0)\, |p'\rangle$$

V_{-n} is a regularized power of V

$$V_{-n}(x,\bar{x}) = \lim_{\epsilon\to 0} V_{-(n-1)}(x,\bar{x})\, V(x+\epsilon,\bar{x}-\epsilon)\, \epsilon^{\alpha\,(n-1)}$$

where $\alpha = \frac{\hbar}{2\pi}$ and we rescale $p \to 4\pi p$.

The regularization provides

$$V_{-n}(x,\bar{x}) = \sum_{m=o}^{n} C_m^n \, \mu_\alpha^{2m} \, J_m(p) e^{-n\phi(x,\bar{x})} \, S^m(x,\bar{x}) \, J_m^*(p)$$

where

$$C_m^n = \prod_{l=1}^{m} \frac{\sin \pi(n-l+1)\alpha}{\sin \pi l \alpha}$$

are the deformed binomial coefficients and

$$J_m(p) = \prod_{l=1}^{m} \frac{1}{2 \sinh \pi(p + il\alpha)}$$

deformed powers of the sinh-term.

The vacuum expectation of $V_n(0,0)$

$$\langle p|V_{-n}|p'\rangle = \sum_{m=0}^{n} \binom{n}{m} \left(\mu_\alpha^2 \, \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}\right)^m \times$$

$$V_m^n(p,p')\,\delta(p-p'-i(n-2m)\alpha)$$

where

$$V_m^n(p,p') = \prod_{l=1}^{m} \frac{\Gamma(ip-l\alpha)\,\Gamma(-ip'-l\alpha)}{\Gamma(1-ip+l\alpha)\,\Gamma(1+ip'+l\alpha)} \times$$

$$\frac{\Gamma(1+(n-l+1)\,\alpha)\Gamma(1-l\alpha)}{\Gamma(1+(l-n-1)\,\alpha)\,\Gamma(1+l\alpha)}$$

How to continue from discrete $\beta = -n$

to arbitrary β?

Using that

$$\frac{\Gamma\left(\lambda+ik\right)\Gamma\left(\lambda-ik\right)}{2\pi\Gamma(2\lambda)} \quad \longleftrightarrow \quad \sum_{m=0}^{n}\binom{n}{m}\delta(k-i(n/2-m))$$

and the integral representation of $\log\Gamma$-function, we find DOZZ 3-point function for $\lambda>0$

(Dorn and Otto 1992),

(Zamolodchikov and Zamolodchikov (1995)

$$\langle p|\,V_{2\lambda}|p'\rangle = \frac{\Gamma\left(\lambda+ik\right)\,\Gamma\left(\lambda-ik\right)}{4\pi\alpha\Gamma(2\lambda)}\left(m^2\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}\right)^{ik-\lambda}\;V^{-2\lambda}_{ik-\lambda}(p,p')$$

(G.J., G. Weigt 2003)

Conclusions

- S-Matrix for the periodic case (the strip)

- V-operator for the strip $(*)$

- The spectrum of s for negative s

- The correlation functions for the strip

- Geometric quantization of ω $(*)$

Conclusions