Supplementary material for
"A simple method of constructing binary black hole initial data"

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THE SIGN OF THE PRINCIPAL COEFFICIENT IN THE PARABOLIC EQUATION

In the main text considerations were restricted to binaries comprised by individual black holes with speeds, displacements and spins aligned parallel to the x, y and z-axis, respectively (as indicated on Fig. 1) [1]. Foliating then Σ by $z = const$ level surfaces, and determining the function $\overset{\star}{K}$ by making use of (14) by a direct calculation $\overset{\star}{K}$ can be seen to have the form

$$\overset{\star}{K} = -z \cdot \overset{+}{K}\,, \tag{SM.1}$$

where $\overset{+}{K}$ is a strictly positive function. It is also follows from the pertinent form of (3) that the sign of $\overset{\star}{K}$ determines whether the parabolic-hyperbolic system evolves in the positive or negative z-direction. Indeed, the coupled system is known to propagate aligned $(\partial_z)^i$ for positive $\overset{\star}{K}$, while anti-aligned for negative $\overset{\star}{K}$.

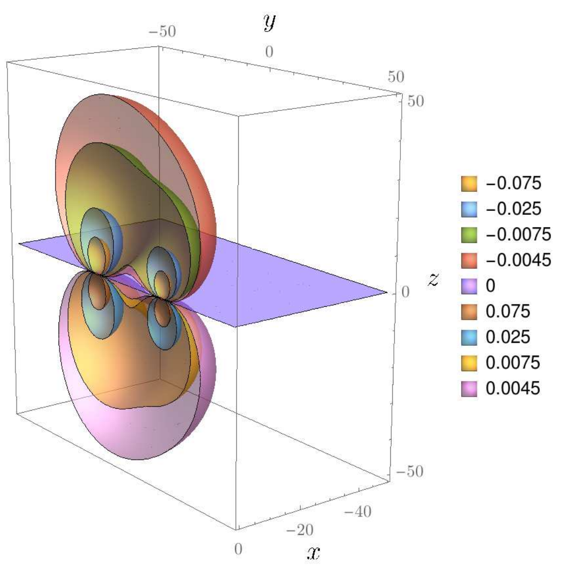

FIG. SMF.1. (color online). $\overset{\star}{K} = const$ level surfaces are depicted in the $x < 0$ half of the cube with edges $2A = 100$. The parameters of the relevant binary are: $M^{[1]} = 1, d^{[1]} = 20, v^{[1]} = 0.5, a^{[1]} = 0.6$ and $M^{[2]} = 2, d^{[2]} = -10, v^{[2]} = -0.25, a^{[2]} = -0.8$. The positive and negative $\overset{\star}{K} = const$ level surfaces are well separated by the $z = 0$ plane that also coincides with $\overset{\star}{K} = 0$ level surface.

As an immediate consequence of (SM.1) we have that $\overset{\star}{K}$ is positive everywhere below the $z = 0$ plane while it is negative above that plane. This assertion is also supported by Fig. SMF.1, where $\overset{\star}{K} = const$ level surfaces are plotted for a specific choice of physical parameters.

Notable, the absolute value of $\overset{\star}{K}$ is increased the $\overset{\star}{K} = const$ level surfaces are more and more concentrated on smaller and smaller neighborhoods of the (point-like or ring-like) singularities that, for the considered class of binaries, are confined to the $z = 0$ plane.

ON THE EXISTENCE OF UNIQUE C^2 SOLUTIONS ON THE CLOSURE OF THE UNION OF Σ^+ AND Σ^-

As discussed in the main text, for the considered class of binary black hole configurations the parabolic-hyperbolic system (3)-(5) has to be solved as an initial-boundary value problem on the disjoint domains, Σ^+ and Σ^-, located above and below the $z = 0$ plane. Though the corresponding initial-boundary value problem is known to be well-posed locally [8] to verify that (at least) C^2 solutions exist on the closure of the union of Σ^+ and Σ^- the followings have to be guaranteed:

(i) First, it has to be shown that solutions to the considered initial-boundary value problem exist on the closure of Σ^+ and Σ^-, separately. This means that, for the specific choice of the initial-boundary values, fields $\widehat{N}$, $\mathbf{k}_i$ and $\mathbf{K}^l{}_l$ exist (apart from singularities that, in the present case, are confined to the $z = 0$ plane) on disjoint domains comprised by $z = const$ level surfaces with $0 < z \leq A$ or $-A \leq z < 0$, where $A > 0$, such that they also satisfy the parabolic-hyperbolic system (3)-(5).

(ii) Second, it has also to be verified that the fields $\widehat{N}$, $\mathbf{k}_i$ and $\mathbf{K}^l{}_l$, and at least their first two z-derivatives match through the $z = 0$ plane. More specifically, to guarantee the existence of at least C^2 solutions the constrained fields, along with their derivatives, have to possess well-defined values (apart from singularities) in the $z \to 0$ limit.

Verifying point (i)

The proof of the existence of global solutions to the parabolic-hyperbolic system is centered on iterations. It is proved that solutions to the separated parabolic and hyperbolic equations consistently determine a sequence that, via a contraction mapping argument, converge to a fixed point of the iteration that gives then a global solution to the coupled parabolic-hyperbolic system.

In carrying out the indicated iteration process the initial-boundary data for the constrained fields $\widehat{N}$, $\mathbf{k}_i$ and $\mathbf{K}^l{}_l$ will be held fixed, as determined by (14). In each iterative step first we solve the parabolic equation (3), as an initial boundary value problem, for $\widehat{N}$ by replacing the variables $\mathbf{k}_i$ and $\mathbf{K}^l{}_l$ by the solution of the hyperbolic system (4)-(5) yielded by the previous level of iteration, with the exception of the initial step where they are replaced by $\mathbf{k}_i$ and $\mathbf{K}^l{}_l$ derived from (14). The iterative steps are completed by solving the hyperbolic system (4)-(5), as an initial boundary value problem, for $\mathbf{k}_i$ and $\mathbf{K}^l{}_l$ with insertion of the solution $\widehat{N}$ just received into (4)-(5).

The entire iteration process relays heavily on the global existence of solutions to the initial boundary value problem of the Bernoulli type parabolic equation (3) with given globally defined variables $\mathbf{k}_i$ and $\mathbf{K}^l{}_l$ on Σ^+ or Σ^-, respectively. To overcome the main technical difficulties in solving (3) we generalize and extend known results concerning the Bernoulli type parabolic equations covered by [2, 12] (see also [10]). In particular, it will be argued below that (3) can be put into the form where not only the existence of unique solutions for all positive or negative z may be proven but, in addition, in the $z \to 0$ limit (apart from singularities) $\widehat{N} \equiv 1$ can be derived.

In doing so it is rewarding to introduce instead of z a new independent variable ζ given as

$$\zeta = z^{-1} \,. \tag{SM.2}$$

This allows us to put (3), for the dependent variable $\breve{N}(x, y, \zeta) = \widehat{N}(x, y, z(\zeta))$, into the form of (4) in [12], to which—by making use of results covered in [10]—the desired global existence and uniqueness, and also suitable asymptotic behavior results of [2, 12] can be applied.

To see this note that by applying the Leibnitz rule $\partial_z \widehat{N} = \partial_z \zeta \cdot \partial_\zeta \breve{N}$, along with (SM.2), we get $-z \cdot \partial_z \widehat{N} = \zeta \cdot \partial_\zeta \breve{N}$. It follows then from (SM.1) and from the latter relation that the first term on the left hand side of (3) can be written as

$$\overset{\star}{K} \cdot \partial_z \widehat{N} = \overset{+}{K} \cdot [-z \cdot \partial_z \widehat{N}] = \overset{+}{K} \cdot \zeta \cdot \partial_\zeta \breve{N} \,, \tag{SM.3}$$

where $\overset{+}{K}$ is the strictly positive function in (SM.1).

Not also that as a result of the replacement (SM.2) the second term on the left hand side of (3) picks up only a factor z, or rather ζ^{-1}, which tends to zero while approaching the $z = 0$ plane.

Taking all the above observations into account and by dividing the yielded equation by the strictly positive function $\overset{+}{K}$ it is straightforward to verify that (3) indeed takes the form of the Bernoulli type parabolic equation to which existence and uniqueness of global solutions could be deduced in [2, 12]. Note, however, that there are two important additional technical issues to be handled here.

First, the $z = const$ level surfaces possess non-empty spatial boundary. This difficulty can be treated by applying results covered in [10] that allow to generalize and extend global existence and uniqueness results of [2, 12] to the case of level surfaces with spatial boundary. The second issue is that $\overset{+}{K}$, along with $|\mathcal{A}|$ and $|\mathcal{B}|$, tends to $+\infty$ while approaching the singularities. This means that suitable lower bounds of coefficient of the two-dimensional Laplace operator, and upper bounds to the coefficients $|\mathcal{A}|/\overset{+}{K}$ and $|\mathcal{B}|/\overset{+}{K}$ can only be given in the complement of small neighborhoods of the singularities. As these singularities are either pointlike or ringlike, and they are confined to the $z = 0$ plane, boundedness of various expressions can be guaranteed in the complement of sufficiently small ball-like or toroidal-like neighborhoods of these singularities. As (14) is known to become extremely good approximation of a solution to Einstein's equations while approaching either of the singularities at the boundaries yielded by the removal of the above mentioned sufficiently small ball-like or toroidal-like neighborhoods the fields $\widehat{N}$, $\mathbf{k}_i$ and $\mathbf{K}^l{}_l$ derived from (14) are suitable to be used as boundary data there. In order to get a solution on both Σ^+ and Σ^- a shrinking sequence of neighborhoods and a sequence of solutions have to be applied in verifying that the substitution $z \to \zeta$ replaces the $z \to 0$ limiting behavior of $\widehat{N}(x, y, z)$ by the investigation of the $\zeta \to \infty$ asymptotic behavior of $\breve{N}(x, y, \zeta) = \widehat{N}(x, y, z(\zeta))$.

This, along with the use of a variant of (3), relevant for the auxiliary variable $m = r/2\,(1 - \widehat{N}^2)$, allows suitable generalization and extension of the asymptotic behavior results in [2, 12] that can be used to conclude that in the $z \to 0$ limit (apart from singularities) $\widehat{N} \equiv 1$.

Note that the global existence of solutions to the initial boundary value problem in solving the hyperbolic system (4)-(5) with a given globally defined function $\widehat{N}$ is much more straightforward. In this case one may refer to the fact that the first order hyperbolic system (4)-(5) is linear in the variables $\mathbf{k}_i$ and $\mathbf{K}^l{}_l$ that allows the use of energy estimate methods, and in turn, to show that, unique smooth global (on Σ^+ or Σ^-) solutions exist to the considered initial-boundary value problem (see, e.g. the argument on pages 198-200 in [5]).

In combining all the above partial results denote first by $\mathfrak{u}_\mathfrak{i}$ the globally defined vector valued variable $(\widehat{N}, \mathbf{k}_i, \mathbf{K}^l{}_l)^T$ comprised by globally existing fields yielded in the $\mathfrak{i}^{th}$ iterative step. These solutions $\mathfrak{u}_\mathfrak{i}$ can be seen to belong to some Banach space $\mathfrak{B}$. Denote by T the map $T : \mathfrak{B} \to \mathfrak{B}$ relating the succeeding elements of the sequence $\{\mathfrak{u}_\mathfrak{i}\}$ as $\mathfrak{u}_{\mathfrak{i}+1} = T(\mathfrak{u}_\mathfrak{i})$. If T is contracting then the sequence $\{\mathfrak{u}_\mathfrak{i}\}$ converges, i.e. it is a Cauchy sequence in $\mathfrak{B}$. In this case $\mathfrak{u}^* = \lim_{\mathfrak{i}\to\infty} \mathfrak{u}_\mathfrak{i}$ is a fixed point of T, i.e. $\mathfrak{u}^* = T(\mathfrak{u}^*)$. This means that $\mathfrak{u}^*$ possesses the same smoothness properties as elements of the sequence $\{\mathfrak{u}_\mathfrak{i}\}$ in $\mathfrak{B}$, and that it is also a solution to the couple parabolic-hyperbolic system (3)-(5). Recall that the contraction property can be demonstrated by using sufficiently small domains. Therefore, as the initial-boundary value problem in case of the considered parabolic-hyperbolic system (3)-(5) is locally well-posed [8] the contraction property of the above defined map T follows.

Verifying point (ii)

Concerning the matching of fields and their derivatives raised in point (ii) above, recall first that there is a significant simplification ensured by the specific choice we made for binary black hole systems—by requiting the speeds, displacements and spins aligned parallel to the x, y and z-axis, respectively—, and also by the choice we made for the freely specifiable part of data by applying (14). Indeed, for the considered class of binary black hole configurations, the auxiliary metric (14) possesses a $z \to -z$ reflection symmetry. This, in particular, guarantees that the physical fields h_{ij} and K_{ij} satisfying the constraint equations (1) and (2), along with their even derivatives, will match at $z = 0$. Most importantly, using this symmetry all the scalar expressions derived from h_{ij} and K_{ij}, including $\widehat{N}$ and $\mathbf{K}^l{}_l$, along with their even derivatives, match at $z = 0$. Similarly, vector variables such as $\widehat{N}^i$ and $\mathbf{k}_i$, deduced from h_{ij} and K_{ij}, do vanish at $z = 0$ as they are odd in $\widehat{n}_i = \widehat{N}\,(\mathrm{d}z)_i$. This guaranties then that the vector variables $\widehat{N}^i$ and $\mathbf{k}_i$ do match at the $z = 0$ plane.

Accordingly, in order to show that (apart from singularities that are confined to the $z = 0$ plane) at least C^2 solutions exist the matching of the first z-derivatives of $\widehat{N}$, $\mathbf{k}_i$ and $\mathbf{K}^l{}_l$, along with the matching of the second z-derivative of $\mathbf{k}_i$, on Σ has to be verified.

In doing so we shall apply what has been verified in the proof of part (i) concerning the functional behavior of $\widehat{N}$, namely that (apart from singularities) $\widehat{N} \equiv 1$ at the $z = 0$ plane. Note also that the fields $\widehat{N}^i, \boldsymbol{\kappa}, \mathring{\mathbf{K}}_{ij}$—determined by (14)—vanish identically at $z = 0$. These latter two observations imply that the Lie derivatives $\mathscr{L}_{\widehat{n}}$ in (4) and (5) can be replaced by ∂_z, and also that it suffices to evaluate the second terms in (4) and (5) to determine the z-derivative of $\mathbf{k}_i$ and $\mathbf{K}^l{}_l$, respectively, in the $z \to 0$ limit.

Then (5), along with the vanishing of $\mathbf{k}_i$ at $z = 0$, gives that $\partial_z \mathbf{K}^l{}_l = 0$ there, whereas from (4) $\partial_z \mathbf{k}_i = \frac{1}{2}\partial_i \mathbf{K}^l{}_l$ follows, in the $z \to 0$ limit. Since $\mathbf{K}^l{}_l$, as a scalar, is an even function in z the derivative $\partial_z \mathbf{k}_i$ is well-defined at $z = 0$. An analogous argument gives then that $\partial_z^2 \mathbf{k}_i = \frac{1}{2}\partial_i(\partial_z \mathbf{K}^l{}_l)$ which, along with the vanishing of $\partial_z \mathbf{K}^l{}_l$ at $z = 0$, verifies that $\partial_z^2 \mathbf{k}_i = 0$ also holds there. Finally, by taking into account that $\overset{\star}{K}$ is an odd function in z, whereas all the terms on the right hand side of (3) are of even parity functions in z, we get, in virtue of the vanishing of $\widehat{N}^i$, in the $z \to 0$ limit, that $\partial_z \widehat{N}$ must also vanish in approaching the $z = 0$ plane which completes the proof that the considered solutions have indeed to be at least C^2. (The detailed proof will be published elsewhere.)

GLOBAL ADM CHARGES

It is remarkable that the new proposal provides a control on the full set of global ADM charges. Indeed, the ADM mass, center of mass, the linear and angular momenta can be given in terms of the input parameters of the construction.

To see this note first that though the superposed Kerr-Schild metric (14) does not satisfy Einstein's equations it is an asymptotically flat metric, and also it was constructed by adding contributions of individual black hole metrics to a Minkowski metric. As the ADM quantities are linear in deviation from flat Euclidean space at infinity we get then, by the superposition principle, that the global ADM mass, centre of mass, linear and angular momenta of the auxiliary system are determined by the rest masses, velocities, spins and displacements of the individual black holes. In particular, the relations

$$M^{\,ADM} = \gamma^{[1]} M^{[1]} + \gamma^{[2]} M^{[2]} \tag{SM.4}$$

$$M^{\,ADM} \vec{d}^{\,ADM} = \gamma^{[1]} M^{[1]} \vec{d}^{\,[1]} + \gamma^{[2]} M^{[2]} \vec{d}^{\,[2]} \tag{SM.5}$$

$$\vec{P}^{\,ADM} = \gamma^{[1]} M^{[1]} \vec{v}^{\,[1]} + \gamma^{[2]} M^{[2]} \vec{v}^{\,[2]} \tag{SM.6}$$

$$\begin{aligned} \vec{J}^{\,ADM} = {} & \gamma^{[1]} \left\{ M^{[1]} \vec{d}^{\,[1]} \times \vec{v}^{\,[1]} + M^{[1]} a^{[1]} \vec{s}_\circ^{\,[1]} \right\} \\ & + \gamma^{[2]} \left\{ M^{[2]} \vec{d}^{\,[2]} \times \vec{v}^{\,[2]} + M^{[2]} a^{[2]} \vec{s}_\circ^{\,[2]} \right\} \end{aligned} \tag{SM.7}$$

hold, where $\vec{d}^{\,[n]}$, $\vec{v}^{\,[n]}$ and $a^{[n]} \vec{s}_\circ^{\,[n]}$ denote the centre of mass, speed and spin of the individual black holes.

Consider now a solution to the parabolic-hyperbolic system (3)–(5) with free data chosen as dictated by the superposed Kerr-Schild metric (14), and with initial and boundary data that is chosen at "infinity", in accordance with the asymptotic behavior of (14). This solution is asymptotically flat in the sense that, beside the free data $\widehat{N}^i, \widehat{\gamma}_{ij}, \boldsymbol{\kappa}, \mathring{\mathbf{K}}_{ij}$, the constrained fields $\widehat{N}$, $\mathbf{k}_i$ and $\mathbf{K}^l{}_l$ are also guaranteed to fall off sufficiently fast, and that the associated fields h_{ij} and K_{ij} do satisfy the Regge-Teitelboim parity conditions which guarantee then that well-defined ADM quantities exists for this solution likewise they exist for (14).

It is of obvious interest to know then what is the relation between ADM quantities relevant for the true physical solution and for the auxiliary metric. This is the second point where the choice we made for the free data provides considerable payback. Indeed, as the leading order behavior of the constrained fields $\widehat{N}$, $\mathbf{k}_i$ and $\mathbf{K}^l{}_l$ and that of the corresponding auxiliary fields concur it can be verified (for more details see [11]) that the two sets of ADM quantities are also pairwise equal to each other. This verifies then that the ADM mass, centre of mass, linear and angular momenta of the true physical solution of the constraints can also be given by (SM.4)–(SM.7) which require the use of the input parameters of the construction, such as the rest masses, displacements, speeds and spins of the individual black holes.

It is important to be mentioned here that some of the other initial data constructions determine at least some of the ADM quantities. In particular, the proposal of Bowen and York provides a control on the ADM linear and angular momenta. Note, however, that it does not allow any further control on the rest of the ADM quantities as use of the Bowen-York method requires h_{ij} to be conformally flat and the mean curvature to vanish [3, 4]. It is also somewhat odd that these conditions are so restrictive that they exclude even the Kerr black hole solution from the outset [7, 9]. Notably by applying the gluing techniques [6] one cannot either have a full control of the ADM quantities apart from the extremal limit demanding the black holes to be infinitely separated [6, 11].

[1] Hereafter numbered objects (the figure and the equations) of the main text will be refereed by applying the vary same numbers, whereas the figure and the equations introduced in this Supplemental Material will always be referred as (SMF.*) and (SM.*), respectively.

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