Lectures on gauge-gravity duality

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I. Review of AdS/CFT

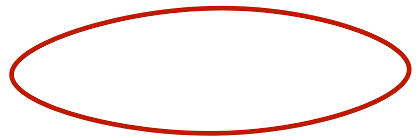
i. D-branes: open and closed string picture
ii. AdS/CFT conjecture
iii. generating function and correlators

II. Extensions and applications

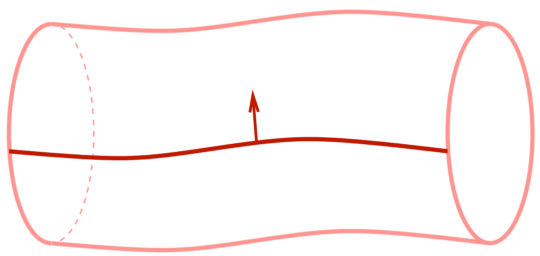
i. Finite T and thermal aspects
ii. Wilson-loop
iii. Confinement-deconfinement transition and QCD
iv. Non-relativistic CFTs

Open-closed duality

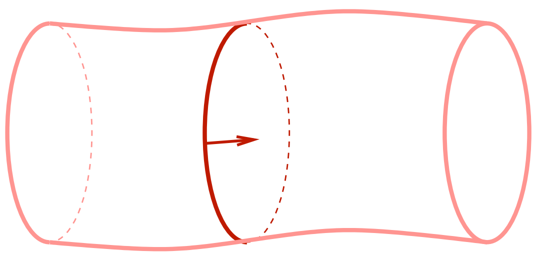
open



closed

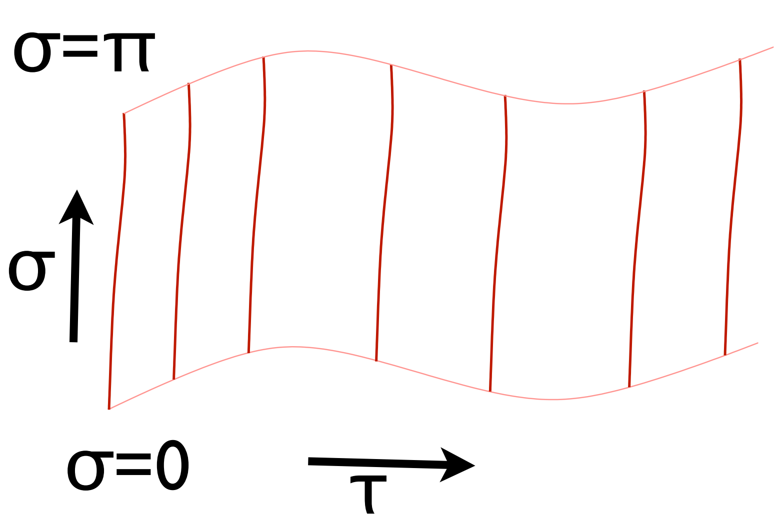


open loop



closed propagator

b.c. on open strings:



Polyakov action

$$M = 0, \ldots, D-1$$

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\ \partial_a x^M \partial^a x_M$$

$$\delta S = -\frac{1}{2\pi\alpha'}\int d^2\sigma\ \partial_a x^M \partial^a \delta x_M$$

$$\partial_\sigma x^M \delta x_M \Big|_{\sigma\,=\,0,\,\pi} = 0$$

D-branes in open string picture

b.c.

Neumann

$$\partial_\sigma x^M = 0$$

Dirichlet

$$\delta x^M = 0$$

$$x^M = c^M = \text{fixed}$$

Dp-branes: $\partial_\sigma x^\mu = 0 \qquad \mu = 0, \ldots, p$

$x^i = c^i \qquad i = p+1, \ldots, D-1$

D0-brane: particle
D1-brane: string
D2-brane: membrane
...
D9-brane: space-filling brane

- superstrings: carries RR-charge

IIA-strings: even Dp-branes
IIB-strings: odd Dp-branes

D-brane action

DBI action

$$S_{\mathrm{DBI}} = -T_p \int d^{p+1}\xi e^{-\varphi} \sqrt{-\det(\gamma_{ab} + 2\pi\alpha' F_{ab} + B_{ab})}$$

$a = 0 \dots p$

Induced metric $\gamma_{ab} = \frac{\partial x^M}{\partial \xi_a} \frac{\partial x^N}{\partial \xi_b} g_{MN}$

φ : dilaton

B : B-field

Low-energy expansion: $\alpha' \to 0$

$$S = -(2\pi\alpha')^2 T_p \int d^{D+1}\xi \; e^{-\varphi} \left(\underbrace{\frac{1}{4} F_{ab} F^{ab}}_{\text{YM theory on brane}} + \frac{1}{2} \partial_a \phi^i \partial^a \phi^i + \dots \right)$$

D-branes from closed strings

- massive charged objects in II string theory

$$S \sim \int d^{10}x\sqrt{-g}\left(e^{-2\varphi}\left(R+4(\nabla\varphi)^2\right)-\frac{2}{(8-p)!}F_{p+2}^2\right)$$

look for solutions $\displaystyle\int_{S^{8-p}} *F_{p+2} = N$

3-brane solution

$$ds^2 = H^{-1/2}dx_\mu dx^\mu + H^{1/2}(dr^2 + r^2 d\Omega_s^2)$$
$$H = 1 + \frac{R^4}{r^4}, \qquad R^4 = 4\pi g_s \alpha'^2 N$$

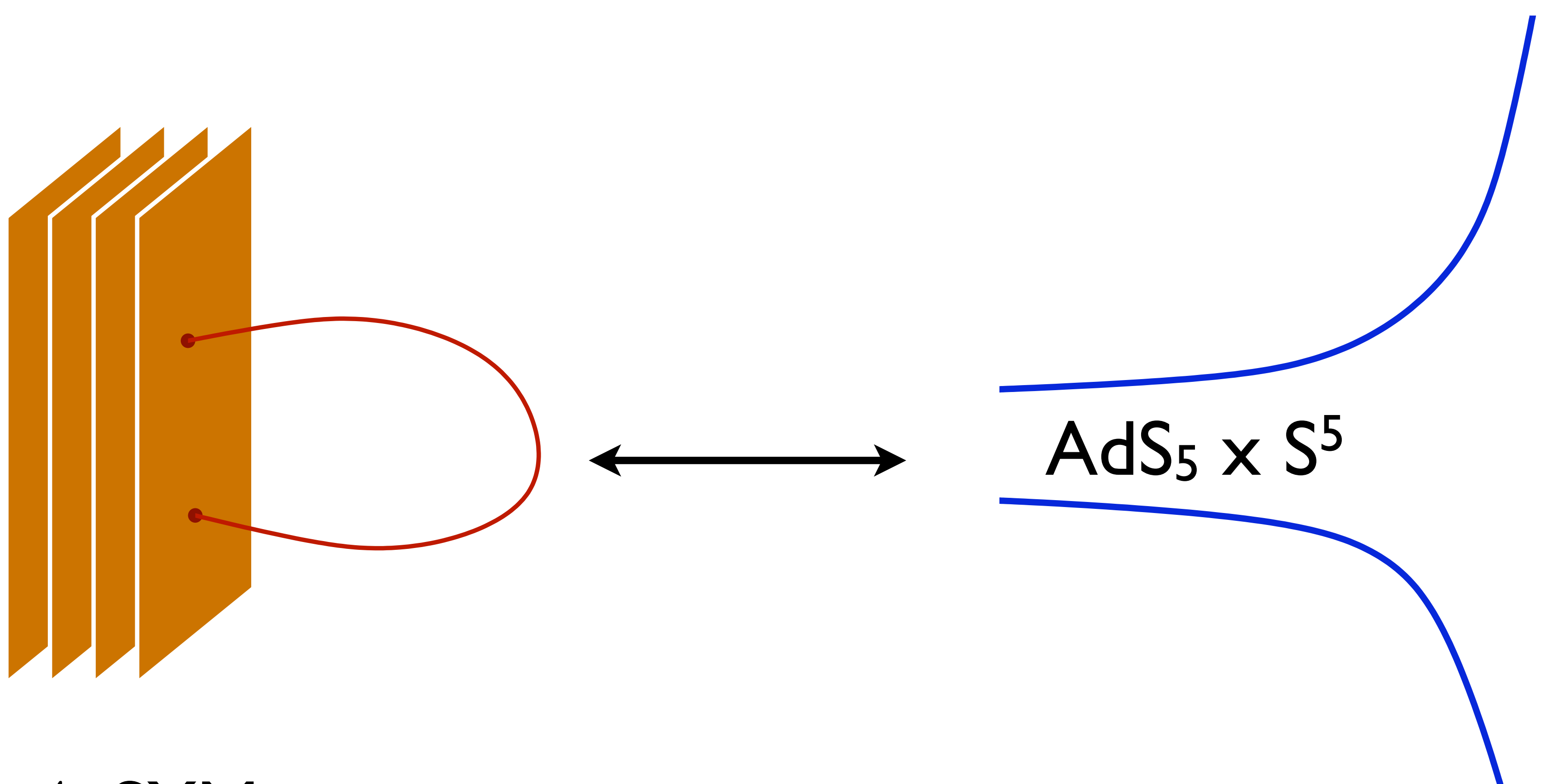
- D-brane and p-brane are same objects

Near-horizon limit

$r \to 0$ $\qquad$ $AdS_5 \times S^5$

$$ds^2 = \frac{r^2}{R^2} dx_\mu \, dx^\mu + R^2 \frac{dr^2}{r^2} + R^2 d\Omega_5^2$$

AdS/CFT:



$\mathcal{N} = 4$ SYM

II string theory on $AdS_5 \times S^5$

Parameters

closed | **open**

string theory on AdS5 | SYM

g_S: string coupling

ℓ_S: string scale

R : radius of AdS5 and S5

$$R^4 = 4\pi g_s \alpha'^2 N$$

N: rank of gauge group

$g_{\rm YM}$: coupling const.

$$g^2_{\rm YM} = 4\pi g_S$$

$$\lambda = g^2_{\rm YM} N$$

$$\lambda = 4\pi g_S N = \frac{R^4}{\alpha'^2} = \left(\frac{R}{\ell_S}\right)^4$$

SUGRA valid:

$$R \gg \ell_S \quad \Leftrightarrow \quad \lambda \gg 1$$

strongly coupled field theory

string pert. theory valid:

$$g_S \ll 1 \quad \Leftrightarrow \quad N \gg 1$$

large N field theory

$$g_s \sim 1/N \quad \alpha' \sim 1/\sqrt{\lambda}$$

Generating function

string theory on AdS$_5$×S^5 ⟷ $\mathcal{N} = 4$ SYM theory

$S(g_{\mu\nu}, A^{(4)}, \varphi, \ldots)$

$\Phi(x, r)$

CFT operators $\mathcal{O}(x)$
on boundary

boundary coupling: $\int d^4x\, \Phi(x)\mathcal{O}(x)$

Poincare coordinates: $ds^2 = \frac{R^2}{z^2}(dx_\mu\, dx^\mu + dz^2)$

boundary: $z \to 0$

generating function:

$$Z_{\text{gauge}} = \left\langle e^{\int d^4x\, \Phi(x)\mathcal{O}(x)} \right\rangle_{\text{CFT}} = Z_{\text{string}}\left(\Phi(x,z)|_{z=0} = \Phi(x)\right)$$

SUGRA regime

$$Z_{\text{string}}(\Phi) = e^{-I_{\text{SUGRA}}(\Phi)}$$

field in AdS_5 ⇔ operator in CFT

eg. $$\int d^4x\sqrt{g}(g_{\mu\nu}T^{\mu\nu} + A_\mu J^\mu + \varphi F_{\mu\nu}F^{\mu\nu} + \dots)$$

- match of symmetries
  bosonic SO(2,4)×SO(6) (supergroup PSU(2,4,4))

- massless spectrum of string theory ⇔ BPS ops in gauge theory

  massive string modes ⇔ non-BPS-operators

$$\Delta \sim R/l_s \sim (g_{\text{YM}}^2 N)^{1/4}$$

Correlators - 1

- eg. massive scalar field in AdS

$$I(\phi) = \int d^4x\, dz\, \sqrt{g}(\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2)$$

$$(\nabla^2 - m^2)\phi = 0$$

$$ds^2 = \frac{R^2}{z^2}(dx_\mu dx^\mu + dz^2)$$

$$\Rightarrow \quad \frac{1}{z^3}\partial_\mu \partial^\mu \phi + \partial_z \left(\frac{1}{z^3} \partial_z \phi \right) - m^2 \frac{R^2}{z^5}\phi = 0$$

$$z \longrightarrow 0$$

$$\phi \sim z^a$$

$$a(a-4) - m^2 R^2 = 0 \qquad \Rightarrow \qquad m^2 R^2 = a(a-4)$$

Correlators - 2

$$m^2 = a(a-4)$$

$$a_\pm = 2 \pm \sqrt{4 + m^2 R^2}$$

$$a_+ = \Delta \geq 4, \qquad a_- = 4 - \Delta$$

near boundary $z \to 0$ $\quad z^{4-\Delta}$ dominates

renormalized b.c.: $\quad \phi(x,\epsilon) = \epsilon^{4-\Delta}\phi_0(x)$

source in generating function

- coordinate rescaling

$$x \to \lambda x, \quad z \to \lambda z$$

$\phi(x,z)$: no change

$\phi_0(x)$: dim $\Delta - 4$, $\mathcal{O} : \Delta = 2 + \sqrt{4 + m^2 R^2}$

Two-point functions

Conformal invariance:

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_{\Delta'}(x')\rangle \sim \frac{c\delta_{\Delta\Delta'}}{(x-y)^{2\Delta}}$$

Bulk to boundary propagator:

$$K_\Delta(z,x,x')$$

$$(\nabla_x^2 - m^2)K_\Delta(z,x,x') = 0 \qquad \text{in bulk}$$

$$K_\Delta(z,x,x') \to z^{4-\Delta}\delta(x-x') \qquad \text{at boundary,} \quad z \to 0$$

$$K_\Delta(z,x,x') = c_\Delta \frac{z^\Delta}{(z^2+(x-x')^2)^\Delta} \qquad \text{(m=0)}$$

$$\nabla^2_{\mathrm{AdS}} = \frac{1}{z^3}\partial_\mu\partial^\mu + \partial_z\left(\frac{1}{z^2}\right) - \frac{m^2R^2}{z^5}$$

Solution of Laplace equation I

$$\phi(x,z) = \int d^4x'\, K_\Delta(z,x,x')\phi_0(x')$$

$$\phi(x,\epsilon) = \epsilon^{4-\Delta}\phi_0(x)$$ renormalized b.c.

$$\phi(x,z) = c\int d^4x'\, \frac{z^\Delta}{\left(z^2+(x-x')^2\right)^\Delta}\phi_0(x')$$

generating function:

$$Z_{\text{gauge}} = \left\langle e^{\int d^4x\,\phi_0(x)\mathcal{O}(x)}\right\rangle_{\text{CFT}} = \underset{\lim \epsilon\to 0}{Z_{\text{string}}}\left(\phi(\epsilon,x) = \epsilon^{4-\Delta}\phi_0(x)\right)$$

$$\frac{\delta^2}{\delta\phi_0(x)\delta\phi_0(x')} : \langle \mathcal{O}(x)\mathcal{O}(x')\rangle = \dots$$

Solution of Laplace equation II

$$Z_{\text{string}} = e^{-S_{\text{SUGRA}}}\Big|_{b.c.\phi_0(x)}$$

$$\begin{aligned}
S &= \int d^5x\,\sqrt{g}\partial^\mu\phi\partial_\mu\phi \\
&= \int d^5x\,\partial^\mu(\sqrt{g}\phi\partial_\mu\phi) - \phi\partial_\mu(\sqrt{g}\partial^\mu\phi) \\
&= \int d^5x\,\partial^\mu(\sqrt{g}\phi\partial_\mu\phi) + \int \text{e.om} \\
&= \lim_{\epsilon\to 0}\int d^4x\,(\sqrt{g}\phi\partial^z\phi)\Big|_{z=\epsilon}
\end{aligned}$$

$$\sqrt{g} = \frac{R^5}{z^5} \qquad\qquad g^{zz} = \frac{z^2}{R^2}$$

Two-point function from SUGRA

$$S = \lim_{\epsilon \longrightarrow 0} \int d^4x \, \frac{1}{\epsilon^3} \phi(x,\epsilon) \, \partial_z \phi(x,z)|_{z=\epsilon}$$

$$\lim_{\epsilon \longrightarrow 0} \phi(x,z) = \epsilon^{4-\Delta} \phi_0(x)$$

$$\phi(x,z) = c_\Delta \int d^4x' \, \frac{z^\Delta}{[z^2 + (x-x')^2]^\Delta} \phi_0(x')$$

$$S = c_\Delta \int d^4x \, d^4x' \, \Delta \frac{1}{(x-x')^{2\Delta}} \phi_0(x) \phi_0(x')$$

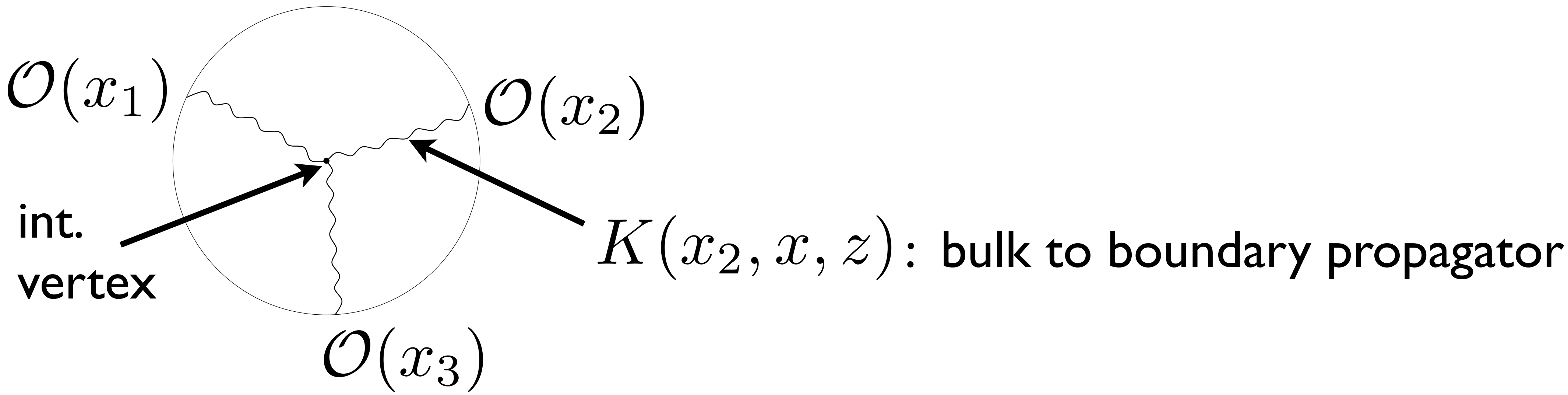
$$\langle \mathcal{O}(x) \mathcal{O}(x') \rangle \sim \frac{1}{(x-x')^{2\Delta}}$$

as expected from CFT

Higher correlators

$$S = \int (\partial\phi)^2 + m^2\phi^2 + b\phi^3$$

Diagrammatic representation (Witten diagrams)



$$\int \phi^3 = \int d^5x \int d^4x_1\, d^4x_2\, d^4x_3\, K_{\Delta_1}(x_1, x, z)$$
$$K_{\Delta_2}(x_2, x, z) K_{\Delta_3}(x_3, x, z) \phi_0(x_1)\phi_0(x_2)\phi_0(x_3)$$

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle =$$
$$= \int \frac{dz}{z^5} \int d^4x K_{\Delta_1}(x_1, x, z) K_{\Delta_2}(x_2, x, z) K_{\Delta_3}(x_3, x, z)$$

Higher correlators - 2

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle \sim$$

$$\sim \frac{c_{\Delta_1\Delta_2\Delta_3}(g_5, N)}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3}\ |x_1 - x_3|^{\Delta_1+\Delta_3-\Delta_2}\ |x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}}$$

in agreement with CFT

Non-renormalization theorem:

$$c_{\Delta_1\Delta_2\Delta_3}(g_S, N)\big|_{\mathrm{AdS}} = c_{\Delta_1\Delta_2\Delta_3}(g^2_{\mathrm{YM}}, N)\big|_{\mathrm{SYM}}$$

Compare SYM : $g_{\mathrm{YM}} \ll 1$

SUGRA AdS : $N \to \infty, \quad \lambda \gg$

Finite T AdS/CFT

- black hole physics
- phase structure of SYM theories
- QCD, hydrodynamics, condensed matter ...

SYM: $S^1 \times \mathbb{R}^3$ (periodic imaginary time)

$$T = \frac{1}{2\pi R}$$

gravity: near-extremal 3-brane solution

$$\begin{aligned} ds^2 &= H^{-1/2}(r)(-f(r)dt^2 + dx_i\, dx^i) \\ &\quad + H^{1/2}(r)(f^{-1}(r)dr^2 + r^2 d\Omega_s^2) \end{aligned} \qquad i = 1, 2, 3$$

$$H(r) = 1 + \frac{R^4}{r^4} \qquad R^4 = 4\pi g_S \alpha'^2 N$$

$$f(r) = 1 - \frac{r_0^4}{r^4}$$

Black 3-brane

Decoupling limit: $r \ll R$

$$ds^2 = \frac{r^2}{R^2}\left(-\left(1-\frac{r_0^4}{r^4}\right)^{-1} dt^2 + dx_i\, dx^i\right)$$

$$+\frac{R^2}{r^2}\left(1-\frac{r_0^4}{r^4}\right)^{-1} dr^2 + R^2 d\Omega_5^2$$

horizon at $r = r_0$

Exercise: compute Hawking T

Find coord. singularity at horizon. Removable with suitable choice of coordinates (i.e. only if Euclidean time is periodic).

$$\beta = \frac{1}{T} = \frac{\pi R^2}{r_0} \qquad \Rightarrow \qquad T_H = \frac{r_0}{\pi R^2}$$

Bekenstein-Hawking entropy

$$ds^2 = \frac{r^2}{R^2}\left(-\left(1-\frac{r_0^4}{r^4}\right)dt^2 + dx_i\,dx^i\right) + \frac{R^2}{r^2}\left(1-\frac{r_0^4}{r^4}\right)^{-1}dr^2$$
$$+R^2 d\Omega_5^2$$

$$A = \left(\frac{r_0}{R}\right)^3 V_3 \overbrace{R^5\pi^3}^{\text{volume of } S^5} = T^3 R^8 V_3 \pi^6$$

spatial volume of D3-branes

$$T = \frac{r_0}{\pi R^2}$$

$$S_{\mathrm{BH}} = \frac{A}{4G_{10}} \longrightarrow \text{holography}$$

G_{10} : 10D Newton constant

YM entropy

$$S_{BH} = \frac{A}{4G_{10}} = \frac{\pi^2}{2} N^2 V_3 T^3$$

cf. $R^4 = 4\pi N g_s \alpha'^2 \qquad G_{10} = 8\pi^6 g_s^2 \alpha'^4$

field theory: $$S = \frac{2\pi^2}{3} N^2 V_3 T^3$$

Agreement up to a factor 4/3 !

$$S_{\rm corr} = \frac{2\pi^2}{3} f(\lambda) N^2 V_3 T^3$$

α' corrections: $\alpha'^3 R^4 + \dots$

leading corrections:

$$f(\lambda) = 1 - \frac{3}{2\pi^2}\lambda + \dots \qquad \lambda \ll$$

diagrammatic methods in pert. finite T field theory

$$f(\lambda) = \frac{4}{3} + \frac{45}{32}\frac{\zeta(3)}{\lambda^{3/2}} + \dots \qquad \lambda \gg$$

leading α' correction in SUGRA

Wilson loop

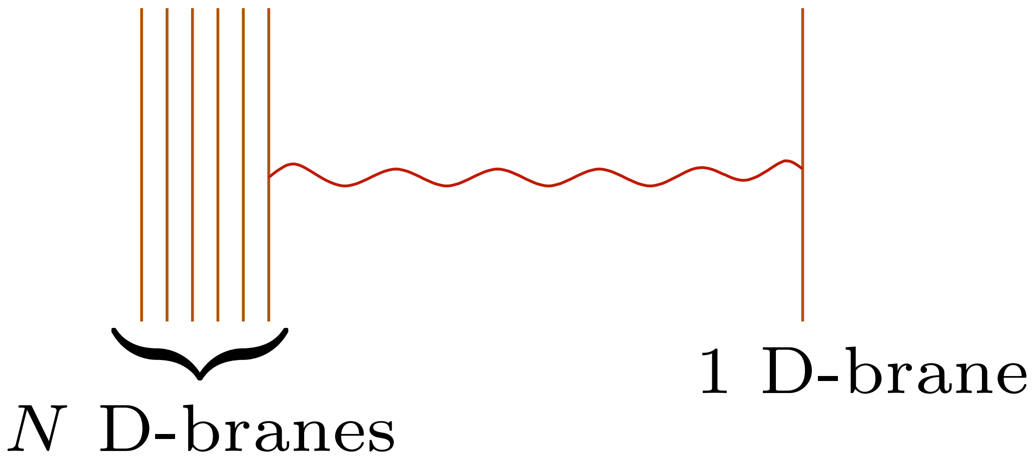
$$W(C) = \mathrm{Tr}\,\left(\underbrace{\mathcal{P}}_{\text{path ordering}}\exp\left(i\oint_{\underbrace{C}_{\text{contour}}} A\right)\right)$$

- order parameter for confining-deconfining phase transition
  confining phase: $\langle W(C)\rangle \sim \exp(-\sigma A_C)$ area law
- computes quark-antiquark potential

ex. show this taking a rectangular loop $T \times L$

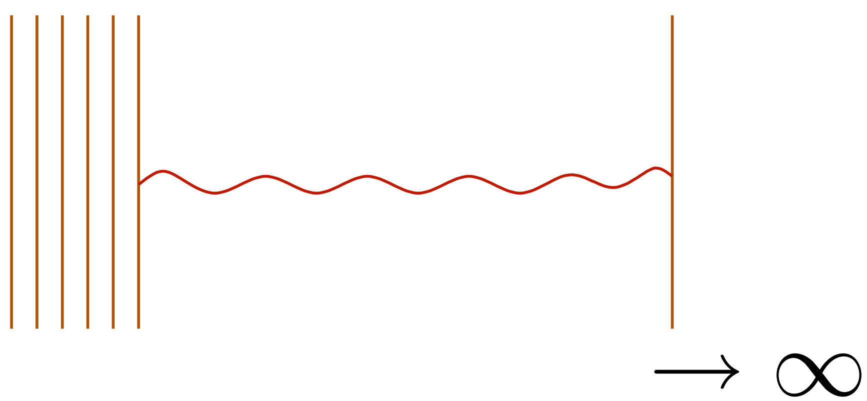
$$V(L) = -\lim_{T\to\infty} \log\langle W(C)\rangle$$

D-brane picture

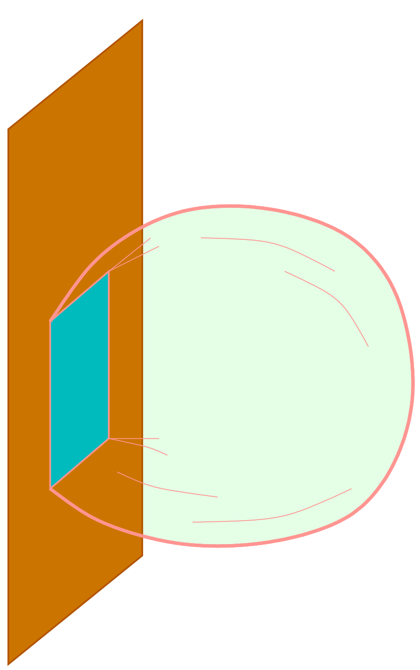


$$U(N+1) \rightarrow U(N) \times U(1)$$

- open massive string states $\sim$ massive quarks in gauge theory

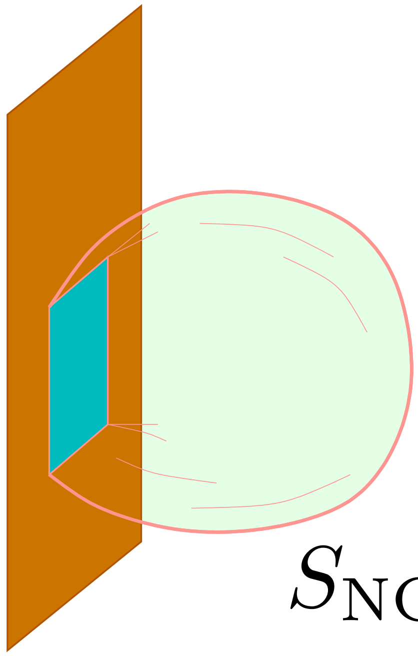


non-dynamical external quark



geometric picture: string worldsheet in AdS

Wilson loop in AdS/CFT



- string chooses minimal area
- in flat 5d spacetime, surface of minimal area with boundary C would lie in boundary

$$S_{\rm NG} = \frac{1}{2\pi\alpha'} \int d\tau\, d\sigma\, \sqrt{\det(g_{MN}\partial_a x^M \partial^a x^N)}$$

AdS metric

$$ds^2 = \frac{r^2}{R^2} dx_\mu\, dx^\mu + R^2 \frac{dr^2}{r^2}$$

string stretching in AdS energetically favored

$\Rightarrow$ quark-antiquark potential can be computed from geometry

Exercise: compute the quark-antiquark potential for a rectangular loop

1. write NG-action for AdS-space
2. extremize to find minimal area
3. regularize area by a cutoff near boundary
4. find energy of two separated quark

$$E = V(L) \sim \frac{\sqrt{\lambda}}{L}$$

"QCD string" $\simeq$ fundametal string in AdS!

Compactification

confining example: $\mathcal{N}=4$ SYM on $\mathbb{R}^{2,1}\times S^1_{R_0}$

$$ds^2_{\mathrm{AdS}_5}=\frac{R^2}{z^2}\left[-\left(1-\frac{z^4}{z_0^4}\right)dt^2+dx^i dx^i+\frac{dz^2}{\left(1-\frac{z^4}{z_0^4}\right)}\right]$$

double Wick rotation $\quad t=iy \quad y=y+2\pi R_0 \qquad x_3=it$

$$ds^2=\frac{R^2}{z^2}\left[-dt^2+dx_1^2+dx_2^2+\left(1-\frac{z^4}{z_0^4}\right)dy^2+\frac{dz^2}{1-\frac{z^4}{z_0^4}}\right]$$

$$z_0=2R_0$$

boundary: $\mathbb{R}^{2,1}\times S^1$ as required

Mass gap

$$ds^2 = \frac{R^2}{z^2}\left[-dt^2 + dx_1^2 + dx_2^2 + \left(1 - \frac{z^4}{z_0^4}\right) dy^2 + \frac{dz^2}{1 - \frac{z^4}{z_0^4}}\right]$$

space terminates $z \geq z_0$

AdS-soliton

warp factor $w(z) \geq w(z_0) = \frac{R}{z_0}$

UV/IR correspondence

$$x^{\mu}_{YM} = \frac{1}{w(z)} x^{\mu}_{\text{proper}}$$

$$E_{YM} = w(z) E_{\text{proper}} = \frac{R}{z_0} E_{\text{proper}}$$

UV in field theory $\longleftrightarrow$ IR in gravity theory

- area law and mass gap

Finite T

finite T: YM theory on $\mathbb{R}^2 \times S^1_\beta \times S^1_{R^0}$

gravity theory: 1. AdS-soliton

$$ds_s^2 = \frac{R^2}{z^2}\left[d\tau^2 + dx_1^2 + dx_2^2 + \left(1 - \frac{z^4}{z_0^4}\right)dy^2 + \frac{dz^2}{1 - \frac{z^4}{z_0^4}}\right]$$

2. black brane

$$ds_{bb}^2 = \frac{R^2}{z^2}\left[\left(1 - \frac{z^4}{z_0^4}\right)d\tau^2 + dx_1^2 + dx_2^2 + dy^2 + \frac{dz^2}{(1 - \frac{z^4}{z_0^4})}\right]$$

$$\tau \leftrightarrow y \quad \beta \leftrightarrow 2\pi R_0$$

But Lorentzian different!

Confinement-deconfinement transition

- P.I. sum over geometries with same boundary
- dominated by lowest action (infinite volume)
- phase transition: $\beta_c = 2\pi R_0 \quad (T_c = 1/2\pi R_0)$

(Hawking-Page phase transition)

field theory: confinement-deconfinement phase transition

$T < T_c$: AdS-soliton, confined, N-independent spectrum

$T > T_c$: Black-brane, deconfined, $\mathcal{O}(N^2)$ states

not real QCD: KK-modes do not decouple $\Lambda_{\mathrm{QCD}} \sim \dfrac{1}{R_0}$

AdS/CFT and QCD

advantage:

- can examine QCD-like theories (but not QCD...)
- qualitative geometric interpretation of
  - Wilson-loop
  - confinement-deconfinement transition
- easy to compute at finite T
  cf. hydrodynamics and quark-gluon plasma

minus:

- $N \to \infty$, $\lambda \to \infty$: handles strong coupling, large N region
  $\Rightarrow$ exact solutions, integrability
- weakly coupled gauge theory $\Leftrightarrow$ highly curved string background

  at weak coupling, KK-modes important

Non-relativistic CFTs

- invariant under Galilean transformation
- invariant under non-relativistic scale invariance
- many non-relativistic CFTs govern physical systems in
  - condensed matter physics
    e.g. fermions at unitarity
- gravity duals?
- holographic dictonary?

Symmetry algebra

rotations $\{M_{ij}\}$
translations $\{P_i\}$
Galilean boost $\{K_i\}$
time translations $\{H\}$
dilatations $\{D\}$

Schrodinger-algebra

"dynamical exponent":

$$t \to \lambda^z t, \quad \vec{x} \to \lambda \vec{x}$$

$$[D, M_{ij}] = 0, \qquad [D, P_i] = iP_i, \qquad [D, H] = izH$$

$$[D, K_i] = i(1-z)K_i \quad [D, N] = i(2-z)N \quad [P_i, K_j] = -\delta_{ij}N$$

$N:$ number operator (eg. fermion number) $\qquad i, j = 1 \ldots d$

$z = 2:$ special conformal transformations C

$$[D, C] = -2iC \quad [H, C] = -iD \quad [D, N] = 0$$

Dual geometry

$$ds^2 = R^2 \left(-\frac{dt^2}{r^{2z}} + \frac{dx^i dx^i + 2d\xi\, dt}{r^2} + \frac{dr^2}{r^2} \right) \qquad i = 1 \ldots d$$

scaling

$$x \sim \lambda x \quad t \sim \lambda^z t \quad \xi \sim \lambda^{2-z} \xi$$

Galilean boost if

$$\xi' = \xi + \frac{1}{2}(2\vec{v}\vec{x} - v^2 t), \quad \vec{x}' = \vec{x} - \vec{v}t$$

$$N = i\partial_\xi$$

ex. Take a scalar field on this background. Find relevant scaling dimension and two point function in NR boundary theory.

String theory embedding

- to quantize N, $\xi \sim \xi + 2\pi L_\xi$ → DLCQ

Null Melvin twist: sequence of boost, T-dualities, and twist

$$(\sigma \to \sigma + \alpha dy)$$

extremal D3-brane solution → Schrodinger geometry, d=2, z=2

⇒ dual field theory is $\mathcal{N} = 4$ SYM twisted by an R-charge

$$SU(4) \to SU(3) \times U(1)$$

non-extremal D3-brane solution → finite T black hole, asymptotically Schrodinger

- thermodynamics, shear viscosity

Summary

AdS/CFT based on

- open-closed duality
- holography
- large N expansion

range of applicability from fundamental string questions to condensed matter systems...