Introduction to the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring

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- String sigma model

- Light cone gauge, quantization, symmetry algebra

$\mathrm{AdS}_5 \times \mathrm{S}^5$ max. symm. II.B SUGRA solution (like M_{10})
$\exists$ RR flux (sd) $\longrightarrow$ NSR formalism problematic (coupling to background nonlocal)

GS formalism: any SUGRA background manifest space time SUSY
local fermionic symmetry "κ -symm."
problem: bosonic sol. $\longrightarrow$ full II.B superfield not known in general

M_{10} "coset" GS: WZ type non linear Σ model on coset superspace
super Poincare naturally WZ guarantees κ symmetry

GS on $\mathrm{AdS}_5 \times \mathrm{S}^5$ Σ model with target space $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times \mathrm{SO}(5)}$

bosonic $\mathrm{SU}(2,2) \times \mathrm{SU}(4) \sim \mathrm{SO}(4,2) \times \mathrm{SO}(6)$ is in $\mathrm{PSU}(2,2|4)$
$\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ local Lorentz transformations

$\mathrm{PSU}(2,2|4)$ by left multiplication isometry group of $\mathrm{AdS}_5 \times \mathrm{S}^5$ superspace

superconformal algebra $\mathfrak{psu}(2,2|4)$

superalgebra $\mathfrak{sl}(4|4) \equiv \mathscr{G}$ 8×8 matrices M 4×4 blocks

$M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix}$ $\mathrm{str} M \equiv \mathrm{tr}\, m - \mathrm{tr}\, n = 0$ m, n even θ, η odd

$M \in \mathfrak{su}(2,2|4)$ if $M^\dagger H + HM = 0$ $M^\dagger = (M^t)^*$ $H = \begin{pmatrix} \Sigma & 0 \\ 0 & \mathbb{I}_4 \end{pmatrix}$

$\Sigma = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}$ $m \in \mathfrak{u}(2,2)$ $n \in \mathfrak{u}(4)$ also $\mathfrak{u}(1)$-generator $i\mathbb{I}$

bosonic subalgebra (BSA) $\mathfrak{su}(2,2|4)$ is $\mathfrak{su}(2,2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1)$

superalgebra $\mathfrak{psu}(2,2|4)$ is the quotient $\mathfrak{su}(2,2|4)/\mathfrak{u}(1)$

basis for BSA Dirac matrices $\gamma^i\gamma^j + \gamma^j\gamma^i = 2\delta^{ij}$ $i,j = 1,\ldots,5$

$\gamma^5 = -\gamma^1\gamma^2\gamma^3\gamma^4 = \mathrm{diag}(1,1,-1-1) = \Sigma$ $(\gamma^i)^* = (\gamma^i)^t$

$m \sim \mathfrak{su}(2,2) \sim \mathrm{span}_{\mathbb{R}}\left\{\frac{1}{2}\gamma^i,\ \frac{i}{2}\gamma^5,\ \frac{1}{4}[\gamma^i,\gamma^j],\ \frac{i}{4}[\gamma^5,\gamma^j]\right\}$ $i,j = 1,\ldots,4$

$n \sim \mathfrak{su}(4) \sim \mathrm{span}_{\mathbb{R}}\left\{\frac{i}{2}\gamma^i,\ \frac{1}{4}[\gamma^i,\gamma^j]\right\}$ $i,j = 1,\ldots,5$

conformal $\mathfrak{su}(2,2)$ introduce $\gamma^{ij} = \frac{1}{4}[\gamma^i, \gamma^j]$

$i\gamma^{15}\, i\gamma^{25}\, i\gamma^{35}\, i\gamma^{45}$ together with $\gamma^{1,2,3,4}$ span $\begin{pmatrix} 0 & \bullet \\ \bullet & 0 \end{pmatrix} \subset \mathfrak{su}(2,2)$

γ^{ij} $i,j = 1,\ldots 4$ span $\mathfrak{so}(4) \begin{pmatrix} \mathfrak{su}(2) & 0 \\ 0 & \mathfrak{su}(2) \end{pmatrix} \subset \mathfrak{su}(2,2)$

$\frac{i}{2}\gamma^5$ diagonal   “conformal Hamiltonian"

important $K = -\gamma^2\gamma^4 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ $(\gamma^i)^t = K\gamma^i K^{-1}$

$\mathbb{Z}_4$-grading: endow $\mathscr{G} = \mathfrak{sl}(4|4)$ with graded Lie superalgebra structure

$M = \begin{pmatrix} m & \theta \\ \eta & n \end{pmatrix}$ $M^{st} = \begin{pmatrix} m^t & -\eta^t \\ \theta^t & n^t \end{pmatrix}$ fourth order automorphism Ω

$$M \to \Omega(M) = -\mathcal{K} M^{st} \mathcal{K}^{-1} \qquad \mathcal{K} = \mathrm{diag}(K, K)$$

$\Omega(M_1 M_2) = -\Omega(M_2)\Omega(M_1)$

$\mathscr{G}^{(k)} = \left\{ M \in \mathscr{G},\ \Omega(M) = i^k M \right\}$ $\mathscr{G} = \mathscr{G}^{(0)} \oplus \mathscr{G}^{(1)} \oplus \mathscr{G}^{(2)} \oplus \mathscr{G}^{(3)}$

for any $M \in \mathscr{G}$ its projection $M^{(k)} \in \mathscr{G}^{(k)}$

$$M^{(k)} = \frac{1}{4}\Big(M + i^{3k}\Omega(M) + i^{2k}\Omega^2(M) + i^k\Omega^3(M)\Big)$$

$M^{(0)}$ and $M^{(2)}$ even (bosonic) $M^{(1)}$ and $M^{(3)}$ odd (fermionic)

$$M^{(0)} = \tfrac{1}{2}\begin{pmatrix} m - Km^tK^{-1} & 0 \\ 0 & n - Kn^tK^{-1} \end{pmatrix} \qquad [\gamma^i, \gamma^j] = -K[\gamma^i, \gamma^j]^t K^{-1}$$

$$M^{(2)} = \tfrac{1}{2}\begin{pmatrix} m + Km^tK^{-1} & 0 \\ 0 & n + Kn^tK^{-1} \end{pmatrix}$$

$\mathscr{G}^{(0)}$ coincides with $\mathfrak{so}(4,1) \oplus \mathfrak{so}(5) \subset \mathfrak{su}(2,2) \oplus \mathfrak{su}(4)$

$\mathscr{G}^{(2)}$ spanned by $\{\gamma^{1,2,3,4}, i\gamma^5\} \in \mathfrak{su}(2,2)$ and $\{i\gamma^i\} \in \mathfrak{su}(4)$ $i = 1, \ldots, 5$

Lie algebra generators corresponding to directions

$\mathsf{SU}(2,2) \times \mathsf{SU}(4)/\mathsf{SO}(4,1) \times \mathsf{SO}(5) = \mathsf{AdS}_5 \times \mathsf{S}^5$

central element $i\mathbb{I} \in \mathfrak{su}(2,2|4)$ also in $M^{(2)}$

Lagrangian and symmetries

dimensionless string tension $g = \frac{R^2}{2\pi\alpha'}$ R radius of S^5 AdS/CFT $g = \frac{\sqrt{\lambda}}{2\pi}$

σ τ world sheet coordinates cylinder $-r \leq \sigma \leq r$

$\mathfrak{g} \in \mathrm{SU}(2,2|4)$ $A = -\mathfrak{g}^{-1}\mathrm{d}\mathfrak{g} = A^{(0)} + A^{(2)} + A^{(1)} + A^{(3)} \in \mathfrak{su}(2,2|4)$

vanishing curvature $\partial_\alpha A_\beta - \partial_\beta A_\alpha - [A_\alpha, A_\beta] = 0$

$\epsilon^{\tau\sigma} = 1$ $\gamma^{\alpha\beta} = h^{\alpha\beta}\sqrt{-h}$ Weyl-invariant combination

superstring in AdS$_5 \times$ S^5: $\mathcal{L} = -\frac{g}{2}\Big[\gamma^{\alpha\beta}\mathrm{str}\Big(A_\alpha^{(2)}A_\beta^{(2)}\Big) + \kappa\,\epsilon^{\alpha\beta}\mathrm{str}\Big(A_\alpha^{(1)}A_\beta^{(3)}\Big)\Big]$

$$\Theta_3 = \mathrm{str}\Big(\mathcal{A}^{(2)} \wedge \mathcal{A}^{(3)} \wedge \mathcal{A}^{(3)} - \mathcal{A}^{(2)} \wedge \mathcal{A}^{(1)} \wedge \mathcal{A}^{(1)}\Big) = \mathrm{d}\,\mathrm{str}\Big(\mathcal{A}^{(1)} \wedge \mathcal{A}^{(3)}\Big)/2$$

$\mathfrak{g} \to \mathfrak{g}\mathfrak{h}(\sigma,\tau)$ $\mathfrak{h} \in \mathrm{SO}(4,1)\times\mathrm{SO}(5)$ $A^{(0)}$ gauge $A^{(i)}$ $i = 1, \ldots 3$ rotation

$\mathcal{L}$ invariant $\to$ depends on G/H coset element only

global symmetries PSU(2,2|4): $G:\ \mathfrak{g} \to \mathfrak{g}'$ where $G \cdot \mathfrak{g} = \mathfrak{g}'\,\mathfrak{h}$

restricted to bosonic variables usual Polyakov action in AdS$_5 \times$ S^5

eq. of motion

$$\delta\mathcal{L} = -\mathrm{str}(\delta\mathcal{A}_\alpha\,\Lambda^\alpha) \qquad \Lambda^\alpha = g\Big[\gamma^{\alpha\beta}\mathcal{A}^{(2)}_\beta - \tfrac{1}{2}\kappa\,\epsilon^{\alpha\beta}(\mathcal{A}^{(1)}_\beta - \mathcal{A}^{(3)}_\beta)\Big]$$

$\partial_\alpha\Lambda^\alpha - [\mathcal{A}_\alpha, \Lambda^\alpha] = 0$ in $\mathfrak{psu}(2,2|4)$ $\mathbb{Z}_4$ projection $\mathscr{G}^{(0)}$ vanishes

$\mathscr{G}^{(2)}$: $\partial_\alpha(\gamma^{\alpha\beta}\mathcal{A}^{(2)}_\beta) - \gamma^{\alpha\beta}[\mathcal{A}^{(0)}_\alpha, \mathcal{A}^{(2)}_\beta] + \frac{1}{2}\kappa\epsilon^{\alpha\beta}\Big([\mathcal{A}^{(1)}_\alpha, \mathcal{A}^{(1)}_\beta] - [\mathcal{A}^{(3)}_\alpha, \mathcal{A}^{(3)}_\beta]\Big) = 0$

$\mathscr{G}^{(1)}$: $\gamma^{\alpha\beta}[\mathcal{A}^{(3)}_\alpha, \mathcal{A}^{(2)}_\beta] + \kappa\epsilon^{\alpha\beta}[\mathcal{A}^{(2)}_\alpha, \mathcal{A}^{(3)}_\beta] = 0$ $\mathscr{G}^{(3)}$ similar

Noether current $J^\alpha = \mathfrak{g}\,\Lambda^\alpha\mathfrak{g}^{-1}$ $\partial_\alpha J^\alpha = 0$ conserved charge

$$\mathrm{Q} = \int_{-r}^{r} \mathrm{d}\sigma\, J^\tau = g\int_{-r}^{r}\mathrm{d}\sigma\,\mathfrak{g}\Big[\gamma^{\tau\tau}A^{(2)}_\tau + \gamma^{\tau\sigma}A^{(2)}_\sigma - \frac{\kappa}{2}(A^{(1)}_\sigma - A^{(3)}_\sigma)\Big]\mathfrak{g}^{-1}$$

Virasoro constraints $\mathrm{str}(\mathcal{A}^{(2)}_\alpha\mathcal{A}^{(2)}_\beta) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\rho\delta}\mathrm{str}(\mathcal{A}^{(2)}_\rho\mathcal{A}^{(2)}_\delta) = 0$

κ symmetry

right local action $\quad \mathfrak{g} \cdot \exp(\epsilon(\tau,\sigma)) = \mathfrak{g}' \, \mathfrak{h} \quad \epsilon \quad$ in $\mathfrak{psu}(2,2|4)$

$\delta_\epsilon \mathcal{A} = -\mathrm{d}\epsilon + [\mathcal{A}, \epsilon]$ $\quad \mathbb{Z}_4$-decomposition with $\epsilon = \epsilon^{(1)} + \epsilon^{(3)}$

$$-\frac{2}{g}\,\delta_\epsilon \mathcal{L} = \delta\gamma^{\alpha\beta} \mathrm{str}\Big(\mathcal{A}^{(2)}_\alpha \mathcal{A}^{(2)}_\beta\Big) - 4\,\mathrm{str}\Big([\mathcal{A}^{(1),\alpha}_+, \mathcal{A}^{(2)}_{\alpha,-}]\epsilon^{(1)} + [\mathcal{A}^{(3),\alpha}_-, \mathcal{A}^{(2)}_{\alpha,+}]\epsilon^{(3)}\Big)$$

$V^\alpha_\pm = \mathsf{P}^{\alpha\beta}_\pm V_\beta \qquad \mathsf{P}^{\alpha\beta}_\pm \mathcal{A}_{\beta,\mp} = 0 \qquad \mathcal{A}_{\tau,\pm} = -\frac{\gamma^{\tau\sigma} \mp \kappa}{\gamma^{\tau\tau}} \mathcal{A}_{\sigma,\pm}$

$\epsilon^{(1)} = \mathcal{A}^{(2)}_{\alpha,-} \kappa^{(1),\alpha}_+ + \kappa^{(1),\alpha}_+ \mathcal{A}^{(2)}_{\alpha,-} \qquad \epsilon^{(3)} = \mathcal{A}^{(2)}_{\alpha,+} \kappa^{(3),\alpha}_- + \kappa^{(3),\alpha}_- \mathcal{A}^{(2)}_{\alpha,+}$

$\kappa^{(i),\alpha}_\pm$ independent parameters, homogeneous of degree $i = 1,3$ under Ω

$\delta_\epsilon \mathcal{L}$ vanishes if $\quad \delta\gamma^{\alpha\beta} = \frac{1}{2}\mathrm{tr}\Big([\kappa^{(1),\alpha}_+, \mathcal{A}^{(1),\beta}_+] + [\kappa^{(3),\alpha}_-, \mathcal{A}^{(3),\beta}_-]\Big)$

exploited $\quad \mathsf{P}^{\alpha\beta}_\pm$ orthogonal projectors $\quad \kappa = \pm 1 \quad$ must hold on shell rank of κ symmetry: how many fermions can be gauged away

in LCG $A^{(2)} = \begin{pmatrix} ix\gamma^5 & 0 \\ 0 & iy\gamma^5 \end{pmatrix}$ Vir constraint $\mathrm{str}(A^{(2)}_{\alpha,-}A^{(2)}_{\beta,-}) = 0 \quad x = \pm y$

$$\epsilon^{(1)} = 2ix \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon^\dagger\Sigma & 0 \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} \varkappa_{11} & \varkappa_{12} & 0 & 0 \\ \varkappa_{21} & \varkappa_{22} & 0 & 0 \\ 0 & 0 & -\varkappa_{33} & -\varkappa_{34} \\ 0 & 0 & -\varkappa_{43} & -\varkappa_{44} \end{pmatrix}$$

$\varkappa$ belongs to $\mathscr{G}^{(1)} \to \epsilon^{(1)}$ depends on 8 real fermionic parameters ($\epsilon^{(3)}$ also)

generic odd element of $\mathfrak{su}(2,2|4)$

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \bullet & \bullet & 0 & 0 \\ 0 & 0 & 0 & 0 & \bullet & \bullet & 0 & 0 \\ \hline 0 & 0 & \bullet & \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & \bullet & \bullet & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Classical integrability of the superstring

integrability (Lax pair); spectral parameter z $\Psi(\sigma,\tau,z)$ r comp. vector
$\frac{\partial\Psi}{\partial\sigma} = L_\sigma(\sigma,\tau,z)\Psi$ $\frac{\partial\Psi}{\partial\tau} = L_\tau(\sigma,\tau,z)\Psi$ L_σ L_τ $r\times r$ matrix
$L_\alpha = (L_\tau, L_\sigma)$ zero curvature ZCC $\partial_\alpha L_\beta - \partial_\beta L_\alpha - [L_\alpha, L_\beta] = 0$
ZCC invariant under $L_\alpha \to L'_\alpha = hL_\alpha h^{-1} + \partial_\alpha h h^{-1}$

conservation laws: mondromy matrix $\mathsf{T}(z) = \overleftarrow{\exp} \int_0^{2\pi} d\sigma\, L_\sigma(z)$
$\partial_\tau \mathsf{T}(z) = [L_\tau(0,\tau,z), \mathsf{T}(z)]$ $\mathcal{H}^{(n)} = \mathrm{tr}(\mathsf{T}^n(\tau,z))$ independent of τ
eigenvalues $\Gamma(z,\mu) \equiv \det(\mathsf{T}(z) - \mu\mathbb{I}) = 0$

example: principal chiral model $\mathfrak{g} \equiv \mathfrak{g}(\sigma,\tau)$ $\mathcal{L} = -\frac{1}{2}\gamma^{\alpha\beta}\mathrm{tr}\Big(\partial_\alpha \mathfrak{g}\mathfrak{g}^{-1}\partial_\beta \mathfrak{g}\mathfrak{g}^{-1}\Big)$
e.o.m.: $A_l^\alpha = -\gamma^{\alpha\beta}\partial_\beta \mathfrak{g}\mathfrak{g}^{-1}$ $A_r^\alpha = -\gamma^{\alpha\beta}\mathfrak{g}^{-1}\partial_\beta \mathfrak{g}$ $\partial_\alpha A_l^\alpha = 0 = \partial_\alpha A_r^\alpha$
flatness: $\partial_\alpha A_\beta - \partial_\beta A_\alpha \pm [A_\alpha, A_\beta] = 0$ $+$ for $A = A^l$ $-$ for $A = A^r$
ansatz: $L_\alpha = \ell_1 A_\alpha + \ell_2 \gamma_{\alpha\beta}\epsilon^{\beta\rho}A_\rho$ ZCC satisfied $\ell_1^2 - \ell_2^2 \pm \ell_1 = 0$

$$L_\alpha^l = \frac{z^2}{1-z^2} A_\alpha^l + \frac{z}{1-z^2}\gamma_{\alpha\beta}\epsilon^{\beta\rho}A_\rho^l$$

Lax pair for superstring:

$$L_\alpha = \ell_0 \mathcal{A}_\alpha^{(0)} + \ell_1 \mathcal{A}_\alpha^{(2)} + \ell_2 \gamma_{\alpha\beta}\epsilon^{\beta\rho} \mathcal{A}_\rho^{(2)} + \ell_3 \mathcal{A}_\alpha^{(1)} + \ell_4 \mathcal{A}_\alpha^{(3)}$$

project ZCC to $\mathscr{G}^{(k)}$ exploit flatness of $\mathcal{A}^{(k)}$ and string e.o.m.

$\mathscr{G}^{(0)}$: $\ell_0 = 1$ $\quad \ell_1^2 - \ell_2^2 = 1$ $\quad \ell_3 \ell_4 = 1$

$\mathscr{G}^{(2)}$: $\frac{\ell_3^2 - \ell_1}{\ell_2} = -\kappa$ $\quad \frac{\ell_4^2 - \ell_1}{\ell_2} = \kappa$ $\quad 2\ell_1 = \ell_3^2 + \ell_4^2$

$\mathscr{G}^{(1,3)}$: $\frac{\ell_1 \ell_4 - \ell_3}{\ell_2 \ell_4} = \kappa$ $\quad \frac{\ell_4 - \ell_1 \ell_3}{\ell_2 \ell_3} = \kappa$

consistency: $\kappa^2 = 1$ integrability $\leftrightarrow$ κ symmetry ! solution

$$\ell_0 = 1 \quad \ell_1 = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right) \quad \ell_2 = -\frac{1}{2\kappa}\left(z^2 - \frac{1}{z^2}\right) \quad \ell_3 = z \quad \ell_4 = \frac{1}{z}$$

automorphism Ω: $\Omega(L_\alpha(z)) = L_\alpha(iz)$ explicitly $\mathcal{K} L_\alpha^{st}(z) \mathcal{K}^{-1} = -L_\alpha(iz)$

at $z = 1$ $\quad L = -\mathfrak{g}^{-1} \mathrm{d}\mathfrak{g}$

can be gauged away $h = \mathfrak{g}$ changing $\mathcal{A}^{(i)} \rightarrow a^{(i)} = \mathfrak{g} \mathcal{A}^{(i)} \mathfrak{g}^{-1}$

$$\ell_0 = 0 \quad \ell_1 = \frac{1}{2z^2}(z^2 - 1)^2 \quad \ell_2 = -\frac{1}{2\kappa}\left(z^2 - \frac{1}{z^2}\right) \quad \ell_3 = z - 1 \quad \ell_4 = \frac{1}{z} - 1$$

at $w = z - 1 \quad L_\alpha = \frac{2w}{\kappa}\mathcal{L}_\alpha + \dots \quad \mathcal{L}_\alpha = \gamma_{\alpha\beta}\epsilon^{\beta\rho}a_\rho^{(2)} - \frac{\kappa}{2}(a_\alpha^{(1)} - a_\alpha^{(3)})$

ZCC in first order in w

$$\partial_\alpha \mathcal{L}_\beta - \partial_\beta \mathcal{L}_\alpha = 0 \quad \Longrightarrow \quad \partial_\alpha\left(\epsilon^{\alpha\beta}\mathcal{L}_\beta\right) = 0$$

$J^\alpha = g\,\epsilon^{\alpha\beta}\mathcal{L}_\beta$ Noether current corresponding to global $\mathfrak{psu}(2,2|4)$ symmetry

Integrability and symmetries

reprametrization plus κ symmetries $\rightarrow$ not all d.o.f of L are physical

physical subspace (Vir constraint and gauge fixed) is interesting

Vir constraint is not from ZCC

Theorem: Lax connection keeps ZCC after κ symm. iff Vir satisfied

diffeomorphism $\sigma^\alpha \to \sigma^\alpha + f^\alpha(\sigma) \quad \sigma^\alpha = (\sigma, \tau)$

δL_α gauge transformation with parameter $f^\beta L_\beta$

Coset parametrization

$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)} \in \mathrm{SU}(2,2|4)$ various embeddings by field redefinitions

most suitable for LCG $\mathfrak{g} = \mathfrak{g}_{\mathfrak{f}}\mathfrak{g}_{\mathfrak{b}}$

$$\mathfrak{g}_{\mathfrak{b}} = \exp \frac{1}{2} \begin{pmatrix} it\gamma^5 + z^i\gamma^i & 0 \\ 0 & i\phi\gamma^5 + iy^i\gamma^i \end{pmatrix} \qquad \mathfrak{g}_{\mathfrak{f}} = \exp \chi \qquad \chi = \begin{pmatrix} 0 & \Theta \\ -\Theta^\dagger\Sigma & 0 \end{pmatrix}$$

t, z^i cover AdS_5 ϕ, y^i cover S^5 $0 \leq \phi < 2\pi$

fermions linearly under global bosonic symmetries $\mathfrak{g}_{\mathfrak{f}} \to G\, \mathfrak{g}_{\mathfrak{f}}\, G^{-1} = \exp G\chi G^{-1}$

also under shifts in t, ϕ in LCG we need neutral ones

$$\Lambda(t,\phi) = \exp \begin{pmatrix} \frac{i}{2}t\gamma^5 & 0 \\ 0 & \frac{i}{2}\phi\gamma^5 \end{pmatrix} \qquad \mathfrak{g}(\mathbb{X}) = \exp \mathbb{X} \qquad \mathbb{X} = \begin{pmatrix} \frac{1}{2}z^i\gamma^i & 0 \\ 0 & \frac{i}{2}y^i\gamma^i \end{pmatrix}$$

$\mathfrak{g} = \Lambda(t,\phi)\,\mathfrak{g}(\chi)\,\mathfrak{g}(\mathbb{X})$ shifts $t \to t + a$, $\phi \to \phi + b$ identified with $\Lambda(a,b)$

$$G \cdot \mathfrak{g} = \Lambda(a,b)\Lambda(t,\phi)\,\mathfrak{g}(\chi)\,\mathfrak{g}(\mathbb{X}) = \Lambda(t+a,\phi+b)\,\mathfrak{g}(\chi)\,\mathfrak{g}(\mathbb{X})$$

$\mathfrak{g}$ adapted to LCG only a subgroup of bosonic symmetries realized linearly the centralizer of the shifts $\mathfrak{C} = \mathfrak{so}(4) \oplus \mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$

$G \in \exp \mathfrak{C} \quad G\,\Lambda(t,\phi)\,G^{-1} = \Lambda(t,\phi)$ both fermions and bosons

$$\chi \to \chi' = G\,\chi\,G^{-1} \qquad \mathbb{X} \to \mathbb{X}' = G\,\mathbb{X}\,G^{-1}$$

$G = \mathrm{diag}(\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, \mathfrak{g}_4)$ $\quad \mathfrak{g}_i$ four independent $\quad \mathrm{SU}(2)$

$$\mathbb{X} = \begin{pmatrix} 0 & Z & 0 & 0 \\ Z^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & iY \\ 0 & 0 & iY^\dagger & 0 \end{pmatrix} \qquad \chi = \begin{pmatrix} 0 & 0 & \Theta_1 & \Theta_2 \\ 0 & 0 & \Theta_3^\dagger & \Theta_4 \\ -\Theta_1{}^\dagger & \Theta_3 & 0 & 0 \\ -\Theta_2^\dagger & \Theta_4{}^\dagger & 0 & 0 \end{pmatrix}$$

$Z = \frac{1}{2}\begin{pmatrix} z_3 - iz_4 & -z_1 + iz_2 \\ z_1 + iz_2 & z_3 + iz_4 \end{pmatrix}$ $\qquad Y : z_i \to y_i$ block structure preserved

Z, Y, Θ_2, Θ_3 bifundamental repr. of $\mathrm{SU}(2)$

$\alpha = 3,4$ for $\mathfrak{g}_1$ $\quad \dot\alpha = \dot 3, \dot 4$ for $\mathfrak{g}_2$ $\quad a = 1,2$ for $\mathfrak{g}_3$ $\quad \dot a = \dot 1, \dot 2$ for $\mathfrak{g}_4$

$$Z = \begin{pmatrix} Z^{3\dot 4} & -Z^{3\dot 3} \\ Z^{4\dot 4} & -Z^{4\dot 3} \end{pmatrix} \qquad \Theta_2 = \begin{pmatrix} \eta^{3\dot 2} & -\eta^{3\dot 1} \\ \eta^{4\dot 2} & -\eta^{4\dot 1} \end{pmatrix}$$

dynamical variables: the fields $Z^{\alpha\dot\alpha}$, $\quad Y^{a\dot a}$, $\quad \theta^{a\dot\alpha}$, $\quad \eta^{a\dot\alpha}$

Light cone gauge and quantization

fix reparametrization invariance uniform light cone gauge LCG

string $\rightarrow$ $2D$ field theory on cylinder with circumference P_+

$H = H(g, P_+)$ string states carry P_+ l.c. momentum

physical states: level matching $p_{\text{ws}} = 0$

Quantization: $P_+ \rightarrow \infty$ decompactification limit cylinder $\rightarrow$ plane
g fixed (but large) $p_{\text{ws}} = p/g$ p fixed
droping level matching $\rightarrow$ “off shell” theory symmetries enhanced
perturbation th. in $1/g$ leading order 8 massive bosons and fermions
perturbative S-matrix symmetry algebra in LCG

Bosonic string in uniform LCG

background with two Abelian isometries: shifts in t time ϕ angle

$$S = -\frac{g}{2}\int_{-r}^{r} \mathrm{d}\sigma\mathrm{d}\tau\,\gamma^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N G_{MN} \qquad X^M = \{t,\phi,x^\mu\}$$

G_{MN} independent of t and ϕ $\qquad p_M$ conjugate to X^M

$p_M = \frac{\delta S}{\delta \dot{X}^M} = -g\,\gamma^{0\beta}\partial_\beta X^N G_{MN}$ $\qquad$ first order form

$S = \int_{-r}^{r} \mathrm{d}\sigma\mathrm{d}\tau\,\left(p_M\dot{X}^M + \frac{\gamma^{01}}{\gamma^{00}}C_1 + \frac{1}{2g\,\gamma^{00}}C_2\right)$ $\qquad C_i$ Vir constraints

shift invariance $\rightarrow$ two conserved charges $\quad E = -\int_{-r}^{r}\mathrm{d}\sigma\,p_t \qquad J = \int_{-r}^{r}\mathrm{d}\sigma\,p_\phi$

light-cone coordinates $x_\pm$ and momenta $p_\pm$ $\qquad a$ arbitrary const.

$$x_- = \phi - t\,,\ x_+ = (1-a)\,t + a\,\phi\,,\ p_- = p_\phi + p_t\,,\ p_+ = (1-a)p_\phi - a\,p_t$$

$$t = x_+ - a\,x_-\,,\ \phi = x_+ + (1-a)x_-\,,\ p_t = (1-a)\,p_- - p_+\,,\ p_\phi = p_+ + a\,p_-$$

$$P_- = \int_{-r}^{r}\mathrm{d}\sigma\,p_- = J - E \qquad P_+ = \int_{-r}^{r}\mathrm{d}\sigma\,p_+ = (1-a)\,J + a\,E$$

$$S = \int_{-r}^{r} \mathrm{d}\sigma \mathrm{d}\tau \left(p_- \dot{x}_+ + p_+ \dot{x}_- + p_\mu \dot{x}^\mu + \frac{\gamma^{01}}{\gamma^{00}} C_1 + \frac{1}{2g\,\gamma^{00}} C_2 \right)$$

$C_1 = p_+ x'_- + p_- x'_+ + p_\mu x'^\mu$ $\quad C_2 = C_2(x_\pm, p_\pm, p_\mu, x^\mu)$ complicated

light-cone gauge: $x_+ = \tau + a\,m\,\sigma \quad p_+ = 1$

$r = \frac{1}{2} P_+ \quad x^\mu(r) = x^\mu(-r)$ cylinder's circumference $2r = P_+$

m winding number $\quad \phi(r) - \phi(-r) = 2\pi m \quad m \in \mathbb{Z}$

gauged fixed action: solve $C_1 = 0$ for x'_- then $C_2 = 0$ for p_-

$S = \int_{-r}^{r} \mathrm{d}\sigma \mathrm{d}\tau \ (p_\mu \dot{x}^\mu - \mathcal{H}) \quad \mathcal{H} = -p_-(p_\mu, x^\mu, x'^\mu)$ indep. of P_+

world sheet Hamiltonian $H = \int_{-r}^{r} \mathrm{d}\sigma\, \mathcal{H} = E - J$

physical states level-matching condition

$\Delta x_- = \int_{-r}^{r} \mathrm{d}\sigma\, x'_- = amH - \int_{-r}^{r} \mathrm{d}\sigma\, p_\mu x'^\mu = 2\pi m$

shifting σ symmetry $p_{\mathrm{ws}} = -\int_{-r}^{r} \mathrm{d}\sigma\, p_\mu x'^\mu$ conserved world sheet mom.

zero winding $m = 0 \quad \Delta x_- = p_{\mathrm{ws}} = 0$

BMN limit $g \to \infty$ $P_+ \to \infty$ keeping g/P_+ fixed

decompactification limit $P_+ \to \infty$ g fixed

$2d$ massive model on a plane $\to$ $\exists$ asymptotic states and S matrix

LCG Σ model solitons giant magnons

GS string in LCG

Lie-algebra valued auxiliary field π

$$\mathcal{L} = -\text{str}\left(\pi\, A_0^{(2)} + \kappa\frac{g}{2}\epsilon^{\alpha\beta}A_\alpha^{(1)}A_\beta^{(3)} + \frac{\gamma^{01}}{\gamma^{00}}\pi\, A_1^{(2)} - \frac{1}{2g\gamma^{00}}\left(\pi^2 + g^2(A_1^{(2)})^2\right)\right)$$

Vir. constraints $C_1 = \text{str}\,\pi\, A_1^{(2)} = 0$ $C_2 = \text{str}\left(\pi^2 + g^2(A_1^{(2)})^2\right) = 0$

κ symmetry and light cone gauge fixing $\mathcal{L}_{gf} = \mathcal{L}_{kin} - \mathcal{H}$

$$\mathfrak{g}(x) = \mathfrak{g}_+\mathbb{I}_8 + \mathfrak{g}_-\Upsilon + \mathfrak{g}_\mu\Sigma_\mu\,, \quad \mathfrak{g}(x)^2 = G_+\mathbb{I}_8 + G_-\Upsilon + G_\mu\Sigma_\mu$$

$\Upsilon = \text{diag}(\mathbb{I}_4, -\mathbb{I}_4)$ $\Sigma_k = \text{diag}(\gamma_k, 0_4)$ $\Sigma_{4+k} = \text{diag}(0_4, i\gamma_k)\; k = 1\ldots4$

$$\mathfrak{g}^{-1}(\chi)\partial_\alpha\mathfrak{g}(\chi) = B_\alpha + F_\alpha$$

$$\mathcal{L}_{kin} = p_\mu \dot{x}_\mu - \frac{i}{2}\mathrm{str}\left(\Sigma_+ \chi \partial_\tau \chi\right) + \frac{1}{2}\mathfrak{g}_\nu \pi_\mu \,\mathrm{str}\left([\Sigma_\nu, \Sigma_\mu]\, B_\tau\right)$$
$$- \; i\kappa \frac{g}{2}(G_+^2 - G_-^2)\,\mathrm{str}\left(F_\tau \mathcal{K} F_\sigma^{st} \mathcal{K}\right) + i\kappa \frac{g}{2} G_\mu G_\nu \,\mathrm{str}\left(\Sigma_\nu F_\tau \Sigma_\mu \mathcal{K} F_\sigma^{st} \mathcal{K}\right)$$

$\mathcal{H} = -\mathbf{p}_- + \mathcal{H}_{WZ}$ $\qquad$ $\mathbf{p}_- = \frac{i}{2}\mathrm{str}\left(\pi \Sigma_+ \mathfrak{g}(x)(1+2\chi^2)\mathfrak{g}(x)\right)$

$$\mathcal{H}_{WZ} = -\kappa \frac{g}{2}(G_+^2 - G_-^2)\,\mathrm{str}\left(\Sigma_+ \chi \sqrt{1+\chi^2} \mathcal{K} F_\sigma^{st} \mathcal{K}\right)$$
$$-\kappa \frac{g}{2} G_\mu G_\nu \,\mathrm{str}\left(\Sigma_+ \Sigma_\nu \chi \sqrt{1+\chi^2} \Sigma_\mu \mathcal{K} F_\sigma^{st} \mathcal{K}\right)$$

$\mathcal{L}_{\mathrm{gf}}$ $\quad g$ $\quad$ dependence exact $\qquad$ independent of $\quad P_+$

decompactification limit $\quad P_+ \to \infty \quad g \quad$ fixed

on cylinder $\qquad$ highly non linear $2d$ model

bosonic fermionic fields periodic b.c.

action $\quad S_{gf} = \int_{-r}^{r} \mathrm{d}\sigma \mathrm{d}\tau \, \mathcal{L}_{gf}$ $\quad$ depends on P_+ only through $\quad r = P_+/2$

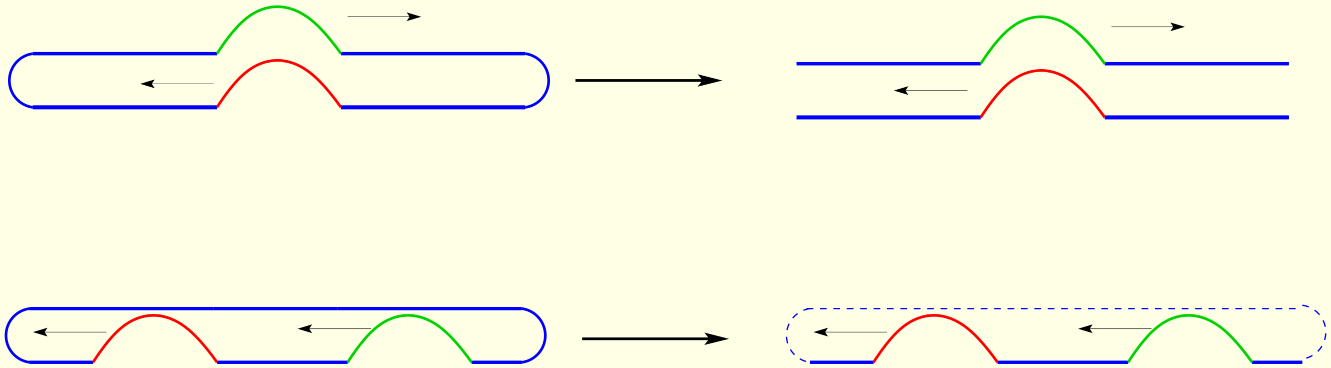
in d.c. limit $\quad$ cylinder $\quad \to \quad$ plane $\qquad$ periodic b.c. $\quad \to \quad$ vanishing b.c.

interested in string states with finite $H = E - J \quad \rightarrow \quad E,\, J \rightarrow \infty$

$P_+ = (1-a)J + aE$ non Lorentz inv. model but massive

solitons quantum integrable

particles with arbitrary mom. drop level matching



Solitonic excitations of a closed string in the decompactification limit

Giant magnon

classical solution bosonic fields only

simplest with finite energy $y_1 \in S^5 \quad z = \frac{y_1}{1+\frac{y_1^2}{4}}$

string in $\mathbb{R} \times \mathrm{S}^2 \in \mathrm{AdS}_5 \times \mathrm{S}^5$ $ds^2_{\mathrm{S}^2} = \frac{dz^2}{1-z^2} + (1-z^2)d\phi^2$

rescale $\sigma \to g\sigma$ $S = g\int_{-\infty}^{\infty} \mathrm{d}\sigma \mathrm{d}\tau \; (p_z \dot{z} - \mathcal{H}(z, z', p_z))$

Lagrangian description $p_z \to \dot{z} = \frac{\partial \mathcal{H}}{\partial p_z}$ $\mathcal{L} = \mathcal{L}(z, z', \dot{z})$

one soliton: propagating wave $z = z(\sigma - v\tau)$ v velocity

$\mathcal{L} \to L_{red}(z, z')$ one particle model σ as time

$\pi_z = \frac{\partial L_{red}}{\partial z'}$ $H_{red} = \pi_z z' - L_{red}$ conserved w.r.t. σ

z vanishing b.c. $z(\pm\infty) = z'(\pm\infty) = 0$ $\to$ $H_{red} = 0$ solve for z'

$$z'^2 = \left(\frac{1-z^2}{1-(1-a)z^2}\right)^2 \frac{z^2}{1-v^2-z^2}$$

finite energy $0 \le a \le 1\,,$ $0 \le |v| \le 1$ $z \in (0, z_{max} = \sqrt{1-v^2})$

on the solution $\frac{\mathcal{H}}{|z'|} = \frac{z}{\sqrt{z_{max}^2 - z^2}}$

then $E - J = g\int_{-\infty}^{\infty} \mathrm{d}\sigma\, \mathcal{H} = 2g\int_0^{z_{max}} \mathrm{d}z \frac{\mathcal{H}}{|z'|} = 2g\sqrt{1-v^2}$

dispersion relation   world sheet momentum

$$p_{\mathrm{ws}} = -\int_{-\infty}^{\infty} \mathrm{d}\sigma p_z z' = 2\int_0^{z_{max}} \mathrm{d}z |p_z|$$

since $p_z = \frac{vz}{(1-z^2)\sqrt{z_{max}^2 - z^2}}$ $p_{\mathrm{ws}} = 2 \arccos v$

giant magnon dispersion relation $E - J = 2g \left| \sin \frac{p_{\mathrm{ws}}}{2} \right|$

- non relativistic
- independent of gauge parameter a
- resembles of lattice
- classical $\longrightarrow$ valid for large g
- gets quantum corrections for finite g

Large g expansion and quantization

rescalings $\quad \sigma \to g\sigma \quad x_\mu \to x_\mu/\sqrt{g}\,, \quad p_\mu \to p_\mu/\sqrt{g}\,, \quad \chi \to \chi/\sqrt{g}$

$$S_{gf} = \int \mathrm{d}\sigma \mathrm{d}\tau \left(\mathcal{L}_2 + \frac{1}{g}\mathcal{L}_4 + \frac{1}{g^2}\mathcal{L}_6 + \cdots \right) \qquad p_{\mathrm{ws}} = -\int \mathrm{d}\sigma \left(p_\mu x'_\mu + \cdots \right) = \frac{1}{g}\, p$$

$g \to \infty \qquad p$ fixed "near plane wave limit"

$\mathcal{L}_2 = p_\mu \dot{x}_\mu - \frac{i}{2}\mathrm{str}\left(\Sigma_+ \chi \dot{\chi}\right) - \mathcal{H}_2$ standard Poisson structure

8 massive bosons and fermions with equal masses

$\mathcal{L}_4$ two unpleasant properties

- terms with time derivative $\rightarrow$ removed by field redefinition order by order in $1/g$
- bosonic terms p^2x^2 $\rightarrow$ removed by canonical transformation

$P_{a\dot{a}} \quad P_{\alpha\dot{\alpha}}$ conjugate to $\quad Y^{a\dot{a}} \quad Z^{\alpha\dot{\alpha}}$

$$\mathcal{L}_2 = P_{a\dot{a}}\dot{Y}^{a\dot{a}} + P_{\alpha\dot{\alpha}}\dot{Z}^{\alpha\dot{\alpha}} + i\,\eta^\dagger_{\alpha\dot{a}}\dot{\eta}^{\alpha\dot{a}} + i\,\theta^\dagger_{a\dot{\alpha}}\dot{\theta}^{a\dot{\alpha}} - \mathcal{H}_2$$

standard (anti)comm. rel. relativistic dispersion relation superindices $M=(a|\alpha)$ $\dot{M}=(\dot{a}|\dot{\alpha})$ a even α odd $\omega_p=\sqrt{1+p^2}$

$$[\,a^{M\dot{M}}(p,\tau)\,,\,a^\dagger_{N\dot{N}}(p',\tau)\,\}=\delta^M_N\,\delta^{\dot{M}}_{\dot{N}}\,\delta(p-p')$$

Q-particle state $|\Psi\rangle=a^\dagger_{M_1\dot{M}_1}(p_1)\,a^\dagger_{M_2\dot{M}_2}(p_2)\cdots a^\dagger_{M_Q\dot{M}_Q}(p_Q)\,|0\rangle$

energy $\mathbb{H}_2|\Psi\rangle=E|\Psi\rangle$ $E=\sum_i\omega_{p_i}$ $\mathbb{H}_2=\int\mathrm{d}p\sum_{M,\dot{M}}\omega_p\,a^\dagger_{M\dot{M}}(p)a^{M\dot{M}}(p)$

world sheet mom. $\mathbb{P}\equiv p_{\mathrm{ws}}$ $\mathbb{P}\,|\Psi\rangle=\sum_i p_i$ level matching $=0$

16 one particle states (OPS) S-matrix $(16\times16)^2$ non diagonal

closed sectors: OPS scattering among themselves

16 OPS charged under $(\mathfrak{su}(2))^4$

two $\mathfrak{su}(2)$ belong to $\mathfrak{su}(4)\subset\mathfrak{psu}(2,2|4)$ act on $a,b,\dot{a},\dot{b},\ldots$ $1,2,$ $\dot{1},\dot{2}$

two $\mathfrak{su}(2)$ belong to $\mathfrak{su}(2,2)\subset\mathfrak{psu}(2,2|4)$ act on $\alpha,\beta,\dot{\alpha},\dot{\beta},\ldots$ $3,4,$ $\dot{3},\dot{4}$

$\mathfrak{su}(2)$ sector: bosonic particles $a^\dagger_{1\dot{1}}$

Q-particle state $|\Psi_{\mathfrak{su}(2)}\rangle=a^\dagger_{1\dot{1}}(p_1)\,a^\dagger_{1\dot{1}}(p_2)\cdots a^\dagger_{1\dot{1}}(p_Q)\,|0\rangle$

maximal charge $Q/2$ $a^\dagger_{1\dot{1}}$ charge 1 under $\mathfrak{u}(1)$ rotating in y_1,y_2 plane field theory operators dual to these states (with $p_{\text{ws}} = 0$)

$P_+ = \frac{1}{2}(E + J)$ large but finite J also large

recall J: • $U(1)$ generating ϕ translation on S^5 • all $a^\dagger_{M\dot{M}}$ neutral

J assigned to light cone vacuum $|\Psi_{\mathfrak{su}(2)}\rangle$ lightest with J and Q

dual to the $\mathcal{N} = 4$ SYM operators $O_{\mathfrak{su}(2)} = \text{tr}\left(Z^J X^Q + \text{permutations}\right)$

$\mathfrak{sl}(2)$ sector: bosonic particles $a^\dagger_{3\dot{3}}$

Q-particle state $|\Psi_{\mathfrak{sl}(2)}\rangle = a^\dagger_{3\dot{3}}(p_1)\, a^\dagger_{3\dot{3}}(p_2) \cdots a^\dagger_{3\dot{3}}(p_Q)\, |0\rangle$

dual to $O_{\mathfrak{sl}(2)} = \text{tr}\left(D_-^Q Z^J + \text{permutations}\right)$ D_- cov. der.

many other closed sectors

Perturbative world sheet S-matrix

$\mathbb{H} = \mathbb{H}_0 + \mathbb{V}$ interaction repr. $\mathbb{S} = \mathscr{T} \exp\left(-i \int_{-\infty}^{\infty} \text{d}\tau\, \mathbb{V}\left(a^\dagger_{\text{in}}(\tau), a_{\text{in}}(\tau)\right)\right)$

leading term in $1/g$: $\mathbb{S} = \mathbb{I} + i\,\frac{1}{g}\,\mathbb{T}$ $\mathbb{T} = -g \int_{-\infty}^{\infty} \text{d}\tau\, \mathbb{V}(\tau) + \cdots$

factorization $\mathbb{T} = \mathscr{T} \otimes \mathbb{I} + \mathbb{I} \otimes \dot{\mathscr{T}}$ consistent with $\mathbb{S} = \mathcal{S} \otimes \dot{\mathcal{S}}$

factorization $a^\dagger_{M\dot{M}}(p) \sim a^\dagger_M(p)\, a^\dagger_{\dot{M}}(p)$ $a^\dagger_a\, a^\dagger_{\dot{a}}$ bosonic $a^\dagger_\alpha\, a^\dagger_{\dot{\alpha}}$ fermionic

one-particle states tensor product $|a^\dagger_{M\dot{M}}(p)\rangle \sim |a^\dagger_M(p)\rangle \otimes |a^\dagger_{\dot{M}}(p)\rangle$
two-particle states

$$|a^\dagger_{M\dot{M}}(p_1)a^\dagger_{N\dot{N}}(p_2)\rangle \sim (-1)^{\epsilon_{\dot{M}}\epsilon_N}|a^\dagger_M(p_1)a^\dagger_N(p_2)\rangle \otimes |a^\dagger_{\dot{M}}(p_1)a^\dagger_{\dot{N}}(p_2)\rangle$$

$\mathcal{S}$ and $\dot{\mathcal{S}}$ act in the usual way 16×16 matrices

$$\mathcal{S} \cdot |a^\dagger_M(p_1)a^\dagger_N(p_2)\rangle = \mathcal{S}^{PQ}_{MN}(p_1, p_2)|a^\dagger_P(p_1)a^\dagger_Q(p_2)\rangle$$

explicit form of $\mathcal{S}^{PQ}_{MN}(p_1, p_2)$ determined

Symmetry algebra

Noether charge Q in terms of momenta $\pi = g\,\gamma^{\tau\beta}A^{(2)}_\beta$

$$A^{(1)}_\sigma - A^{(3)}_\sigma = i\mathfrak{g}(x)\mathcal{K}F^{st}_\sigma\mathcal{K}\mathfrak{g}(x)^{-1} \quad F_\sigma \text{ odd component of } \mathfrak{g}^{-1}(\chi)\partial_\sigma\mathfrak{g}(\chi)$$

$\mathsf{Q} = \int_{-r}^{r} \mathrm{d}\sigma \, \Lambda \, \mathfrak{g}(\chi)\mathfrak{g}(x) \left(\pi - ig\frac{\kappa}{2}\mathfrak{g}(x)\mathcal{K}F_{\sigma}^{st}\mathcal{K}\mathfrak{g}(x)^{-1} \right) \mathfrak{g}(x)^{-1}\mathfrak{g}(\chi)^{-1}\Lambda^{-1}$

independent of $\gamma^{\alpha\beta}$ schematic form in the $a = 1/2$ gauge

$$\mathsf{Q} = \int_{-r}^{r} \mathrm{d}\sigma \, \Lambda \, U(x, p, \chi) \, \Lambda^{-1} \quad \Lambda = e^{\frac{i}{2}x_+\Sigma_+ + \frac{i}{4}x_-\Sigma_-} \quad \Sigma_{\pm} = \mathrm{diag}(\pm\Sigma, \Sigma)$$

LCG: $x_+ = \tau$ $\quad x'_- = -\frac{1}{g}\left(p_M x'_M - \frac{i}{2}\mathrm{str}(\Sigma_+ \chi\chi') \right) + \cdots$

zero mode becomes a central element in decomp. limit

rotation dilatation SUSY etc. generators $\mathsf{Q}_{\mathcal{M}} = \mathrm{str}\,(\mathsf{Q}\mathcal{M})$ $\quad \mathcal{M}$ 8×8 matrix

$H = -\frac{i}{2}\mathrm{str}\,(\mathsf{Q}\,\Sigma_+)$ $\quad P_+ = \frac{i}{4}\mathrm{str}(\mathsf{Q}\,\Sigma_-)$ $\quad \mathsf{Q}_{\mathcal{M}}(x_+ \equiv \tau, x_-)$ classified

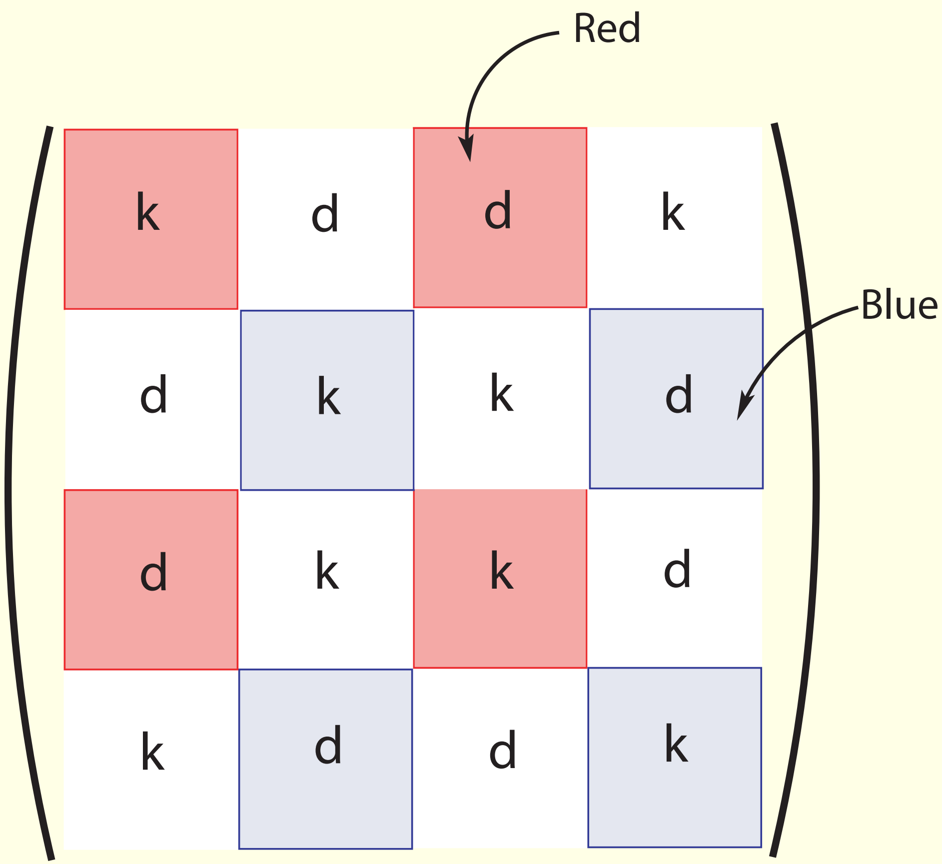
kinematical (x_- indep.) dynamical (x_- dep.) explicitly τ indep./dep.

Hamiltonian setting conservation $\frac{d\mathsf{Q}_{\mathcal{M}}}{d\tau} = \frac{\partial \mathsf{Q}_{\mathcal{M}}}{\partial\tau} + \{H, \mathsf{Q}_{\mathcal{M}}\} = 0$

τ indep. $\mathsf{Q}_{\mathcal{M}}$-s commute with H

form algebra $\mathcal{J}$ with H as central element

structure of Q and $\mathcal{J}$: $\quad \mathcal{M}$ in terms of 2×2 blocks

$\mathcal{M} =$ 

The distribution of the kinematical and dynamical charges in the $\mathcal{M}$ supermatrix. The red (dark) and blue (light) blocks correspond to the subalgebra $\mathcal{J}$ of $\mathfrak{psu}(2,2|4)$

$\mathcal{J} = \mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \oplus \Sigma_+ \oplus \Sigma_-$ for $P_+ \to \infty$ Σ_- decouples

transformation properties of charges under $\mathfrak{C}$; the centrally extended $\mathfrak{su}(2|2)$ algebra time independent charges

$$\mathrm{Q_{sym}} = \begin{pmatrix} \mathbb{R} & 0 & -\mathbb{Q}^\dagger & 0 \\ 0 & \mathring{\mathbb{R}} & 0 & \mathring{\mathbb{Q}} \\ \mathbb{Q} & 0 & \mathbb{L} & 0 \\ 0 & \mathring{\mathbb{Q}}^\dagger & 0 & \mathring{\mathbb{L}} \end{pmatrix} \quad \mathbb{R}, \mathring{\mathbb{R}} \in \mathfrak{su}(2,2) \quad \mathbb{L}, \mathring{\mathbb{L}} \in \mathfrak{su}(4)$$

Hamiltonian $\mathrm{Q}_{\mathbb{H}} = -\frac{i}{4}\mathbb{H}\mathrm{diag}\,(-\mathbb{I}, \mathbb{I}, \mathbb{I}, -\mathbb{I})$ LC mom. $\mathrm{Q}_{\mathbb{P}_+} = \frac{i}{2}\mathbb{P}_+\mathrm{diag}\,(\mathbb{I}, -\mathbb{I}, \mathbb{I}, -\mathbb{I})$

2×2 blocks $\mathbb{R}, \mathbb{L}, \mathbb{Q}, \mathbb{Q}^\dagger$ two-index entries $\mathbb{L}^{ab}, \mathbb{R}^{\alpha\beta}, \mathbb{Q}^{\alpha b}, \mathbb{Q}^\dagger_{a\beta}$

$\mathbb{L}_a{}^b = \epsilon_{ac}\, \mathbb{L}^{cb} \; \ldots \qquad \mathbb{Q}_b^{\dagger\alpha} = \epsilon^{\alpha\gamma}\, \mathbb{Q}^\dagger_{b\gamma}$

bosonic $\mathbb{L}_a{}^b, \mathbb{R}_\alpha{}^\beta$ SUSY $\mathbb{Q}_\alpha{}^a, \mathbb{Q}_a^{\dagger\alpha}$ $\mathbb{H}, \mathbb{C}$ and $\mathbb{C}^\dagger$ centrally extended $\mathfrak{su}(2|2)$

$$[\mathbb{L}_a{}^b, \mathbb{J}_c] = \delta_c^b \mathbb{J}_a - \frac{1}{2}\delta_a^b \mathbb{J}_c \qquad [\mathbb{R}_\alpha{}^\beta, \mathbb{J}_\gamma] = \delta_\gamma^\beta \mathbb{J}_\alpha - \frac{1}{2}\delta_\alpha^\beta \mathbb{J}_\gamma$$

$$[\mathbb{L}_a{}^b, \mathbb{J}^c] = -\delta_a^c \mathbb{J}^b + \frac{1}{2}\delta_a^b \mathbb{J}^c \qquad [\mathbb{R}_\alpha{}^\beta, \mathbb{J}^\gamma] = -\delta_\alpha^\gamma \mathbb{J}^\beta + \frac{1}{2}\delta_\alpha^\beta \mathbb{J}^\gamma$$

$$\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_b^{\dagger\beta}\} = \delta_b^a \mathbb{R}_\alpha{}^\beta + \delta_\alpha^\beta \mathbb{L}_b{}^a + \frac{1}{2}\delta_b^a \delta_\alpha^\beta \mathbb{H}$$

$$\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_\beta{}^b\} = \epsilon_{\alpha\beta}\epsilon^{ab}\, \mathbb{C} \qquad \{\mathbb{Q}_a^{\dagger\alpha}, \mathbb{Q}_b^{\dagger\beta}\} = \epsilon_{ab}\epsilon^{\alpha\beta}\, \mathbb{C}^\dagger$$

$$p_{\mathrm{ws}} \equiv \mathbb{P} \qquad \mathbb{C} = \frac{i}{2} g\,(e^{i\mathbb{P}} - 1)e^{2i\xi} \qquad \mathbb{C}^\dagger = -\frac{i}{2} g\,(e^{-i\mathbb{P}} - 1)e^{-2i\xi}$$

deriving $\mathbb{C}$ $\mathbb{C}^\dagger$ "hybrid expansion" $\mathbb{Q}_A{}^B = \int \mathsf{d}\sigma\, e^{i\alpha x_-}\Omega(x,p,\chi;g)$
$\alpha = 1/2$ for $\mathbb{Q}$, $\mathring{\mathbb{Q}}$ $\alpha = -1/2$ for $\mathbb{Q}^\dagger$, $\mathring{\mathbb{Q}}^\dagger$ expand only Ω

field redef. $\mathbb{Q}_A{}^B = \int \mathsf{d}\sigma\, e^{i\alpha x_-}\chi \cdot \Big(\Upsilon_1(x,p) + \frac{1}{g}\Upsilon_3(x,p) + \cdots\Big) + \mathcal{O}(\chi^3)$

$$\{\mathbb{Q}_1, \mathbb{Q}_2\} \sim \int_{-\infty}^{\infty} \mathsf{d}\sigma\, e^{i(\alpha_1+\alpha_2)x_-}\left(\Upsilon_1^{(1)}(x,p)\Upsilon_1^{(2)}(x,p) + \mathcal{O}(\tfrac{1}{g})\right)$$

for $\alpha_1 = \alpha_2 = \pm 1/2$ $\Upsilon_1^{(1)}(x,p)\Upsilon_1^{(2)}(x,p) \sim g\, x'_- + \frac{d}{d\sigma} f(x,p)$ f local

since $g\, x'_- e^{\pm i x_-} \sim g \frac{d}{d\sigma} e^{\pm i x_-}$ in the central charges

$$\int_{-\infty}^{\infty} \mathsf{d}\sigma\, \frac{d}{d\sigma} e^{\pm i x_-} = e^{\pm i x_-(\infty)} - e^{\pm i x_-(-\infty)} = e^{\pm i x_-(-\infty)}\left(e^{\pm i p_{\mathsf{ws}}} - 1\right)$$

identifying $x_-(-\infty) \equiv \xi$ $\mathbb{C}$ $\mathbb{C}^\dagger$ obtained
checked $\mathcal{O}(\frac{1}{g})$ vanishes consistent $\mathbb{C}$ $\mathbb{C}^\dagger$ exact