Bethe equations for AdS/CFT

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Plan of the lecture

- **Reminder**
- $SU(2)$ **sector at 1-loop order (XXX Heisenberg chain)**
- $SU(N)$ **spin chain and the Nested Bethe Ansatz**
- $SU(2)$ **sector at higher orders and the asymptotic Bethe Ansatz**
- $SU(N)$ **spin chain with long range interactions**
- **Bethe equations for AdS/CFT**

Reminder

SYM operators $\longleftrightarrow$ Spin chain $\longleftrightarrow$ Strings on AdS

anomalous dimensions $\longleftrightarrow$ energies $\longleftrightarrow$ energies

mixing matrix $\longleftrightarrow$ Hamilton operator

Spectrum of an integrable spin chain $\longrightarrow$ Bethe Ansatz

The purpose of the lecture is to give an introduction into the Coordinate Bethe Ansatz technique that is necessary to diagonalize the spin chains appearing in AdS/CFT.

$SU(2)$ sector at 1-loop order

In the $SU(2)$ sector the mixing matrix at 1-loop order is given by the Hamiltonian of the isotropic $XXX_{1/2}$ Heisenberg chain (Minahan, Zarembo '03)

Identification: $\uparrow \quad \longleftrightarrow \quad Z \qquad \downarrow \quad \longleftrightarrow \quad W$

States: $|\downarrow\downarrow\uparrow\uparrow\uparrow \ldots \downarrow\uparrow\rangle \quad \longleftrightarrow \quad \mathrm{Tr}(ZZWWW...ZW)$

Hamilton operator: $\hat{H} = \sum_{l=1}^{L} H_{l,l+1}$

Nearest neighbour interaction: $H_{l,l+1} = I_{l,l+1} - P_{l,l+1}$

Example: $H_{12}|\uparrow\uparrow\rangle = H_{12}|\downarrow\downarrow\rangle = 0 \qquad H_{12}|\uparrow\downarrow\rangle = |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$

The Tr is cyclic $\longrightarrow$ periodic boundary condition +
translation invariance: $e^{iP} = 1$.

<u>Task:</u> search for the eigenvalues of $\hat{H}$: $\hat{H}|\psi\rangle = E|\psi\rangle$.

<u>Observation:</u> $\hat{H}$ preserves the number of $\uparrow$ and $\downarrow$ spins separately.
$n_\uparrow + n_\downarrow = L$ number of lattice sites.

Why do we need Bethe Ansatz?

If we diagonalize in the subsector where $n_\uparrow =$ fixed, then
a matrix of size $\begin{pmatrix} L \\ n_\uparrow \end{pmatrix} \times \begin{pmatrix} L \\ n_\uparrow \end{pmatrix}$ must be diagonalized.

E.g.: $L = 20 \quad n_\uparrow = 10$ then $\begin{pmatrix} L \\ n_\uparrow \end{pmatrix} \approx 200000$

<u>Intelligent procedure: Coordinate Bethe Ansatz</u> (Bethe 1931)

<u>Consists of 4 main steps:</u>

1.) Find a trivial eigenstate. (vacuum)
2.) Search for the 1-particle excitations
3.) Investigate the 2-particle problem and find
the 2-particle S-matrix.
4.) 2-particle S-matrix $\longrightarrow$ m-particle wavefunction.

1. step: trivial vacuum: $|\uparrow \dots \uparrow\rangle = |F\rangle$.
Completely ferromagnetic state: $\hat{H}|F\rangle = 0$.

1-particle states: $[\hat{H}, \hat{n}_\downarrow] = 0 \longrightarrow$ 1 spin turned down

$$|\psi\rangle = \sum_{n=1}^{L} A(n)\sigma_n^-|F\rangle \qquad \sigma_n^-|F\rangle = |\uparrow \dots \uparrow \underbrace{\downarrow}_{n.} \uparrow \dots \uparrow\rangle$$

Eigenvalue equation: $\hat{H}|\psi\rangle = E_1|\psi\rangle$.

$$2A(n) - A(n-1) - A(n+1) = E_1 A(n).$$

Solution: $$A(n) = e^{ikn}$$

$$E_1 = 4\sin^2\tfrac{k}{2}.$$

Energy must be real: $k \in \mathbb{R}$

The wavefunction is identical for all momenta $k \longrightarrow k + 2\pi\ell, \quad \ell \in \mathbb{Z}$, thus $-\pi < k \le \pi$. This is the consequence of the discreteness.

Periodic boundary conditions: $\sigma_{L+n}^- = \sigma_n^-$

$A(n+L) = A(n) \quad \longrightarrow \quad e^{ikL} = 1$ is the simplest Bethe equation.

L solution: $k_\ell = \frac{2\pi\ell}{L}, \quad -\frac{L}{2} < \ell \le \frac{L}{2}, \quad \ell \in \mathbb{Z}$

Energy: $E_\ell = 4\sin^2\frac{k_\ell}{2}$ Momentum: $P_\ell = k_\ell$

SYM: The condition $e^{iP_\ell} = 1$ must be imposed, so only the solution $\ell = 0, \quad k = 0$ is acceptable.

$$E_{\ell=0} = 0 \quad \Longrightarrow \quad \delta D = 0.$$

2-particle states

$$|\psi\rangle = \sum_{n_1<n_2} A(n_1,n_2)\sigma^-_{n_1}\sigma^-_{n_2}|F\rangle \qquad |\uparrow \ldots \underbrace{\downarrow}_{n_1.} \ldots \underbrace{\downarrow}_{n_2.} \ldots \uparrow\rangle$$

$$\hat{H}|\psi\rangle = E_2|\psi\rangle$$

If $n_1 < n_2 - 1$:

$$\begin{aligned} 2A(n_1,n_2) - A(n_1-1,n_2) - A(n_1+1,n_2) \quad &+ \\ 2A(n_1,n_2) - A(n_1,n_2-1) - A(n_1,n_2+1) \quad &= \quad E_2 A(n_1,n_2) \end{aligned}$$

If $n_1 = n_2 - 1$:

$$2A(n_1,n_1+1) - A(n_1-1,n_1+1) - A(n_1,n_1+2) = E_2 A(n_1,n_1+1).$$

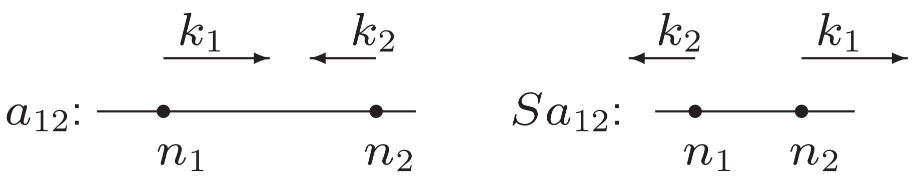
Ansatz: $A(n_1,n_2) = a_{12}e^{ik_1n_1+ik_2n_2} + a_{21}e^{ik_2n_1+ik_1n_2}$

From the 1st equation: $E_2 = E_1(k_1) + E_1(k_2)$

From the 2nd equation: $a_{21} = S(k_1,k_2)a_{12}$

"scattering phase": $S(k_1,k_2) = -\frac{e^{i(k_1+k_2)} - 2e^{ik_2} + 1}{e^{i(k_1+k_2)} - 2e^{ik_1} + 1}$

Physical picture:

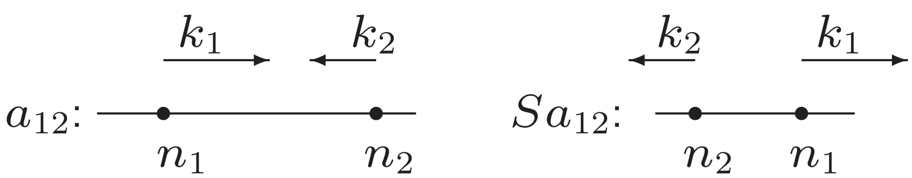


Scattering: particles change momenta and repulse each other

In an other way: reshuffling the wavefunction:

$$|\psi\rangle = \sum_{n_1,n_2} \tilde{A}(n_1,n_2)\sigma^-_{n_1}\sigma^-_{n_2}|F\rangle$$

$$\tilde{A}(n_1,n_2) = \{a_{12}\Theta(n_2-n_1) + a_{21}\Theta(n_1-n_2)\}e^{ik_1n_1+ik_2n_2}$$



Scattering: particles change place

Periodic boundary conditions:

$$\tilde{A}(0,n_2) = \tilde{A}(L,n_2) \longrightarrow e^{ik_1L} = S(k_2,k_1)$$

$$\tilde{A}(n_1,0) = \tilde{A}(n_1,L) \longrightarrow e^{ik_2L} = S(k_1,k_2)$$

Identity: $S(k_1,k_2)S(k_2,k_1) = 1$

SYM: cyclicity: $e^{i(k_1+k_2)} = 1$

Let: $k_1 = -k_2 = k$

Bethe equation: $e^{ikL} = S(-k, k)$

Solution: $k_\ell = \frac{2\pi\ell}{L-1} \qquad \ell \in \{1, ..., L-1\}$

Energy: $E_\ell = 8\sin^2\frac{\pi\ell}{L-1}$.

Generalization to m-particles

$$|\psi\rangle = \sum_{n_1,\ldots,n_m} \tilde{A}(n_1,\ldots,n_m)\sigma^-_{n_1}\ldots\sigma^-_{n_m}|F\rangle$$

$$\tilde{A}(n_1,\ldots,n_m) = \sum_P a_P\ \Theta(n_p)\ \exp\left(i\sum_{j=1}^{m} k_j n_j\right)$$

P is a permuatation of the elements (1,2,...m)

$$\Theta(n_P) = \begin{cases} 1\ , & \text{if} \quad n_{P_1} < n_{P_2} < \ldots < n_{P_m} \\ 0 & \text{otherwise} \end{cases}$$

The S-matrix makes a connection between amplitudes belonging to different orders of the particles. If

$$P = (p_1, .., i, j, \ldots p_m)$$

$$P' = (p_1, .., j, i, \ldots p_m)$$

differ only by a transposition of 2 neighbouring elements, then:

$$a_{P'} = S(k_i, k_j)a_P$$

Let $P_0 = (1, 2, .., m)$, then from the amplitude a_{P_0} all the other amplitudes a_P can be determined uniquely.

E.g. 3-particle wavefunction

$$\tilde{A}(n_1, n_2, n_3) = e^{ik_1n_1+ik_2n_2+ik_3n_3}\{a_{123}\Theta(123) + a_{213}\Theta(213) + a_{132}\Theta(132) + a_{231}\Theta(231) + a_{312}\Theta(312) + a_{321}\Theta(321)\}$$

$a_{123} = 1$ normalization

$(213) = \text{"}(123) \longrightarrow (213)\text{"}$ $\qquad a_{213} = S(k_1, k_2)$

$(132) = \text{"}(123) \longrightarrow (132)\text{"}$ $\qquad a_{132} = S(k_2, k_3)$

$(231) = \text{"}(123) \longrightarrow (213) \longrightarrow (231)\text{"}$

$$a_{231} = S(k_1, k_2)S(k_1, k_3)$$

$(312) = \text{"}(123) \longrightarrow (132) \longrightarrow (312)\text{"}$

$$a_{312} = S(k_2, k_3)S(k_1, k_3)$$

$(321) = \text{"}(123) \longrightarrow (213) \longrightarrow (231) \longrightarrow (321)\text{"}$

$$a_{321} = S(k_1, k_2)S(k_2, k_3)S(k_1, k_3)$$

Bethe equations: $$e^{ik_l L} = \prod_{j=1, j\neq l}^{m} S(k_j, k_l)$$

Energy: $$E = \sum_{j=1}^{m} E_1(k_j)$$

Cyclicity: $$\prod_{j=1}^{m} e^{ik_j} = 1$$

Rapidity variables: $u_j = \frac{1}{2}\cot\frac{k_j}{2} \qquad j = 1, .., m$

Bethe equations: $$\left(\frac{u_k - \frac{i}{2}}{u_k + \frac{i}{2}}\right)^L = \prod_{j=1, j\neq k}^{m} \frac{u_j - u_k + i}{u_j - u_k - i}$$

Energy: $$E = \sum_{j=1}^{m} \left(\frac{i}{u_k + \frac{i}{2}} - \frac{i}{u_k - \frac{i}{2}}\right)$$

Cyclicity: $$\prod_{j=1}^{m} \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} = 1$$

Main point: the 2-particle S-matrix determines uniquely the m-particle wavefunction.

Integrability: the m-particle wavefunction can be characterized by m conserved momenta $\{k_\ell\}_{\ell=1}^{m}$. Not only the entire momentum, but the 1-particle momenta are also conserved.
m-particles $\longrightarrow$ m conserved quantities.

$SU(N)$ spin chain

Fundamental representation: N spin orientations are possible:

$$|0\rangle, |1\rangle, ..., |N-1\rangle$$

Hamilton operator: $\hat{H} = \sum_{j=1}^{L} H_{j,j+1} \qquad H_{ij} = I_{ij} - P_{ij}$

Elementary excitations have flavour indices as well .

Observation: $\hat{H}$ does not change the number of the individual flavours.

Coordinate Bethe Ansatz

Trivial vacuum: $|\underbrace{00...0...0}_{Ldb}\rangle \equiv |0\rangle$

1-particle states:

$$|\psi^{(a)}\rangle = \sum_{n=1}^{L} A_a(n)\, \sigma_n^{+a}\, |0\rangle \qquad\qquad |0...\underbrace{a}_{n.}...0\rangle$$

Solution: $\qquad A_a(n) = e^{ikn}$

$$E_1^a(k) = 4\,\sin^2 \frac{k}{2} \qquad a \in \{1, 2, ..., N-1\}$$

Periodic boundary condition: $\qquad e^{ikL} = 1$

2-particle states:

wavefunction: depends on the positions and flavours of the two particles. .

$\hat{H}$ acts on the wavefunction as a linear difference operator.

2-particle wavefunction: $\psi_{a_1 a_2}(n_1, n_2) =$

$$e^{ik_1 n_1 + ik_2 n_2}\big(A_{a_1 a_2}(12)\,\Theta(n_1 < n_2) \quad + \quad A_{a_1 a_2}(21)\,\Theta(n_2 < n_1)\big)$$
$$e^{ik_2 n_1 + ik_2 n_2}\big(A_{a_2 a_1}(12)\,\Theta(n_1 < n_2) \quad + \quad A_{a_2 a_1}(21)\,\Theta(n_2 < n_1)\big)$$

$$A_{a_1 a_2}(21) = S_{a_1 a_2}^{b_1 b_2}(k_1, k_2)\, A_{b_1 b_2}(12)$$

$S(k_1, k_2)$ is a $(N-1)^2 \times (N-1)^2$ matrix, because particles can change place and quantum numbers.

In rapidity variables:

$$S^{12} \equiv S(k_1, k_2) = \frac{1}{u_1 - u_2 + i}\big((u_1 - u_2)\, I_{12} - i\, P_{12}\big)$$

$$(I_{12})_{a_1 a_2}^{b_1 b_2} = \delta_{a_1}^{b_1}\, \delta_{a_2}^{b_2} \qquad (P_{12})_{a_1 a_2}^{b_1 b_2} = \delta_{a_1}^{b_2}\, \delta_{a_2}^{b_1}$$

2-particle energy: $E_2 = E_1(k_1) + E_1(k_2)$

Periodic boundary condition:

$$\psi_{a_1a_2}(0,n_2)=\psi_{a_1a_2}(L,n_2)\longrightarrow\left(e^{ik_1L}\,S(k_1,k_2)-1\right)A(12)=0$$

$$\psi_{a_1a_2}(n_1,0)=\psi_{a_1a_2}(n_1,L)\longrightarrow\left(S(k_1,k_2)-e^{ik_2L}\right)A(12)=0$$

$A(12)$ is a vector, and these are eigenvalue equations.

The eigenvalues of the S-matrix determine the quantizations of the 1-particle momenta.

<u>M-particle wavefunction</u>

$$\psi_{a_1\ldots a_M}(n_1,\ldots,n_M)=\sum_P e^{i\sum_{j=1}^{M}k_jn_{p_j}}\sum_Q A_{a_{p_1},\ldots,a_{p_M}}(Q)\,\Theta(x_Q)$$

P and Q are permutations of (1,2,...,M)

$$\Theta(x_Q)=\begin{cases}1\ , & \text{ha}\quad n_{Q_1}<n_{Q_2}<\ldots<n_{Q_M}\\ 0 & \text{otherwise}\end{cases}$$

If Q and Q' differ by the transposition of 2 neighbouring elements:

$$Q:\qquad n_{Q_1}<n_{Q_2}<\ldots<n_{Q_\ell}<n_{Q_{\ell+1}}<\ldots<n_{Q_M}$$

$$Q' : \qquad n_{Q_1} < n_{Q_2} < \ldots < n_{Q_{\ell+1}} < n_{Q_\ell} < \ldots < n_{Q_M}$$

$$Q_\ell = i \qquad Q_{\ell+1} = j \qquad Q' = P_{ij} Q$$

then: $A(Q') = S^{ij} A(Q)$

$$A_{a_1 \ldots a_i \ldots a_j \ldots a_M}(Q') = (S^{ij})_{a_i a_j}^{b_i b_j} A_{a_1 \ldots b_i .. b_j \ldots a_M}(Q)$$

$$S^{ij} \equiv S(k_i, k_j) = \frac{1}{u_i - u_j + i}\big((u_i - u_j)\, I_{ij} - i\, P_{ij}\big)$$

Consistency conditions:

"inversion": $S^{ij} S^{ji} = 1$

"independence": $S^{ij} S^{kl} = S^{kl} S^{ij} \qquad i \neq j \neq k \neq l$

Yang-Baxter: $S^{jk} S^{ik} S^{ij} = S^{ij} S^{ik} S^{jk}$

$$\begin{array}{ccc} n_1 < n_2 < n_3 & \xrightarrow{S^{12}} & n_2 < n_1 < n_3 \\ \downarrow S^{23} & & \downarrow S^{13} \\ n_1 < n_3 < n_2 & & n_2 < n_3 < n_1 \\ \downarrow S^{13} & & \downarrow S^{23} \\ n_3 < n_1 < n_2 & \xrightarrow{S^{12}} & n_3 < n_2 < n_1 \end{array}$$

Periodic boundary conditions:

$$\psi_{a_1 \dots a_M}(n_1, \dots, n_\ell = 0, \dots, n_M) = \psi_{a_1 \dots a_M}(a_1, \dots, n_\ell = L, \dots, n_M)$$

$$Q_1 : n_\ell < n_{Q_2} < n_{Q_3} < \dots < n_{Q_M}$$

$$Q_2 : n_{Q_2} < n_{Q_3} < \dots < n_{Q_M} < n_\ell$$

$$A(Q_1) = e^{ik_\ell L} A(Q_2)$$

$$A(\ell, 1, 2, \dots, \ell{-}1, \ell{+}1, .., M) = e^{ik_\ell L}\, A(1, 2, \dots \ell{-}1, \ell{+}1, \dots M, \ell)$$

$$S^{1\ell} S^{2\ell} \dots S^{\ell-1,\ell}\, A(Q_0) = e^{ik_\ell L}\, S^{\ell M} S^{\ell, M-1} \dots S^{\ell,\ell+1}\, A(Q_0)$$

$$Q_0 = (1, 2, 3, .., M)$$

$$e^{ik_\ell L}\, \hat{Z}_\ell\, A(Q_0) = A(Q_0) \qquad \ell = 1, \dots, M$$

$$\hat{Z}_\ell = S^{\ell,\ell-1} S^{\ell,\ell-2} \dots S^{\ell 1} S^{\ell M} S^{\ell, M-1} \dots S^{\ell,\ell+1}$$

$$\hat{Z}_\ell\, \hat{Z}_k = \hat{Z}_k\, \hat{Z}_\ell \qquad \forall \ell, k$$

We have got N eigenvalue equations

Z_ℓ can be considered as Hamiltonians depending on M parameters $(k_1, k_2, ..., k_M)$ and they act on the M-folded tensor product of the $N-1$ dimensional flavour space.

Hilbert space: $N-1$ state spin chain of length M.

Coordinate Bethe ansatz:

trivial vacuum: $|\underbrace{11...1...1}_{M db}\rangle \equiv |0\rangle$

1-particle,2-particles, M_2-particles $\longrightarrow N-2$ state spin chain of length M_2.

Coordinate Bethe ansatz:

trivial vacuum: $|\underbrace{22...2...2}_{M_2 db}\rangle \equiv |0\rangle$

1-particle,2-particles, M_3-particle $\longrightarrow N-3$ state spin chain of length M_3 etc.

until we get a 2 state spin chain which leads to scalar quantization equations.

The procedure consists of $N-1$ steps.

$N-1$ steps $\longrightarrow N-1$ types of roots

$$u_{\ell,k} \longrightarrow \begin{cases} \ell = & 1, ..., N-1 \\ k = & 1, 2, ..., M_\ell \end{cases}$$

Nested Bethe Ansatz

$$\left(\frac{u_{1k} + \frac{i}{2}}{u_{1k} - \frac{i}{2}}\right)^L = \prod_{j=1, j\neq k}^{M} \frac{u_{1k} - u_{1j} + i}{u_{1k} - u_{1j} - i} \prod_{j=1}^{M_2} \frac{u_{1k} - u_{2j} - \frac{i}{2}}{u_{1k} - u_{2j} + \frac{i}{2}}$$

$$k = 1, ..., M$$

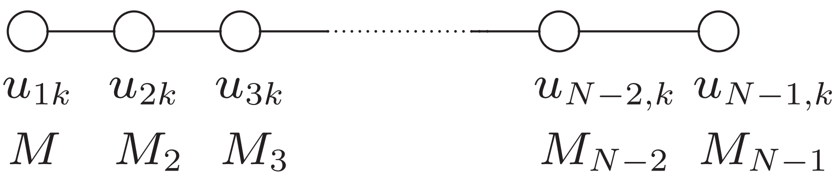
$$1 = \prod_{j=1}^{M_{\ell-1}} \frac{u_{\ell,k} - u_{\ell-1,j} - \frac{i}{2}}{u_{\ell,k} - u_{\ell-1,j} + \frac{i}{2}} \prod_{j=1, j\neq k}^{M_\ell} \frac{u_{\ell,k} - u_{\ell,j} + i}{u_{\ell,k} - u_{\ell,j} - i} \times$$

$$\prod_{j=1}^{M_{\ell+1}} \frac{u_{\ell,k} - u_{\ell+1,j} - \frac{i}{2}}{u_{\ell,k} - u_{\ell+1,j} + \frac{i}{2}} \qquad \ell = 2, ..., N-2, \qquad k = 1, .., M_\ell$$

$$1 = \prod_{j=1}^{M_{N-1}} \frac{u_{N-1,k} - u_{N-2,j} - \frac{i}{2}}{u_{N-1,k} - u_{N-2,j} + \frac{i}{2}} \prod_{j=1, j\neq k}^{M_{N-2}} \frac{u_{N-1,k} - u_{N-1,j} + i}{u_{N-1,k} - u_{N-1,j} - i}$$

$$k = 1, ..., M_{N-1}$$

Demonstration: $SU(N) \longrightarrow A_{N-1}$ Dynkin-diagram



The $SU(N)$ quantum numbers of the Bthe states can be expressed by numbers of the different species of Bethe roots.

Bethe equations in terms of Lie-algebra data:

$$\left(\frac{u_{\ell,k}-\frac{i}{2}V_\ell}{u_{\ell,k}+\frac{i}{2}V_\ell}\right)^L=\prod_{m=1}^{N-1}\prod_{j=1}^{M_m}\frac{u_{\ell,k}-u_{\ell,j}-\frac{i}{2}C_{\ell m}}{u_{\ell,k}-u_{\ell,j}+\frac{i}{2}C_{\ell m}}$$

$C_{\ell m}=-2\frac{\langle\alpha_\ell|\alpha_m\rangle}{\langle\alpha_\ell|\alpha_\ell\rangle}$ Cartan matrix

V_ℓ is the weight of the representation

Cyclicity: $1=\prod_{\ell=1}^{N-1}\prod_{k=1}^{M_\ell}\frac{u_{\ell,k}+\frac{i}{2}V_\ell}{u_{\ell,k}-\frac{i}{2}V_\ell}$

Energy: $E=\sum_{\ell=1}^{N-1}\sum_{k=1}^{M_\ell}\frac{V_\ell}{u_{\ell k}^2+\frac{1}{2}V_\ell^2}$

Long range interactions and the perturbative Bethe Ansatz technique

Example: $SU(2)$ sector at 3-loop order:

$$\hat{H} = \sum_{j=1}^{L} \left\{ H_0(j) + g^2\, H_2(j) + g^4\, H_4(j) + \ldots \right\}$$

$$H_0(j) = I_{j,j+1} - P_{j,j+1}$$

$$H_2(j) = -2I_{j,j+1} + 3P_{j,j+1} - \frac{1}{2}(P_{j,j+1}\, P_{j+1,j+2} + P_{j+1,j+2}\, P_{j,j+1})$$

$$\begin{aligned} H_4(j) &= \frac{15}{2}\, I_{j,j+1} - 13\, P_{j,j+1} + \frac{1}{2} P_{j,j+1}\, P_{j+2,j+3} \\ &+ 3\,(P_{j,j+1}\, P_{j+1,j+2} + P_{j+1,j+2}\,) \\ &- \frac{1}{2}(P_{j,j+1}P_{j+1,j+2}P_{j+2,j+3} + P_{j+2,j+3}P_{j+1,j+2}P_{j,j+1}) \end{aligned}$$

The model is perturbatively integrable up to order of g^4 , i.e. there exist charges Q_n $\quad n = 1, .., L$ which satisfy:

$$[Q_n, Q_m] = [Q_n, \hat{H}] = O(g^6) \qquad \forall n, m$$

Coordinate Bethe Ansatz

Trivial vacuum: $|\uparrow \ldots \uparrow\rangle \equiv |0\rangle$

1-particle states: $|\psi\rangle = \sum_n e^{ikn} \sigma_n^- |0\rangle$

Energy: $E_1(k) = 4\sin^2\frac{k}{2} - 8g^2 \sin^4\frac{k}{2} + 32\, g^4 \sin^6\frac{k}{2} + \ldots$

"Physicist's intuition": $E_1(k) \approx \frac{1}{g^2}\left(\sqrt{1+8g^2\sin^2\frac{k}{2}} - 1\right)$

2-particle states: usual Bethe Ansatz does not work

New Ansatz $\longrightarrow$ perturbative Bethe Ansatz (Staudacher '04)

$$|\psi\rangle = \sum_{n_1<n_2} A(n_1,n_2)\, \sigma_{n_1}^- \sigma_{n_2}^- |0\rangle$$

$$A(n_1,n_2) = \big(1 + B(n_2-n_1,k_1,k_2,g)\big)\, e^{ik_1n_1+ik_2n_2} + \big(1 + C(n_2-n_1,k_1,k_2,g)\big)\, S(k_1,k_2)\, e^{ik_2n_1+ik_1n_2}$$

$$\begin{aligned} B(n_2-n_1,k_1,k_2,g) &= B_2(n_2-n_1,k_1,k_2)g^{2|n_2-n_1|} + \\ &+ B_4(n_2-n_1,k_1,k_2)g^{2+2|n_2-n_1|} + \ldots \\ C(n_2-n_1,k_1,k_2,g) &= C_2(n_2-n_1,k_1,k_2)g^{2|n_2-n_1|} + \\ &+ C_4(n_2-n_1,k_1,k_2)g^{2+2|n_2-n_1|} + \ldots \end{aligned}$$

Everything is calculated only up to g^4 order.

If $|n_2 - n_1| > 3 =$ range of the interaction, then

$$A(n_1, n_2) \approx e^{ik_1 n_1 + ik_2 n_2} + S(k_1, k_2)\, e^{ik_2 n_1 + ik_1 n_2}$$

The S-matrix gets $O(g^2)$ corrections.

m-particle states:

Asymptotic Bethe equations: $e^{ik_\ell L} = \prod_{j=1, j\neq\ell}^{m} S(k_\ell, k_j)$

Validity: $L >$ range of the interaction=order of perturbation theory

Energy: $E = \sum_{j=1}^{m} E_1(k_j)$

Momentum: $P = \sum_{j=1}^{m} k_j$

Rapidity: $u = \frac{1}{2} \cot \frac{k}{2} \sqrt{1 + 8\, g^2 \, \sin^2 \frac{k}{2}}$

$$e^{ik} = \frac{x(u + i/2)}{x(u - i/2)} \qquad x(u) = \frac{1}{2}\, u \left(1 + \sqrt{1 - \frac{2g^2}{u^2}}\right)$$

$$\left(\frac{x(u_\ell + i/2)}{x(u_\ell - i/2)}\right)^L = \prod_{j=1, j\neq\ell}^{m} \frac{u_j - u_\ell - i}{u_j - u_\ell + i}$$

$SU(N)$ spin chain with long range interactions

$$\hat{H} = \sum_{\ell} \{ H_{\ell,\ell+1} + g^2\, H_{\ell,\ell+1,\ell+2} + g^4\, H_{\ell,\ell+1,\ell+2,\ell+3} + \ldots \}$$

Assume: $\exists \quad [Q_n, Q_m] = [Q_n, \hat{H}] = 0, \qquad n, m \in \{1, 2, .., L\}$

Due to long range interactions:

1.) dispersion relation changes (rapidity map): $e^{ik} = \frac{x(u+i/2)}{x(u-i/2)}$

2.) The S-matrix can be modified by a scalar factor:

$$S(u_1, u_2) \longrightarrow e^{2i\,\theta(u_1,u_2)}\, S(u_1, u_2)$$

$$\theta(u_1, u_2) = -\theta(u_2, u_1)$$

Bethe equations:

$$\left[\frac{x(u_{1k} + i/2)}{x(u_{1k} - i/2)}\right]^L = \prod_{j=1, j\neq k}^{M} \left(\frac{u_{1k} - u_{1j} + i}{u_{1k} - u_{1j} - i}\, e^{2i\,\theta(u_{1k},u_{1j})} \right) \times$$

$$\cdot \prod_{j=1}^{M_2} \frac{u_{1k} - u_{2j} - i/2}{u_{1k} - u_{2j} + i/2}$$

Other eqs. agree with those of the previous case..

Asymptotic Bethe equations for AdS/CFT

Bethe equations (Perturbative Nested Bethe Ansatz): (Beisert, Staudacher '05)

$$1 = \left(\frac{x_{4k}^-}{x_{4k}^+}\right)^L \prod_{j=1,j\neq k}^{K_4} \left(\frac{u_{4k}-u_{4j}+i}{u_{4k}-u_{4j}-i}\, e^{2i\,\theta(x_{4k},x_{4j})}\right) \prod_{j=1}^{K_3} \frac{x_{4k}^- - x_{3j}}{x_{4k}^+ - x_{3j}}$$

$$\cdot \prod_{j=1}^{K_5} \frac{x_{4k}^- - x_{5j}}{x_{4k}^+ - x_{5j}} \prod_{j=1}^{K_1} \frac{1-\frac{g^2}{2\,x_{4k}^-\,x_{1j}}}{1-\frac{g^2}{2\,x_{4k}^+\,x_{1j}}} \prod_{j=1}^{K_7} \frac{1-\frac{g^2}{2\,x_{4k}^-\,x_{7j}}}{1-\frac{g^2}{2\,x_{4k}^+\,x_{7j}}} \qquad k = 1\ldots, K_4$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{1k}-u_{2k}+i/2}{u_{1k}-u_{2k}-i/2} \prod_{j=1}^{K_4} \frac{1-\frac{g^2}{2\,x_{1k}\,x_{4j}^+}}{1-\frac{g^2}{2\,x_{1k}\,x_{4j}^-}} \qquad k = 1\ldots, K_1$$

$$1 = \prod_{j=1,j\neq k}^{K_2} \frac{u_{2k}-u_{2j}-i}{u_{2k}-u_{2j}+i} \prod_{j=1}^{K_3} \frac{u_{2k}-u_{3j}+i/2}{u_{2k}-u_{3j}-i/2}$$

$$\cdot \prod_{j=1}^{K_1} \frac{u_{2k}-u_{1j}+i/2}{u_{2k}-u_{1j}-i/2} \qquad k = 1, .., K_2$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3k}-u_{2j}+i/2}{u_{3k}-u_{2j}-i/2} \prod_{j=1}^{K_4} \frac{x_{3k}-x_{4j}^+}{x_{3k}-x_{4j}^-} \qquad k = 1, .., K_3$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5k} - u_{6j} + i/2}{u_{5k} - u_{6j} - i/2} \prod_{j=1}^{K_4} \frac{x_{5k} - x_{4j}^+}{x_{5k} - x_{4j}^-} \qquad k = 1, .., K_5$$

$$1 = \prod_{j=1, j\neq k}^{K_6} \frac{u_{6k} - u_{6j} - i}{u_{6k} - u_{6j} + i} \prod_{j=1}^{K_5} \frac{u_{6k} - u_{5j} + i/2}{u_{6k} - u_{5j} - i/2}$$

$$\cdot \prod_{j=1}^{K_1} \frac{u_{6k} - u_{7j} + i/2}{u_{6k} - u_{7j} - i/2} \qquad k = 1, .., K_6$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{7k} - u_{6k} + i/2}{u_{7k} - u_{6k} - i/2} \prod_{j=1}^{K_4} \frac{1 - \frac{g^2}{2\, x_{7k}\, x_{4j}^+}}{1 - \frac{g^2}{2\, x_{7k}\, x_{4j}^-}} \qquad k = 1..., K_7$$

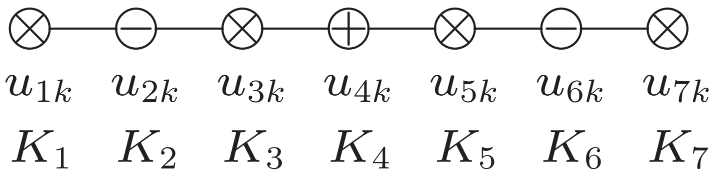
Cyclicity: $1 = \prod_{j=1}^{K_4} \frac{x_{4j}^+}{x_{4j}^-}$

where: $x^\pm(u) = x(u \pm i/2) \qquad x(u) = \frac{1}{2}\, u \left(1 + \sqrt{1 - \frac{2g^2}{u^2}}\right)$

Dressing phase (Beisert, Eden, Staudacher '07, Dorey, Hofmann, Maldacena '07): $\theta(x_1, x_2) = \sum_{r,s=\pm} r \cdot s \cdot \chi(x_1^r, x_2^s)$

$$\chi(x_1, x_2) = -i \oint_{|z_1|=1} \frac{dz_1}{2\pi} \frac{1}{x_1 - z_1} \oint_{|z_2|=1} \frac{dz_2}{2\pi} \frac{1}{x_2 - z_2} \cdot$$

$$\ln \Gamma \left(1 + i\, g \left(z_1 + \frac{1}{z_1} - z_2 - \frac{1}{z_2}\right)\right)$$



Energy or anomalous dimension:

$$\delta D = g^2 \, E = g^2 \sum_{j=1}^{K_4} \left(\frac{i}{x_{4k}^{+}} - \frac{i}{x_{4k}^{-}} \right)$$