Exact S-matrix of AdS/CFT

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Based on: G. Arutyunov and S. Frolov, J. Phys. **A42** (2009), `arXiv;hep-th/0901.4937v2`

Outline

Problems in AdS/CFT correspondence

$\mathcal{N} = 4$ SYM side

- Dynamical spin chains: the length is changing
- Asymptotic Bethe ansatz: works only when the spins are separated well
- Wrapping problem: the size of spin chain J should be infinite for the infinite order perturbations

String side

- Full Quantization of the string theory: only perturbatively in α'
- Classical solitons in Finite-size system: for finite angular momentum

SOLUTION

- Exact S-matrix on the worldsheet

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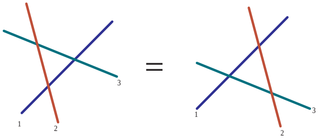
Yang-Baxter Equation

- Due to the infinite # of conserved charges: momenta are preserved
- Multiparticle scatterings are factorized into products of two-body S-matrices

$$S(p_1, p_2, \ldots, p_N) = \prod_{i \leq j}^{N} S_{ij}(p_i, p_j)$$

- Consistency in the order of factorization: YBE

$$S_{12} S_{13} S_{23} = S_{23} S_{13} S_{12}, \quad S_{12} = S \otimes \mathbf{1}, S_{23} = \mathbf{1} \otimes S, \ldots$$



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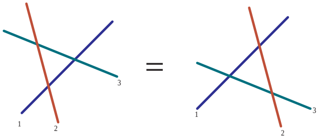

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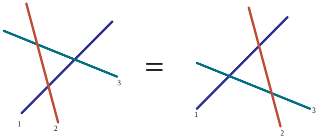

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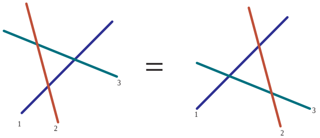

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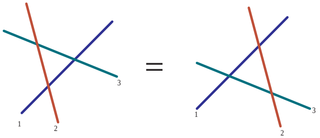

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Symmetries of AdS/CFT

Full superconformal symmetry

- $SO(2,4) \cong SU(2,2)$: Lorentz $L_{\mu\nu}$, trans. P_μ, Dilatation D, spec. conf. K_μ
- R-symmetry: $SO(6) \cong SU(4)$ due to $\mathcal{N} = 4$ SUSY
- Combined: $SU(2,2|4)$ which includes Poincare and Conformal SUSY charges:

$$\left(\begin{array}{c|c} SU(2,2) & Q, S \\ \hline \overline{Q}, \overline{S} & SU(4) \end{array}\right)$$

Ferromagnetic SYM Composite operators

- Most general composite fields: $\mathrm{Tr}\,[\ldots Z \ldots Z \chi_1 Z \ldots Z \chi_2 Z \ldots]$
- $\chi_i = \Phi_i, \Psi_{\alpha\beta}, \Psi_{\dot{\alpha}\dot{\beta}}, D_\mu$
- Symmetries are broken to $[SO(4) \cong SU(2) \times SU(2), SU(2,2) \supset SU(2) \times SU(2)]$

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Chiral sector $SU(2|2)$

- Generators:

$$\left(\begin{array}{c|c} \mathbf{R}_\alpha{}^\beta & \mathbf{Q}_\alpha{}^a \\ \hline \mathbf{S}_b{}^\beta & \mathbf{L}_b{}^a \end{array}\right)$$

- Commutation Relations and SUSY algebra:

$$\begin{aligned} \left[\mathbf{L}_a{}^b, \mathbf{J}_c\right] &= \delta_c^b \mathbf{J}_a - \frac{1}{2}\delta_a^b \mathbf{J}_c\,, \qquad \left[\mathbf{R}_\alpha{}^\beta, \mathbf{J}_\gamma\right] = \delta_\gamma^\beta \mathbf{J}_\alpha - \frac{1}{2}\delta_\alpha^\beta \mathbf{J}_\gamma\,, \\ \left\{\mathbf{Q}_\alpha{}^a, \mathbf{Q}_b^{\dagger\beta}\right\} &= \delta_b^a \mathbf{R}_\alpha{}^\beta + \delta_\alpha^\beta \mathbf{L}_b{}^a + \frac{1}{2}\delta_b^a \delta_\alpha^\beta \mathbf{H}\,, \\ \left\{\mathbf{Q}_\alpha{}^a, \mathbf{Q}_\beta{}^b\right\} &= \epsilon_{\alpha\beta}\epsilon^{ab}\,\mathbf{C}\,, \qquad \left\{\mathbf{Q}_a^{\dagger\alpha}, \mathbf{Q}_b^{\dagger\beta}\right\} = \epsilon_{ab}\epsilon^{\alpha\beta}\,\mathbf{C}^\dagger \end{aligned}$$

- Central Charges: Energy: $\mathbf{H}$, Momentum: $\mathbf{C} = ig(e^{i\mathbf{P}} - 1)$

Spectrum

- fundamental representation

$$\boxslash \equiv \begin{pmatrix} \phi_a \\ \psi_\alpha \end{pmatrix}, \quad a = 1,2, \quad \alpha = 3,4. \qquad \boxslash = \overbrace{(\square, \mathbf{1})}^{\phi_a} \oplus \overbrace{(\mathbf{1}, \square)}^{\psi_\alpha}.$$

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$$\left(\begin{array}{c|c} \mathbf{R}_\alpha{}^\beta & \mathbf{Q}_\alpha{}^a \\ \hline \mathbf{S}_b{}^\beta & \mathbf{L}_b{}^a \end{array}\right)$$

- Commutation Relations and SUSY algebra:

$$\begin{aligned} \left[\mathbf{L}_a{}^b, \mathbf{J}_c\right] &= \delta_c^b \mathbf{J}_a - \frac{1}{2}\delta_a^b \mathbf{J}_c\,, \qquad \left[\mathbf{R}_\alpha{}^\beta, \mathbf{J}_\gamma\right] = \delta_\gamma^\beta \mathbf{J}_\alpha - \frac{1}{2}\delta_\alpha^\beta \mathbf{J}_\gamma\,, \\ \left\{\mathbf{Q}_\alpha{}^a, \mathbf{Q}_b^{\dagger\,\beta}\right\} &= \delta_b^a \mathbf{R}_\alpha{}^\beta + \delta_\alpha^\beta \mathbf{L}_b{}^a + \frac{1}{2}\delta_b^a \delta_\alpha^\beta \mathbf{H}\,, \\ \left\{\mathbf{Q}_\alpha{}^a, \mathbf{Q}_\beta{}^b\right\} &= \epsilon_{\alpha\beta}\epsilon^{ab}\,\mathbf{C}\,, \qquad \left\{\mathbf{Q}_a^{\dagger\,\alpha}, \mathbf{Q}_b^{\dagger\,\beta}\right\} = \epsilon_{ab}\epsilon^{\alpha\beta}\,\mathbf{C}^\dagger \end{aligned}$$

- Central Charges: Energy: $\mathbf{H}$, Momentum: $\mathbf{C} = ig(e^{i\mathbf{P}} - 1)$

Spectrum

- fundamental representation

$$\boxslash \equiv \begin{pmatrix} \phi_a \\ \psi_\alpha \end{pmatrix}, \quad a = 1, 2, \quad \alpha = 3, 4. \qquad \boxslash = \overbrace{(\square, \mathbf{1})}^{\phi_a} \oplus \overbrace{(\mathbf{1}, \square)}^{\psi_\alpha}\,.$$

Full sector $SU(2|2) \times SU(2|2)$

Fields Contents

$$\Phi_i \equiv (\phi_a; \phi_{\dot a}) \,, \quad D_\mu \equiv (\psi_\alpha; \psi_{\dot\alpha}) \,, \quad \Psi_{\alpha\beta} \equiv (\phi_a; \psi_{\dot\alpha}) \,, \quad \Psi_{\dot\alpha\dot\beta} \equiv (\psi_\alpha; \phi_{\dot a})$$

- Tensor Product:

$$(\boxslash; \boxslash) = (\square, \mathbf{1}; \square, \mathbf{1}) \oplus (\square, \mathbf{1}; \mathbf{1}, \square) \oplus (\mathbf{1}, \square; \square, \mathbf{1}) \oplus (\mathbf{1}, \square; \mathbf{1}, \square)$$

-

Fields	$SU(2)_{S^5,L}$	$\times$ $SU(2)_{AdS_5,R}$	$\times$ $SU(2)_{S^5,R}$	$\times$ $SU(2)_{AdS_5,L}$
Z	($\mathbf{1}$	, $\mathbf{1}$	; $\mathbf{1}$	, $\mathbf{1}$)
$\bar Z$	($\mathbf{1}$	, $\mathbf{1}$	; $\mathbf{1}$	, $\mathbf{1}$)
Φ_i	($\square$	, $\mathbf{1}$	; $\square$	, $\mathbf{1}$)
D_μ	($\mathbf{1}$	, $\square$	; $\mathbf{1}$	, $\square$)
$\Psi_{\alpha\beta}$	($\square$	, $\mathbf{1}$	; $\mathbf{1}$	, $\square$)
$\Psi_{\dot\alpha\dot\beta}$	($\mathbf{1}$	, $\square$	; $\square$	, $\mathbf{1}$)

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Fields	$SU(2)_{\mathrm{S}^5,\mathrm{L}}$	$\times\ SU(2)_{\mathrm{AdS}_5,\mathrm{R}}$	$\times\ SU(2)_{\mathrm{S}^5,\mathrm{R}}$	$\times\ SU(2)_{\mathrm{AdS}_5,\mathrm{L}}$
Z	($\mathbf{1}$,	$\mathbf{1}$;	$\mathbf{1}$,	$\mathbf{1}$)
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Φ_i	($\square$,	$\mathbf{1}$;	$\square$,	$\mathbf{1}$)
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Zamolodchikov-Faddeev algebra Approach

- One particle state: $|e_i\rangle = |\phi_a;\psi_\alpha\rangle = \mathbf{A}_i^\dagger|0\rangle$
- Multi-particle state:

$$|e_{i_1}(p_1)e_{i_2}(p_2)\dots e_{i_n}(p_n)\rangle_{\rm in} = \mathbf{A}_{i_1}^\dagger(p_1)\mathbf{A}_{i_2}^\dagger(p_2)\dots\mathbf{A}_{i_n}^\dagger(p_n)|0\rangle, \qquad p_1 > p_2 > \dots > p_n$$

- S-matrix: $S\cdot|e_i(p_1)e_j(p_2)\rangle_{\rm in} = \mathbf{S}_{ij}^{kl}(p_1,p_2)|e_l(p_2)e_k(p_1)\rangle_{\rm out}$
- ZF algebra: $\mathbf{A}_i^\dagger(p_1)\mathbf{A}_j^\dagger(p_2) = \mathbf{S}_{ij}^{kl}(p_1,p_2)\mathbf{A}_l^\dagger(p_2)\mathbf{A}_k^\dagger(p_1)$

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SUSY transformation on particle states

SUSY on particle states

$$\mathbf{Q}_{\alpha}{}^{a}|\phi_b\rangle = \mathbf{a}\,\delta^a_b|\psi_\alpha\rangle\,, \qquad \mathbf{Q}_{\alpha}{}^{a}|\psi_\beta\rangle = \mathbf{b}\,\epsilon_{\alpha\beta}\epsilon^{ab}|\phi_b\rangle$$
$$\mathbf{Q}^{\dagger}_{a}{}^{\alpha}|\psi_\beta\rangle = \mathbf{d}\,\delta^\alpha_\beta|\phi_a\rangle\,, \qquad \mathbf{Q}^{\dagger}_{a}{}^{\alpha}|\phi_b\rangle = \mathbf{c}\,\epsilon_{ab}\epsilon^{\alpha\beta}|\psi_\beta\rangle$$

- Non-zero elements:

$$\mathbf{Q}_3{}^{a}|\phi_a\rangle = \mathbf{a}\,|\psi_3\rangle\,, \quad \mathbf{Q}_4{}^{a}|\phi_a\rangle = \mathbf{a}\,|\psi_4\rangle\,, \quad \mathbf{Q}^{\dagger}_{1}{}^{\alpha}|\psi_\alpha\rangle = \mathbf{d}\,|\phi_1\rangle\,, \quad \mathbf{Q}^{\dagger}_{2}{}^{\alpha}|\psi_\alpha\rangle = \mathbf{d}\,|\phi_2\rangle\,,$$
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SUSY algebra

$$\mathbf{H} = \mathbf{ad} + \mathbf{bc}, \quad \mathbf{C} = \mathbf{ab}, \quad \mathbf{C}^{\dagger} = \mathbf{cd} \quad \& \quad \mathbf{ad} - \mathbf{bc} = 1$$

Shortening relation

$$\mathbf{H}^2 - 4\mathbf{C}\mathbf{C}^{\dagger} = 1$$

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SUSY transformation on particle states

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Unitary Representation

- Unitary Representation:

$$\mathbf{a} = \eta e^{i\xi}, \quad \mathbf{b} = -\eta \frac{e^{-ip/2}}{x^-} e^{i\xi}, \quad \mathbf{c} = -\eta \frac{e^{-i\xi}}{x^+}, \quad \mathbf{d} = \eta e^{-ip/2} e^{-i\xi}$$

- spectral parameters:

$$\eta = e^{ip/4} \sqrt{ig(x^- - x^+)}$$

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g} \quad \& \quad \frac{x^+}{x^-} \equiv e^{ip}$$

$$\rightarrow x^\pm = \frac{e^{\pm ip/2}}{4g \sin\frac{p}{2}} \left(1 + \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}}\right)$$

- Central charges:

$$\mathbf{H} = 1 + \frac{2ig}{x^+} - \frac{2ig}{x^-} = \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}}, \quad \mathbf{C} = ige^{2i\xi}\left(\frac{x^+}{x^-} - 1\right)$$

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Non-local property

- Gauge fix $\xi \equiv 0$ for one-particle state
- Act **C** on multi-particle states:

$$\mathbf{C}|e_{i_1}(p_1)e_{i_2}(p_2)\dots e_{i_n}(p_n)\rangle = ig[e^{i(p_1+\dots+p_n)}-1]|e_{i_1}(p_1)e_{i_2}(p_2)\dots e_{i_n}(p_n)\rangle$$

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$$\xi_1 = 0, \quad \xi_2 = \frac{p_1}{2}, \dots, \xi_n = \frac{1}{2}(p_1+\dots+p_{n-1})$$

- This means a nontrivial braiding relation in Zamolodchikov-Faddeev algebra

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Zamolodchikov-Faddeev algebra

Commutation Relations with $SU(2|2)$ generators

- From SUSY transformation:

$$\mathbf{L}_a{}^b \mathbf{A}^\dagger(p) = \mathbf{A}^\dagger(p)\, L_a^b + \mathbf{A}^\dagger(p)\, \mathbf{L}_a{}^b\,, \quad \mathbf{R}_\alpha{}^\beta A^\dagger(p) = \mathbf{A}^\dagger(p)\, R_\alpha^\beta + \mathbf{A}^\dagger(p)\, \mathbf{R}_\alpha{}^\beta\,,$$
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- Commutativity with $SU(2|2)$ determines the S-matrix

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S-Matrix Structure

$$
\left(\begin{array}{cccc|cccc|cccc|cc}
a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_{1+2} & 0 & 0 & -a_2 & 0 & 0 & 0 & 0 & 0 & 0 & -a_7 & 0 & 0 \\
0 & 0 & a_5 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 \\
\hline
0 & -a_2 & 0 & 0 & a_{1+2} & 0 & 0 & 0 & 0 & 0 & 0 & a_7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_5 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5 & 0 & 0 & 0 & 0 & 0 & a_9 \\
\hline
0 & 0 & a_{10} & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_{10} & 0 & 0 & a_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 \\
0 & -a_8 & 0 & 0 & a_8 & 0 & 0 & 0 & 0 & 0 & 0 & a_{3+4} & 0 & 0 \\
\hline
0 & 0 & 0 & a_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{10} & 0 & 0 & 0 & 0 & 0 & a_6 \\
0 & a_8 & 0 & 0 & -a_8 & 0 & 0 & 0 & 0 & 0 & 0 & -a_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

S-Matrix Elements

$$a_1 = \frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\frac{\eta_1\eta_2}{\tilde{\eta}_1\tilde{\eta}_2}, \qquad a_2 = \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\frac{\eta_1\eta_2}{\tilde{\eta}_1\tilde{\eta}_2}, \qquad a_3 = -1,$$

$$a_4 = \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}, \qquad a_5 = \frac{x_2^- - x_1^-}{x_2^+ - x_1^-}\frac{\eta_1}{\tilde{\eta}_1}, \qquad a_6 = \frac{x_1^+ - x_2^+}{x_1^- - x_2^+}\frac{\eta_2}{\tilde{\eta}_2},$$

$$a_7 = i\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^- x_2^-)\tilde{\eta}_1\tilde{\eta}_2}, \qquad a_8 = i\frac{x_1^- x_2^-(x_1^+ - x_2^+)\eta_1\eta_2}{x_1^+ x_2^+(x_1^- - x_2^+)(1 - x_1^- x_2^-)},$$

$$a_9 = \frac{x_1^+ - x_1^-}{x_1^- - x_2^+}\frac{\eta_2}{\tilde{\eta}_1}, \qquad a_{10} = \frac{x_2^+ - x_2^-}{x_1^- - x_2^+}$$

Charge conjugation

Torus parametrization

$$p = 2\mathrm{am}z, \quad \sin\frac{p}{2} = \mathrm{sn}(z,k), \quad H = \mathrm{dn}(z,k), \quad k = -16g^2$$
$$2\omega_1 = 4K(k), \quad 2\omega_2 = 4iK(1-k) - 4K(k)$$
$$x^{\pm} = \frac{1}{g}\left(\frac{\mathrm{cn}z}{\mathrm{sn}z} \pm i\right)(1 + \mathrm{dn}z)$$

Charge conjugation

$$E \to -E, \qquad p \to -p; \quad x^+ \to \frac{1}{x^+}, \qquad x^- \to \frac{1}{x^-}; \quad z \to z + \omega_2$$

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$$\mathbf{B}_i^{\dagger}(p) \equiv C_{ij}\mathbf{A}^j(-p), \qquad C = \begin{pmatrix} \sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}$$

Zamolodchikov-Faddeev algebra for anti-particles

$$\mathbf{A}_i^{\dagger}(p_1)\mathbf{A}^j(p_2) = \mathbf{A}^{j'}(p_2)\mathbf{S}_{j'i}^{ji'}(p_2,p_1)\mathbf{A}_{i'}^{\dagger}(p_1) + \delta(p_1,p_2)\delta_i^j$$
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$$\mathbf{A}_i^\dagger(p_1)\mathbf{A}^j(p_2) = \mathbf{A}^{j'}(p_2)\mathbf{S}_{j'i}^{ji'}(p_2,p_1)\mathbf{A}_{i'}^\dagger(p_1) + \delta(p_1,p_2)\delta_i^j$$

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Charge conjugation

Torus parametrization

$$p = 2\mathrm{am}z, \quad \sin\frac{p}{2} = \mathrm{sn}(z,k), \quad H = \mathrm{dn}(z,k), \quad k = -16g^2$$

$$2\omega_1 = 4K(k), \quad 2\omega_2 = 4iK(1-k) - 4K(k)$$

$$x^\pm = \frac{1}{g}\left(\frac{\mathrm{cn}z}{\mathrm{sn}z} \pm i\right)(1 + \mathrm{dn}z)$$

Charge conjugation

$$E \to -E, \qquad p \to -p; \quad x^+ \to \frac{1}{x^+}, \qquad x^- \to \frac{1}{x^-}; \quad z \to z + \omega_2$$

Anti-particle operator

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Crossing relation

Component relation

$$\mathbf{S}_{kj'}^{k'j}(p_1,p_2)C_{ik'}(p_1)=C_{i'k}(p_1)\mathbf{S}_{j'i}^{ji'}(p_2,-p_1)$$

Matrix relation

$$C_1^{-1}\mathbf{S}^{t_1}(z_1,z_2)C_1=\mathbf{S}^{-1}(z_1+\omega_2,z_2)$$

Full S-matrix

$$\mathbf{S}(z_1,z_2)=\Sigma^2(z_1,z_2)\mathbf{S}_{SU(2|2)}(z_1,z_2)\otimes\mathbf{S}_{SU(2|2)}(z_1,z_2)$$

Functional relations

$$\Sigma(z_1,z_2)\Sigma(z_1+\omega_2,z_2)=\Sigma(z_1,z_2)\Sigma(z_1,z_2-\omega_2)=\frac{(x_1^- - x_2^+)\left(1-\frac{1}{x_1^- x_2^-}\right)}{(x_1^+ - x_2^+)\left(1-\frac{1}{x_1^+ x_2^-}\right)}$$

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BES, Asymptotic Bethe ansatz, and others

- Solution of the functional relation leads to Beisert-Eden-Staudacher dressing phase
- S-matrix leads to Beisert-Staudacher asymptotic Bethe ansatz
- Similar S-matrix has been derived for $\mathcal{N} = 6$ super Chern-Simons theory
- Consistent with the large coupling limit
- Consistent with the weak coupling limit
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Coordinate Bethe-ansatz

one-loop $SO(6)$ Hamiltonian

$$\mathbf{H} = \sum_{l=1}^{L} \left(1 - \mathbf{P}_{l,l+1} + \frac{1}{2} \mathbf{K}_{l,l+1} \right)$$

- **K** acts

$$\mathbf{K}\, \phi_1 \otimes \phi_2 = \begin{cases} 0 & \text{if } \phi_1 \neq \bar{\phi}_2 \\ X \otimes \bar{X} + \bar{X} \otimes X + Y \otimes \bar{Y} + \bar{Y} \otimes Y + Z \otimes \bar{Z} + \bar{Z} \otimes Z & \text{if } \phi_1 = \bar{\phi}_2 \end{cases} .$$

- two-particle states

$$|x_1, x_2\rangle_{\phi_1 \phi_2} = |\overset{1}{\underset{\downarrow}{}}{Z} \cdots \overset{x_1}{\underset{\downarrow}{}}{\phi_1} \cdots \overset{x_2}{\underset{\downarrow}{}}{\phi_2} \cdots \overset{L}{\underset{\downarrow}{}}{Z}\rangle$$

when $\phi_1 = \phi_2$

$$|\psi\rangle = \sum_{x_1 < x_2} \left[e^{i(p_1 x_1 + p_2 x_2)} + S(p_2, p_1)\, e^{i(p_2 x_1 + p_1 x_2)} \right] |x_1, x_2\rangle_{\phi\phi}$$

$$S(p_2, p_1) = \frac{u_2 - u_1 + i}{u_2 - u_1 - i}, \qquad u(p) = \frac{1}{2} \cot \frac{p}{2}$$

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Coordinate Bethe-ansatz

when $\phi_1 \neq \bar{\phi}_2$

$$|\psi\rangle = \sum_{x_1<x_2} \left\{ f_{\phi_1\phi_2}(x_1, x_2)\, |x_1, x_2\rangle_{\phi_1\phi_2} + f_{\phi_2\phi_1}(x_1, x_2)\, |x_1, x_2\rangle_{\phi_2\phi_1} \right\}$$

$$f_{\phi_i\phi_j}(x_1, x_2) = A_{\phi_i\phi_j}(12)\, e^{i(p_1x_1+p_2x_2)} + A_{\phi_i\phi_j}(21)\, e^{i(p_2x_1+p_1x_2)}$$

- Solution

$$\begin{pmatrix} A_{\phi_1\phi_2}(21) \\ A_{\phi_2\phi_1}(21) \end{pmatrix} = \begin{pmatrix} R(p_2\,, p_1) & T(p_2\,, p_1) \\ T(p_2\,, p_1) & R(p_2\,, p_1) \end{pmatrix} \begin{pmatrix} A_{\phi_1\phi_2}(12) \\ A_{\phi_2\phi_1}(12) \end{pmatrix},$$

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Coordinate Bethe-ansatz

when $\phi_1 = \bar{\phi}_2$

$$|\psi\rangle = \sum_{x_1<x_2} \sum_{\phi=X,Y} \left\{ f_{\phi\bar{\phi}}(x_1,x_2)\,|x_1,x_2\rangle_{\phi\bar{\phi}} + f_{\bar{\phi}\phi}(x_1,x_2)\,|x_1,x_2\rangle_{\bar{\phi}\phi} \right\} + \sum_{x_1} f_{\bar{Z}}(x_1)|x_1\rangle_{\bar{Z}}$$

- Solution

$$\begin{pmatrix} A_{X\bar{X}}(21) \\ A_{\bar{X}X}(21) \\ A_{Y\bar{Y}}(21) \\ A_{\bar{Y}Y}(21) \end{pmatrix} = \begin{pmatrix} R(p_2,p_1) & T(p_2,p_1) & S(p_2,p_1) & S(p_2,p_1) \\ T(p_2,p_1) & R(p_2,p_1) & S(p_2,p_1) & S(p_2,p_1) \\ S(p_2,p_1) & S(p_2,p_1) & R(p_2,p_1) & T(p_2,p_1) \\ S(p_2,p_1) & S(p_2,p_1) & T(p_2,p_1) & R(p_2,p_1) \end{pmatrix} \begin{pmatrix} A_{X\bar{X}}(12) \\ A_{\bar{X}X}(12) \\ A_{Y\bar{Y}}(12) \\ A_{\bar{Y}Y}(12) \end{pmatrix},$$

$$T(p_2,p_1) = \frac{(u_2-u_1)^2}{(u_2-u_1-i)(u_2-u_1+i)}, \quad R(p_2,p_1) = \frac{-1}{(u_2-u_1-i)(u_2-u_1+i)},$$

$$S(p_2,p_1) = \frac{-i(u_2-u_1)}{(u_2-u_1-i)(u_2-u_1+i)}$$

Coordinate Bethe-ansatz

when $\phi_1 = \bar{\phi}_2$

$$|\psi\rangle = \sum_{x_1<x_2} \sum_{\phi=X,Y} \left\{ f_{\phi\bar{\phi}}(x_1,x_2)\,|x_1,x_2\rangle_{\phi\bar{\phi}} + f_{\bar{\phi}\phi}(x_1,x_2)\,|x_1,x_2\rangle_{\bar{\phi}\phi} \right\} + \sum_{x_1} f_{\bar{Z}}(x_1)|x_1\rangle_{\bar{Z}}$$

- Solution

$$\begin{pmatrix} A_{X\bar{X}}(21) \\ A_{\bar{X}X}(21) \\ A_{Y\bar{Y}}(21) \\ A_{\bar{Y}Y}(21) \end{pmatrix} = \begin{pmatrix} R(p_2,p_1) & T(p_2,p_1) & S(p_2,p_1) & S(p_2,p_1) \\ T(p_2,p_1) & R(p_2,p_1) & S(p_2,p_1) & S(p_2,p_1) \\ S(p_2,p_1) & S(p_2,p_1) & R(p_2,p_1) & T(p_2,p_1) \\ S(p_2,p_1) & S(p_2,p_1) & T(p_2,p_1) & R(p_2,p_1) \end{pmatrix} \begin{pmatrix} A_{X\bar{X}}(12) \\ A_{\bar{X}X}(12) \\ A_{Y\bar{Y}}(12) \\ A_{\bar{Y}Y}(12) \end{pmatrix},$$

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Comparison with exact S-matrix

$SU(2|2)$ elements

$$S^{a\,a}_{a\,a}(p_1,p_2) = A\,,\quad S^{a\,b}_{a\,b}(p_1,p_2) = \frac{1}{2}(A-B)\,,\quad S^{b\,a}_{a\,b}(p_1,p_2) = \frac{1}{2}(A+B)$$

$$A = \frac{x_2^- - x_1^+}{x_2^+ - x_1^-} \to \frac{u_1 - u_2 + i}{u_1 - u_2 - i}\,,$$

$$B = -\left[\frac{x_2^- - x_1^+}{x_2^+ - x_1^-} + 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\right] \to -1$$

Dressing phase

$$\Sigma(p_1\,,p_2)^2 = \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\,\frac{1 - \frac{1}{x_1^+ x_2^-}}{1 - \frac{1}{x_1^- x_2^+}}\,\sigma(p_1\,,p_2)^2 \to \frac{u_1 - u_2 - i}{u_1 - u_2 + i}$$

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Comparison with exact S-matrix

- The same type:

$$S(p_1,p_2) \equiv \left(\Sigma(p_1,p_2)\,\widehat{S}^{a\,a}_{a\,a}(p_1,p_2)\right)^2 = S_0^2\,A^2 \rightarrow \frac{u_1-u_2+i}{u_1-u_2-i}$$

- $\phi_1 \neq \bar{\phi}_2$ type:

$$T(p_1,p_2) = \frac{1}{2}S_0^2A(A-B) \rightarrow \frac{u_1-u_2}{u_1-u_2-i}\,, \qquad R(p_1,p_2) = \frac{1}{2}S_0^2A(A+B) \rightarrow \frac{i}{u_1-u_2-i}$$

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\end{aligned}$$