New analytic solutions of the non-relativistic hydrodynamical equations

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Abstract

New solutions are found for the non-relativistic hydrodynamical equations. These solutions describe expanding matter in Gaussian space distribution. In the simplest case, thermal equilibrium is maintained *without any interaction*, the energy is conserved, and the process is isentropic. More general solutions are also obtained that describe explosions driven by heat production, or contraction of the matter caused by energy loss.

Introduction. The equations of hydrodynamics correspond to local conservation of some charges as well as energy and momentum. The equations are scale-invariant, hence can be applied to phenomenological description of physical phenomena from collisions of heavy nuclei to collisions of galaxies. Recently, a lot of experimental and theoretical effort went into the exploration of hydrodynamical behaviour of strongly interacting hadronic matter in non-relativistic as well as in relativistic heavy ion collisions, see for example [1, 3, 4, 5, 6]. Due to the non-linear nature of the equations of hydrodynamics, exact solutions of these equations are rarely found, see e.g. [1] for an exact solution of hydrodynamics of expanding fireballs. The purpose of this Letter is to present and analyze such a solution of the non-relativistic hydrodynamical equations, with a generalization to heat production or loss (e.g. due to radiation). We hope that the results presented herewith may be utilized to access analytically the time-evolution of the hydrodynamically behaving strongly interacting matter as probed by non-relativistic heavy ion collisions [3, 4]. The results presented in this letter are, however, rather general in nature and they can be applied to any physical phenomena where the non-relativistic hydrodynamical description is valid.

Adiabatic expansion. Consider a non-relativistic hydrodynamical system described by the Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\nabla\right) f(\mathbf{r},\, \mathbf{p},\, \mathbf{t}) \;=\; \mathcal{S}(\mathbf{r},\, \mathbf{p},\, t). \qquad (1)$$

Suppose that the $\mathcal{S}$ source function or emission function describes the creation of all the particles, so the boundary condition is $f(\mathbf{r},\, \mathbf{p},\, t \to -\infty) = 0$. The emission function $\mathcal{S}$ is related to the temporal change of the non-relativistic Wigner function of the system. The single-particle momentum distribution can be obtained by integrating the emission function over time and space:

$$N_1(\mathbf{p},\, t) \;=\; \int d^3x f(\mathbf{r},\, \mathbf{p},\, t) \;=\; \int\limits_{-\infty}^{t} dt' \int d^3x\, \mathcal{S}(\mathbf{r},\, \mathbf{p},\, t') \tag{2}$$

Suppose that the emission takes place within an instant, the emission function is spherically symmetric, Gauss-like, and incorporates a linear radial flow:

$$\mathcal{S}(\mathbf{r},\, \mathbf{p},\, t) \;=\; C \exp\left(-\frac{r^2}{2R_0^2} - \frac{\left(\mathbf{p} - \frac{m}{\tau}\mathbf{r}\right)^2}{2mT_0} \right) \delta(t - t_0). \tag{3}$$

Let us assume that there is no collision after the particle emission, so the system described by equations (1) and (3) after t_0 can be equivalently described by the free-streaming equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\nabla \right) f(\mathbf{r},\, \mathbf{p},\, \mathbf{t}) \;=\; 0 \tag{4}$$

with the boundary condition

$$f(\mathbf{r},\, \mathbf{p},\, t_0) \;=\; C \exp\left(-\frac{r^2}{2R_0^2} - \frac{\left(\mathbf{p} - \frac{m}{\tau}\mathbf{r}\right)^2}{2mT_0} \right). \tag{5}$$

The solution of (4-5) is

$$\begin{aligned} f(\mathbf{r},\, \mathbf{p},\, t) \;&=\; f\left(\mathbf{r} - \frac{t - t_0}{m}\mathbf{p},\, \mathbf{p},\, t_0 \right) \\ &=\; C \exp\left(-\frac{\left(\mathbf{r} - \frac{t-t_0}{m}\mathbf{p}\right)^2}{2R_0^2} - \frac{\left((1 + \frac{t-t_0}{\tau})\mathbf{p} - \frac{m}{\tau}\mathbf{r}\right)^2}{2mT_0} \right), \end{aligned} \tag{6}$$

or equivalently

$$f(\mathbf{r},\, \mathbf{p},\, t) \;=\; C \exp\left(-\frac{r^2}{2R^2(t)} - \frac{(\mathbf{p} - m\mathbf{v}(\mathbf{r},\, t))^2}{2mT(t)} \right), \tag{7}$$

where $\mathbf{v}(\mathbf{r}, t) = \beta(t)\mathbf{r}$ and we have three new quantities that depend only on time: the *radius of the expanding matter* $R(t)$, the *local temperature* $T(t)$ and $\beta(t)$, the coefficient of the local *flow velocity* $\mathbf{v}(\mathbf{r},\, t)$. They can be determined from the equality of (6) and (7). After some calculations, we get the following expressions:

$$T(t) \;=\; \frac{T_0}{\varphi(t)}, \qquad R^2(t) \;=\; R_0^2\varphi(t), \qquad \beta(t) \;=\; \frac{\dot{\varphi}(t)}{2\varphi(t)}, \tag{8}$$

where

$$\varphi(t) \;=\; \left(1+\frac{t-t_0}{\tau}\right)^2 + \frac{T_0}{T_G}\left(\frac{t-t_0}{\tau}\right)^2, \tag{9}$$

and $T_G = mR_0^2/\tau^2$ is the "geometrical temperature".

Let us determine the normalization factor in (7). By integrating f over coordinate and momentum space, we get N, the *total number of particles emitted*. Then the normalization factor can be expressed in terms of N and the other parameters,

$$C \;=\; \frac{N}{(2\pi)^3(mT(t)R^2(t))^{3/2}}.$$

Since the system is considered after t_0, the normalization coefficient C is constant because $T(t)R^2(t) = T_0R_0^2 = const.$

Note also that one can introduce the flow parameter $u_0 = R_0/\tau$ — the radial flow at the mean radius — so that the geometrical temperature $T_G = mu_0^2$ carries the flow contribution to the effective temperature. With this notation,

$$\varphi(t) \;=\; \left(1+\frac{u_0}{R_0}(t-t_0)\right)^2 + \frac{T_0}{mR_0^2}(t-t_0)^2. \tag{10}$$

There are two limiting cases. The beginning

$$T(t_0) \;=\; T_0, \qquad R(t_0) \;=\; R_0, \qquad \mathbf{v}(\mathbf{r},\, t_0) \;=\; \frac{\mathbf{r}}{\tau}, \tag{11}$$

and the $t - t_0 \gg \tau$ limit:

$$T(t) \simeq \left(\frac{\tau}{t-t_0}\right)^2 \frac{T_0T_G}{T_0+T_G}, \quad R(t) \simeq \frac{t-t_0}{\tau}\sqrt{R_0^2+R_T^2}, \quad \mathbf{v}(\mathbf{r},\, t) \simeq \frac{\mathbf{r}}{t-t_0},$$

where $R_T^2 = \tau^2 T_0/m = \frac{T_0}{mu_0^2}R_0^2$. So we have $R_T^2/R_0^2 = T_0/T_G$.

Without rescattering and other final state interactions after particle emission, the *momentum spectrum* is independent of time. It is obvious from equation (2). But it can also be checked directly by calculating the single particle spectra $N_1(\mathbf{p}, t)$. We did it and got the following result:

$$N_1(\mathbf{p},\, t) \;=\; \frac{N}{(2\pi mT_*)^{3/2}} \exp\left(-\frac{p^2}{2mT_*}\right), \tag{12}$$

where $T_* = T_0 + T_G$.

The *particle density*, the local *energy density*, the *pressure* and the *entropy density* can be evaluated from kinetic theory in a straightforward manner:

$$n(\mathbf{r},\, t) \;=\; \int d^3p\, f \;=\; \frac{N}{(2\pi R^2(t))^{3/2}} \exp\left(-\frac{r^2}{2R^2(t)}\right), \tag{13}$$

$$\varepsilon(\mathbf{r},\, t) \;=\; \int d^3p\, \frac{(\mathbf{p}-m\mathbf{v})^2}{2m} f \;=\; \frac{3}{2}n(\mathbf{r},\, t)T(t), \tag{14}$$

$$
\begin{aligned}
P(\mathbf{r},\, t) &= \int d^3p\, \frac{(p_x - mv_x)^2}{m} f \;=\; n(\mathbf{r},\, t)T(t), && (15) \\
s(\mathbf{r},\, t) &= -\int d^3p\, (\ln f - 1) f \\
&= \left(\frac{r^2}{2R^2(t)} - \ln N + \frac{3}{2}\ln(4\pi^2 m T(t) R^2(t)) + \frac{5}{2} \right) n(\mathbf{r},\, t). && (16)
\end{aligned}
$$

Note that the local phase-space distribution function $f(\mathbf{r}, \mathbf{p}, t)$ maintains its locally thermalized shape without any collisions, due to the special choice of the initial conditions and the invariance of the Gaussian shape under convolution. We find a non-vanishing pressure maintained without collisions; the interpretation of this result is that any wall or bubble inserted into this expanding Knudsen gas would feel a pressure that arises due to the random, locally disordered motion of the free-streaming particles in any part of space. Such a pressure arising from a collisionless gas of photons is well-known in cosmology as a source of gravity.

By introducing the "effective volume" $V_G = (2\pi)^{3/2} R_0^3$ and integrating the entropy density over space, we find that the total entropy is

$$
S(t) \;=\; \left(\ln \frac{V_G}{N} + \frac{3}{2} \ln T_0 + \frac{3}{2} \ln(2\pi m) + 4 \right) N \;=\; S_{\rm ideal} + \frac{3}{2} N.
$$

As it was shown in [2], one can modify the thermodynamical definition of the entropy by adding terms linear in extensives, without changing the thermodynamics. And now we got the interesting result that the entropy is almost the same as $S_{\rm ideal}$, the entropy of an ideal gas at temperature T_0 in volume V_G — the difference is the extensive quantity $\frac{3}{2}N$, hence the thermodynamics of the system considered is the same as that of an ideal gas. Note that the total entropy is independent of time.

It is worthwhile to evaluate the total energy in local disordered motion (heat energy, denoted by $E_{\rm heat}$), the total energy in ordered motion (flow energy, denoted by $E_{\rm flow}$) and the total energy $E_{\rm tot}$. One obtains that

$$
\begin{aligned}
E_{\rm tot} &= \frac{3}{2} N m u_0^2 + \frac{3}{2} N T_0, && (17) \\
E_{\rm heat} &= \frac{3}{2} N T(t), && (18) \\
E_{\rm flow} &= E_{\rm tot} - E_{\rm heat}. && (19)
\end{aligned}
$$

The time dependence of the local temperature is given by eqs. (8, 10). Of course, the total energy is conserved.

It is straightforward to verify that equations (13-15) together with (8) solve the *continuity equation*, *Euler's equation* — we also have to use eq. (9) for this —, and satisfy the *energy conservation*:

$$
\frac{\partial n}{\partial t} + \nabla(\mathbf{v} n) \;=\; 0, \qquad (20)
$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\nabla\right)\mathbf{v} \quad = \quad -\frac{\nabla P}{mn}, \tag{21}$$

$$\frac{\partial \varepsilon}{\partial t} + \nabla(\mathbf{v}\varepsilon) \quad = \quad -P\nabla\mathbf{v}. \tag{22}$$

From equations (8), (16) and (20) it is easy to verify that the process is *locally isentropic*:

$$\frac{\partial s}{\partial t} + \nabla(\mathbf{v}s) \; = \; 0. \tag{23}$$

Hence, we found a new solution of equations (20-21), the basic equations of non-relativistic hydrodynamics. This solution is given by eqs. (8-10) and eqs. (13-15). The equation of state is that of an ideal gas (15). The system maintains *local thermal equilibrium without any collisions.* This is due to the special initial conditions.

It is also worthwhile to make some remarks about the intensity correlations. We proved that the single-particle spectra are time independent (12). Now we will prove that the *two-particle spectra* $N_2(\mathbf{p}_1, \mathbf{p}_2, t_1, t_2)$ are also independent of time if $t_1, t_2 > t_0$. The two particle distribution function can be written in terms of the emission function S in the following way:

$$\begin{aligned} N_2(\mathbf{p}_1,\, \mathbf{p}_2,\, t_1,\, t_2) \quad &= \quad \int_{-\infty}^{t_1} dt_1' \int d^3r_1 \int_{-\infty}^{t_1} dt_2' \int d^3 r_2 \,(S(\mathbf{r}_1,\, \mathbf{p}_1,\, t_1')S(r_2,\, \mathbf{p}_2,\, t_2') \\ &\pm \quad S(\mathbf{r}_1,\, \mathbf{K}/2,\, t_1')S(\mathbf{r}_2,\, \mathbf{K}/2,\, t_2')\cos\left[\mathbf{q}(\mathbf{r}_1 - \mathbf{r}_2)\right], \end{aligned}$$

where $\mathbf{K} = (\mathbf{p}_1 + \mathbf{p}_2)/2$, $\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_2$ and the "+" sign refers to bosons, "-"to fermions, if the final state Coulomb and other interactions are negligible. S vanishes for $t_{1,2}' > t_0$ so the the time integration is independent of the upper bound $t_{1,2}$ as long as $t_{1,2} > t_0$. Therefore N_2 is independent of t_1 and t_2.

Because both the single and the two-particle spectra are independent of time, the two-particle *correlation function*

$$C_2(\mathbf{p}_1,\, \mathbf{p}_2) \;=\; \frac{N_2(\mathbf{p}_1,\, \mathbf{p}_2)}{N_1(\mathbf{p}_1)N_2(\mathbf{p}_2)} \;=\; 1 + e^{-Q^2 R_*^2}$$

must also be *independent of time.* So the radius parameter of the two-particle correlation function, R_* is a constant. We can calculate it at t_0, the time of particle production. The result

$$\frac{1}{R_*^2} \;=\; \frac{1}{R_T^2} + \frac{1}{R_0^2} \;=\; \frac{1}{R_0^2}\left(1 + \frac{m u_0^2}{T_0}\right)$$

is a generalization for arbitrary u_0 of the result in ref. [3].

The general adiabatic solution. We are seeking for the solutions of the continuity equation (20), Euler's equation (21) and the ideal gas equation of state (15) with three basic assumptions:

1. The particle density is a Gaussian specified by eq. (13), where $R(t)$ is an unknown function.

2. The time dependence of the volume-like quantity $V(t) = (2\pi R^2(t))^{3/2}$ is related to the temperature by $V^{2/3}(t)T(t) = const$, which means that the process is *isentropic* for pointlike particles.

   This relation is equivalent to the first two relations in (8), where T_0 and R_0 are constants and $\varphi(t)$ is an arbitrary function of time.

3. The flow is linear in space coordinates: $\mathbf{v}(\mathbf{r},\, t) = \mathbf{r}\beta(t)$, where $\beta(t)$ is an arbitrary function.

   Now if we try to solve the continuity equation (20) by substituting the results in the previous two paragraphs, we get $\beta(t) = \frac{\dot{\varphi}(t)}{2\varphi(t)}$ which leads to the third relation in (8).

So far we proved that our assumptions lead to equations (8), where $\varphi(t)$ is an arbitrary function. To get restrictions for $\varphi(t)$, we have to solve Euler's equation. The substitution of the expressions in (8), the particle density (13) and the pressure (15) into Euler's equation (21), leads to this differential equation:

$$\ddot{\varphi}(t)\varphi(t) - \frac{1}{2}\dot{\varphi}^2(t) \;=\; \frac{2T_0}{mR_0^2}. \tag{24}$$

The general solution of this equation is a quadratic polynomial with two arbitrary constants, denoted by c_0 and c_1. Additionally, a third coefficient c_2 can be determined from c_0 and c_1. Our result is

$$\varphi(t) \;=\; c_0 + c_1 t + \left(\frac{T_0}{mR_0^2 c_0} + \frac{c_1^2}{4c_0}\right) t^2. \tag{25}$$

Now define the two "new" variables t_0 and τ with the following relations:

$$c_0 \;=\; \left(1 - \frac{t_0}{\tau}\right)^2 + \frac{T_0}{mR_0^2} t_0^2, \qquad c_1 \;=\; \frac{2}{\tau}\left(1 - \frac{t_0}{\tau}\left(1 + \frac{T_0}{T_G}\right)\right).$$

If we substitute these expressions into (25), we get eq. (9). So we proved that eqs. (8,9) yield the most general solution of the non-relativistic hydrodynamical equations for a Gaussian density profile and linear flow profile ansatz, for an ideal gas equation of state. As this hydro solution corresponds to a solution of the collisionless Boltzmann equation as well, we find that in this case the hydro equations are equivalent with a collisionless Boltzmann equation.

More general solutions. The above presented solution of the non-relativistic hydrodynamical equation can be generalized in many ways. For instance, a straightforward way would be to introduce a location dependent temperature profile [8]. Such a temperature profile is present in a known analytic solution of non-relativistic hydrodynamic equations, see ref. [1]. However, in the present study we will choose another way to generalize the hydro solution presented above. Namely, we will investigate the possibility in our model to describe the effects of some local heat production or heat loss. A simple model is introduced to mimic the effects of heat production by chemical or nuclear reactions, or the cooling of the system by radiation that decreases the local energy density. The microscopic details of these processes are not sought for and the sources of heat production, or the radiated energies are not part of the system under consideration, so heat production of radiation changes the 'total' energy.

Let us introduce an additional term into the energy balance eq. (22). Let us assume that this new term is proportional to the particle density:

$$\frac{\partial \varepsilon}{\partial t} + \nabla(\mathbf{v}\varepsilon) + P\nabla\mathbf{v} \;=\; \frac{3}{2} j(t) n(\mathbf{r},\, t) T(t). \tag{26}$$

This new term is the simplest possible model of heat production ($j(t) > 0$) or energy loss e.g. due to radiation ($j(t) < 0$). The heat loss or heat production is assumed to be proportional to the local internal energy.

Now we have — in place of eqs. (8) — a more general Gaussian solution of the continuity equation:

$$T(t) \;=\; T_0 \frac{h(t)}{\varphi(t)}, \qquad R^2(t) \;=\; R_0^2 \frac{\varphi(t)}{h_0}, \qquad \mathbf{v}(\mathbf{r},\, t) \;=\; \mathbf{r}\,\frac{\dot{\varphi}(t)}{2\varphi(t)}, \tag{27}$$

where $h(t)$ is an arbitrary function. This parameterization satisfies the boundary conditions for the temperature and the radius (11) if $\varphi(t_0) = h(t_0) = h_0$.

By substituting the expressions of the particle density (13), the energy density (14) and the pressure (15) into the modified energy equation (26), and using the parameterization (27), we get the following differential equation:

$$\dot{h}(t) \;=\; h(t) j(t). \tag{28}$$

Note that adding source terms to the energy balance equation results in a deviation from the isentropic expansion or contraction (23). From (16), (20), (27) and (28), the local entropy production is

$$\frac{\partial s}{\partial t} + \nabla(\mathbf{v}s) \;=\; \frac{3}{2} j n. \tag{29}$$

The solution of eq. (28) is

$$h(t) \;=\; h_0 e^{\int_{t_0}^{t} dt' j(t')}. \tag{30}$$

The function $\varphi(t)$ can be determined from the Euler equation which now becomes

$$\ddot{\varphi}(t)\varphi(t) - \frac{1}{2}\dot{\varphi}^2(t) \;=\; \frac{2T_0}{mR_0^2} h(t). \tag{31}$$

Let us assume that $j(t) = j$ is a nonzero constant. Then we can solve equation (31) analytically. The result is the following:

$$\varphi(t) \;=\; h(t) \;=\; h_0 e^{j\cdot(t-t_0)}, \qquad h_0 \;=\; \frac{4T_0}{mR_0^2 j^2}. \tag{32}$$

In terms of temperature, radius and velocity, the solution reads as:

$$T(t) \ = \ T_0, \qquad R^2(t) \ = \ R_0^2 e^{j(t-t_0)}, \qquad \mathbf{v}(\mathbf{r},\, t) \ = \ \frac{j}{2}\mathbf{r}.$$

The case of an exponential expansion is described by an energy source term, $j > 0$, which means that the expansion is driven by an external energy source in such a way that the temperature is kept constant. Such energy sources may physically correspond e.g. to the release of the latent heat during a strong first order phase transition. The total energy of the system (which corresponds to the produced phase in the case of a strong first order phase transition) is increasing linearly in time during the period of $j = const > 0$. Another case, the exponential contraction of the matter, may be caused by the the continuous emission of its energy by some radiation. It is described by an energy loss term, $j < 0$, and the total energy is decreasing linearly in time.

Algorithm to generate new solutions: With the method above, one can generate even infinitely many new analytical solutions to the equations of non-relativistic hydrodynamics with energy producing or radiative processes. This algorithm reads as follows:

1) Fix the value of all but one of these parameters: m, t_0, T_0 and R_0.

2) Assume a functional form for $\varphi(t)$.

3) Determine the function $h(t)$ and the value of the non-fix parameter (m, t_0, T_0 or R_0) from eq. (31) and the condition $\varphi(t_0) = h(t_0)$.

4) Find the energy source function $j(t)$ from eq. (30), or equivalently from

$$j(t) = \frac{\dot{h}(t)}{h(t)}. \tag{33}$$

This way, a solution of eqs. (26, 20-21) is generated. The only requirement for consistency arises from $T(t) > 0$, which results in $h(t)/\varphi(t) > 0$, which has to be checked explicitly. The solution is given by eq. (27) with $\varphi(t)$, $h(t)$ and $j(t)$ generated by steps 1-4).

Summary. We found a new class of solutions for the non-relativistic hydrodynamical equations. Our initial result applies to a spherically expanding, collisionless Knudsen gas with time dependent but location independent temperature.

Then we incorporated into our formalism the possibility to describe the effects of the emission or absorption of some secondary radiation. We did this by introducing an energy source or loss term. Then we found an analytic solution, in which the system expands or contracts exponentially, its temperature is kept constant, and its total energy is a linear function of time.

We presented an algorithm that can be used to generate infinitely many new analytical solutions of the non-relativistic hydrodynamical equations with energy sources.

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