Three qubit entanglement

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Plan of the talk

1. SLOCC classification of three-qubit entanglement

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5. The geometry of three-qubit entanglement
6. Three-qubit canonical forms
7. STU black hole entropy as a three-qubit entanglement measure

Three qubit entanglement

An arbitrary three-qubit pure state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ is characterized by 8 complex numbers ψ_{lkj} with $l, k, j = 0, 1$

$$|\psi\rangle = \sum_{l,k,j} \psi_{lkj} |lkj\rangle \quad |lkj\rangle \equiv |l\rangle_C \otimes |k\rangle_B \otimes |j\rangle_A$$

In a class of quantum information protocols the parties can manipulate their qubits *reversibly* with some probability of success by performing *local* manipulations assisted by classical communication between them. Such protocols are yielding special transformations of the states, called stochastic local operations and classical communication (SLOCC).

$$|\psi\rangle \mapsto (\mathcal{C} \otimes \mathcal{B} \otimes \mathcal{A})|\psi\rangle, \quad \mathcal{C} \otimes \mathcal{B} \otimes \mathcal{A} \in GL(2, \mathbb{C})^{\otimes 3}$$

Classification of entanglement amounts to classifying the SLOCC orbits.

The SLOCC classification of three-qubit entanglement

The basic result is due to W. Dür, G. Vidal and J. I. Cirac (2000).
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3. $(B)(AC)$ **biseparable**
4. $(C)(AB)$ **biseparable**
5. **W-class**, eg. $|001\rangle + |010\rangle + |100\rangle$
6. **GHZ-class**, eg. $|000\rangle + |111\rangle$

Cayley's hyperdeterminant as the three-tangle

There are polynomial invariants characterizing these entanglement classes. The most important one is the $SL(2,\mathbb{C})^{\otimes 3}$ and permutation (triality) invariant three-tangle related to Cayley's hyperdeterminant (1845).

$$
\begin{aligned}
D(|\psi\rangle) \quad \equiv \quad & \psi_{000}^2\psi_{111}^2 + \psi_{001}^2\psi_{110}^2 + \psi_{010}^2\psi_{101}^2 + \psi_{011}^2\psi_{100}^2 \\
- \quad & 2(\psi_{000}\psi_{001}\psi_{110}\psi_{111} + \psi_{000}\psi_{010}\psi_{101}\psi_{111} \\
+ \quad & \psi_{000}\psi_{011}\psi_{100}\psi_{111} + \psi_{001}\psi_{010}\psi_{101}\psi_{110} \\
+ \quad & \psi_{001}\psi_{011}\psi_{110}\psi_{100} + \psi_{010}\psi_{011}\psi_{101}\psi_{100}) \\
4 \quad & (\psi_{000}\psi_{011}\psi_{101}\psi_{110} + \psi_{001}\psi_{010}\psi_{100}\psi_{111})
\end{aligned}
$$

$$\tau_{ABC} \equiv 4|D(|\psi\rangle)| \leq 1$$

The two classes containing **genuine** tripartite entanglement are the **W** and **GHZ** classes having $\tau_{ABC}(|W\rangle) = 0$ and $\tau_{ABC}(|GHZ\rangle) \neq 0$.

Attaching special role to qubits

By chosing the first, second or third qubit one can introduce three sets of complex four vectors, e.g. by chosing the first we can define

$$\xi_I^{(A)} = \begin{pmatrix} \psi_{000} \\ \psi_{010} \\ \psi_{100} \\ \psi_{110} \end{pmatrix}, \quad \eta_J^{(A)} = \begin{pmatrix} \psi_{001} \\ \psi_{011} \\ \psi_{101} \\ \psi_{111} \end{pmatrix} \quad I, J = 1, 2, 3, 4$$

Similarly we can define the four-vectors $\xi^{(B)}, \eta^{(B)}$ and $\xi^{(C)}, \eta^{(C)}$. Alternatively one can define three bivectors $P^{(A)} = \xi^{(A)} \wedge \eta^{(A)}$ with components (Plücker coordinates)

$$P_{IJ}^{(A)} = \xi_I^{(A)} \eta_J^{(A)} - \xi_J^{(A)} \eta_I^{(A)}$$

The structure of the three-tangle

Then we have

$$\tau_{ABC} = 2|P^{(A)}_{IJ}P^{(A)IJ}| = 2|P^{(B)}_{IJ}P^{(B)IJ}| = 2|P^{(C)}_{IJ}P^{(C)IJ}|$$

where indices are raised with respect to the $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ invariant metric $g = \varepsilon \otimes \varepsilon$

$$g^{IJ} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Since the Plücker coordinates are $SL(2,\mathbb{C})$ invariant the expression above shows the $SL(2,\mathbb{C})^{\otimes 3}$ and triality invariance at the same time. Notice that the three-tangle can also be written in the form

$$\tau_{ABC} = 4|(\xi \cdot \xi)(\eta \cdot \eta) - (\xi \cdot \eta)^2| = 4| - D(|\psi\rangle)|$$

with $\xi \cdot \eta \equiv g(\xi, \eta) = g^{IJ}\xi_I \eta_J$. Hence $\xi, \eta \in (\mathbb{C}^4, g)$.

Unitary invariants, needed for a finer classification

One and two partite reduced density matrices are defined as

$$\rho_A = \mathrm{Tr}_{BC}|\psi\rangle\langle\psi|, \quad \rho_{BC} = \mathrm{Tr}_A|\psi\rangle\langle\psi|$$

The quantity $\tau_{A(BC)}$ called the *squared-concurrence* between the subsystems A and BC is

$$\tau_{A(BC)} = 4\mathrm{Det}\rho_A = 2\sum_{I,J=1}^{4} \overline{P}_{IJ}^{(A)} P_{IJ}^{(A)}$$

We can alternatively write

$$\tau_{A(BC)} = 4(\langle\xi|\xi\rangle\langle\eta|\eta\rangle - |\langle\xi|\eta\rangle|^2) \leq 1$$

$\tau_{A(BC)} = 0$ if and only if $\xi^{(A)}$ and $\eta^{(A)}$ are linearly dependent. In this case the corresponding reduced density matrix ρ_A has rank one a condition equivalent to $A(BC)$ separability.

Two-partite correlations inside a three-qubit state

A useful measure for the two-qubit mixed-state entanglement is

$$\tau_{AB} = (\max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\} \leq 1)^2$$

where $\lambda_i, i = 1, 2, 3, 4$ is the nonincreasing sequence of the square-roots of the eigenvalues for the nonnegative matrix

$$\rho\tilde{\rho} \equiv \rho(\varepsilon \otimes \varepsilon)\overline{\rho}(\varepsilon \otimes \varepsilon)$$

In the three-qubit context the two-qubit density matrices are of rank two hence we have merely **two nonzero eigenvalues** $\lambda_{1,2}$. As a result of this the invariants discussed above are not independent, they are subject to the important relations

$$\tau_{A(BC)} = \tau_{AB} + \tau_{AC} + \tau_{ABC}$$

The Coffman-Kundu-Wootters relation (2000)

We can write the above relation as

$$\tau_{AB} + \tau_{AC} \leq \tau_{A(BC)}$$

A consequence of this is that if two qubits are maximally entangled with each other then neither of them can be at all entangled with the third one. This fact is sometimes called the **monogamy** of entanglement.

According to T. J. Osborne and F. Verstraete (2006) this relation also holds for an arbitrary number of qubits a special case is e.g.

$$\tau_{AB_1} + \tau_{AB_2} + \dots \tau_{AB_n} \leq 1$$

Hence qubit A has a limited amount of entanglement to share. Any amount of entanglement that it has with qubit B_1 reduces the amount available for the rest of the qubits.

Principal null directions

Consider a pure three-qubit state which is not $A(BC)$ or $(A)(B)(C)$ separable. Then the bivectors

$$P = \xi \wedge \eta \leftrightarrow P_{IJ} = \xi_I \eta_J - \xi_J \eta_I$$

are giving rise to the **planes** in $(\mathbb{C}^4, g)$

$$a\xi_I + b\eta_I, \qquad I, J = 1, 2, 3, 4 \qquad a, b \in \mathbb{C}$$

Let us solve the quadratic equation for the ratio $\frac{a}{b}$, $b \neq 0$

$$a^2(\xi \cdot \xi) + 2ab(\xi \cdot \eta) + b^2(\eta \cdot \eta) = 0$$

The discriminant of this equation is just Cayley's hyperdeterminant. The solutions $u_\pm \equiv a\xi + b\eta$ are the **principal null directions**. Assuming $\xi \cdot \xi \neq 0$ and solving the quadratic equations for the ratio $\frac{a}{b}$, the PNDs are:

Real states

$$u_I^{\pm} = -P_{IJ}\xi^J \pm \sqrt{D}\xi_I$$

or alternatively assuming $\eta \cdot \eta \neq 0$ and solving for the ratio $\frac{b}{a}$

$$v_I^{\pm} = P_{IJ}\eta^J \pm \sqrt{D}\eta_I$$

One can show that

$$P_I^J u_J^{\pm} = \mp\sqrt{D}u_J^{\pm}, \quad P_I^J v_J^{\pm} = \pm\sqrt{D}v_J^{\pm}$$

States with nonvanishing τ_{ABC} which are $SU(2)^{\otimes 3}$ equivalent to ones with **real amplitudes** are called **real states**. There are **two classes of real states** according to whether the PNDs are **real** or **complex conjugate to each other**. Later we will need the unitary invariant

$$\sigma_{ABC} = ||u_+||^2 + ||u_-||^2 + ||v_+||^2 + ||v_-||^2$$

The basic objects of our geometric picture are **pairs of complex four vectors**. These vectors span **planes** in $(\mathbb{C}^4, g)$. It is convenient to switch to the projective picture and use the projective space which is $\mathbb{C}P^3$. In this space our pairs of complex four-vectors define **complex lines**.

We describe three-qubit entanglement from the viewpoint of one of the parties. The vectors we use are then ξ and η.

For $\xi, \eta \in (\mathbb{C}^4, g)$ we have

$$g(\xi, \eta) \equiv \xi \cdot \eta = \xi_1\eta_4 + \xi_4\eta_1 - \xi_2\eta_3 - \xi_3\eta_2$$

The vectors $\zeta \in \mathbb{C}^4$ with $g(\zeta, \zeta) = 0$ define a quadric surface $\mathcal{Q}$ in $\mathbb{C}P^3$.

Let us now consider a complex line corresponding to a three-qubit state in $\mathbb{C}P^3$ of the form

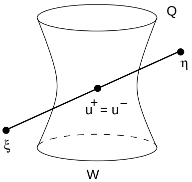
$$w\xi + \eta \quad w \in \mathbb{C}^*$$

with ξ and η are **non null**.

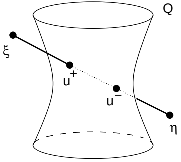
Intesection properties of lines with $\mathcal{Q}$

When the equation $g(w\xi + \eta, w\xi + \eta) = 0$ has two solutions for w the line intersects $\mathcal{Q}$ in **two different points**. The sufficient and necessary condition for this to happen is just $D \neq 0$ or $\tau_{ABC} \neq 0$.
States belong to the GHZ class iff the representative lines intersect $\mathcal{Q}$ in two points.
If the equation $g(w\xi + \eta, w\xi + \eta) = 0$ has merely one solution the line is tangent to the quadric $\mathcal{Q}$ at this particular point.This can happen iff $D = 0$ i.e. $\tau_{ABC} = 0$.
States belong to the W-class iff the corresponding lines are tangent to the quadric $\mathcal{Q}$.
Note, however that in these two cases of **genuine three-qubit entanglement** the points through which the lines were defined are themselves *not* lying on $\mathcal{Q}$.

Geometric representation of the W class



Geometric representation of the GHZ class



The geometry of separable states

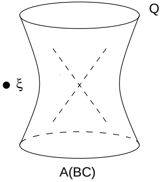
For $A(BC)$ separable states $\tau_{A(BC)} = 0$ and the vectors ξ and η are dependent. Our line degenerates to a point not lying on the quadric $\mathcal{Q}$. We can represent the corresponding situation by drawing a point off the quadric.

When the lines themselves are lying inside the quadric $\mathcal{Q}$ we have *isotropic* lines with respect to $\mathcal{Q}$. There are exactly two families of lines on a nondegenerate quadric $\mathcal{Q}$ in $\mathbb{C}P^3$. Two lines belonging to the same family do not intersect; whereas, two lines belonging to the opposite families intersect at a single point on $\mathcal{Q}$. Hence any nondegenerate quadric in $\mathbb{C}P^3$ is isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$.

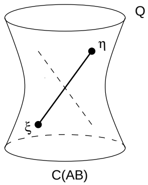
One can prove that

The two different classes of isotropic lines correspond precisely to $B(AC)$ and $C(AB)$ separable states.

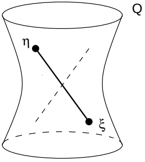
Geometric representation of the A(BC) biseparable class



Geometric representation of the C(AB) biseparable class



Geometric representation of the B(AC) biseparable class



A sketch of proof

Let $\tau_+ = \tau_{C(AB)}$ and $\tau_- = \tau_{B(AC)}$. Then we have

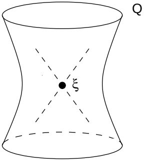
$$\tau_\pm = |\xi \cdot \xi|^2 + 2|\xi \cdot \eta|^2 + |\eta \cdot \eta|^2 + (P^{IJ} \mp *P^{IJ})\overline{P}_{IJ}$$

where

$$*P_{IJ} \equiv \frac{1}{2}\varepsilon_{IJKL}P^{KL}$$

Isotropic lines satisfy the relations $\xi \cdot \xi = \eta \cdot \eta = \xi \cdot \eta = 0$, moreover such lines are necessarily self-dual or anti-self-dual. Hence for isotropic lines we have either $\tau_+ = 0$ or $\tau_- = 0$. Conversely, using the positivity of the terms the vanishing of $\tau_\pm$ implies that the corresponding lines are isotropic. Finally, states of the form $(A)(B)(C)$ are represented by **points** since they are $A(BC)$ separable, moreover they have to lie on the quadric since due to $C(AB)$ and $B(AC)$ separability they are parts of isotropic lines.

Geometric representation of the (A)(B)(C) separable class



(A)(B)(C)

Three-qubit canonical form

In order to obtain a **finer** classification we can use **local unitary operations**. Performing such transformation on the **first** qubit we get:

$$\xi' = a\xi + b\eta, \qquad \eta' = c\xi + d\eta, \qquad U_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2)$$

$$\xi_{lk} \equiv \psi_{lk0}, \qquad \eta_{lk} \equiv \psi_{lk1}$$

Let us chose the complex numbers a and b such that

$$\mathrm{Det}(\xi') = 0 \leftrightarrow \xi' \cdot \xi' = 0$$

This means that we have used a local unitary acting on the first qubit to rotate one of the four-vectors ξ to one of the principal null directions.

Three-qubit canonical form

We can use the remaining local unitary transformations to get

$$\xi' \mapsto U_3\xi' U_2 = \begin{pmatrix} \lambda_0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \lambda_0 \in \mathbb{R}^+$$

$$\eta' \mapsto U_3\eta' U_2 = \begin{pmatrix} \lambda_1 e^{i\varphi} & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix}, \qquad \lambda_1, \dots \lambda_4 \in \mathbb{R}^+$$

Hence we have a canonical form containing **five positive real numbers** and a **phase**:

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\varphi}|001\rangle + \lambda_2|101\rangle + \lambda_3|011\rangle + \lambda_4|111\rangle$$

It can be shown (A. Acin et.al. (2001)) that with the restriction $0 < \varphi < \pi$ this canonical form is **unique**.

Three-qubit canonical form in terms of unitary invariants

$$(\lambda_0^{\pm})^2 = \frac{\sigma_{ABC} \pm \sqrt{\Delta}}{2(\tau_{AB} + \tau_{ABC})}$$

$$(\lambda_2^{\pm})^2 = \frac{\tau_{AC}(\tau_{AB} + \tau_{ABC})}{2(\sigma_{ABC} \pm \sqrt{\Delta})}$$

$$(\lambda_3^{\pm})^2 = \frac{\tau_{BC}(\tau_{AB} + \tau_{ABC})}{2(\sigma_{ABC} \pm \sqrt{\Delta})}$$

$$(\lambda_4^{\pm})^2 = \frac{\tau_{ABC}(\tau_{AB} + \tau_{ABC})}{2(\sigma_{ABC} \pm \sqrt{\Delta})}$$

$$\cos\varphi^{\pm} = \frac{(\lambda_1^{\pm}\lambda_4^{\pm})^2 + (\lambda_2^{\pm}\lambda_3^{\pm})^2 - \tau_{BC}/4}{2\lambda_1^{\pm}\lambda_2^{\pm}\lambda_3^{\pm}\lambda_4^{\pm}}$$

$$\Delta = \sigma_{ABC}^2 - (\tau_{AB} + \tau_{ABC})(\tau_{BC} + \tau_{ABC})(\tau_{AC} + \tau_{ABC})$$

The STU model. Black Hole Entropy as a three-tangle

Ungauged $N = 2$ supergravity in $d = 4$ coupled to 3 vector multiplets. The bosonic part of the action is ($\hbar = c = 1$)

$$\begin{aligned} \mathcal{S} &= \frac{1}{8\pi G_N}\int d^4x\sqrt{|g|}\{-\frac{R}{2} + G_{a\overline{b}}\partial_\mu z^a \partial_\nu \overline{z}^{\overline{b}} g^{\mu\nu} \\ &+ (\mathrm{Im}\mathcal{N}_{IJ}\mathcal{F}^I\mathcal{F}^J + \mathrm{Re}\mathcal{N}_{IJ}\mathcal{F}^{I*}\mathcal{F}^J)\} \end{aligned}$$

Here $\mathcal{F}^I$, and ${}^*\mathcal{F}^I$, $I = 0, 1, 2, 3$ refer to field strengths $\mathcal{F}^I_{\mu\nu}$ of 4 $U(1)$ gauge-fields and their duals. The z^a $a = 1, 2, 3$ are complex scalar fields taking values in the manifold for the STU model: $SL(2,\mathbb{R})/U(1) \times SL(2,\mathbb{R})/U(1) \times SL(2,\mathbb{R})/U(1)$. Conventionally: $z^1 \equiv S$, $z^2 = T$ and $z^3 = U$.

The coupling matrices

For the STU model the scalar dependent vector couplings $\mathrm{Re}\mathcal{N}_{IJ}$ and $\mathrm{Im}\mathcal{N}_{IJ}$ take the following form $(z_a = x_a - iy_a)$

$$\mathrm{Re}\mathcal{N}_{IJ} = \begin{pmatrix} 2x_1x_2x_3 & -x_2x_3 & -x_1x_3 & -x_1x_2 \\ -x_2x_3 & 0 & x_3 & x_2 \\ -x_1x_3 & x_3 & 0 & x_1 \\ -x_1x_2 & x_2 & x_1 & 0 \end{pmatrix}$$

$$\mathrm{Im}\mathcal{N}_{IJ} = -y_1y_2y_3 \begin{pmatrix} 1 + \left(\frac{x_1}{y_1}\right)^2 + \left(\frac{x_2}{y_2}\right)^2 + \left(\frac{x_3}{y_3}\right)^2 & -\frac{x_1}{y_1^2} & -\frac{x_2}{y_2^2} & -\frac{x_3}{y_3^2} \\ -\frac{x_1}{y_1^2} & \frac{1}{y_1^2} & 0 & 0 \\ -\frac{x_2}{y_2^2} & 0 & \frac{1}{y_2^2} & 0 \\ -\frac{x_3}{y_3^2} & 0 & 0 & \frac{1}{y_3^2} \end{pmatrix}$$

String theoretical interpretation of electric and magnetic charges

When type IIA string theory is compactified on a T^6 of the form $T^2 \times T^2 \times T^2$ one recovers $N = 8$ supergravity in $d = 4$ with 28 vectors and 70 scalars . This $N = 8$ model has the STU model as a consistent $N = 2$ truncation with 4 vectors and 6 scalars. In this truncation the $D0 - D2 - D4 - D6$ branes wrapping the various T^2 give rise to four electric and magnetic charges defined as

$$P^I = \frac{1}{4\pi}\int_{S^2}\mathcal{F}^I, \qquad Q_I = \frac{1}{4\pi}\int_{S^2}\mathcal{G}_I, \quad I = 0, 1, 2, 3$$

where

$$\mathcal{G}_I = \overline{\mathcal{N}}_{IJ}\mathcal{F}^{+J}, \qquad \mathcal{F}^{\pm I}_{\mu\nu} = \mathcal{F}^I_{\mu\nu} \pm \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}\mathcal{F}^{I\rho\sigma}$$

These charges can be organized into pairs

$$\Gamma \equiv (P^I, Q_J)$$

Static, spherically symmetric extremal black hole solutions

For the 4d space-time metric the Euler-equations for the Lagrangian admit static, spherically symmetric, extremal black hole solutions. These are of Reissner-Nordström type. Solutions of both supersymmetric (BPS) and not supersymmetric (non-BPS) types are known. The leading order term in the macroscopic black hole entropy for such solutions can be calculated using the Bekenstein-Hawking formula (K. Behrndt et.al. 1996). The surprising result is that the values of the scalar fields at the black hole horizon can be expressed in terms of the electric and magnetic charges. This stabilization of the scalar fields is called the **attractor mechanism** (R. Kallosh, A. Strominger, S. Ferrara 1995). Hence for the STU model the black hole entropy is only depending on 8 charges: 4 electric and 4 magnetic.

The Black Hole Qubit Correspondence

The leading term (Bekenstein-Hawking) of the STU black hole entropy is

$$S = \frac{\pi}{G_N}\sqrt{|I_4(\Gamma)|}$$

$$I_4(\Gamma) = 4Q_0P^1P^2P^3 - 4P^0Q_1Q_2Q_3 - (P_IQ^I)^2 + 4\sum_{m<n} P^mQ_mP^nQ_n$$

Let us reorganize the 8 charges to a three-qubit "state" as follows:

$$|\Gamma\rangle = \sum_{l,k,j=0,1} \Gamma_{lkj}|lkj\rangle \qquad |lkj\rangle \equiv |l\rangle_C \otimes |k\rangle_B \otimes |j\rangle_A$$

where

$$\begin{pmatrix} P^0, & P^1, & P^2, & P^3 \\ -Q_0, & Q_1, & Q_2, & Q_3 \end{pmatrix} = \begin{pmatrix} \Gamma_{000}, & \Gamma_{001}, & \Gamma_{010}, & \Gamma_{100} \\ \Gamma_{111}, & \Gamma_{110}, & \Gamma_{101}, & \Gamma_{011} \end{pmatrix}$$

Now one can check that (M. J. Duff, 2006)

$$I_4(\Gamma) = -D(|\Gamma\rangle)$$

hence

$$S = \frac{\pi}{2G_N}\sqrt{\tau_{ABC}}$$

Notice that for BPS black holes we have $D(|\Gamma\rangle) < 0$, and for non-BPS ones $D(|\Gamma\rangle) > 0$ (R. Kallosh, A. Linde, 2006). These two classes of real states are precisely the two inequivalent types of real states embedded into the complex ones (P. Lévay, 2006). Recall that for these classes the principal null directions are **real** or **complex conjugate** to each other.

Notice that the classical symmetry group of the STU model $SL(2,\mathbb{R})^{\otimes 3}$ is broken down to the **U-duality** subgroup $SL(2,\mathbb{Z})^{\otimes 3}$ due to quantum corrections (Dirac-Zwanziger quantization of electric and magnetic charges). The action of this subgroup on the "charge states" is

$$|\Gamma\rangle \mapsto (\mathcal{C}\otimes\mathcal{B}\otimes\mathcal{A})|\Gamma\rangle, \qquad \mathcal{C},\mathcal{B},\mathcal{A}\in SL(2,\mathbb{Z})$$

Note also that the "charge state" belonging to the GHZ-class

$$|\Gamma\rangle = P^0|000\rangle - Q^0|111\rangle$$

independent of the signs of the charges represents a **non-BPS** solution. (A D0-D6 system in the type IIA duality frame.)

The Black Hole Qubit Correspondence

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