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A General Telegraph-type Model for Heat Conduction with Self-similar Behaviour of Solutions

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In our former studies we introduced a modified Fourier-Cattaneo law and derived a non-autonomous telegraph-type heat conduction equation which has desirable self-similar solution. Now we present a detailed in-depth analysis of this model and discuss additional analytic solutions for different parameters. The solutions have a very rich and interesting mathematical structure due to various special functions.

Keywords: Heat transfer, heat conduction, telegraph-type, self-similar solution

1 INTRODUCTION

The heat equation propagates perturbation with infinite velocity, which is a well-known theoretical problem from a long time ago and has been assessed [1-3]. According to Gurtin and Pipkin [3], the most general form of the flux in linear heat conduction and diffusion related to the flux, q, expressed in one space dimension *via* an integral over the history of the temperature gradient:

$$q = -\int_{-t}^{t} Q(t-t')\frac{\partial T(x,t)}{\partial x} df \qquad (1)$$

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where $Q(t\text{-}t')$ is a positive, decreasing relaxation function that tends to zero as $t\text{-}t'$ goes to infinity and $T(x,t)$ is the temperature distribution.

There are two notable relaxation kernel functions: if $Q_1(s) = k\delta(s)$ where $\delta(s)$ is a Dirac delta function, then we will get back the original Fourier law. The second one leads to the Cattaneo law. In our former study [4] we introduced the

$$Q(t-t') = \frac{\kappa\tau^l}{(t-t'+\omega)^l} \tag{2}$$

time dependent kernel which leads to a non-autonomous telegraph-equation:

$$\varepsilon \frac{\partial^2 T(x,t)}{\partial t^2} + \frac{a}{t}\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2} \tag{3}$$

which has a self-similar solution of the form of

$$T(x,t) = t^\alpha f\left(\frac{x}{t^\beta}\right) = t^\alpha f(\eta). \tag{4}$$

The similarity exponents α and β are of primary physical importance since α represents the rate of decay of the magnitude $T(x,t)$, while β is the rate of spread (or contraction if $\beta < 0$) of the space distribution as time goes on.

The two parameters a and ε are similar to the diffusion coefficient and to the inverse square root of the wave propagation velocity, and helps us to investigate the effects of the two terms. We will see later, that the ratio a/ε will be the crucial parameter of all the solutions. Substituting Equation (4) into Equation (3) and after some work we obtain an ordinary differential equation (ODE) for $f(\eta)$ for when $\alpha + 2= \alpha + 2\beta$ (universality relation). In the following we will investigate two cases. The most important when $\alpha = \beta = 1$ and the α =-2 and the $\beta = 1$ case.

2 SELF-SIMILAR SOLUTIONS

2.1 The $\alpha = \beta = 1$ case

The ODE for η is

$$(\varepsilon f \eta^2 - f)'' = a(\eta f)' \tag{5}$$

where the prime stands for derivation with respect to η. This equation can be integrated directly, if the first integration constant (c_1) is set to zero than the solution is

$$f(\eta) = (1-\varepsilon\eta^2)^{a/2\varepsilon-1} \tag{6}$$

The solution is globally bounded and positive in the domain $\{(x,t) : 1\text{-}\varepsilon\eta\eta > 0\}$.

The self-similar solution of Equation (3) is

$$T(x,t) = \frac{1}{t}(1 - \varepsilon \frac{x^2}{t^2}).^{a/2\varepsilon - 1} \tag{7}$$

In our former work [4] we analysed this solution which has desirable compact support with vanishing derivatives and ends at this point. However, if the first integration constant *(c_1)* is not zero then the solution has a much complex form; namely,

$$\begin{aligned} f(\eta) = [c_1(signum(\varepsilon\eta^2 - 1)^{-a/2\varepsilon}) \times (-signum(\varepsilon\eta^2 - 1))^{a/2\varepsilon} \cdot \eta \cdot \times \\ {}_2F_1(1/2, a/2\varepsilon; 3/2; \varepsilon\eta\eta) + c_2] \times (\varepsilon\eta^2 - 1)^{a/2\varepsilon - 1} \end{aligned} \tag{8}$$

where ${}_2F_1$ is the hypergeometric function [5]. It is well known that many elementary functions can be expressed via the hypergeometric function. Fortunately, in our case if $a/2\varepsilon$ is a positive (or negative) integer or half-integer number the hypergeometric series breaks down to a finite sum of terms. There are four basic cases which have crucial importance:

$$\frac{a}{2\varepsilon} = 0, \rightarrow {}_2F_1(0, 1/2; 3/2; \varepsilon\eta^2) = 1 \tag{9a}$$

and

$$\frac{a}{2\varepsilon} = 1, \rightarrow {}_2F_1(1, 1/2; 3/2; \varepsilon\eta^2) = \frac{1}{2\sqrt{\varepsilon}\eta} \times \ln(\frac{1 + \sqrt{\varepsilon}\eta}{1 - \sqrt{\varepsilon}\eta}) \tag{9b}$$

and for half-integer values:

$$\frac{a}{2\varepsilon} = \frac{1}{2}, \rightarrow {}_2F_1(1/2, 1/2; 3/2; \varepsilon\eta^2) = \frac{\arccos(\sqrt{\varepsilon}\eta)}{\sqrt{\varepsilon}\eta} \tag{10}$$

and

$$\frac{a}{2\varepsilon} = 1, \rightarrow {}_2F_1(1/2, 1/2; 3/2; \varepsilon\eta^2) = \frac{1}{\sqrt{(1 - \varepsilon\eta^2)}} \tag{10b}$$

With the following recursion relation all the other cases can be evaluated:

$$\begin{aligned} (c - a)\, {}_2F_1(a - 1, b; c; z) + (2a - c - az + z) \times \\ {}_2F_1(a, b; c; z) + a(z - 1)\, {}_2F_1(a + 1, b; c; z) = 0 \end{aligned} \tag{11}$$

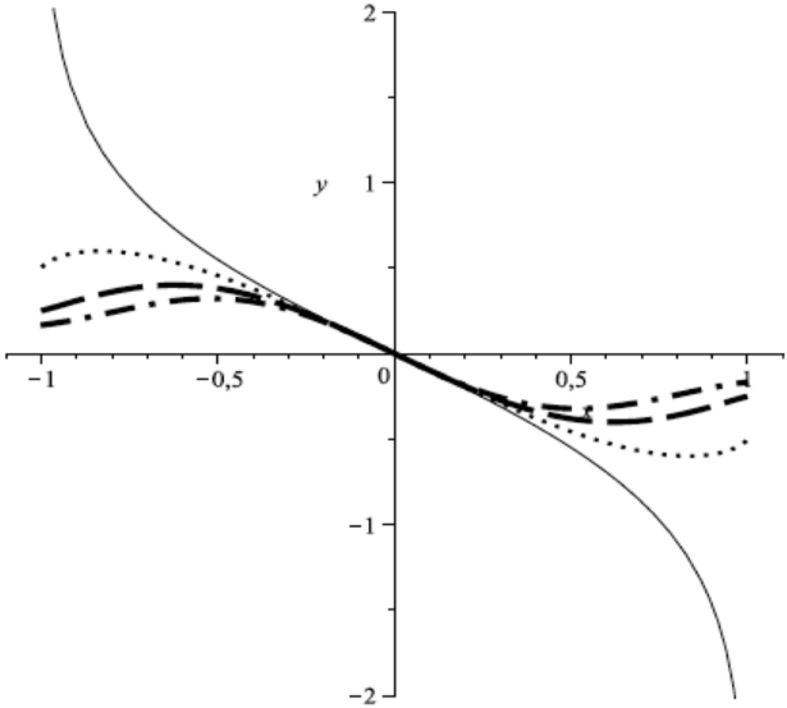

FIGURE 1
Solutions for Equation (8) for zero and positive integer $a/2\varepsilon$ values. The thin solid line is for 0, the thin dotted line is for 1, the thick dashed line is for 2 and the thick dash dotted line is for 3.

We will examine four different cases. For physical reasons the inverse square root of the propagation velocity, ε, will always have to be positive, we take the plus unity value, and changing only the diffusion coefficient *a* parameter. For the first case let us take the $a/2\varepsilon > 0$ with integer values. The resulting curves are presented on Figure 1. We can see, that for positive integer $a/2\varepsilon$ values the solution is compact but has a finite jump at ±1. Figure 2 shows the same results, but for when $a/2\varepsilon$ is equal to -1, -2, -3 and -4. All figures are continuous at the [-1:1] and have singular asymptotic values at the boundaries. Figure 3 presents the results for positive half-integer $a/2\varepsilon$ values. All the curves are odd functions. Figure 4 presents curves for negative half integer $a/2\varepsilon$ numbers. All the results are odd functions and have asymptotes at ±1.

All of these 16 curves represent analytic solutions, but the equations are all tedious to analyse. The self-similar solution $T(x,t)$ can be calculated from Equation (8) just setting $\eta = x/t$. Unlike to the first and most important solution of Equation (7), all these functions have singularities or a non-convergent improprious integrals.

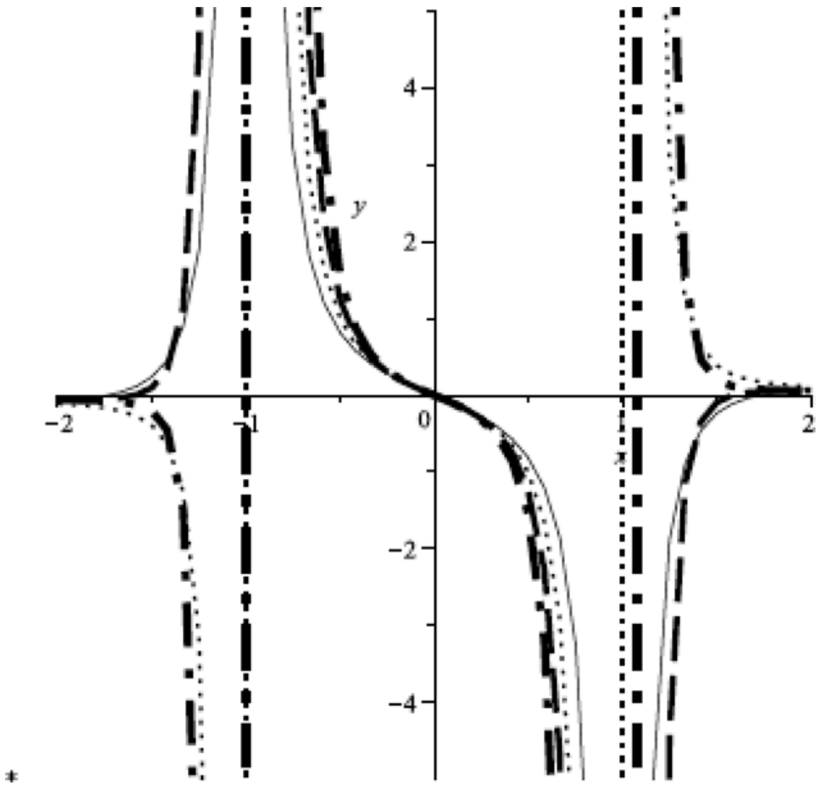

FIGURE 2
Solutions for Equation (8) for negative integer $a/2\varepsilon$ values. The thin solid line is for -1, the thin dotted line is for -2, the thick dashed line is for -3 and the thick dash dotted line is for -4.

2.2 THE α = -2 AND β = +1 CASE

We can see from the universality relations that β always has to be unity; however, α can be arbitrary. We analyse α = -2, other αs have the same type of Legendre functions as solutions but the details are much more complicated to understand. Now, the ODE has the following form:

$$f''\cdot[\varepsilon\eta^2-1]-f'\cdot\eta\cdot[2\varepsilon+a]+2f\cdot[\varepsilon+a]=0 \tag{12}$$

and the solutions are

$$f(\eta)=[c_1P_{a/2\varepsilon-1}^{a/2\varepsilon+2}(\sqrt{\varepsilon}\eta)+c_2Q_{a/2\varepsilon-1}^{a/2\varepsilon+2}(\sqrt{\varepsilon}\eta)]\times(\varepsilon\eta^2-1)^{a/4\varepsilon+1} \tag{13}$$

where P and Q are the associated Legendre functions of the first and second kind [5]. The subscript of the functions is the order and the superscript is the degree.

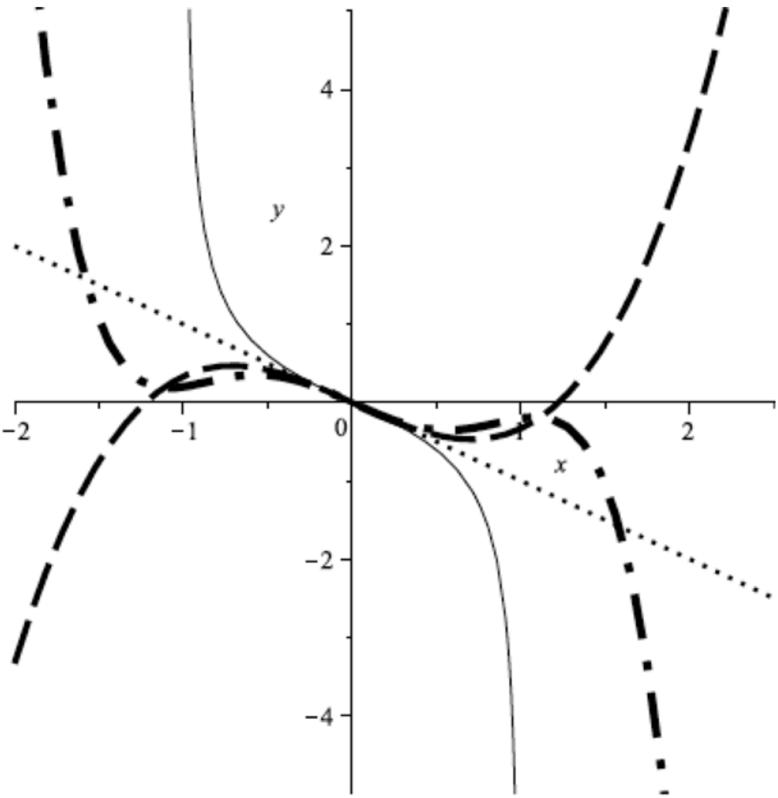

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FIGURE 3
Solutions for Equation (8) positive half-integer $a/2\varepsilon$ values. Thin solid line is for 1/2, the thin dotted line is for 3/2, the thick dashed line is for 5/2 and the thick dash dotted line is for 7/2.

It is well known that if the order and the degree are integer numbers and the order is larger that the degree than associated Legendre functions of the first kind became the associated Legendre polynomials. These polynomials span a Hilbert space with the following orthonormalization condition

$$\int_{-1}^{+1} P_{l_1}^m(x)\cdot P_{l_2}^m(x)dx = 0 \quad \text{for} \quad \left(l_1 \neq l_2\right) \tag{14a}$$

and

$$\int_{-1}^{+1} P_{l_1}^m(x)\cdot P_{l_2}^m(x)dx = \frac{2}{2l_1+1}\frac{(l_1+m)!}{(l_1-m)!} \quad \text{for} \quad \left(l_1 = l_2\right). \tag{14b}$$

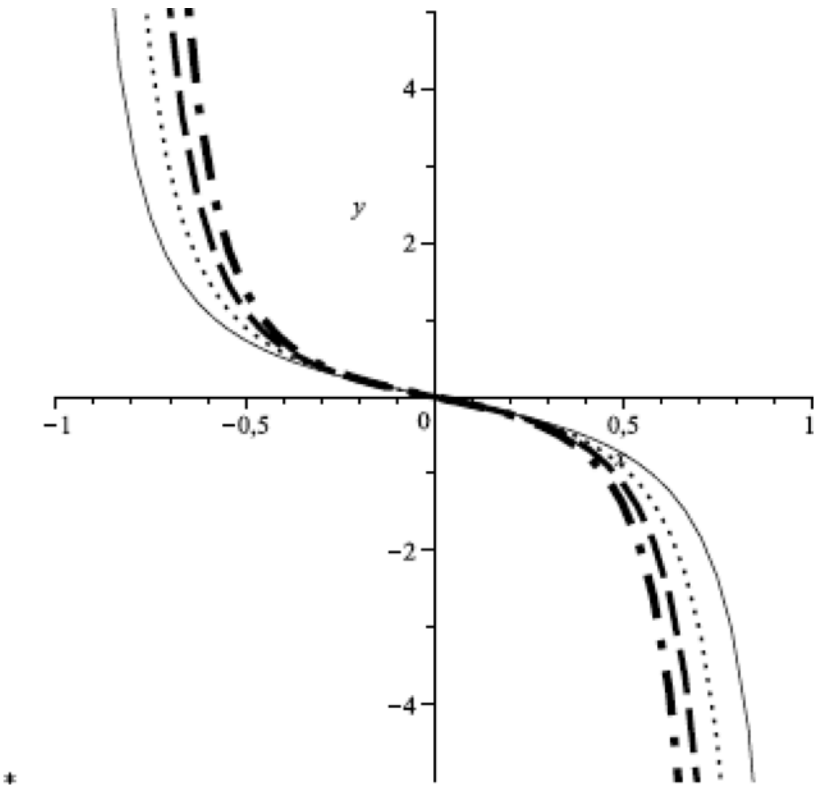

FIGURE 4
Solutions of Equation (8). The thin solid line is for $a/2\varepsilon$ =-1/2, the thin dotted line is for -3/2, the thick dashed line is for -5/2 and the thick dash dotted line is for -7/2.

Similar relationships are available for the second order associated Legendre polynomials as well. If the degree and the order are non-integer values than the Legendere functions can be evaluated with the help of hyperbolic functions [5]. The equations are very complicated to mention at this point.

There are four groups of solutions again: the $a/2\varepsilon$ quantity can be positive and negative integer and half value. Some of them, however, cannot be defined because of other properties of ${}_2F_1$. After using additional properties of the ${}_2F_1$ it comes out that the solutions for the first order Legendre functions P are basically a second order parabolas with some extra feature. For some exponents $a/4\varepsilon+1$ of the weighting function $(\varepsilon\eta-1)$ can cause a cut at $\eta = -1$ or together with the Legendre function can make the solution indefinite for at $\eta < -1$. Figure 5. presents solution for $a/2\varepsilon = 3/2$ and for -3/2. Figure 6 presents solutions of Equations (13) for P. It is worth noting that these are also second order parabolas. The structure of the irregular Q Leg-

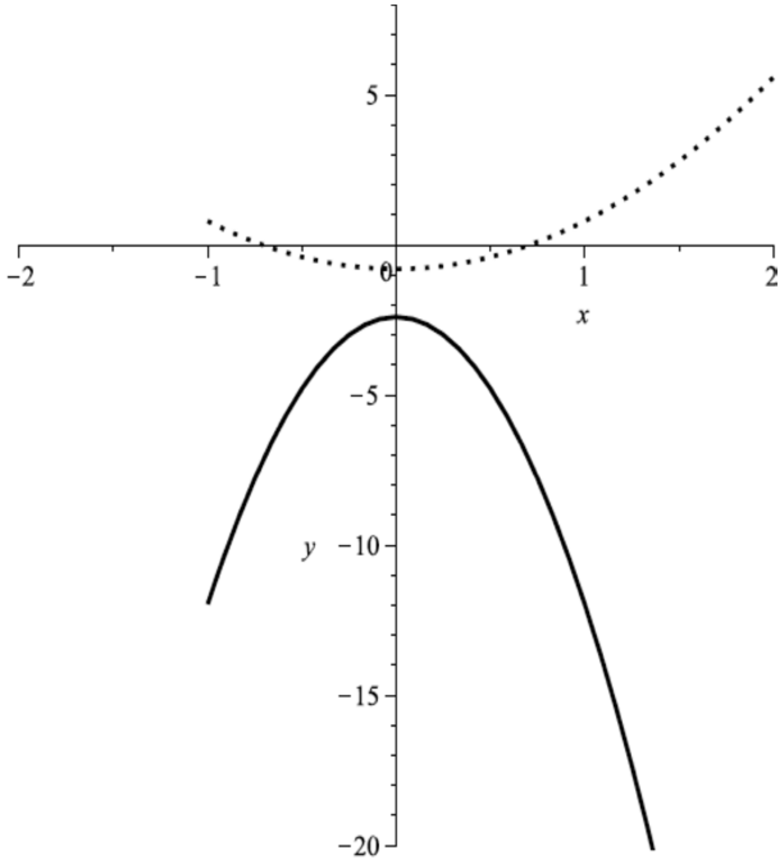

FIGURE 5
Solutions of Equation (13) for *P*. The solid line is for $a/2\varepsilon$ = 3/2 and the dotted line is for $a/2\varepsilon$ = -3/2.

endre function is even more complex, and cannot be simplified at all. A more defined description of Q can be found elsewhere [6]. Some real solutions of the irregular Legendre function Q are presented in Figure 7. The cuts are due to the exponent of the weighting function as mentioned above. At last, we mention than all the solutions can be written in the form of the product of two travelling waves propagating in opposite directions. If we insert $c^2 = 1/\varepsilon$ (the wave-propagation speed) into a regular solution like

$$T(x,t) = [t^2 P_1^4(\sqrt{\varepsilon} x/t)] \times [\varepsilon(x/t)^2 - 1]^2 \tag{15}$$

then after some algebraic manipulation we get

$$T(x,t) = t^2 P_1^4(x/ct) U(x-ct) U(x+ct) \tag{16}$$

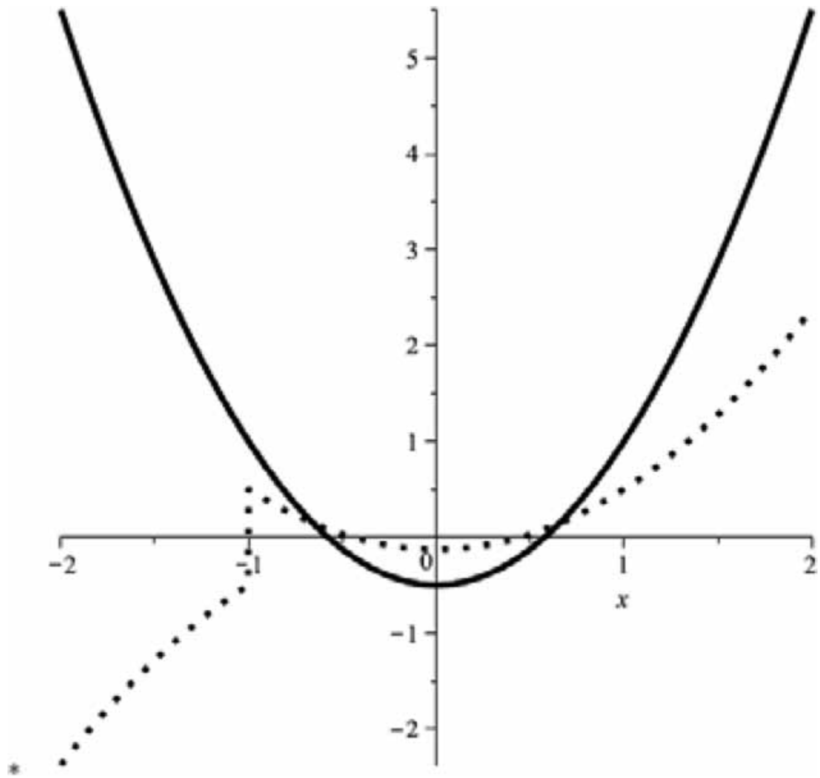

FIGURE 6
Solutions of Equation (13) for P. The solid line is for $a/4\varepsilon+1 = -1$ and the dotted line is for $a/4\varepsilon +1 = -1/2$.

which means a distorted wave solution with a non-trivial weight function.

3 CONCLUSIONS

In the presented study we gave an in-depth analysis of a time-dependent telegraph-type equation for heat propagation. The form of the relaxation function is given which might be understood from microscopic modelling and leads to a dynamical equation which has self-similar solution a very desirable property for all transport processes. All the cases for different parameter ranges are carefully examined and analysed. Different special functions came into play which may serve as a strong hint for further relevance of this equation. We hope that our equation can help us to investigate two dimensional turbulent flows, or even some quantum mechanical problems with the quantum telegraph equation [7].

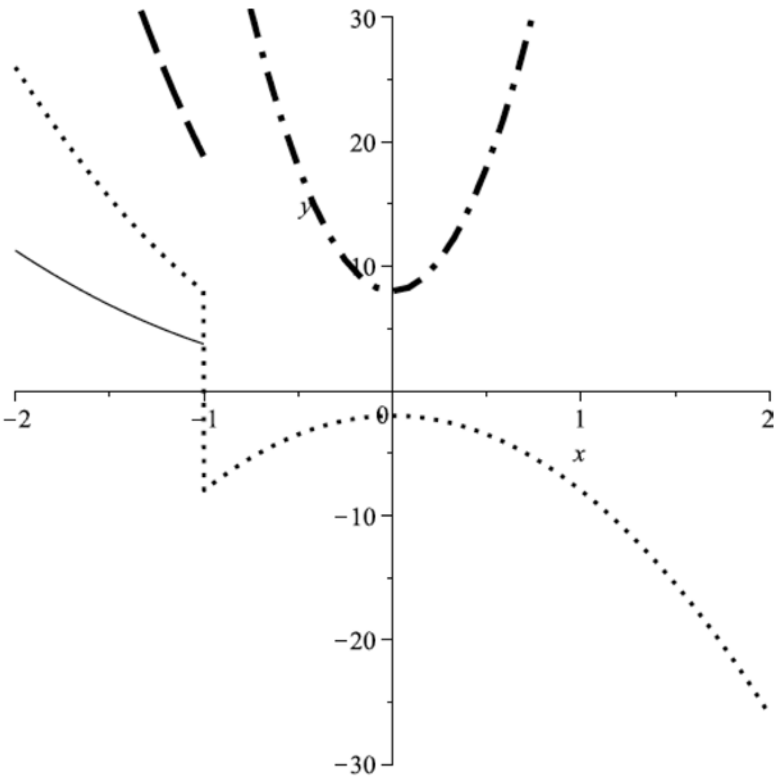

FIGURE 7
The results for the irregular Legendre function Q. The solid line is for $a = 1$, the dotted line is for $a = 2$ the dashed line is for $a = 3$ and the dot-dashed is for $a = 4$.

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