Analytical investigation of the rotating and stratified hydrodynamical problem

Cite as: AIP Conference Proceedings **2425**, 290002 (2022); https://doi.org/10.1063/5.0081575
Published Online: 06 April 2022

Imre Ferenc Barna, Gabriella Bognár, László Mátyás, et al.

Analytical Investigation of the Rotating and Stratified Hydrodynamical Problem

Imre Ferenc Barna[1], Gabriella Bognár[2,a)], László Mátyás[3] and Krisztián Hriczó[2]

[1]*Wigner Research Centre for Physics*
Konkoly-Thege Miklós út 29 - 33, H-1121 Budapest, Hungary

[2]*University of Miskolc, Miskolc-Egyetemváros 3515, Hungary*

[3]*Department of Bioengineering, Faculty of Economics, Socio-Human Sciences and Engineering, Sapientia Hungarian University of Transylvania Libertătii sq. 1, 530104 Miercurea Ciuc, Romania*

[a)]Corresponding author: v.bognar.gabriella@uni-miskolc.hu

Abstract. In this paper, the two-dimensional incompressible rotating and stratified equations are analytically investigated. We examine rotating and stratified Euler equations and the normal Euler equations with the self-similar Ansatz. The analytic solutions are available for our all models for velocity, density and pressure fields. In general, they have a rich mathematical structure involving compound power-law dependent functions. Some of them show unphysical explosive properties, others are physically acceptable and have finite numerical values with power law decays. For a better transparency we present some figures for the most complicated velocity and pressure fields.

INTRODUCTION

There is no need to prove the evidence that both geophysics and meteorology have crucial importance for human society and civilization. Part of it is the special interest in science. Overwhelm problems in meteorology are hydrodynamical in origin. This statement is partially true for geophysics as well. On the surface of Earth due to axial rotation and the additional gravity the question of stratified flows play an important role. Various hydrodynamical models of such kind for meteorology and geophysics can be found in numerous monographs [1, 2, 3, 4, 5].

In this study we investigate the time-dependent dispersive self-similar solutions [6, 7] (not the blow-up type) of these kind of multidimensional Euler-type equations. The form of the original one-dimensional Ansatz reads as follows

$$V(x,t) = t^{-\alpha} f(x/t^{\beta}) = t^{-\alpha} f(\eta), \tag{1}$$

where $V(x,t)$ is the dynamical variable, $f(\eta)$ is the shape function with the reduced variable η and α, β are the self-similar exponents. Usually $\alpha, \beta > 0$ present physically relevant power-law decaying physical solutions of the problem. This transformation is based on the assumption that a self-similar solution exists, i.e., every physical parameter preserves its shape during the expansion. Self-similar solutions usually describe the asymptotic behavior of an unbounded or a far-field problem; the time t and the space coordinate x appear only in the combination of $\eta = x/t^{\beta}$. It means that the existence of self-similar variables implies the lack of characteristic lengths and times. The geometrical and physical interpretations of this Ansatz were exhaustively explained in all our former studies [8, 9, 10], therefore we skip it here.

This study is part of our long-time program which systematically goes over fundamental hydrodynamical systems. Till now we published about half a dozen papers [8, 9] and a book chapter [10] in this field. To the best of our knowledge, there are no such time-dependent self-similar solutions known, presented and analyzed in the scientific literature for these systems. The structure of this paper is the following: to give a broader overview we investigate and compare the solutions of two dimensional rotating and stratified Euler equations with just stratified, just rotating and pure Euler equations. So four different flow systems will be discussed, We already published studies with the similar

International Conference of Numerical Analysis and Applied Mathematics ICNAAM 2020
AIP Conf. Proc. 2425, 290002-1–290002-4; https://doi.org/10.1063/5.0081575
Published by AIP Publishing. 978-0-7354-4182-8/$30.00

logic where several cases were investigated like the surface growth KPZ equation with numerous different noise terms [11] or the compressible one dimensional Euler equations where various equation-of-states were applied [12].

THEORY

To have a complex analysis for all four cases we present the corresponding original partial differential equation (PDE) systems, the applied Ansatz with the obtained self-similar exponents, the obtained coupled ordinary differential equation (ODE) system and the solutions for the dynamical variables, the velocity and pressure fields (in two cases even for the densities). For a better transparency and for a clearer understanding we present figures for the most complicated solutions. These non-trivial shape functions and the corresponding final dynamical variables (velocity and pressure) are plotted and analyzed. We think that it is unnecessary to plot all shape functions and all dynamical variables for all four models for trivial solutions.

THE ROTATING AND STRATIFIED SYSTEM

We start our study with the most complex flow. The stability or turbulence of such systems were extensively studied by Koba [13] and Davidson [14]. According to the book of Dolzhansky [2] the rotating stratified fluid equations in two Cartesian dimensions in vectorial notation read as follows:

$$\nabla \mathbf{v} = 0, \tag{2}$$

$$\rho_t + (\mathbf{v}\nabla)\rho = 0, \tag{3}$$

$$\mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v} + 2\mathbf{\Omega}_0 \times \mathbf{v} = -\frac{\nabla p}{\rho_0} + \frac{G}{\rho_0}\rho, \tag{4}$$

where $\mathbf{v}, \rho, p, \mathbf{\Omega_0}, G$ denote respectively the two-dimensional velocity field, density, pressure, angular velocity and an external force (now gravitation) of the investigated fluid. In the following ρ_0, is one physical parameter of the flow. For a better overview we use the coordinate notation $\mathbf{v}(x, y, t) = u(x, y, t), v(x, y, t)$ for the velocity and $p(x, y, t)$ for the scalar pressure field. To have a trivial rotation contribution we consider the $\mathbf{\Omega_0} = (0, 0, \Omega_0^z(x, y, t))$ angular velocity vector. The direct form, (coordinate form) of the equations are:

$$u_x + v_y = 0, \tag{5}$$

$$\rho_t + u\rho_x + v\rho_y = 0, \tag{6}$$

$$u_t + uu_x + vu_y - 2v\Omega_0 = -\frac{p_x}{\rho_0}, \tag{7}$$

$$v_t + uv_x + vv_y + 2u\Omega_0 = -\frac{p_y}{\rho_0} + \frac{G}{\rho_0}\rho, \tag{8}$$

where the subscripts mean partial derivations with respect to time and spatial coordinates. (For the following three models we skip the vectorial form, and just write out all the coordinates.) Let's consider the self-similar Ansatz for the variables in the form of:

$$\rho(x, y, t) = t^{-\alpha} f(\eta), \quad u(x, y, t) = t^{-\delta} g(\eta),$$
$$v(x, y, t) = t^{-\epsilon} h(\eta), \quad p(x, y, t) = t^{-\gamma} i(\eta), \tag{9}$$

with the new variable $\eta = \frac{x+y}{t^\beta}$. All the exponents $\alpha, \beta, \gamma, \delta, \epsilon$, are real numbers. (Solutions with integer exponents are called self-similar solutions of the first kind, non-integer exponents generate self-similar solutions of the second kind.) The shape functions f, g, h, i could be any continuous functions and will be evaluated later on. The logic, the physical and geometrical interpretation of the Ansatz were exhaustively analyzed in all our former publications [8, 9, 10, 11, 12] therefore we neglect it.

To have consistent coupled ODE system for the shape functions the exponents have to have the following values of

$$\alpha = 3/2, \quad \beta = \delta = \epsilon = 1/2, \quad \gamma = 1. \tag{10}$$

Note, that all exponents have a fixed numerical value, which clearly defines the solutions. Each exponent is positive so the solutions are expected to be physical (will have power law decay at large times). It is important to emphasize, that only the $\mathbf{\Omega}_0^z = \omega_0/t$ angular velocity function (which is trivial from dimensional consideration) leads the following clean-cut ordinary differential equation (ODE) system

$$
\begin{aligned}
f' + g' &= 0, && (11)\\
-\frac{3}{2}f - \frac{1}{2}\eta f' + gf' + hf' &= 0, && (12)\\
-\frac{1}{2}g - \frac{1}{2}\eta g' + gg' + hg' - 2h\omega_0 &= -\frac{i'}{\rho_0}, && (13)\\
-\frac{1}{2}h - \frac{1}{2}\eta h' + gh' + hh' + 2g\omega_0 &= -\frac{i'}{\rho_0} + \frac{G}{\rho_0}f. && (14)
\end{aligned}
$$

From the first (continuity) equation we automatically get $f + g = c_0$, where c_0 is proportional with the constant mass flow rate.

From equations 11 and 12 the solution is almost trivial

$$f = \frac{c_1}{(2c_0 - \eta)^3}, \tag{15}$$

where c_1 stands for the usual integration constant. The density has a power-law decay for large times which is physically desirable

$$\rho(x, y, t) = \frac{1}{t^{\frac{3}{2}}} \cdot \frac{c_1}{2c_0 - \frac{(x+y)^3}{t^{\frac{3}{2}}}} \simeq \frac{c_1}{t^{\frac{3}{2}}}. \tag{16}$$

Extracting equation 13 the ODE for the shape function of the velocity component $v(x, y, t)$ is of the form

$$h = \frac{\eta(-c_0 - 4c_0\omega_0)}{2c_0 - \eta} - \frac{Gc_1}{\rho_0(\eta - 2c_0)^3} + \frac{c_2}{2c_0 - \eta}. \tag{17}$$

The function has a singularity at $\eta = 2c_0$ and it it strictly monotonous growing for all positive η where $\eta > 2c_0$. The solution is the sum of a shifted first and third order hyperbola. All the parameters are responsible for the scaling and the shift of the singularity. It is straightforward to show that the asymptotic behavior of the velocity field is $v(x, y, t) \simeq t^{-\frac{1}{2}}$, which makes it a physically acceptable solution.

The equations 13 and 14 the solution for i can be easily evaluated with quadrature

$$
\begin{aligned}
i = \quad & 2\omega_0 ln(\eta - 2c_0)(c_2 - 4\omega_0 c_0 - \rho_0 c_0^2) + \tfrac{Gc_1}{2(\eta-2c_0)^2}\left(\tfrac{1}{2} - \omega_0\right) + \\
& \eta\left(\tfrac{1}{2}\omega_0\rho_0 c_0 - 4\rho_0\omega_0^2 c_0 - \tfrac{c_0\rho_0}{4}\right) + c_3. && (18)
\end{aligned}
$$

To exhibit the general properties of the pressure, Figure 1 and 2 present the ten-based logarithm of the solution for two different angular velocities. In bouth cases the pressure functions have clear asymptotic values.

WITHOUT ROTATION AND STRATIFICATION

The solution to system of equations 2, 4 and 5 with $\omega_0 = 0$ can be given in closed form.

SUMMARY

We investigated the two-dimensional incompressible rotating and stratified Euler equations and the normal Euler equations applying the self-similar Ansatz. To emphasize the scientific relevance of these equations and disciplines we mentioned numerous textbooks and monographs which were written in the recent decades. We found analytic solutions for all dynamical variables of all four models. The solutions of the rotating stratified flows are much more complex, therefore we presented additional figures to enlighten the details. Overall the physically relevant, power-law time decaying solutions were emphasized. We think that due to the lack of higher order viscous terms in the Euler equations all solutions are quite simple contains no additional internal finer structure e.g. some waves or oscillations.

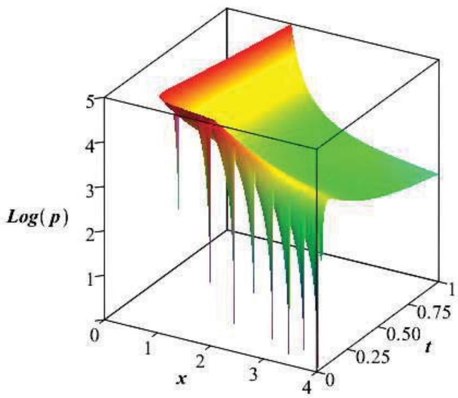

FIGURE 1. The ten-based logarithm of the pressure $Log(p(x, y = 0, t))$ for $\omega_0 = 0.125$ angular velocity, other parameters $c_0 = 0.5, c_1 = 3.25, c_2 = -3.1, c_3 = 15, G = 10, \rho_0 = 3$.

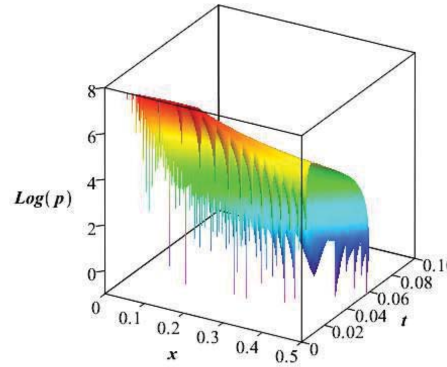

FIGURE 2. The ten-based logarithm of the pressure $Log(p(x, y = 0, t))$ for $\omega_0 = 0.9$ angular velocity, other parameters $c_0 = 0.5, c_1 = 3.25, c_2 = 3.1, c_3 = 15, G = 10, \rho_0 = 3$.

ACKNOWLEDGMENTS

This work was supported by project no. 129257 implemented with the support provided from the National Research, Development and Innovation Fund of Hungary, financed under the K_18 funding scheme.

REFERENCES

[1] K. G. Vallis, *Atmospheric and oceanic fluid dynamics* (Cambridge University Press, 2017).
[2] F. V. Dolzhansky, *Fundamentals of geophysical hydrodynamics*, Vol. 103 (Springer Science & Business Media, 2012).
[3] V. Zeitlin, *Geophysical fluid dynamics: understanding (almost) everything with rotating shallow water models* (Oxford University Press, 2018).
[4] E. Kovács, Numerical Methods for Partial Differential Equations (2020).
[5] E. Kovács, Journal of Computational and Applied Mechanics (JCAM) **15**, 3–13 (2020).
[6] L. I. Sedov, *Similarity and dimensional methods in mechanics* (CRC press, 1993).
[7] G. I. Barenblatt, RaF **19**, 902–931 (1976).
[8] I. F. Barna and L. Mátyás, Fluid Dynamics Research **46**, p. 055508 (2014).
[9] I. F. Barna and L. Mátyás, Chaos, Solitons & Fractals **78**, 249–255 (2015).
[10] D. Campos, *Handbook on Navier-Stokes equations: theory and applied analysis* (Nova Science Publisher's, Incorporated, 2017).
[11] I. F. Barna, G. Bognár, M. Guedda, L. Mátyás, and K. Hriczó, Mathematical Modelling and Analysis **25**, 241–256 (2020).
[12] I. F. Barna and L. Mátyás, Miskolc Mathematical Notes **14**, 785–799 (2013).
[13] H. Koba, *Nonlinear stability of Ekman boundary layers in rotating stratified fluid*, Vol. 228 (American Mathematical Society, 2014).
[14] P. A. Davidson, *Turbulence in rotating, stratified and electrically conducting fluids* (Cambridge University Press, 2013).
[15] C. Ansorge, *Analyses of turbulence in the neutrally and stably stratified planetary boundary layer* (Springer, 2016).
[16] V. Armenio and S. Sarkar, Environmental Stratified Flows, Springer 179–232 (2005).
[17] F. B. Pedersen, *Environmental Hydraulics: Stratified Flows: Stratified Flows*, Vol. 18 (Springer Science & Business Media, 2012).