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Heat conduction: a telegraph-type model with self-similar behavior of solutions

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Abstract
For the heat flux q and temperature T, we introduce a modified Fourier–Cattaneo law $q_t+lq/(t+\tau) = -kT_x$. The consequence of it is a non-autonomous telegraph-type equation. This model already has a typical self-similar solution which may be written as a product of two traveling waves modulo a time-dependent factor and might play a role of intermediate asymptotics.

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It is well known that the heat equation propagates perturbations with infinite velocity. For this contradiction a possible answer is the telegraph equation which is 'obviously hyperbolic'. Generally, other fundamental properties of the parabolic heat equations are forgotten: the existence of self-similar solutions (e.g. the Gaussian kernel or the fundamental solution) and the attracting nature of these special solutions (intermediate asymptotics).

Is is easy to show that the telegraph equation—see (3)—has no self-similar solutions e.g. solutions of the form $t^{-\alpha} f(x/t^{\beta})$ and that even asymptotic self-similarity property is lacking: no solutions of the form $g(t) \cdot f(x/w(t))$ with $g \sim t^{-\alpha}$ and $h \sim t^{\beta}$ for $t \gg 1$.

Since the telegraph equation (possibly with reaction terms) supposed to be relevant not only in heat conduction but also in various diffusion processes, the lack of self-similarity might be a bad sign for the adequacy of the model. Furthermore, in diffusion and heat theory various physical quantities—like fluxes—have to be continuous; therefore, the solutions of this equation cannot be 'too bad'.

According to Gurtin and Pipkin [1–3], the most general form of the flux in linear heat conduction and diffusion is related to the flux q expressed in one space dimension via an integral over the history of the temperature gradient

$$q = -\int_{-\infty}^{t} Q(t-t')\frac{\partial T(x,t')}{\partial x}\,\mathrm{d}t', \tag{1}$$

where $Q(t-t')$ is a positive, decreasing relaxation function that tends to zero as $t-t' \to \infty$ and $T(x, t)$ is the temperature distribution.

There are two notable relaxation kernel functions: if $Q_1(s) = k\delta(s)$, where $\delta(s)$ is a Dirac delta 'function', then we will get back the original Fourier law.

If we define the kernel as $Q_2(s) = \frac{k}{\tau}\,\mathrm{e}^{-s/\tau}$, where $s = t - t'$ and k is the constant of the effective thermal conductivity, we get back to the well-known Cattaneo [4] heat conduction law. The energy conservation law

$$\frac{\partial q}{\partial x} = -\gamma \frac{\partial T(x,t)}{\partial t}, \tag{2}$$

where γ is the heat capacity, gives the heat equation with Q_1 and the telegraph equation with Q_2:

$$\frac{\partial^2 T(x,t)}{\partial t^2} + \frac{1}{\tau}\frac{\partial T(x,t)}{\partial t} = c^2 \frac{\partial^2 T(x,t)}{\partial x^2}, \tag{3}$$

where $c = \sqrt{k/\tau\gamma}$ is the propagation velocity of the transmitted heat wave. The flux q satisfies the same equation.

The thermal diffusivity $\kappa = k/\gamma$ can be defined as the ratio of the effective thermal conductivity k and the heat capacity γ. The key quantity in this equation is the physically well-defined relaxation time τ. This positive number has a distinct physical meaning, namely the time constant of an exponential return of a system to the steady state (whether of thermodynamic equilibrium or not) after a disturbance.

The relaxation time is—however somehow connected to the mean-time collision time t_c of the particles—responsible for the dissipative process, oftentimes erroneously identified with it. In principle, they are different since τ is (conceptually and many times in practice) a macroscopic time, although in some instances it may correspond just to a few t_c. Unfortunately, no general formula linking t_c and τ exists; their relationship depends (in each case) on the system under consideration.

The telegraph equation (3) can be derived in various transport systems, see [2, 3, 5–9].

In the present paper we introduce a new kernel which interpolates the Dirac delta and the exponential kernel having the main properties of both. $Q(s) = 1/s^l$ is such a function which is singular at the origin and has a short range of decay for $l > 1$. Let us consider the following relaxation kernel:

$$Q(t-t') = \frac{k\tau^l}{(t-t'+\omega)^l}, \tag{4}$$

where k is the effective thermal conductivity, τ is relaxation time and $l > 1$ is a parameter, and $-t' + \omega$ is just a time shift which is necessary to regularize the expression.

Using the general form of heat flux (1),

$$q = -\int_{-\infty}^{t} \frac{k\tau^l}{(t-t'+\omega)^l}\frac{\partial T(x,t)}{\partial x}\,\mathrm{d}t'. \tag{5}$$

Derivating (5) with respect to t,

$$\frac{\partial q}{\partial t} = -k\left(\frac{\tau}{\omega}\right)^l \frac{\partial T(x,t)}{\partial x} + l\int_{-\infty}^{t} \frac{1}{t-t'+\omega}\frac{k\tau^l}{(t-t'+\omega)^l}\frac{\partial T(x,t')}{\partial x}\,\mathrm{d}t'. \tag{6}$$

A *formal* application of the integral mean theorem to the second term on the right-hand side and the definition of q leads to a new phenomenological law:

$$\frac{\partial q}{\partial t} = -k\left(\frac{\tau}{\omega}\right)^l \frac{\partial T(x,t)}{\partial x} - \frac{l}{t-t''+\omega}q. \tag{7}$$

The additional energy conservation law is valid, and from (2) and (7) this equation can be obtained:

$$\frac{\gamma}{k}\left(\frac{\omega}{\tau}\right)^l \frac{\partial^2 T(x,t)}{\partial t^2} + \frac{\gamma}{k}\left(\frac{\omega}{\tau}\right)^l \frac{l}{t-t''+\omega}\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2}. \tag{8}$$

For a better transparency, let us call $\epsilon = \frac{\gamma}{k}\left(\frac{\omega}{\tau}\right)^l$ and $a = \frac{\gamma}{k}\left(\frac{\omega}{\tau}\right)^l \cdot l$. The physical meaning of ϵ is still the thermal diffusivity multiplied by a scaling constant which is the renormalized relaxation time (the ratio of an ordinary time shift ω and a well-defined relaxation time τ). The exponential l is a real number which describes the non-locality in time which can be called memory effects of the heat conduction phenomena. Larger l means shorter memory. The physical meaning of a is approximately the thermal diffusivity multiplied by another time-scaling factor. In the following one can see that the role of a, ϵ or l will be crucial in the structure of the solutions. A new time variable is introduced as $t = t - t'' + \omega$. Now the telegraph-type equation reads

$$\epsilon \frac{\partial^2 T(x,t)}{\partial t^2} + \frac{a}{t}\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2}. \tag{9}$$

Note that the a/t factor appearing in front of the first time derivative makes the equation time reversible, which cannot be true for diffusion or heat propagation processes, at the same time when the a/t factor makes the equation irregular at the origin. To avoid these problems the singularity was shifted to a negative time $a/(t+\tau)$ where τ still can be any kind of relaxation time with well-founded physical interpretation. Physically it is clear that if a process has a well-defined time-scale, then the reverse process cannot run back in time more than the physically relevant time. Now, in this sense, for the positive time ($t > 0$) the equation can be used to describe diffusion-like processes. Below we see that no self-similar solution of (9) exists for $at^{-\gamma}$ with $\gamma \neq 1$.

In the classical telegraph equation (3), the diffusion in a sense takes over after some time, although not uniformly. In the presented case, the $1/t$ factor in front of diffusion inhibits the diffusive effect in long times in comparison with the hyperbolic effect: the existence of the self-similarity is exactly a manifestation of the fact that propagation and diffusion are on the same footing, so they will coexist at all times.

Wave properties (like dispersion phenomena) can be investigated if equation (9) is considered as a nonlinear wave equation.

Inserting the standard plain wave approximation $T(x,t) = \mathrm{e}^{\mathrm{i}(\tilde{k}x + \tilde{\omega}t)}$ into (9), the dispersion relation and the attenuation distance can be obtained. These are the followings:

$$v_p = \frac{\tilde{\omega}}{\mathrm{Re}(\tilde{k})} = \sqrt{\frac{2}{\epsilon}}\tilde{\omega}\left(1 + \sqrt{1 + \left(\frac{l}{t}\right)^2}\right)^{-1/2} \tag{10}$$

$$\tilde{\alpha} = \frac{1}{\mathrm{Im}(\tilde{k})} = \frac{2t}{\epsilon l}\frac{1}{v_p}. \tag{11}$$

Equation (9) is time dependent; hence both the dispersion relation and the attenuation distance have a time dependence. Note that v_p has a very weak time dependence, basically only till $t \leqslant l$. The properties of the attenuation distance is even more interesting; it is divergent in time and has a $1/\tilde{\omega}$ angular frequency. However, if the angular frequency and the time go infinite with the same speed, then the attenuation distance has a strong decay. This is like the skin effect when high frequency electrons can only propagate on the surface of a metal. The solution of (9) is of the form

$$T(x,t) = t^{-\alpha} f\left(\frac{x}{t^\beta}\right) := t^{-\alpha} f(\eta). \tag{12}$$

The similarity exponents α and β are of primary physical importance since α represents the rate of decay of the magnitude $T(x,t)$, while β is the rate of spread (or contraction if $\beta < 0$)

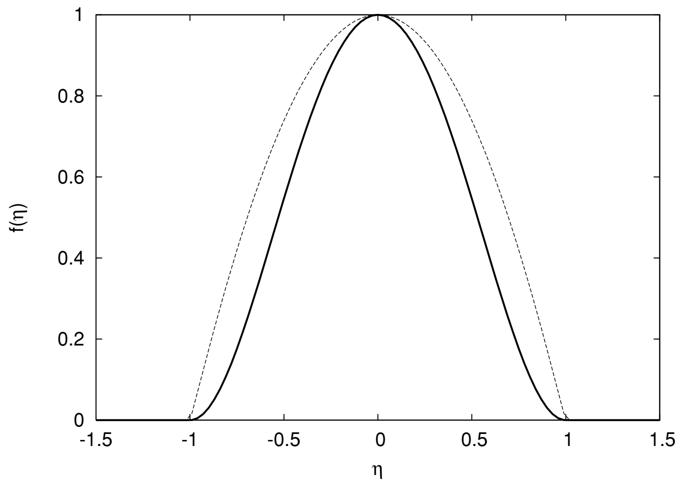

Figure 1. Equation (19): thick solid line is for $l = 6.2$ and the thin dashed line is for $l = 4.1$.

of the space distribution as time goes on. Substituting this into (9),

$$f''(\eta)t^{-\alpha-2}[\epsilon\beta^2\eta^2] + f'(\eta)\eta t^{-\alpha-2}[\epsilon\alpha\beta - \epsilon\beta(-\alpha-\beta-1) - \beta a] + f(\eta)t^{-\alpha-2}[-\epsilon\alpha(-\alpha-1) - a\alpha] = f''(\eta)t^{-\alpha-2\beta}, \tag{13}$$

where prime denotes differentiation with respect to η.

One can see that this is an ordinary differential equation (ODE) if and only if $\alpha+2 = \alpha+2\beta$ (*the universality relation*). So it has to be

$$\beta = 1, \tag{14}$$

while α can be any number. The corresponding ODE is

$$f''(\eta)[\epsilon\eta^2 - 1] + f'(\eta)\eta(2\epsilon\alpha + 2\epsilon - a) + f(\eta)\alpha(\epsilon\alpha + \epsilon - a) = 0. \tag{15}$$

In pure heat conduction–diffusion processes—no sources or sinks—the heat mass is conserved: the integral of $T(x, t)$ with respect to x does not depend on time t. For $T(x, t)$ this means

$$\int T(x,t)\,\mathrm{d}x = t^{-\alpha}\int f\left(\frac{x}{t}\right)\mathrm{d}x = t^{-\alpha+1}\int f(\eta)\,\mathrm{d}\eta = \text{const} \tag{16}$$

if and only if $\alpha = 1$. This case is investigated only. Clearly, (15) can be written as

$$(\varepsilon f\eta^2 - f)'' = a(\eta f)', \tag{17}$$

which after integration and supposing $f(\eta_0) = 0$ for some η_0 gives

$$\frac{\mathrm{d}f}{f} = \frac{a\eta\,\mathrm{d}\eta}{\varepsilon\eta^2 - 1}. \tag{18}$$

From this equation two qualitatively different solutions are obtained. The first solution is globally bounded and positive in the domain $\{(x, t) : 1 - \varepsilon\eta^2 > 0\}$ and has the form

$$f = (1 - \varepsilon\eta^2)_+^{\frac{a}{2\varepsilon}-1}, \tag{19}$$

where $(f)_+ = \max(f, 0)$. See figure 1.

The corresponding self-similar solution is

$$T(x,t) = \frac{1}{t}\left(1 - \varepsilon\frac{x^2}{t^2}\right)_+^{\frac{a}{2\varepsilon}-1}. \tag{20}$$

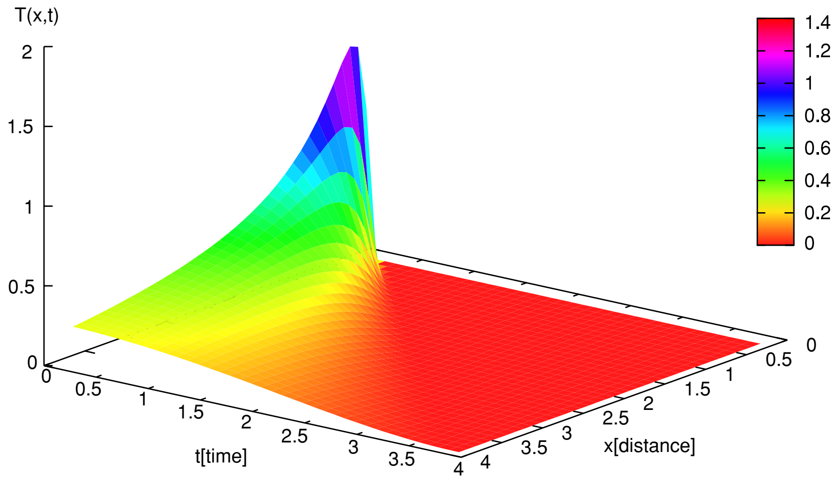

Figure 2. Solution (20) for the parameter $l = 6.2$.
(This figure is in colour only in the electronic version)

This solution is positive in the cone $t^2 > \varepsilon x^2$ and is zero outside it, see figure 2. Note that only the $x > 0$ and $t > 0$ quarter of the plane is presented because of its physical relevance.

On the (x, t) plane there are two fronts $x(t) = \pm\frac{t}{\sqrt{\varepsilon}}$ separating these domains. Because the function $T(x, t)$ does not always have continuous derivatives entering to (9), we have to make clear what we mean by 'solution'. With the physical background in mind, we ask the continuity of T_t, T_x, q_t and q_x so that in (2) and (3) all functions were continuous.

In our case this means that

$$\frac{a}{2\varepsilon} - 1 = \frac{l-2}{2} \tag{21}$$

has to be greater than 1, i.e. $a/\varepsilon = l > 4$, which we will suppose further on. In the case of $4 \leqslant l \leqslant 6$ the second derivatives entering into (9) are not continuous. If we multiply (9) by a test function $\varphi(x, t)$ and integrate it by parts (which is possible because T_x and T_t are continuous and we can see that (20) is a well-defined weak solution to (9)).

If $l > 6$, the solution is classical. In figure 1, we compare the solutions with $l = 4.1$ and $l = 6.2$. The thick solid line represents the solution for $l = 6.2$ and the thin dashed line however shows the solution for $l = 4.1$.

Remark 1. Solution (20) is of *source type,* i.e. $\lim_{t\to 0} T(x, t) = K\delta(x)$, where δ is the Dirac measure, $K > 0$. One can calculate the second initial condition $\lim_{t\to 0} T_t(x, t)$ too.

Remark 2. One can write (20) in the form of the*product* of two traveling waves propagating in opposite direction (divided by a time factor):

$$T(x, t) = \frac{1}{t^{l-1}}(t - \sqrt{\varepsilon}x)_+^{\frac{l}{2}-1}(t + \sqrt{\varepsilon}x)_+^{\frac{l}{2}-1}, \tag{22}$$

which is a new type of purely hyperbolic wave; the typical solution of the wave equation is the *sum* of two such waves: $g(x - ct) + g(x + ct)$.

It is known that another possible answer to contradiction connected with the infinite speed of propagation is the nonlinear Fourier law ($\tau = 0, k = k_0 T^{m-1}$ in (2)), which leads to a nonlinear heat equation:

$$T_t = (T^m)_{xx}, \qquad m > 1. \tag{23}$$

Zeldovich and Kompaneets [11] have found the fundamental solution T_1 of this equation which we write in the following form:

$$T_1^{m-1} = t^{-\alpha(m-1)}(A^2 - B^2x^2t^{-2\beta})_+ = \frac{1}{t}(At^\beta - Bx)_+(At^\beta + Bx)_+, \tag{24}$$

where A is constant and

$$\alpha = \beta = \frac{1}{m+1}, \qquad B^2 = \frac{m-1}{2m(m+1)}. \tag{25}$$

One can see that this solution has bounded support in x for any $t > 0$, which is a hyperbolic property. Using the comparison principle for such equations one can show this finite speed property for any initial condition having compact support. However, the fronts are not straight lines: $x(t) = \pm\frac{A}{B}t^\beta$, $\beta < 1$, so the speed of propagation $\dot{x}(t)$ goes to zero if t goes to infinity. One can also see that T_1 is of source type: $T_1(x, 0) = K_1\delta(x)$.

The most intrinsic property of T_1 is that it plays the role of *intermediate asymptotic*: any solution of (23) corresponding to the initial data $t(x, 0)$ with $\int t(x, 0)\, dx = K_1$ converges to T_1 as $t \to \infty$. This was conjectured earlier but was shown only in 1973 by Kamin, see [10].

It would be important and interesting to understand whether or not our special solution $T(x, t)$ had this attractive property. If 'yes', in what sense: we recall that there is a second initial condition too.

In summary, we introduced a new phenomenological law for heat flux which in some sense 'interpolates' between Fourier and Cattaneo laws. The consequence of it is a non-autonomous model, a telegraph-type partial differential equation. It already has, unlike the classical telegraph equation, self-similar solutions, the presence of which is desirable in the theory of heat propagation free from sources and absorbers.

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References

[1] Gurtin M E and Pipkin A C 1968 *Arch. Ration. Mech. Anal.* **31** 113
[2] Joseph D D and Preziosi L 1989 *Rev. Mod. Phys.* **61** 41
[3] Joseph D D and Preziosi L 1990 *Rev. Mod. Phys.* **62** 375
[4] Cattaneo C 1948 Sulla conduzione del calore *Atti. Sem Mat. Fis. Univ. Modena* **3** 83
[5] Goldstein S 1951 *Q. J. Mech. Appl. Math.* **4** 129
[6] Okubo A Application of the telegraph equation to oceanic diffusion: another mathematical model *Technical Report 69* N00014-67-A-0163-0006 NR 083-016, Chesapeake Bay Institute, John Hopkins University (http://dspace.udel.edu:8080/dspace/handle/19716/1439)
[7] Müller I and Ruggeri T 1998 *Rational Extended Thermodynamics* (Berlin: Springer)
[8] Othmer H G, Dunbar S R and Alt W 1988 *J. Math. Biol.* **26** 263
[9] Gilding B H and Kersner R 2004 *Travelling Waves in Nonlinear Diffusion–Convection Reactions* (*Progress in Nonlinear Differential Equations and Their Applications*) (Basel: Birkhäuser)
[10] Kamin Sh 1973 *Isr. J. Math.* **14** 76
[11] Zeldovich Y B and Kompaneets A S 1950 *On the theory of propagation of heat with the conductivity depending upon the temperature* (Collection Dedicated to the 70th Birthday of AF Joffe) *Izdat. Akad. Nauk SSSR* p 61–71 (Russian)